1. Consider a two agent model with three alternatives \( \{a, b, c\} \). Table 1 shows two preference profiles of preferences. Suppose \( f(P_1, P_2) = a \). Show that if \( f \) is strategy-proof then \( f(P_1', P_2') = b \). You are allowed to use the result that for any preference profile \( (\bar{P}_1, \bar{P}_2) \), \( f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\} \) (but do not use any other result from the lectures).

\[
\begin{array}{c|cc}
P_1 & P_2 & P_1' & P_2' \\
\hline
a & c & b & c \\
b & b & a & a \\
c & a & c & b \\
\end{array}
\]

Table 1: Two Preference Profiles

2. Let \( X \) be a set of projects. A social choice function chooses a non-empty subset of projects. Agent \( i \) has a linear ordering \( P_i \) over the set of projects \( X \). Agent \( i \) evaluates subsets of projects by extending \( P_i \) in the following manner: for any pair of subsets of projects \( S, T \subseteq X \), \( S \) is preferred to \( T \) if the highest ranked project in \( S \) (according to \( P_i \)) is better than the highest ranked project in \( T \) - if these two projects are the same, then \( S \) and \( T \) are indifferent.

Suppose \( |X| \geq 2 \). Will the Gibbard-Satterthwaite result apply here? Discuss your answer.

3. Consider the unanimous SCF \( f \) defined as follows. If \( P_1(1) = \ldots = P_n(1) = a \), then \( f(P_1, \ldots, P_n) = a \). Else, \( f(P_1, \ldots, P_n) = b \) for some alternative \( b \in A \). In other words, \( f \) satisfies unanimity wherever possible and picks a “status-quo” alternative \( b \) otherwise. Argue how \( f \) can be manipulated if there are at least three alternatives?

4. Let \( A \) be a finite set of alternatives and \( f : \mathcal{P}^n \rightarrow A \) be a social choice function that is unanimous and strategy-proof. Suppose \( |A| \geq 3 \).

Now, consider another social choice function \( g : \mathcal{P}^2 \rightarrow A \) defined as follows. The scf \( g \) only considers profiles of two agents, denote these two agents as 1 and 2. For any \( (P_1, P_2) \in \mathcal{P}^2 \), let

\[
g(P_1, P_2) = f(P_1, P_2, P_1, \ldots, P_1),
\]

i.e., the outcome of \( g \) at \( (P_1, P_2) \) coincides with the outcome of \( f \) at the profile where agents 1 and 2 have types \( P_1 \) and \( P_2 \) respectively, and all other agents have type \( P_1 \). Show that \( g \) is a dictatorship scf.
5. Let the number of alternatives be $m$. Show that the number of single-peaked preference orderings with respect to $<$ (an exogenously given ordering of alternatives) is $2^{m-1}$.

6. Consider the single-peaked domain model. A social choice function $f$ is manipulable by a group of agents $K \subseteq N$ if for some preference profile $(P_K, P_{-K})$ there exists some preference profile $P'_K$ of agents in $K$ such that $f(P'_K, P_{-K}) P_i f(P_K, P_{-K})$ for all $i \in K$. A social choice function $f$ is **group strategy-proof** if cannot be manipulated by any group of agents. Is the median voter SCF group strategy-proof?

7. Let $A = [0, 1]$ and assume that agents have single peaked preferences over $A = [0, 1]$. Consider the following social choice function.

**Definition 1** A social choice function $f$ is a **generalized median voter social choice function** if there exists weights $y_S$ for every $S \subseteq N$ satisfying

(a) $y_\emptyset = 0$, $y_N = 1$ and
(b) $y_S \leq y_T$ for all $S \subseteq T$

such that for all preference profile $P$, $f(P) = \max_{S \subseteq N} z(S)$, where $z(S) = \min\{y_S, P_i(1) : i \in S\}$.

Show that a generalized median voter SCF is strategy-proof.

8. Let $A$ be a finite set of alternatives and $\succ$ be a linear order over $A$. Suppose $a_L, a_R \in A$ be two alternatives such that $a \succ a_L$ for all $a \in A \setminus \{a_L\}$ and $a_R \succ a$ for all $a \in A \setminus \{a_R\}$ - in other words, $a_L$ is the “left-most” alternative and $a_R$ is the “right-most” alternative with respect to $\succ$.

Let $S$ be the set of all possible single-peaked strict orderings over $A$ with respect to $\succ$. An SCF $f : S^n \rightarrow A$ maps the set of preference profiles of $n$ agents to $A$.

Let $P_i(1)$ denote the peak of agent $i$ in $P_i$. Suppose $f$ satisfies the following property (call it property II). There is an alternative $a^* \in A$ such that for any preference profile $(P_1, \ldots, P_n) \in S^n$, where $P_i(1) \in \{a_L, a_R\}$ for all $i \in N$ with at least one agent’s peak at $a_L$ and at least one agent’s peak at $a_R$, $f(P_1, \ldots, P_n) = a^*$.

(a) Suppose $f$ is strategy-proof, efficient, anonymous, and satisfies property II. Then, give a precise (simplified) description of $f$ (using $a^*$), i.e., for every preference profile $P$, what is $f(P)$?

(b) Can $f$ be strategy-proof, anonymous, and satisfy property II, but not efficient (give a formal argument or an example)?