Theory of mechanism design

Final Examination

April 26, 2019; Duration: 3 hours; Total marks: **40**

Write your answers clearly without unnecessary arguments.

1. Consider a single object allocation model with one agent. Suppose the agent's preference over allocation and transfer satisfies quasilinearity, and the value of the agent is drawn from $V \equiv [0,1]$. Consider an implementable allocation rule $f : V \to [0,1]$ such that $f(v) \notin \{0,1\}$ for some $v \in V$.

   (a) Construct two other (different from f) implementable allocation rules $f' : V \to [0,1]$ and $f'' : V \to [0,1]$ such that for all $v \in V$
   $$f(v) = \frac{1}{2}\Big[f'(v) + f''(v)\Big].$$
   Hint. The following function may be useful. Define a function $h : V \to [0,1]$ as follows. For all $v \in V$
   $$h(v) = \begin{cases} 1 - f(v) & \text{if } f(v) > 0.5 \\ f(v) & \text{otherwise} \end{cases}$$
   [7 marks]

   Answer. Let $f'(v) = f(v) + h(v)$ for all $v \in [0,1]$. Then, $f'(v) = 2f(v)$ if $f(v) \leq 0.5$ and $f'(v) = 1$ otherwise. This is clearly monotone since f is monotone (f is implementable). Similarly, define $f''(v) = f(v) - h(v)$ for all $v \in [0,1]$. This gives us $f''(v) = 0$ if $f(v) \leq 0.5$ and $f''(v) = 2f(v) - 1$ if $f(v) > 0.5$. Again, f'' is monotone since f is monotone. Hence, f' and f'' are implementable. Further, $f(v) = \frac{1}{2}(f'(v) + f''(v))$ for all v.

   (b) What does the above result say about the set of implementable allocation rules? **[3 marks]**

   Answer. Notice that $f \neq f''$ and $f \neq f'$ when $f(v) \in (0,1)$ for some v (i.e., f is a random allocation rule), this means that every **random allocation rule** can be expressed as a **convex combination** of two other implemtable allocation rules. This is not true if f itself is a deterministic rule – in that case, $h(v) = 0$ for all v. Hence, every *extreme point* of the set of implementable allocation rules is a deterministic implementable allocation rule, where by extreme point we mean that it cannot be expressed as a convex combination of two different implementable allocation rules.

2. Let A be a finite set of alternatives and $T \subseteq \mathbb{R}^{|A|}$ be a **polygonally connected type space**, i.e., for any pair of types $s, t \in T$, there exists a finite sequence of types $(s = s^0, s^1, \ldots, s^k, s^{k+1} = t)$, such that for each $j \in \{0, 1, \ldots, k\}$, the line segment connecting s^j and s^{j+1} (denoted by $L(s^j, s^{j+1})$) lies in T.

   Suppose $f : T \to A$ is an implementable allocation rule. Consider a payment rule $p : T \to \mathbb{R}$ such that for every $s, t \in T$, (f, p) **restricted to** $L(s, t)$ is incentive compatible. Show that (f, p) is incentive compatible on the whole of T. [**10 marks**]

   Answer. There are many ways to show this. Here is one way. Take any $s, t \in T$, and we need to show that the incentive constraint between s and t holds in (f, p). By polygonally connectedness property, there is a finite sequence of types $(s = s^0, s^1, \ldots, s^k, s^{k+1} = t)$, such that for each $j \in \{0, 1, \ldots, k\}$, the line segment connecting s^j and s^{j+1} (denoted by $L(s^j, s^{j+1})$) lies in T. Hence, the incentive constraints along each of the line segment $L(s^j, s^{j+1})$ for $j \in \{0, 1, \ldots, k\}$ hold. Further, since f is implementable, there is a mechanism (f, q) which is incentive compatible. But for the mechanism (f, q), the incentive constraints on $L(s^j, s^{j+1})$ hold for all $j \in \{0, 1, \ldots, k\}$. But each $L(s^j, s^{j+1})$ is a convex set, and revenue equivalence holds in such a type space. Hence, by restricting (f, p) and (f, q) to $L(s^j, s^{j+1})$, we obtain that $p(s^j) - p(s^{j+1}) = q(s^j) - q(s^{j+1})$ for all $j \in \{0, 1, \ldots, k\}$. Adding over all j, we get a telescopic sum on each side, which gives us $p(s) - p(t) = q(s) - q(t)$. But since (f, q) is (globally) incentive compatible, we know that $s \cdot f(s) - s \cdot f(t) \geq q(s) - q(t) = p(s) - p(t)$. Hence, incentive constraint $s \to t$ holds for (f, p). An identical argument shows incentive constraint $t \to s$ also holds

3. Consider the strategic voting model where agents have single peaked preferences. Let N be the set of n agents and A be a set of finite alternatives ordered according to an ordering $\succ$. Let $\mathcal{S}$ be the set of all single peaked preferences with respect to $\succ$. Each agent i has a preference in $\mathcal{S}$.

For every agent $i \in N$, let $P_i(1)$ denote the peak alternative in preference P_i. Let P^0 denote the single peaked preference where an agent has the peak at the lowest alternative with respect to $\succ$ and P^1 denote the single peaked preference where an agent has the peak at the highest alternative with respect to $\succ$.

Suppose $f : \mathcal{S}^n \to A$ be a strategy-proof and peaks-only social choice function (note: f need not be unanimous or anonymous). Show that for every $i \in N$ and at every preference profile P_{-i} of other agents, the following holds:

$$f(P_i, P_{-i}) = \text{median}(f(P^0, P_{-i}), P_i(1), f(P^1, P_{-i})).$$

[10 marks]

Answer. For such proofs, it is better to draw figures for each step and see what is going on.

Fix agent i and P_{-i}, and denote $a_0 = f(P^0, P_{-i})$ and $a_1 = f(P^1, P_{-i})$. First, we show that $a_1 = a_0$ or $a_1 \succ a_0$. If $a_0 \succ a_1$, then agent i will manipulate at (P^0, P_{-i}) to (P^1, P_{-i}) - a contradiction to strtaegy-proofness.

Next, choose any preference ordering P_i such that $a_0 = P_i(1)$ or $a_0 \succ P_i(1)$. We claim that $f(P_i, P_{-i}) = a_0$. Denote $f(P_i, P_{-i}) = a$. If $a_0 \succ a$, then agent i manipulates at (P^0, P_{-i}) to (P_i, P_{-i}). If $a \succ a_0$, then agent i manipulates at (P_i, P_{-i}) to (P^0, P_{-i}). Hence, strategy-proofness implies that $a_0 = a$. This also means that $f(P_i, P_{-i}) = a_0 = med(P_i(1), a_0, a_1)$.

An analogous argument establishes that $f(P_i, P_{-i}) = a_1$ if $a_1 = P_i(1)$ or $P_i(1) \succ a_1$. This also means that $f(P_i, P_{-i}) = a_1 = med(P_i(1), a_0, a_1)$.

So, we are left with the case when $a_1 \succ P_i(1) \succ a_0$. Notice that $med(a_0, P_i(1), a_1) = P_i(1)$. Let $f(P) = a$. Assume for contradiction $a \neq P_i(1)$. Then, there are two cases, $a \succ P_i(1)$ or $P_i(1) \succ a$. We give a proof for $a \succ P_i(1)$ - the other case is analogous. If $a \succ P_i(1)$, then $a \succ a_0$. Consider a preference ordering P'_i of agent i such that $a_0 \; P'_i \; a$ and $P'_i(1) = P_i(1)$ - this is possible in the single peaked domain since a_0 and a lie in opposite sides of $P_i(1)$. By peaks-only property, $f(P'_i, P_{-i}) = f(P_i, P_{-i}) = a$. But $a_0 \; P'_i \; a$ implies that agent i will manipulate at (P'_i, P_{-i}) to (P^0, P_{-i}).

4. Consider the one-sided matching problem with 4 agents and 4 objects $\{a, b, c, d\}$. Consider a preference profile $\succ \equiv (\succ_1, \succ_2, \succ_3, \succ_4)$ defined as follows:

$$\begin{aligned} a \succ_j b \succ_j c \succ_j d \qquad & \text{if } j \in \{1, 2\} \\ b \succ_j a \succ_j d \succ_j c \qquad & \text{if } j \in \{3, 4\} \end{aligned}$$

Consider the uniform randomization over the 24 priorities of agents and the corresponding uniform random priority rule. Let $Q(\succ)$ be the random matching produced by the uniform priority rule at this profile.

(a) Write down the bistochastic matrix $Q(\succ)$. [**3 marks**]

There are 24 possible priorities. Out of it, 6 of them agent 1 is ranked first. So, he gets a. Another 4 priorties, agent 3 or 4 are ranked first and he is ranked second. So, he gets a in them too. He gets a in 10 priorities. If he is ranked second and agent 2 is ranked first, he gets b, and this happens 2 priorities. If he is ranked third, he gets c, which happens in 6 priorities. If he is ranked fourth, agent 2 is not ranked third, agent 1 gets c also. This happens in 4 priorities. So, he gets c in 10 priorities. The rest 2 priorities he gets d. The calculation for agent 2 is identical. Calculations for agents 3 and 4 are identical with the role of a and b switched and c and d switched.

Consider the following bistochastic matrix where rows are for agents (respectively, 1,2,3,4) and columns are objects (a, b, c, d).

$$Q(\succ) = \begin{bmatrix} \frac{10}{24} & \frac{2}{24} & \frac{10}{24} & \frac{2}{24} \\ \frac{10}{24} & \frac{2}{24} & \frac{10}{24} & \frac{2}{24} \\ \frac{2}{24} & \frac{10}{24} & \frac{2}{24} & \frac{10}{24} \\ \frac{2}{24} & \frac{10}{24} & \frac{2}{24} & \frac{10}{24} \end{bmatrix}$$

(b) Construct a random matching (a bistochastic matrix) Q' such that for every agent i, Q'_i first-order-stochastic-dominates $Q_i(\succ)$, where Q'_i denotes the probability distribution over objects for agent i in matching Q'. [**4 marks**]

Answer. Consider the following bistochastic matrix Q'.

$$Q' = \begin{bmatrix} \frac{11}{24} & \frac{1}{24} & \frac{10}{24} & \frac{2}{24} \\ \frac{11}{24} & \frac{1}{24} & \frac{10}{24} & \frac{2}{24} \\ \frac{1}{24} & \frac{11}{24} & \frac{2}{24} & \frac{10}{24} \\ \frac{1}{24} & \frac{11}{24} & \frac{2}{24} & \frac{10}{24} \end{bmatrix}$$

Clearly Q'_i FOSD $Q_i(\succ)$ for each i according to the preference $\succ_i$.

(c) Does this suggest that the uniform random priority rule may not generate an "efficient" random assignment? Clearly define the notion of efficiency used here. **[3 marks]**

Answer. This suggest a notion of efficiency, usually called **ordinal efficiency**, which is violated here. We say a matching Q is ordinally efficient at $\succ$ if there **does not exist** another matching Q' such that Q'_i FOSD Q_i according to $\succ_i$ for each i. The above example shows that uniform random priority is **not ordinally efficient**.