Assignment 1

Instructor: **Debasis Mishra**

August 19, 2024

All questions assume independent private values model in the single object auction with risk neutral bidders.

1. Does the following strategy profile constitute a Nash equilibrium in the second-price auction (assume valuations lie in $[0, 1]$)? If a bidder has value more than 0.5, she bids her value. If bidder 1 has value less than or equal to 0.5, she bids 0.5, and any other bidder $i \neq 1$ has value less than or equal to 0.5 she bids 0.

   Answer. If bidder i has a value v_i, bidding v_i is a weakly dominant strategy. So, if $v_i > 0.5$, the prescribed strategy (which asks to bid value for $v_i > 0.5$) is a best response to any strategy of others.

   If bidder i has value $v_i \leq 0.5$, there are two cases to consider.

   $i = 1$: In this case, she bids 0.5. If any other bidder bids more than 0.5 she looses. If she bids something else and wins, she has to pay second-highest bid, which is more than 0.5, which leads to negative payoff. Hence, bidding 0.5 is optimal. If no other bidder bids more than 0.5, they all bid zero according to prescribed strategy. Hence, bidder 1 wins by bidding 0.5 and pays zero. Clearly, this is optimal.

   $i \neq 1$: In this case, she bids 0 and looses for sure (either bidder 1 bids 0.5 or some other bidder bids more than 0.5). Hence, her payoff is zero. To win, she needs to bid at least 0.5 and pay at least 0.5. Since $v_i \leq 0.5$ this cannot give more than zero payoff. Hence, bidding 0 is optimal.

2. Consider a third-price auction (where highest bidder wins and pays the third-highest price) in a symmetric environment with n bidders and values drawn using cdf F and density f from $[0, a]$. Suppose $F(x)/f(x)$ is increasing in x (that is F is *log concave*). Show that the following strategy is a symmetric Bayesian equilibrium:

$$s(x) = x + \frac{F(x)}{f(x)(n-2)} \; \forall \; x$$

Compare this equilibrium bid with the equilibrium in the first-price auction.

Answer. Denote the distribution of the k-th highest among K draws using F as F_k^K with density f_k^K. Note that $F_1^{(n-2)}(x) = [F(x)]^{n-2}$ and $f_1^{(n-2)}(x) = (n-2)[F(x)]^{n-3}f(x) = (n-2)\frac{f(x)}{F(x)}F_1^{(n-2)}$. Hence, we get

$$s(x) = x + \frac{F(x)}{f(x)(n-2)} = x + \frac{F_1^{(n-2)}(x)}{f_1^{(n-2)}(x)} \qquad \forall\ x$$

Since third price auction is a standard auction and s is strictly increasing, to show that this is a Bayesian equilibrium, it is enough to show that the expected payment of a bidder of type x is same as in the first-price auction.

Since the bidding strategy is monotone (given F/f is monotone), the probability of winning for a bidder with value x is $G(x) = F_1^{(n-1)}(x)$. If a bidder with value x wins, then she makes a payment equal to the bid of the second highest of the remaining bidders. One way to think about this is the following: suppose the highest of the remaining $(n-1)$ bidders have value z, then the highest of remaining $(n-2)$ bidders can have value from 0 to z. The conditional density of the highest of $(n-2)$ bidders having value less than value z is $\frac{f_1^{(n-2)}(y)}{F_1^{(n-2)}(z)}$. Hence, the expected payment collected if highest of $(n-1)$ values is z is

$$\begin{aligned}\frac{1}{F_1^{(n-2)}(z)}\int_0^z \left[y + \frac{F_1^{(n-2)}(y)}{f_1^{(n-2)}(y)}\right] f_1^{(n-2)}(y)dy &= \frac{1}{F_1^{(n-2)}(z)}\int_0^z \left[y f_1^{(n-2)}(y) + F_1^{(n-2)}(y)\right]dy \\ &= \frac{1}{F_1^{(n-2)}(z)}\int_0^z d[yF_1^{(n-2)}(y)] \\ &= z\end{aligned}$$

Now, the conditional density of highest of $(n-1)$ values being less than x is $\frac{f_1^{(n-1)}(z)}{F_1^{(n-1)}(x)}$. As a result, the expected payment made by winning for a value x bidder is

$$\frac{1}{F_1^{(n-1)}(x)}\int_0^x z f_1^{(n-1)}(z)dz$$

Since the probability of winning is $F_1^{(n-1)}(x)$, we get that the total expected payment of a bidder with value x is

$$\int_0^x z f_1^{(n-1)}(z)dz = \int_0^x z g(z)dz$$

where we used the fact that $g \equiv f_1^{(n-1)}$. This is the same expected payment of a bidder in the first-price or second-price auction or in any standard auction. Hence, by our theorem on standard auctions, this strategy must be a Bayesian equilibrium.

3. Consider the all-pay auction (bidder with highest value wins and pays her bid) with n bidders in a symmetric environment with values drawn from $[0, a]$ with cdf F and density f. Show that the following strategy is a symmetric Bayesian equilibrium:

$$s(x) = xG(x) - \int_0^x G(y)dy \qquad \forall\ x \in [0, a]$$

Can you show that this is the unique symmetric equilibrium in an all-pay auction? Compare this equilibrium with the equilibrium in the first-price auction.

Answer. To show that this is an equilibrium, note that s is increasing (derivative of $s(x)$ is $xg(x)$). As a result, probability of winning for a bidder with bid $s(y)$ is $G(y)$ (if others follow s). So, if a bidder has value x and bids as if her value is y, gets a payoff equal to

$$xG(y) - s(y) = xG(y) - yG(y) + \int_0^y G(z)dz$$

Derivative of RHS with respect to y gives $xg(y) - yg(y)$, which is positive if $y < x$ and negative if $y > x$. Hence, the maximum of the payoff is achieved at $y = x$, establishing Bayesian equilibrium. (Ignoring the imitation lemma that we need to show.)

To establish uniqueness, note that if s is an equilibrium, a bidder with value x cannot pretend to a bidder with value y:

$$
\begin{aligned}
& xG(x) - s(x) \geq xG(y) - s(y) = (x-y)G(y) + yG(y) - s(y) \\
\iff & u(x) - u(y) \geq (x-y)G(y)
\end{aligned}
$$

where u is the utility of bidder by playing the strategy. We know the solution to this equation is unique and is given by

$$
\begin{aligned}
u(x) = xG(x) - s(x) = \int_0^x G(y)dy \\
\implies s(x) = xG(x) - \int_0^x G(y)dy
\end{aligned}
$$

Assignment 2

Instructor: **Debasis Mishra**

September 09, 2024

Let X be a continuous random variable with an absolutely continuous distribution function F and a density f. Let Y be a continuous random variable with an absolutely continuous distribution function G and a density g. For convenience, we will assume f and g are positive, and whenever needed F, f, G, g are all differentiable (as many times as we need). The exercise is about comparing X and Y, i.e., comparing two random variables. This has applications in economic theory, in particular, in auction theory. The most basic way is first-order stochastic dominance.

Definition **1** *Random variable Y* **stochastic dominates** *X if*

$$G(x) \leq F(x) \qquad \forall\, x \in \Re$$

We denote this as $Y \succ_{\text{st}} X$.

The definition says that $\text{Prob}(X > x) \leq \text{Prob}(Y > x)$ for all x, i.e., X is "smaller than" Y.

The next definition involves hazard rate of a random variable. For random variable X, the hazard rate at $t \in \Re$ is

$$\frac{f(t)}{1 - F(t)} = \lim_{\Delta t \downarrow 0} \frac{\text{Prob}\Big(t < X < t + \Delta t : X > t\Big)}{\Delta t}$$

If the random variable X represented the life of a product, survival probability at time t is $1 - F(t)$. Hence, the hazard rate is the intensity of survival at time t.

Definition **2** *Random variable Y* **hazard-rate dominates** *X if*

$$\frac{g(x)}{1 - G(x)} \leq \frac{f(x)}{1 - F(x)} \qquad \forall\, x \in \Re$$

We denote this as $Y \succ_{\text{hr}} X$.

A related concept is reverse hazard rate. The reverse hazard rate of random variable X is the ratio $\frac{f(x)}{F(x)}$ at every $x \in \Re$. This indicates the intensity of failure (rather than survival) at any time instance.

Definition **3** *Random variable* Y **reverse hazard-rate dominates** X *if*

$$\frac{g(x)}{G(x)} \geq \frac{f(x)}{F(x)} \qquad \forall\, x \in \Re$$

We denote this as $Y \succ_{\mathrm{rh}} X$.

Notice the direction of inequality is reversed from hazard-rate order.

Finally, the following ordering of random variables is often used in information economics.

Definition **4** *Random variable* Y **likelihood ratio dominates** X *if*

$$\frac{g(x)}{f(x)} \geq \frac{g(y)}{f(y)} \qquad \forall\, x \geq y$$

We denote this as $Y \succ_{\mathrm{lr}} X$.

Prove the following relationships between these orders.

1. If $Y \succ_{\mathrm{hr}} X$, then $Y \succ_{\mathrm{st}} X$
2. If $Y \succ_{\mathrm{rh}} X$, then $Y \succ_{\mathrm{st}} X$
3. If $Y \succ_{\mathrm{lr}} X$, then $Y \succ_{\mathrm{hr}} X$ and $Y \succ_{\mathrm{rh}} X$