Game Theory - Midterm Examination 1

Date: September 1, 2018

Total marks: **24**

Duration: 10:00 AM to Noon

Note: Answer all questions clearly using pen. Please avoid unnecessary discussions. In all the questions, unless mentioned explicitly, **do not** consider mixed strategies (mixed extension) of a game.

1. There are two agents: $N = \{1, 2\}$. They need to travel from point a to b. Consider the left (road) network in Figure 1, where there are two possible paths: (1) Path U: $a \to x \to b$ and (2) Path D: $a \to y \to b$. The right network in Figure 1 has an additional path; Path M: $a \to x \to y \to b$. The costs of travel in these paths is sum of costs of travel in each of the edges on these paths. In particular, cost of each edge is $c(\alpha)$, where $\alpha \in \{0, \frac{1}{2}, 1\}$ is the fraction of agents who travel on these edges. The respective costs of edges are shown in Figure 1. The value of travel from a to b (independent of the path) is 2 units for both the agents. So, the payoff from traveling from a to b to both the agents is the value of travel minus the cost of travel.

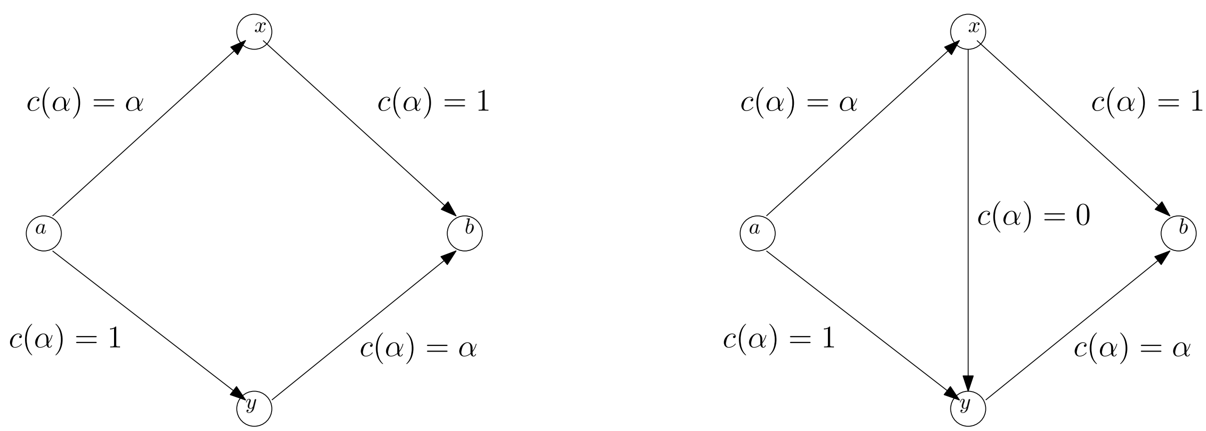

Figure 1: Two possible networks

(a) Describe the underlying strategic form games for both the networks. **(2 marks)**

Answer. For both the networks, $N = \{1, 2\}$ are the set of agents. The strategies in the left network to both the agents are $\{U, D\}$, where U and D refers to the paths as defined before. The strategies in the right network to both the agents

	U	D
U	$(0,0)$	$(0.5,0.5)$
D	$(0.5,0.5)$	$(0,0)$

Table 1: Strategic form game of left network

	U	D	M
U	$(0,0)$	$(0.5,0.5)$	$(0,0.5)$
D	$(0.5,0.5)$	$(0,0)$	$(0,0.5)$
M	$(0.5,0)$	$(0.5,0)$	$(0,0)$

Table 2: Strategic form game of right network

are $\{U, D, M\}$, where M is the new path defined as before. The payoffs are as shown in Table 1 and Table 2.

(b) What are the Nash equilibria in each of these games? For every agent, compare her worst payoff among all equilibria of the game in the left network and the game in the right network. **(3 marks)**

Answer. In the game of Table 1, two Nash equilibria: (U, D) and (D, U) - both give a payoff of 0.5 to each agent. In game of Table 2, everything except (U, U) and (D, D) are Nash equiliria. In particular, (M, M) is a Nash equilibrium giving the worst possible payoff of zero to each agent.

(c) What are the Nash equilibria in the mixed extension of the game in the right network? **(3 marks)**

Answer. Since the game is symmetric, both the agents have the same best response maps. It is easy to see that M is always in the best response map (a weakly dominant strategy). Now, U is in the best response map if the opponent plays U with zero probability and D is in the best response map if the opponent plays D with zero probability. This best response map (same for both the players) is shown in Figure 2.

We can use this to compute all mixed strategy Nash equilibria - pure strategies have been already shown. We consider all cases for mixed strategy Nash equilibria.

- Equilibria where one player mixes *only* U and D. The best response map shows that this can only happen if the other player plays *pure* M. If a players mixes U and D, the best response is indeed M. Hence, one player plays M

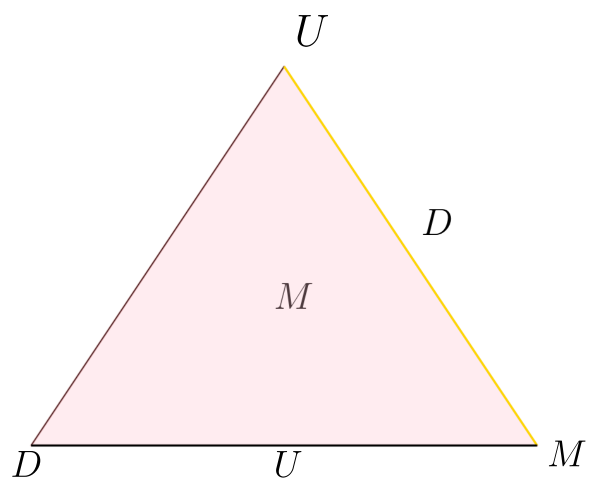

Figure 2: Best response map for game in right network

and the other player mixes U and D (in any probability) is a mixed strategy Nash equilibrium.

- Equilibria where one player mixes *only* U and M. This happens if the other player mixes D and M. Conversely, if a player mixes U and M, the other player's best response includes both D and M. Hence, one player mixing U and M (in any probability) and the other player mixing D and M (in any probability) is a mixed strategy Nash equilibrium.
- Equilibria where one player mixes *only* D and M. This is the same as the previous case.
- Equilibria where one player mixes *all* U, M, D. This can only happen if the other player plays M. Further, if a player mixes all U, M, D, then the other player has M as the unique best response. So, one player playing M and the other player mixing U, M, D in any probability is a mixed strategy Nash equilirium.

(d) Consider the following strategy profiles (a strategy profile is a pair of paths) for the game in the right network of Figure 1:

$$\mathcal{C} := \{(U,U), (U,D), (U,M), (D,U), (M,U)\}.$$

Describe all the correlated equilibria whose support is exactly $\mathcal{C}$. **(3 marks)**

Answer. There are no correlated equilibria whose support is exactly $\mathcal{C}$. To see this, since the support is exactly $\mathcal{C}$, let $p(U,U) > 0$ be the probability of (U,U) being recommended. When an agent observes U, his conditional belief that others received U, D, M are respectively denoted by π_U, π_D, π_M. His expected payoff from obeying is $0.5\pi_D$. By deviating to M, the agent gets $0.5(\pi_U + \pi_D) > 0.5\pi_D$ since $\pi_U > 0$. So, no such correlated equilibrium can exist.

(e) What is the largest set of rationalizable strategies for the agents in the game in the right network of Figure 1? Explain your answer. **(3 marks)**

Answer. From the best response maps of Figure 2, it is clear that every strategy of an agent is a best response with respect to some belief over others strategy. Hence, largest set of rationalizable strategies for each player is $\{U, D, M\}$.

2. Suppose $\Gamma = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ is a finite strategic form game. Show that for each Player $i \in N$, her payoff in any correlated equilibrium of Γ is greater than or equal to her max-min value in the mixed extension of Γ (i.e., $\max_{\sigma_i\in\Delta S_i} \min_{\sigma_{-i}\in\Delta S_{-i}} U_i(\sigma_i, \sigma_{-i})$).

Hint. You may use the fact

$$\max_{\sigma_i\in\Delta S_i} \min_{\sigma_{-i}\in\Delta S_{-i}} U_i(\sigma_i, \sigma_{-i}) = \max_{\sigma_i\in\Delta S_i} \min_{s_{-i}\in S_{-i}} U_i(\sigma_i, s_{-i}).$$

(5 marks)

Answer. Let $p \in \Delta \prod_{i\in N} S_i$ be a correlated equilibrium. Notations: we will write for every $i \in N$, for every $s_i \in S_i$,

$$\pi_i(s_i) := \sum_{s_-\in S_{-i}} p(s_i, s_{-i}),$$

and

$$p(s_i|s_{-i}) := \frac{p(s_i, s_{-i})}{\pi_i(s_i)} \ \forall\ s_{-i}.$$

Now, fix a player $i \in N$. By definition of the correlated equilibrium, for every $s_i, s_i' \in S_i$, we have

$$\sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i})u_i(s_i', s_{-i})$$

Dividing throughout by $\pi_i(s_i)$, we get for every $s_i, s_i' \in S_i$, we have

$$\sum_{s_{-i}\in S_{-i}} p(s_i|s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} p(s_i|s_{-i})u_i(s_i', s_{-i})$$

Now, pick any mixed strategy σ_i and let $supp(\sigma_i)$ be the set of pure strategies in S_i such that $\sigma_i(s_i) > 0$. The above inequality holds for all $s_i' \in supp(\sigma_i)$ for a given s_i. Hence, we can write

$$\begin{aligned}\sum_{s_i' \in supp(\sigma_i)} \sigma_i(s_i') \sum_{s_{-i} \in S_{-i}} p(s_i|s_{-i}) u_i(s_i, s_{-i}) &\geq \sum_{s_{-i} \in S_{-i}} p(s_i|s_{-i}) \sum_{s_i' \in supp(\sigma_i)} \sigma_i(s_i') u_i(s_i', s_{-i}) \\ &= \sum_{s_{-i} \in S_{-i}} p(s_i|s_{-i}) U_i(\sigma_i, s_{-i}).\end{aligned}$$

Using $\sum_{s_i' \in supp(\sigma_i)} \sigma_i(s_i') = 1$, this gives us that for all s_i and for all σ_i, we have

$$\sum_{s_{-i} \in S_{-i}} p(s_i|s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_i|s_{-i}) U_i(\sigma_i, s_{-i}).$$

Using the fact that $\sum_{s_{-i} \in S_{-i}} p(s_i|s_{-i}) = 1$ and $U_i(\sigma_i, s_{-i}) \geq \min_{s_{-i} \in S_{-i}} U_i(\sigma_i, s_{-i})$, the above inequality simplifies to

$$\sum_{s_{-i} \in S_{-i}} p(s_i|s_{-i}) u_i(s_i, s_{-i}) \geq \min_{s_{-i} \in S_{-i}} U_i(\sigma_i, s_{-i}).$$

This further implies that for every $i \in N$ and every $s_i \in S_i$, we have

$$\sum_{s_{-i} \in S_{-i}} p(s_i|s_{-i}) u_i(s_i, s_{-i}) \geq \max_{\sigma_i \in \Delta S_i} \min_{s_{-i} \in S_{-i}} U_i(\sigma_i, s_{-i}) = \max_{\sigma_i \in \Delta S_i} \min_{\sigma_{-i} \in \Delta S_{-i}} U_i(\sigma_i, \sigma_{-i}),$$

where we used our hint for the equality. Then, the payoff of a player i in this correlated equilibrium is given by:

$$\begin{aligned}\sum_{s \in \prod_{i \in N} S_i} p(s) u_i(s) &= \sum_{s_i \in S_i} \pi_i(s_i) \sum_{s_{-i} \in S_{-i}} p(s_i|s_{-i}) u_i(s_i, s_{-i}) \\ &\geq \sum_{s_i \in S_i} \pi_i(s_i) \max_{\sigma_i \in \Delta S_i} \min_{\sigma_{-i} \in \Delta S_{-i}} U_i(\sigma_i, \sigma_{-i}) \\ &= \max_{\sigma_i \in \Delta S_i} \min_{\sigma_{-i} \in \Delta S_{-i}} U_i(\sigma_i, \sigma_{-i}),\end{aligned}$$

where the last equality follows from the fact that $\sum_{s_i \in S} \pi_i(s_i) = 1$.

3. Suppose $\Gamma = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ is finite strategic form game. A Nash equilibrium s^* of Γ is strict if $u_i(s^*) > u_i(s_i, s_{-i}^*)$ for all $i \in N$ and for all $s_i \in S_i \setminus \{s_i^*\}$. Suppose we iteratively eliminate strictly dominated strategies from Γ (a strategy s_i of Player i is strictly dominated if there exists another strategy s_i' such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for all s_{-i}) and this leads to a *unique* strategy profile s^*. Show that s^* is a strict Nash equilibrium, and it is the unique Nash equilibrium of Γ. **(5 marks)**

Answer. Suppose s^* is not a strict Nash equilibrium. Then, there is some player i and s_i such that $u_i(s_i, s^*_{-i}) = \max_{s'_i \in S_i} u_i(s'_i, s^*_{-i}) \geq u_i(s^*_i, s^*_{-i})$. Since s^* is the unique surviving strategy profile, there is an iteration where s_i is eliminated and the entire strategy profile s^* is present in that iteration. Since s_i is eliminated, there is a strategy s''_i such that $u_i(s''_i, s^*_{-i}) > u_i(s_i, s^*_{-i})$. But this contadicts the fact s_i is a best response to s^*_{-i}. Hence, s^* is a strict Nash equilibrium.

Now, pick a strict Nash equilibrium s^*. By definition, for every $i \in N$ and for every $s_i \neq s^*_i$, we have $u_i(s^*_i, s^*_{-i}) > u_i(s_i, s^*_{-i})$. Hence, s^*_i is a unique best response to s^*_{-i}. Hence, in no iteration of iterated elimination, s^*_i is strictly dominated. This implies that s^* survives iterated elimination of strictly dominated strategies. Since the set of strategies that survive iterated elimination of strictly dominated strategies is unique, there is a unique strict Nash equilibrium.