Game Theory 2 - Mid Term Examination Solution

September, 2013; Duration: 3 hours; Total marks: 50.

Explain your answers, but avoid unnecessary elaboration.

1. Let A be the set of alternatives and $\succ$ be a linear order over A with $|A| = m$. Suppose there are two agent $N := \{1, 2\}$. The preference ordering of each agent $i \in N$ is single-peaked with respect to $\succ$.

   Consider the following social choice function f. For every profile of single-peaked preferences (P_1, P_2), f chooses the best alternative with respect to $\succ$ amongst the worst alternative in P_1 and the worst alternative in P_2. Formally,

$$f(P_1, P_2) = \begin{cases} P_1(m) & \text{if } P_1(m) \succ P_2(m) \\ P_2(m) & \text{if } P_2(m) \succ P_1(m) \end{cases}$$

   - Is f Pareto efficient? **2 marks**

     Answer. It is clearly not Pareto efficient. Consider any profile where tops of both the agents coincide. f will not choose this alternative, and hence, not Pareto efficient.

   - Is f onto? **2 marks**

     Answer. Since the worst alternative is either the best or the worst alternative according to $\succ$, the range of f contains exactly these two alternatives. Hence f is not onto (assuming there are at least three alternatives).

   - Is f strategy-proof? Either show that f is strategy-proof or give a profile where an agent can manipulate f. **6 marks**

     Answer. f is not strategy-proof. Suppose agent 1 has preference aP_1bP_1c and agent 2 has preference cP_2bP_2a. Assuming $a \succ b \succ c$, we get $f(P_1, P_2) = c$. But, agent 1 can manipulate by reporting $P_1' = P_2$, where the outcome is $f(P_1', P_2) = a$ and aP_1c.

2. Use the following fact for this question.

   Fact: In any domain, if any pair of alternatives can be ranked first and second, then one direction of the Gibbard-Satterthwaite theorem applies - every unanimous and strategy-proof social choice function is a dictatorship.

   Let A be the set of candidates for an election with $|A| = m > 2$. Each agent has a strict preference ordering over the candidates. Assume that each agent can have any strict ordering over the set of candidates, i.e., the type space of each agent consists of all linear orders over A.

A social choice function f chooses k candidates out of A at every preference profile, where $1 \leq k < m$. Let $\mathcal{K}$ be the set of all subsets of A with cardinality k and $\mathcal{P}$ be the set of all linear orders over A. Then

$$f : \mathcal{P}^n \to \mathcal{K},$$

where n is the number of agents.

For any $S, T \in \mathcal{K}$, agent i with preference ordering P_i over A prefers S over T if the best alternative in S is better than the best alternative in T according to P_i.

(a) Define the notion of a dictatorship social choice function in this model and argue that every unanimous and strategy-proof social choice function is a dictatorship. **4 marks**

Answer. A dictatorship must choose an agent i and choose a set of candidates that includes $P_i(1)$. Unanimity here means that if a particular candidate is top of all the agents, then it must be chosen (among the k candidates).

We now show any pair of k candidates can be ranked first and second. To see this, pick two such subsets S, T such that $|S| = |T| = k$. There is an alternative $a \in S \setminus T$ and another alternative $b \in T \setminus S$. Now, choose a preference ordering P_i where a is ranked first and b is ranked second. Clearly, S will be one of the top ranked subsets of candidates and T second ranked in the ordering over subsets of candidates.

Now, we can use our fact to conclude.

(b) Suppose $n = 3$ and $k = 2$ (m can have any value). Describe at least two social choice functions which are unanimous and strategy-proof. **4 marks**

Answer. By our earlier result, it has to be a dictatorship. Two particular forms of dictatorships are as follows.

1) Choose top two candidates of agent i at every profile. This is clearly unanimous. It is strategy-proof since agent i gets his top and other agents cannot change the outcome.

2) Choose top candidate of agent i and top candidate of agent j. If they are the same, then choose top two of agent i. This is clearly unanimous. It is strategy-proof since agents i and j get their top and other agents cannot change the outcome.

(c) How will your answer to the part (a) change if for any $S, T \in \mathcal{K}$, agent i with preference ordering P_i over A prefers S over T if the worst alternative in S is better than the worst alternative in T according to P_i. **2 marks**

Answer. The answer is not clear in this case since we cannot rank any pair of outcomes first and second. For instance suppose $A = \{a, b, c, d\}$ and let $k = 2$ (i.e. we want to choose two candidates). Then, consider the outcomes $\{a, b\}$ and $\{c, d\}$. First, the top ranked outcome is a unique outcome consisting of top two alternatives in P_i. For instance, $\{a, b\}$ is the top, if in the preference ordering P_i, $P_i(1) = a, P_i(2) = b$. Hence, c and d will be below a, b. Suppose $P_i(3) = c, P_i(4) = d$. Then, $\{a, c\}$ is better than $\{c, d\}$ and $\{a, b\}$ is better than $\{a, c\}$. Hence, $\{c, d\}$ cannot be second ranked. Hence, the set of unanimous and strategy-proof social choice functions are difficult to characterize here (we cannot use our Fact).

3. Let f be a randomized social choice function on the unrestricted domain of preferences, i.e., $f : \mathcal{P}^n \to \mathcal{L}(A)$, where A is the set of alternatives, $\mathcal{L}(A)$ is the set of probability distributions over A, and $\mathcal{P}$ is the set of all linear orders over A. Suppose $N = \{1, 2\}$ and $A = \{a, b, c\}$.

   (a) Suppose f is a unilateral. Consider a preference profile (P_1, P_2), where

   $$P_1(1) = a, P_1(2) = b, P_1(3) = c, P_2(1) = c, P_2(2) = a, P_3(3) = b.$$

   Suppose
   $$f_a(P_1, P_2) = 0.3, f_b(P_1, P_2) = 0.2, f_c(P_1, P_2) = 0.5.$$

   Now, consider another preference profile (P'_1, P'_2), where $P'_1 = P_2$ and $P'_2 = P_1$. Determine $f_a(P'_1, P'_2), f_b(P'_1, P'_2), f_c(P'_1, P'_2)$. **5 marks**

   Answer. Agent 2 is the weak dictator here. So, $f_a(P'_1, P'_2) = 0.5, f_b(P'_1, P'_2) = 0.3, f_c(P'_1, P'_2) = 0.2$.

   (b) Suppose f is a unanimous and strategy-proof randomized social choice function. Consider the preference profiles (P_1, P_2) and (P'_1, P'_2) in the previous question. If

   $$f_a(P_1, P_2) = 0.6, f_b(P_1, P_2) = 0, f_c(P_1, P_2) = 0.4,$$

   then determine $f_a(P'_1, P'_2), f_b(P'_1, P'_2), f_c(P'_1, P'_2)$. **5 marks**

   Answer. It must be a random dictatorship. Since $f(P_1, P_2)$ is given, we conclude that the weight for agent 1 is 0.6 and that for agent 2 is 0.4. Hence, $f_a(P'_1, P'_2) = 0.4, f_b(P'_1, P'_2) = 0, f_c(P'_1, P'_2) = 0.6$.

4. Consider the one-sided matching problem of allocating objects to agents. Suppose there are three agents $N = \{1, 2, 3\}$ and three objects $M = \{a, b, c\}$. The preference of agent i over objects in M is denoted by $\succ_i$.

(a) Consider the following social choice function. Agent 1 gets his best object according to $\succ_1$. Agent 2 gets the second best object according to $\succ_1$ and agent 3 gets the remaining object. Is this social choice function strategy-proof? Is this social choice function efficient? **5 marks**

Answer. This is strategy-proof since agent 1 gets his top, but agents 2 and 3 cannot change the outcome. However, it is not Pareto efficient. Suppose agent 1 has preference $a \succ_1 b \succ_1 \succ_1 c$, but agent 2's top is c and agent 3's top is b. Then, the scf allocates agent 1 object a, agent 2 object b, and agent 3 object c. Clearly, giving agent 2 object c and agent 3 object b improves the allocation.

(b) Consider the following preference profile.

$$a \succ_1 b \succ_1 c, a \succ_2 c \succ_2 b, a \succ_3 b \succ_3 c.$$

If the output of the TTC mechanism with fixed endowment is: agent 1 gets object a, agent 2 gets object object c and agent 3 gets object b. Suppose the initial endowment is different from this matching. Then, what is the initial endowment that produces this outcome in the TTC mechanism? **5 marks**

Answer. The initial endowment must give agent 1 object a, agent 2 object b, and agent 3 object c.

5. Consider the two-sided matching problem with m schools and $2m$ students with each school having a capacity of 2. Schools have preferences over students and students have preferences over schools. The only restrictions on any school's preference is: (i) a school likes to have any student over having an empty capacity and (ii) a school prefers a pair of students (a, b) over (a, c) if any only if b is higher in its preference than c.

(a) Define the notion of stability by considering blocking of (student,school) pairs. **2 marks**

Answer. A schools s and a student t blocks a matching μ, if student t likes s over his match in μ and school s likes t over any of the matches in μ (this is possible because of the restriction on preferences we assume).

(b) Describe a modification of the deferred acceptance algorithm that produces a stable matching. **2 marks**

Answer. The usual student proposing DAA with the modification that schools can now accept at most two of their top offers, where evaluation of top two is done using the restriction on preferences.

(c) Show that your modification produces a stable matching. **6 marks**

Answer. The standard argument for existence of stability works. We just need to ensure that when a schools rejects a student, it must have at least one student better than him, and by our restriction of preferences, this ensures that he has a better pair of students after rejection.