1. Consider the single agent problem with a finite set of alternatives $A$. Let $T = \mathbb{R}^{|A|}$ be the type space (i.e., every vector in $\mathbb{R}^{|A|}$ belongs to the type space). Let $f : T \rightarrow A$ be an onto deterministic allocation rule. Suppose $f$ is monotone, i.e., for every $s, t \in T$, we have $s(f(s)) - s(f(t)) \geq t(f(s)) - t(f(t))$. For every $a \in A$, define $T^f_a := \{ t \in T : f(t) = a \}$ and $d^f(a, b) := \inf_{t \in T^f_a} [t(a) - t(b)]$.

Show the following.

(a) For every $a, b \in A$, we have $d^f(a, b) + d^f(b, a) = 0$.

(b) $f$ is implementable (this will prove that in $T = \mathbb{R}^{|A|}$ with deterministic $f$, monotonicity implies cycle monotonicity).

2. Consider a problem with $n$ agents. Let $T = \mathbb{R}^{|A|}$ be the type space of each agent and let $f : T^n \rightarrow A$ be a deterministic allocation rule. Suppose $f$ is monotone.

We denote a type profile by $t \equiv (t_1, \ldots, t_n)$. We say $f$ satisfies **positive association of differences (PAD)** if for every $t$ with $f(t) = a$ and for every $s$ satisfying

$$s_i(a) - t_i(a) > s_i(b) - t_i(b) \quad \forall b \neq a, \forall i \in N,$$

we have $f(s) = a$. Show that $f$ satisfies PAD.  

3. Consider a problem with $n$ agents. Let $T \subseteq \mathbb{R}^{|A|}$ be any type space. Let $f^e : T \rightarrow A$ be an **efficient** allocation rule, i.e., for every $t \in T^n$, we have

$$f^e(t) \in \arg \max_{a \in A} \left[ \sum_{i \in N} t_i(a) \right].$$

Show that $f^e$ satisfies cycle monotonicity (**do not** use the fact that $f^e$ is implementable using Groves payments).

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Roberts (1979) showed in his work that every $f$ satisfying PAD must be an affine maximizer if $|A| \geq 3$. We know that almost every affine maximizer is implementable.