Game Theory - Midterm Examination 2

Date: October 16, 2018

Total marks: **26**

Duration: 3 PM to 5 PM

Answer all questions clearly. Please avoid unnecessary discussions.

1. Two players play the game in Table 1.

	a	b
A	$(2,2)$	$(0,\theta)$
B	$(\theta,0)$	$(1,1)$

Table 1: A Bayesian game

Possible values of θ are $\{0,3\}$.

(a) Find all Nash equilibria (pure) of the game in Table 1 for each $\theta \in \{0,3\}$. **(2 marks)**

(b) Suppose the realized value of θ is **privately known** by Player 1 (i.e., Player 2 does not know its realized value). However, both players know the following probabilities of the realized values:

$$\text{Prob}(\theta = 0) = \alpha, \ \text{Prob}(\theta = 3) = 1 - \alpha,$$

where $\alpha \in [0,1]$.

Consider the following strategies of Player 1.

i. Player 1 plays B for both values of θ. **(2 marks)**

ii. Player 1 plays A for $\theta = 0$ but plays B for $\theta = 3$. **(4 marks)**

For each of the strategy of Player 1, find a strategy for Player 2 and range of values of α such that the resulting strategy profile is a Bayes-Nash equilibrium.

Is there a Bayes-Nash equilibrium of this game where Player 1 plays B for $\theta = 0$ and plays A for $\theta = 3$? Explain your answer. **(2 marks)**.

2. Consider a **contribution game** among two players. In this game, the two players $\{1, 2\}$ can choose contributions $x_1, x_2 \in [0, 1]$. Given a contribution vector (x_1, x_2), the payoff of any player $i \in \{1, 2\}$ is given by

$$u_i(x_1, x_2) = 2x_j - x_i \text{ where } j \neq i.$$

(a) Find all Nash equilibria of this stage game. **(1 mark)**

(b) What are the strictly enforceable action profiles (i.e., which action profiles generate payoff profiles that are strictly enforceable) of this game (use a figure)? **(1 mark)**

(c) Suppose this game is repeated infinitely with a discount factor $\delta \in (0, 1)$.

i. Clearly describe a Nash reversion strategy of the infinitely repeated game. **(2 marks)**

ii. Suppose $(x_1^*, x_2^*) \in [0, 1]^2$ is a strictly enforceable contribution profile. Find a sufficient condition on value of δ such that the Nash reversion strategy sustains contributions (x_1^*, x_2^*) on the equilibrium path of a subgame perfect equilibrium. **(4 marks)**

iii. Consider a carrot and stick strategy, where every deviation (either from normal state or from punishment state) is punished by 0 contribution for **one period**. Clearly describe such a carrot and stick strategy. **(2 marks)**

iv. Suppose $(x_1^*, x_2^*) \in [0, 1]^2$ is a strictly enforceable contribution profile. Find a sufficient condition on value of δ such that the above carrot and stick strategy sustains contributions (x_1^*, x_2^*) on the equilibrium path of a subgame perfect equilibrium. **(4 marks)**

v. Consider a strictly enforceable contribution profile (x_1^*, x_2^*) such that $x_1^* = x_2^*$. For what values of δ can this be sustained along equilibrium path of a subgame perfect equilibrium using (a) a carrot and stick strategy with one-period punishment and (b) using a Nash reversion strategy? **(2 marks)**

Solutions.

1. Possible values of $\theta \in \{0, 3\}$ with α as the probability of $\theta = 0$.

   (a) If $\theta = 0$, then (A, a) and (B, b) are two Nash equilibria. If $\theta = 3$, then (B, b) is a strictly dominant strategy equilibrium (unique Nash).

   (b) Suppose Player 1 plays B for both values of θ. This is a strictly dominant strategy for Player 1 if $\theta = 3$. If $\theta = 0$, then playing B is a best response for Player 1 if and only if Player 2 plays b. Hence, any Bayes-Nash equilibrium where Player 1 plays B for both values must have Player 2 playing b. But if Player 1 plays B, then b is a best response of Player 2 for both values of θ, and hence, **for all** α, the strategy profile where Player 1 plays B for both value of θ and Player 2 plays b is a Bayes-Nash equilibrium.

   Suppose Player 1 plays A for $\theta = 0$ and B for $\theta = 3$. As before Player 1 plays B when $\theta = 3$ since it is a strictly dominant strategy. But he plays A for $\theta = 0$ if and only if Player 2 plays a. Hence, any Bayes-Nash equilibrium must involve Player 2 playing a. Then, a must be a best response of Player 2, which is possible if:

   $$2\alpha \geq 1 - \alpha \text{ or } \alpha \geq \frac{1}{3}.$$

   Since playing B is a strictly dominant strategy for Player 1 when $\theta = 3$, there cannot be a Bayes-Nash equilibrium where Player 1 plays A when $\theta = 0$.

2. Notice that for each Player i, contributing $x_i = 0$ is a strictly dominant strategy. So, $(0, 0)$ is a unique dominant strategy equilibrium of the stage game. This is also the minmax action profile for each player. Hence, any contribution profile (x_1, x_2) generating payoff $2x_j - x_i$ for Player i is strictly enforceable payoff if $2x_j - x_i > 0$ or $\frac{x_i}{x_j} < 2$. Similarly, for Player $j \neq i$, it is $\frac{x_j}{x_i} < 2$. Summarizing, a contribution profile (x_1, x_2) is strictly enforceable if and only if

   $$x_1, x_2 \in (0, 1] \tag{1}$$
   $$\frac{1}{2} < \frac{x_1}{x_2} < 2. \tag{2}$$

   This is shown in Figure 1.

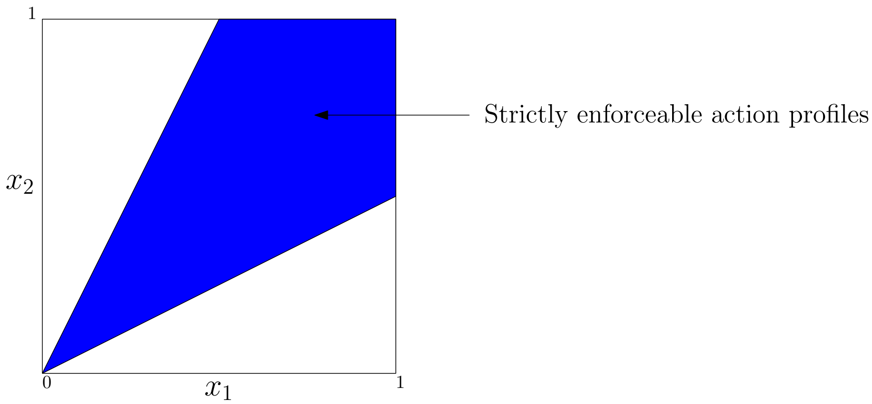

Figure 1: Strictly enforceable action profiles of stage game

A Nash reversion strategy has two states: NORMAL and PUNISH. In normal state, Player i plays action $x_i \in [0,1]$ but in punish state he plays Nash contribution 0. In normal state if a player i does not play x_i, we get to PUNISH state. Once in PUNISH state, we stay forever.

If (x_1^*, x_2^*) is a strictly enforceable contribution profile, then a Nash reversion strategy will ask players to contribute this in NORMAL state. Hence, payoff Player i from this will be: $2x_j^* - x_i^*$. Clearly, no deviation is possible in PUNISH state as players play a stage game Nash equilibrium in each period. We check for deviation from NORMAL state. One-period deviation by Player i gives: $(1-\delta)2x_j^* + 0$. Hence, for no deviation, we must have $2x_j^* - x_i^* \geq (1-\delta)2x_j^*$. This gives us $\delta \geq \frac{x_i^*}{2x_j^*}$. We get a symmetric inequality for Player j also. Hence, we must have

$$\delta \geq \max(\frac{x_1^*}{2x_2^*}, \frac{x_2^*}{2x_1^*}).$$

Notice that by Inequalities 1 and 2, we get that we can choose $\delta < 1$.

A carrot and stick strategy with one-period punishment of 0 is the following. It has two states: NORMAL and PUNISH. For any (x_1, x_2), it says to Player i to play x_i in NORMAL state and 0 in PUNISH state. A NORMAL state becomes PUNISH if any player i does not contribute x_i. A PUNISH state becomes NORMAL if each player contributes 0. Else, a PUNISH state stays PUNISH.

Suppose (x_1^*, x_2^*) is a strictly enforceable contribution profile. Following this gives each Player i a payoff of $2x_j^* - x_i^*$. We check for one-shot deviation at both states. At

NORMAL state, one-shot deviation gives Player i:

$$\begin{aligned}&(1-\delta)\Big[2x_j^* + \delta(0) + \delta^2(2x_j^* - x_i^*) + \delta^3(2x_j^* - x_i^*) + \dots\Big]\\ &= (1-\delta)2x_j^* + \delta^2(2x_j^* - x_i^*).\end{aligned}$$

This gives us a no-deviation condition as follows.

$$\begin{aligned}2x_j^* - x_i^* &\geq (1-\delta)2x_j^* + \delta^2(2x_j^* - x_i^*)\\ \Leftrightarrow (1+\delta)(2x_j^* - x_i^*) &\geq 2x_j^*\\ \Leftrightarrow \delta &\geq \frac{x_i^*}{2x_j^* - x_i^*}.\end{aligned}$$

Now, we check for deviation at the PUNISH state. The payoff from following strategy at PUNISH state is:

$$(1-\delta)\big[0 + \delta(2x_j^* - x_i^*) + \delta^*(2x_j^* - x_i^*) + \dots\big] = \delta(2x_j^* - x_i^*).$$

One-shot deviation is not possible since 0 is a strictly dominant strategy. So, for all values of δ no deviation is possible in PUNISH state.

Summarizing, values of δ that can sustain this as SPE is:

$$1 > \delta \geq \max(\frac{x_2^*}{2x_1^* - x_2^*}, \frac{x_1^*}{2x_2^* - x_1^*}). \tag{3}$$

But this is not possible as this will require both $x_1^* < x_2^*$ and $x_2^* < x_1^*$.

If $x_i^* = x_j^*$, then, the above inequality cannot be satisfied. Hence, carrot and stick strategy with one-period punishment cannot sustain this as subgame perfect equilibrium. On the other hand, $1 > \delta \geq \frac{1}{2}$ ensures that Nash reversion works.