Theory of mechanism design

Final Examination; November 19, 2017; Duration: 3 hours; Total marks: **40**

1. There are two objects to be allocated to a single agent. An allocation rule f either does not allocate any of the objects (alternative 0) or allocates one of the two objects $\{a, b\}$ - see Figure 1 for exact regions of allocation. Type of the agent is a vector $v \in \mathbb{R}^2_+$, where $v(a)$ denotes his value for object a and $v(b)$ denotes his value for object b with value for alternative 0 normalized to zero. Type space is all valuation vectors in $\mathbb{R}^2_+$.

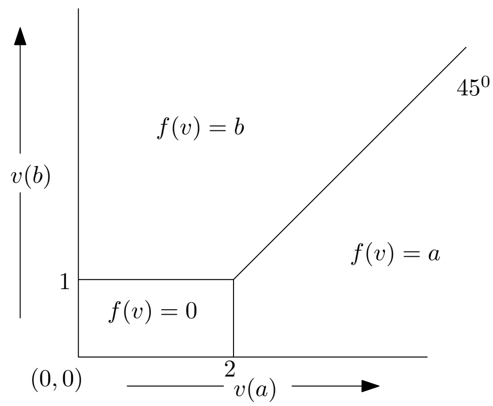

Figure 1: Allocation rule f

Transfers are allowed and preference of the agent is quasilinear.

(a) Show that f is implementable (without deriving an explicit payment rule). Then, find a payment rule that implements f such that payment (by the agent) for receiving object a is zero.**(3+2 marks)**

(b) What does it mean to say that f satisfies revenue equivalence? Provide a short and clear definition. Does f satisfy revenue equivalence? **(2+3 marks)**

2. A single object needs to be allocated to n agents. Each agent i has a budget constraint $B_i \in \mathbb{R}_+$ and a value for the object $v_i \in \mathbb{R}_+$. The cost of a transfer t_i to agent i is defined as follows:

$$C_i(t_i; B_i) = \begin{cases} t_i & \text{if } t_i \leq B_i \\ B_i + (t_i - B_i)(1 + r) & \text{otherwise} \end{cases}$$

Here $r \in (0, 1)$ can be interpreted to be an interest rate. If agent i gets the object and pays t_i, his utility is $v_i - C_i(t_i; B_i)$. If agent i does not get the object and pays t_i, his utility is $-C_i(t_i; B_i)$.

(a) Find the willingness to pay of agent i at transfer $t_i \in \mathbb{R}$. **(4 marks)**

(b) Suppose there are two agents with $v_1 = 5, v_2 = 6, B_1 = B_2 = 4, r = 0.2$. What is the allocation and payment in the generalized Vickrey auction? **(4 marks)**

Answer. Fix agent i and his value v_i (which determines the utility). Note that for willingness to pay (WP), we need to find a solution to the following equation for every t_i:

$$u_i(1, t_i + \delta) = u_i(0, t_i).$$

This will depend on the value of t_i. We consider three cases.

Case 1. If $t_i > B_i$, then $t_i + \delta > B_i$ for all $\delta > 0$. Hence, we need to solve

$$v_i - \Big(B_i + (t_i + \delta - B_i)(1 + r)\Big) = -\Big(B_i + (t_i - B_i)(1 + r)\Big).$$

This gives us a solution:

$$\delta = \frac{1}{1 + r} v_i.$$

Case 2. If $B_i - v_i < t_i \leq B_i$, then we argue that the solution to the WP equation will have $\delta > 0$ such that $t_i + \delta > B_i$. Suppose not, then there is a $\delta > 0$ such that $t_i + \delta \leq B_i$. But $B_i - t_i < v_i$ implies that $\delta < v_i$. But, since δ solves the WP equation, we have

$$v_i - t_i - \delta = -t_i.$$

This implies $\delta = v_i$, a contradiction. Hence, $t_i + \delta > B_i$. This further implies that

$$v_i - \Big(B_i + (t_i + \delta - B_i)(1 + r)\Big) = -t_i.$$

Solving this gives,

$$\delta = \frac{1}{1 + r}\big[v_i + (B_i - t_i)r\big].$$

Case 3. If $t_i \leq B_i - v_i$, then we can argue as in Case 2 that the WP equation will have $\delta > 0$ such that $t_i + \delta \leq B_i$. This implies that $v_i - t_i - \delta = -t_i$, which implies that

$$\delta = v_i.$$

The WP is shown in Figure 2.

In part (b) of the question, we can compute

$$WP_1(v_1, 0) = \frac{1}{1.2}(5 + 0.2 \times 4) = 4.83$$

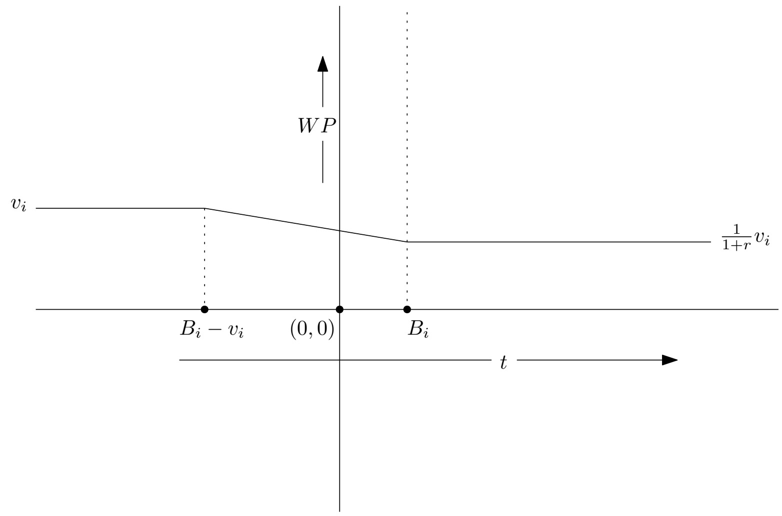

Figure 2: Willingness to pay

$$WP_2(v_2, 0) = \frac{1}{1.2}(6 + 0.2 \times 4) = 5.67.$$

Hence, according to the generalized Vickrey agent 2 wins the object and pays 4.83, whereas agent 1 pays zero.

3. A single object is sold to three agents who have interdependent values for the object. The signals of all the agents are drawn from $[0, 1]$.

   Given a signal profile $s \equiv (s_1, s_2, s_3)$, the valuations of the agents are

   $$v_1(s_1, s_2, s_3) = s_1 + \frac{s_2}{2},\ v_2(s_1, s_2, s_3) = s_2 + \frac{s_3}{2},\ v_3(s_1, s_2, s_3) = s_3 + \frac{s_1}{2}.$$

   (a) Argue that the above valuations satisfy single crossing after clearly defining what single crossing means. **(3 marks)**

   (b) Define the generalized Vickrey auction for this environment. Also, derive the exact allocation and payments of agents when $s_1 = s_2 = 0.8, s_3 = 0.9$. **(4 marks)**

4. Consider a random social choice function $f : \mathcal{P}^n \rightarrow \mathcal{L}(A)$, where $\mathcal{P}$ is the set of all possible strict orderings over A and $\mathcal{L}(A)$ be the set of all probability distributions over A. Fix an agent i and the preference profile of other agents at P_{-i}. Consider two preference orderings of agent i: P_i and P'_i. Suppose $x, y \in A$ are such that x and y are consecutively ranked in both P_i and P'_i with xP_iy and yP'_ix. Suppose for any $a \notin \{x, y\}$, the rank of a in P_i and P'_i is the same. Hence, P'_i is obtained from P_i by swapping only x and y.

   If f is strategy-proof, then show the following:

(a) $f_x(P_i, P_{-i}) + f_y(P_i, P_{-i}) = f_x(P'_i, P_{-i}) + f_y(P'_i, P_{-i})$. **(3 marks)**

(b) for any $a \notin \{x, y\}$, $f_a(P_i, P_{-i}) = f_a(P'_i, P_{-i})$. **(3 marks)**

(c) $f_x(P_i, P_{-i}) \geq f_x(P'_i, P_{-i})$. **(3 marks)**

5. Consider a profile of single peaked preferences $P \equiv (P_1, \ldots, P_n)$, where n is an odd number of agents. For every pair of alternatives $a, b \in A$, we say a beats b at P if

$$|\{i \in N : aP_ib\}| > |\{i \in N : bP_ia\}|.$$

It is known that at every single peaked preference profile P, there will always exist an alternative x such that x beats y at P for every other alternative y. We call such an alternative the **winner** at P and denote it as $\omega(P)$. Consider the social choice function f which picks $w(P)$ at every single peaked preference profile P. Show that f is strategy-proof, unanimous, and anonymous. **(6 marks)**