Game Theory - Midterm Examination

Date: September 14, 2015

Total marks: **42**

Duration: 2:00 PM to 5:00 PM

Note: Answer all questions clearly using pen. Please avoid unnecessary discussions.

For both the questions below, it is important to remind ourselves what the definition of a strictly dominated strategy is.

Definition 1 *A strategy s_i of agent i is* **strictly dominated** *if there exists another strategy s'_i such that for all s_{-i}, we have*

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}).$$

Two important things to note here:

(1) The strategy s'_i is better than s_i **for all** s_{-i}. Hence, you are not allowed to choose different s'_i for different s_{-i}.

(2) Strategy s'_i is **strictly** better than s_i.

We now proceed to the solutions of Q1 and Q3.

1) Ten players are playing the following game. Each player writes down, on a peice of paper, an integer in $\{1, \dots, 100\}$, alongside his identity (name). A *target integer* is the highest integer less than or equal to $\frac{2}{3}$ of the average of all the integers submitted. The winners of the game are all the players who submitted the target integer. Winners equally share a prize of 1000 (assume prize money equals payoff).

   - Describe this as a strategic form game. (**2 marks**)
   - What are the strictly dominated strategies of each player. (**3 marks**)
   - Compute the set of (correlated) rationalizable strategies in this game. (**4 marks**)
   - Find a pure strategy Nash equilibrium of this game. Is this a unique pure strategy Nash equilibrium? (**3 marks**)

   Answer. The description of the strategic game specifies the **set** of players (not the number of players) as $N = \{1, \dots, 10\}$. For each $i \in N$, the set of pure strategies available to Player i is $S_i := \{1, \ldots, 100\}$. At any strategy profile $s \equiv (s_i, \ldots, s_n)$, the target integer is

$$t(s) = \lfloor \frac{2}{3} \frac{\sum_{i \in N} s_i}{10} \rfloor,$$

where the $\lfloor x \rfloor$ indicates the greatest integer less than or equal to x.

Note: It is important to realize that target integer is a function of the strategies chosen. Hence, it should be written as $t(s)$ and not just as t.

Let $W(s) := \{i \in N : s_i = t(s)\}$. Now, for any $i \in N$, $u_i(s) = \frac{1000}{|W(s)|}$ if $i \in W(s)$ and zero otherwise.

Consider Player i. The game has the feature that Player i gets a positive payoff if and only if his strategy is the target integer. If we want to show that strategy $s_i \in \{1, \ldots, 100\}$ is strictly dominated, we need to show that there is some strategy s'_i such that s'_i gives **strictly** more payoff to Player i than s_i **for every** s_{-i} of other players. But the only way some strategy s'_i can give strictly more payoff than s_i for all s_{-i} is that strategy s'_i must ensure positive payoff for all s_{-i}. [1] The only way this is possible is that strategy s'_i equals the target integer for all s_{-i}. This is impossible: choose $s_{-i} \equiv (1, 1, \ldots)$ and $s'_{-i} = (100, 100, \ldots)$, and note that $t(s_i, s_{-i}) \neq t(s'_i, s_{-i})$ for any s'_i.

This implies that **no strategy** of any player is strictly dominated. The set of correlated rationalizable strategies equals the set of strategies that survive iterative elimination of strictly dominated strategies. Since no strategy of any player can be eliminated in the iterative elimination procedure, we conclude that the set of correlated rationalizable strategies of each player is just $\{1, \ldots, 100\}$.

There are multiple Nash equilibria of this game. One of them is all players choose 1 as the strategy. Then, the target interger is 0 and everyone gets zero payoff. By unilaterally deviating, the target integer will always be strictly below the reported integer. Hence, deviation also gives zero payoff for each player. All players except Player 1 chooses 1 and Player 1 chooses 2 is also a Nash equilibrium because of the same reason.

[1] Strategy s_i ensures at least zero payoff for all s_{-i}, and if strategy s'_i gives zero payoff for some s_{-i}, it cannot be **strictly** better than s_i.

3) A Nash equilibrium s^* in a finite strategic form game $\Gamma = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ is a **strict Nash equilibrium** if for every $i \in N$, for every $s_i \in S_i \setminus \{s_i^*\}$,

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*).$$

Prove that if the process of iterative elimination of strictly dominated strategies results in a unique strategy profile s^*, then s^* is a strict Nash equilibrium. (**5 marks**)

Answer. We know that the Nash equilibria of the original game are also the Nash equilibria of the game we derive after iteratively eliminating strictly dominated strategies. Hence, if s^* is the unique strategy profile that survives the process of iterative elimination of strictly dominated strategies, then s^* must be the unique Nash equilibrium of Γ.

We only need to show that s^* is a strict Nash equilibrium. We know that for all $s_i \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

Assume for contradiction that the above inequality is an equality for some $s_i \neq s_i^*$. Then, s_i is eliminated in some stage of iterative elimination of strictly dominated strategy process. Further, since s_{-i}^* survives this process, s_{-i}^* must be present when s_i is eliminated as a strictly dominated strategy. Suppose s_i' is the strategy that gives strictly better payoff than s_i for all the strategies of other players present in this stage. Then, it must be that

$$u_i(s_i', s_{-i}^*) > u_i(s_i, s_{-i}^*) = u_i(s_i^*, s_{-i}^*).$$

But this contradicts the fact that s_i^* is a best response to s_{-i}^*.