Theory of Mechanism Design - Assignment 4

1. Consider a two agent model with three alternatives $\{a, b, c\}$. Table 1 shows two preference profiles of preferences. Suppose $f(P_1, P_2) = a$. Show that if f is strategy-proof then $f(P_1', P_2') = b$. You are allowed to use the result that for any preference profile $(\bar{P}_1, \bar{P}_2)$, $f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\}$ (but do not use any other result from the lectures).

P_1	P_2	P_1'	P_2'
a	c	b	c
b	b	a	a
c	a	c	b

Table 1: Two Preference Profiles

2. Let X be a set of projects. A social choice function chooses a non-empty subset of projects. Agent i has a linear ordering P_i over the set of projects X. Agent i evaluates subsets of projects by extending P_i in the following manner: for any pair of subsets of projects $S, T \subseteq X$, S is preferred to T if the highest ranked project in S (according to P_i) is better than the highest ranked project in T - if these two projects are the same, then S and T are indifferent.

   Suppose $|X| \geq 2$. Will the Gibbard-Satterthwaite result apply here? Discuss your answer.

3. Consider the unanimous SCF f defined as follows. If $P_1(1) = \ldots = P_n(1) = a$, then $f(P_1, \ldots, P_n) = a$. Else, $f(P_1, \ldots, P_n) = b$ for some alternative $b \in A$. In other words, f satisfies unanimity wherever possible and picks a "status-quo" alternative b otherwise. Argue how f can be manipulated if there are at least three alternatives?

4. Let A be a finite set of alternatives and $f : \mathcal{P}^n \to A$ be a social choice function that is unanimous and strategy-proof. Suppose $|A| \geq 3$.

   Now, consider another social choice function $g : \mathcal{P}^2 \to A$ defined as follows. The scf g only considers profiles of two agents, denote these two agents as 1 and 2. For any $(P_1, P_2) \in \mathcal{P}^2$, let

$$g(P_1, P_2) = f(P_1, P_2, P_1, P_1, \ldots, P_1),$$

   i.e., the outcome of g at (P_1, P_2) coincides with the outcome of f at the profile where agents 1 and 2 have types P_1 and P_2 respectively, and all other agents have type P_1.

   Show that g is a dictatorship scf.

5. Let the number of alternatives be m. Show that the number of single-peaked preference orderings with respect to $<$ (an exogenously given ordering of alternatives) is 2^{m-1}.

6. Consider the single-peaked domain model. A social choice function f is manipulable by a group of agents $K \subseteq N$ if for some preference profile (P_K, P_{-K}) there exists some preference profile P'_K of agents in K such that $f(P'_K, P_{-K}) P_i f(P_K, P_{-K})$ for all $i \in K$. A social choice function f is **group strategy-proof** if cannot be manipulated by any group of agents. Is the median voter SCF group strategy-proof?

7. Let $A = [0, 1]$ and assume that agents have single peaked preferences over $A = [0, 1]$. Consider the following social choice function.

   Definition 1 *A social choice function f is a* **generalized median voter** *social choice function if there exists weights y_S for every $S \subseteq N$ satisfying*

   (a) $y_\emptyset = 0$, $y_N = 1$ and

   (b) $y_S \leq y_T$ for all $S \subseteq T$

   such that for all preference profile P, $f(P) = \max_{S \subseteq N} z(S)$, where $z(S) = \min\{y_S, P_i(1) : i \in S\}$.

   Show that a generalized median voter SCF is strategy-proof.

8. Let A be a finite set of alternatives and $\succ$ be a linear order over A. Suppose $a_L, a_R \in A$ be two alternatives such that $a \succ a_L$ for all $a \in A \setminus \{a_L\}$ and $a_R \succ a$ for all $a \in A \setminus \{a_R\}$ - in other words, a_L is the "left-most" alternative and a_R is the "right-most" alternative with respect to $\succ$.

   Let $\mathcal{S}$ be the set of all possible single-peaked strict orderings over A with respect to $\succ$. An SCF $f : \mathcal{S}^n \to A$ maps the set of preference profiles of n agents to A.

   Let $P_i(1)$ denote the peak of agent i in P_i. Suppose f satisfies the following property (call it property Π). There is an alternative $a^* \in A$ such that for any preference profile $(P_1, \ldots, P_n) \in \mathcal{S}^n$, where $P_i(1) \in \{a_L, a_R\}$ for all $i \in N$ with at least one agent's peak at a_L and at least one agent's peak at a_R, $f(P_1, \ldots, P_n) = a^*$.

   (a) Suppose f is strategy-proof, efficient, anonymous, and satisfies property Π. Then, give a precise (simplified) description of f (using a^*), i.e., for every preference profile P, what is $f(P)$?

   (b) Can f be strategy-proof, anonymous, and satisfy property Π, but not efficient (give a formal argument or an example)?