Game Theory - Final Examination

Date: November 05, 2018; Total marks: **30**; Duration: 2 PM to 5 PM

Answer all questions clearly. Please avoid unnecessary discussions.

Unless specified explicitly, consider only **pure** actions in any game.

1. Construct a two-player extensive form game with two decision vertices (but possibly infinite payoff vertices), one for each player, such that it has an infinite number of Nash equilibria but it has no subgame perfect equilibrium. **(3 marks)**

2. Consider the extensive form game shown in Figure 1 with two players: Child and Parent. The Child has two actions in her decision vertex: Behave (B) or Misbehave (M). If the Child behaves, the game ends. If the Child misbehaves, we go to the decision vertex of the Parent, who either Punishes (P) or does not punish (N). The payoffs are as shown in Figure 1 with the first entry representing the payoff of the Child and the second entry representing the payoff of the Parent.

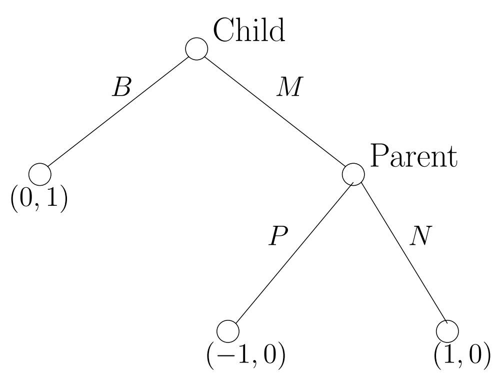

Figure 1: Extensive form game with perfect information

   (a) Find all subgame perfect equilibria (including those where players choose mixed actions at their decision vertices) of this game. **(3 marks)**

   (b) Suppose when the child misbehaves, there is a probability p with which he may be detected - i.e., after the Child plays M, the game will end with payoff $(1, 0)$ (payoff from action profile (M, N)) with probability $1-p$ and the game will continue with the Parent choosing actions P or N as before with probability p.

      i. Draw the game tree representation of this modified game. **(2 marks)**

      ii. Suppose $p = 0.8$. Describe all subgame perfect equilibria of this game where the Child plays M (the Parent may choose mixed actions). **(3 marks)**

3. Let $\Gamma := (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ be a finite strategic form game.

   (a) Give a clear definition of a correlated equilibrium of Γ. **(1 mark)**

   (b) Suppose strategy $s_i \in S_i$ is strictly dominated for Player i. Suppose p is a correlated equilibrium. Is $\sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i}) > 0$? Explain your answer. **(2 marks)**

4. Consider the two extensive form games of imperfect information in Figure 2.

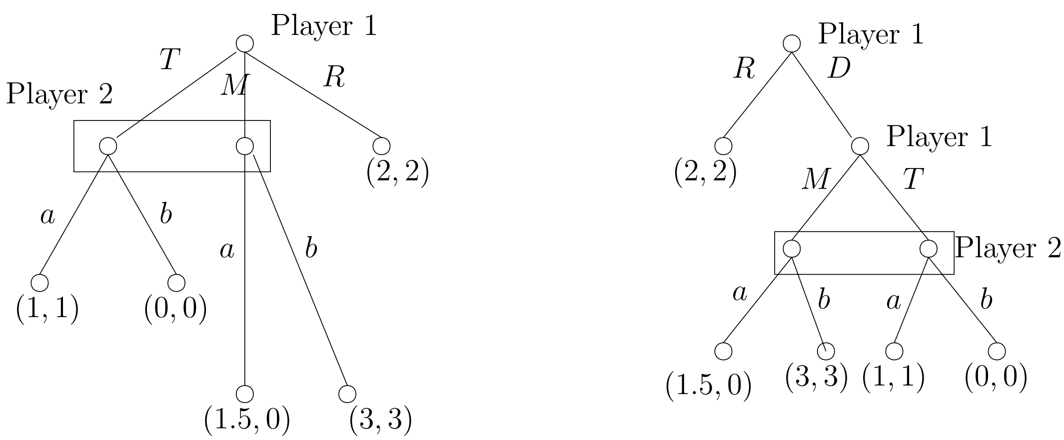

Figure 2: Extensive form game with imperfect information

   (a) Describe a perfect Bayesian equilibrium of the game in left in Figure 2, where Player 1 plays R. Is your perfect Bayesian equilibrium a sequential equilibrium - give a clean argument. **(4+2=6 marks)**

   (b) Describe all perfect Bayesian equilibria of the game in right in Figure 2. **(4 marks)**

5. A seller has an object to sell for which he has zero value (i.e., keeping the object gives him zero utility). There is a buyer whose value v for the object is drawn from $[0, 1]$ using uniform distribution. The seller does not observe v but knows its distribution. The game proceeds as follows. First, the buyer realizes her value privately. Next, the seller posts a price $p \in [0, 1]$. Finally, the buyer either accepts or rejects the price. If the buyer rejects the price both get a payoff of zero. If the buyer accepts the price p, the seller's payoff is p and buyer's payoff is $v - p$.

   (a) Describe this as a sequential game with imperfect information by clearly specifying the information sets of the buyer and the seller. **(2 marks)**

   (b) Compute a perfect Bayesian equilibrium of this game. **(4 marks)**

Answers.

1. In the first decision vertex Player 1 has two actions: Y and N. Action Y ends the game and players get $(1, 1)$. If Player 1 chooses N, then the next decision vertex is Player 2, where the set of actions is $(0, 1)$. If Player 2 takes action $x \in (0, 1)$, then the resulting payoffs are (x, x).

   Notice that Player 2 has no optimal action in her decision vertex. So, this game has no subgame perfect equilibrium. However, for every $x \in (0, 1)$, the strategy profile (Y, x) is a Nash equilibrium - if Player 1 chooses Y, then Player 2's action does not matter and whatever Player 2 chooses, Y is a dominant strategy for Player 1.

2. The Parent is indifferent between P and N. So, we need to look at all possibilities.

   Parent randomizes $\alpha P + (1-\alpha)N$. If $\alpha > \frac{1}{2}$, then Child's optimal action is B. If $\alpha < \frac{1}{2}$, then child's optimal action is M. If $\alpha = \frac{1}{2}$, the Child is indifferent and can choose $\beta B + (1 - \beta)M$, where $\beta \in [0, 1]$.

   So, the set of subgame perfect equilibria can be put in three classes: (a) $(B, \alpha P + (1 - \alpha)N)$ with $\alpha > \frac{1}{2}$; (b) $(M, \alpha P + (1 - \alpha)N)$ with $\alpha < \frac{1}{2}$; (c) $(\beta B + (1 - \beta)M, \frac{1}{2}P + \frac{1}{2}N)$ with $\beta \in [0, 1]$

   When there is probability of detection, the game tree is shown in Figure 3.

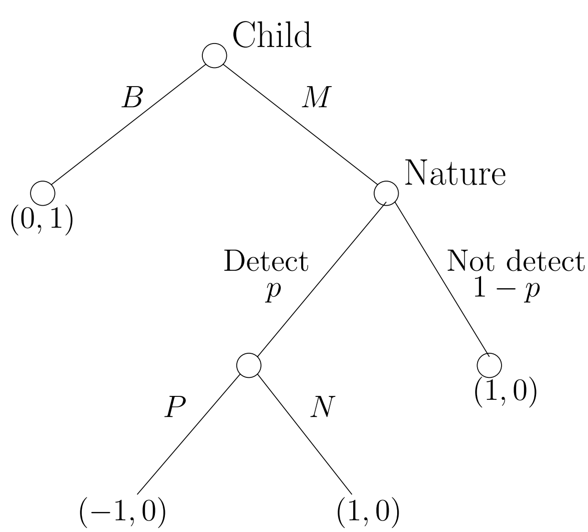

Figure 3: Extensive form game with perfect information with Nature

   Suppose Parent plays $\alpha P + (1 - \alpha)N$ - this is optimal for Parent as she is indifferent. Then, by playing M gives Child a payoff equal to: $(1 - p) + p(1 - 2\alpha)$. For M to be an equilibrium action, we need this payoff to be more than B payoff of zero: $1-p+p-2\alpha p \geq 0$ or $\alpha \leq \frac{1}{2p} = \frac{5}{8}$. Hence, the set of subgame perfect equilibria are: $(M, \alpha P + (1-\alpha)N)$, where $\alpha \in [0, \frac{5}{8}]$.

3. A correlated strategy $p \in \Delta \prod_{i=1}^{n} S_i$ is a correlated equilibrium if for all $i \in N$ and for all $s_i, s_i' \in S_i$,

$$\sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) p(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) p(s_i, s_{-i}).$$

   If s_i is strictly dominated by a strategy $\bar{s}_i$, then

$$\sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) p(s_i, s_{-i}) < \sum_{s_{-i} \in S_{-i}} u_i(\bar{s}_i, s_{-i}) p(s_i, s_{-i}),$$

   if $p(s_i, s_{-i}) > 0$ for some s_{-i}. Hence, s_i cannot be in the support of p, i.e., $p(s_i, s_{-i}) = 0$ for all s_{-i}.

4. If Player 1 plays R, Player 2 can form any belief. Let it be α_M on right vertex (decision vertex following M) and $(1-\alpha_M)$ on left vertex. Then, her payoff from playing a is: $1-\alpha_M$ and playing b is: $3\alpha_M$. Hence, a is preferred if $\alpha_M \leq \frac{1}{4}$ and b is preferred if $\alpha_M \geq \frac{1}{4}$. Notice that if Player 2 plays a, then Player 1 prefers R. So, one (class of) perfect Bayesian equilibrium (assessment) is: $(R, a, \alpha_M \leq \frac{1}{4})$.

   If $\alpha_M < 1$, then we can choose the perturbed strategies of Player 1 as $\epsilon T + \epsilon' M + (1 - \epsilon - \epsilon')R$) such that $\frac{\epsilon'}{\epsilon} = \frac{\alpha_M}{1-\alpha_M}$ and it will satisfy the limiting property of beliefs.

   In the modified game, Player 1 prefers M over T in his second information set. We consider two types of equilibria (in pure actions):

   (a) Player 1 plays D in his first information set. For this to be equilibrium, we see that Player 2 must put probability 1 on the left decision vertex of her information set (since (D, M) is played). This means that her optimal action is b. So, this perfect Bayesian equilibrium is: $(D, M, b, \alpha_M = 1)$, where α_M denotes the probability of the node reached from action M.

   (b) Player 1 plays R in his first information set. For this to be equilibrium, we see that Player 2 can put any belief. But she needs to choose a belief such that Player 1 plays R in equilibrium. From earlier calculation, we know that if $\alpha_M \leq \frac{1}{4}$ she plays a and if $\alpha_M \geq \frac{1}{4}$ she plays b. But Player 2 playing b will imply that Player 1 prefers playing D over R in his first information set. Hence, to sustain R in the first information set, Player 2 needs to play a. This is possible if $\alpha_M \leq \frac{1}{4}$. So, this perfect equilibrium is: $(R, M, a, \alpha_M \leq \frac{1}{4})$. (**Note**. This cannot be a sequential equilibrium since any perturbation will imply that $\alpha_M = 1$.)

5. The game tree is shown in Figure 4.

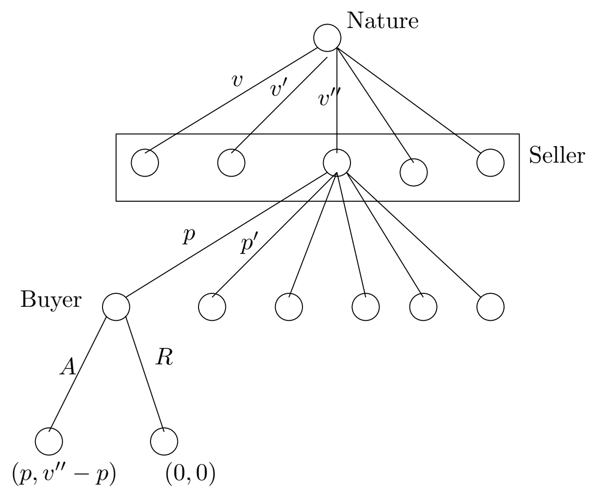

Figure 4: Extensive form game with imperfect information

Buyer accepts any offer with non-negative payoff. Seller's belief is Nature probabilities: probability that he is at a decision vertex with value less than x is x (uniform distribution). Hence, his expected payoff by offering price p is: p times the probability that the price will be accepted. Probability that the price p will be accepted is the probability that value is more than the price (this is when a buyer has non-negative payoff). This gives an expected payoff of $p(1-p)$ for seller. This is maximized at $p = \frac{1}{2}$. Hence, the perfect Bayesian equilibrium is: $(p = \frac{1}{2}, A$ if $v \geq p$ and R if $v < p;$ belief is uniform).