Game Theory - Midterm Examination 2

Date: October 13, 2019

Total marks: **30**

Duration: 10 AM to 12:30 PM

Note: Answer all questions clearly using pen. Please avoid unnecessary discussions.

1. Suppose Γ is a finite strategic form game and p is a correlated equilibrium of Γ. Suppose $p(s) > 0$ for some pure strategy profile s. Then prove or provide a counter-example to the following claim: s survives iterated elimination of strictly dominated pure strategies in $\Delta\Gamma$. **(5 marks)**

   Answer. Fix agent i and let s_i be a strategy of Player i in the support of p. Define $\pi(s_i) := \sum_{t_{-i}} p(s_i, t_{-i})$. By the definition of correlated equilibrium, we know that for all $s_i' \in S_i$, the following holds:

$$\sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) p(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) p(s_i, s_{-i})$$

$$\Leftrightarrow \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \frac{p(s_i, s_{-i})}{\pi(s_i)} \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) \frac{p(s_i, s_{-i})}{\pi(s_i)}$$

   Now define the following belief μ_i of Player i: for every $s_{-i} \in S_{-i}$, let $\mu_i(s_{-i}) = \frac{p(s_i, s_{-i})}{\pi(s_i)}$. Note that $\sum_{s_{-i}} \mu_i(s_{-i}) = \frac{1}{\pi(s_i)} \sum_{s_{-i}} p(s_i, s_{-i}) = 1$, and hence μ_i is a probability distribution over S_{-i}. Thus, we have

$$\sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \mu_i(s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) \mu_i(s_{-i}) \qquad \forall\ s_i' \in S_i.$$

   Hence, strategy s_i is correlated rationalizable. But any strategy which is correlated rationalizable survives iterated elimination of strictly dominated pure strategies in $\Delta\Gamma$. This completes the proof.

2. There are two agents who want to complete a task. Each agent can either work or shirk. So, the possible set of actions for each agent is $\{0, 1\}$, where 0 corresponds to shirking and 1 corresponds to working. The task can be completed if *any* agent works. Working is costly - each agent $i \in \{1, 2\}$ incurs a cost c_i if he works. If the two agents

choose actions (x_1, x_2), where $x_i \in \{0, 1\}$ for each $i \in \{1, 2\}$, then the utility of each agent $i \in \{1, 2\}$ is given by

$$u_i(x_1, x_2) = \begin{cases} 1 - x_i c_i & \text{if } x_1 + x_2 > 0 \\ 0 & \text{if } x_1 + x_2 = 0 \end{cases}$$

(a) Suppose the cost of agent 1 is publicly known and assume that $c_1 > 0$. On the other hand, suppose cost of agent 2 is his private information, but it is commonly known that c_2 is drawn from $[0.5, 1.5]$ using uniform distribution. Describe all Bayes Nash equilibria of this game. Are there values of c_1 for which *all* Bayes Nash equilibria involve Player 2 shirking at all types? **(5+2 marks)**

Answer. A strategy for agent 1 is $s_1 \in \{0, 1\}$. A strategy for agent 2 is a map $s_2 : [0.5, 1.5] \to \{0, 1\}$. If $s_1 = 1$, then all types of agent 2 must choose $s_2(c_2) = 0$ as a best response (since $c_2 > 0$). Hence, in the strategy profile $(s_1 = 1, s_2(c_2) = 0 \;\forall\; c_2)$ the payoff of agent 1 is $1 - c_1$. If agent 2 follows s_2, the payoff of agent 1 by choosing strategy 0 is 0. As a result choosing, $s_1 = 1$ is a best response if $c_1 \leq 1$. So, one Bayes Nash equilibrium is when

$$c_1 \leq 1 : s_1 = 1, s_2(c_2) = 0 \;\; \forall \; c_2.$$

If $s_1 = 0$, then agent 2 gets non-negative payoff if $1 - c_2 \geq 0$ or $c_2 \leq 1$. So, $s_2(c_2) = 1$ if $c_2 < 1$ and $s_2(c_2) = 0$ if $c_2 > 1$ is a best response. If $c_2 = 1$, then agent 2 can choose either action. So, any Bayes Nash equilibrium with $s_1 = 0$ must have this strategy for agent 2. Hence, expected payoff of agent 1 from $s_1 = 0$ is $\frac{1}{2} \times 1 = \frac{1}{2}$, where $\frac{1}{2}$ is the probability with which agent 2 is likely to choose 1. Agent 1's expected payoff from choosing 1 is $1 - c_1$. So, if $s_1 = 0$ is a Bayes Nash equilibrium, then $1 - c_1 \leq \frac{1}{2}$ or $c_1 \geq \frac{1}{2}$. So, another Bayes Nash equilibrium is when

$$c_1 \geq \frac{1}{2} : s_1 = 0, s_2(c_2) = 1 \;\forall\; c_2 \in [0.5, 1), s_2(c_2) = 0 \;\forall\; c_2 \in (1, 1.5].$$

If $c_1 < 0.5$, all equilibria must involve $s_1 = 1$ and $s_2(c_2) = 0$ for all c_2.

(b) Suppose the costs of both the agents are their respective private information. Further, each agent's cost is drawn from $[0.5, 1.5]$ using a uniform distribution. Call a strategy of an agent i a **cutoff** strategy if there is a number $c_i^* \in [0.5, 1.5]$ such that for all types with cost less than c_i^*, i chooses one action and for all types with cost greater than c_i^*, he chooses the other action. Compute all Bayes Nash equilibria of this game where agents use cutoff strategies. **(5 marks)**

Answer. From the earlier part, every Bayes Nash equilibria must involve cutoff strategies. Let (s_1, s_2) be a Bayes Nash equilibria with cutoffs (c_1^*, c_2^*). Then, $\pi(s_i) = c_i^* - \frac{1}{2}$ for each $i \in \{1, 2\}$. The best response condition requires that agent i should be indifferent between choosing 1 or 0 at $c_i = c_i^*$:

$$1 - c_i^* = \pi(s_j) = c_j^* - \frac{1}{2}.$$

That is: $c_1^* + c_2^* = \frac{3}{2}$.

Hence, every Bayes Nash equilibria are with cutoff strategies (c_1^*, c_2^*) with

$$c_1^*, c_2^* \in [0.5, 1.5], c_1^* + c_2^* = 1.5.$$

Finally, we verify that each of them is indeed a Bayes Nash equilibrium. To see this, take agent $i \in \{1, 2\}$. His expected payoff at $c_i < c_i^*$ by choosing 1: $1 - c_i \geq 1 - c_i^* = c_j^* - 0.5$. But $c_j^* - 0.5$ is the payoff of choosing 0. Hence, choosing 1 is best response. Similarly, if $c_i > c_i^*$, identical argument shows that choosing 0 is a best response. At $c_i = c_i^*$, both are best responses. This completes the proof.

3. Consider the stage game G shown in Table 1.

	L	C	R
T	2,2	2,1	0,0
M	1,2	1,1	-1,0
B	0,0	0,-1	-1,-1

Table 1: A Stage game

(a) What is the minmax payoff of each player? What action should Player 1 play to restrict (punish) Player 2 to her minmax payoff? Similarly, what action should Player 2 play to restrict (punish) Player 1 to her minmax payoff? **(3 marks)**

Answer. The minmax payoff each player is 0. Player 1 should play B to minmax Player 2. Player 2 should play R to minmax Player 1.

(b) Describe a strategy profile s^* of G^∞ and a lower bound on discounting value δ such that

(i) s^* is a Nash equilibrium of G^∞ with discounting δ and

(ii) along the equilibrium path of s^*, action profile (M, L) is played in odd periods and action profile (T, C) is played in even periods. **(5 marks)**

Answer. The usual trigger strategy works. Maintain two states: NORMAL and PUNISHMENT. Intial state is NORMAL. In NORMAL state: play (M, L) in odd period and (T, C) in even period. In PUNISHMENT state: Player 1 plays B and Player 2 plays R.

The histories are assigned to states as follows. For every history in period t, there is a history in period $(t-1)$ that leads to this history, called the *predecessor*. A history is normal if (i) its predecessor is NORMAL and (ii) action at predecessor history is (M, L) if the previous period is odd and (T, C) if the previous period is odd. Else, history is PUNISHMENT.

We show that this trigger strategy is a Nash equilibrium. Since the game is symmetric, we only look at Player 1. Suppose Player 2 follows the trigger strategy. The payoff of Player 1 by following trigger strategy is:

$$
\begin{aligned}
(1-\delta)\big[1 + 2\delta + \delta^2 + 2\delta^3 + \delta^4 + \ldots...\big] &= (1-\delta)(1+2\delta)\big[1 + \delta^2 + \delta^4 + \ldots\big] \\
&= (1-\delta)(1+2\delta)\frac{1}{1-\delta^2} \\
&= \frac{1+2\delta}{1+\delta}.
\end{aligned}
$$

Suppose Player 1 deviates. Any period she deviates, she gets punished from the next period and restricted to payoff of zero from thereon. If she deviates in period t and t is even, by playing trigger (which recommends (T, C)) she could have got 2 and by deviating she cannot get more than 2. So, deviating in even period is strictly worse independent of value of δ. If she deviates in period t and t is odd, by playing trigger (which recommends (M, L) in this period) she gets a payoff stream: $(1, 2, 1, 2, 1, 2, \ldots)$. The payoff from period t onwards from this payoff stream is $\frac{1+2\delta}{1+\delta}$.

By deviating, the maximum payoff she can get in period t is 2 (this can happen if she plays T instead of recommended M). After that, she gets punished and gets zero. So, her payoff stream from period t onwards is $(2, 0, 0, 0, 0, \ldots)$. For trigger to be optimal, we need

$$\begin{aligned} \frac{1+2\delta}{1+\delta} - 2(1-\delta) &\geq 0 \\ \Leftrightarrow 1 + 2\delta - 2 + 2\delta^2 &\geq 0 \\ \Leftrightarrow 2\delta^2 + 2\delta - 1 &\geq 0 \\ \Leftrightarrow \delta \geq \frac{\sqrt{3}-1}{2} \end{aligned}$$

(c) Consider the following *carrot-and-stick* strategy. This strategy assigns each history in each period one of two states: (a) NORMAL state (b) PUNISHMENT state. The strategy profile recommends players to play (M, C) in a history which is assigned NORMAL state and to play (B, R) in a history which is assigned PUNISHMENT state. The initial period (with null history) is a NORMAL state.

Now, we can recursively assign the state of every history. For every history in period t, there is a history in period $(t-1)$ that leads to this history, called the *predecessor*. If the predecessor is in NORMAL state, and agents play (M, C), the current history (of period t) becomes a NORMAL state. If the predecessor is in PUNISHMENT state, and agents play (B, R), the current history becomes a NORMAL state. Else, the current history becomes PUNISHMENT state.

Player 1 considers another strategy, which we call the *flip-flop* strategy. In the flip-flop strategy, Player 1 plays T in odd periods and plays B in even periods. Suppose Player 2 follows the carrot-and-stick strategy. Find the values of $\delta \in (0, 1)$ for which the carrot-and-stick strategy is weakly better than the flip-flop strategy for Player 1 (when Player 2 follows the carrot-and-stick strategy). **(5 marks)**

Answer. Player 1 gets a payoff of 1 by following carrot-and-stick. Given that Player 2 is playing carrot-and-stick, this is how the states will change if Player 2 plays flip-flop.

- First period is NORMAL. Resulting action profile (T, C). Payoff for Player 1: 2.
- Second period is PUNISHMENT since Player 1 deviates in first period. Resulting action profile (B, R). Payoff for Player 1: -1.
- Since (B, R) was played in PUNISHMENT state, third period is NORMAL state. Resulting action profile (T, C). Payoff for Player 1: 2.
- Since Player 1 deviated in period 3, state is PUNISHMENT. Resulting action profile (B, R). Payoff for Player 1: -1.

- ...

Summarizing, in odd periods, we will see (T, C) being played with payoff of 2 for Player 1 and in even periods, we will see (B, R) being played with payoff of -1 for Player 1. The resulting payoff stream is $(2, -1, 2, -1, \dots)$ whose value is

$$
\begin{aligned}
(1-\delta)\big(2-\delta+2\delta^2-\delta^3+\dots\big) &= (1-\delta)(2-\delta)(1+\delta^2+\delta^4+\dots) \\
&= (1-\delta)(2-\delta)\frac{1}{1-\delta^2} \\
&= \frac{2-\delta}{1+\delta}
\end{aligned}
$$

For carrot-and-stick to do better than flip-flop, we will need, $2-\delta \leq 1+\delta$ or $\delta \in [\frac{1}{2}, 1)$.