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Two-sided matching with indifferences ☆

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Abstract

Most of the two-sided matching literature maintains the assumption that agents are never indifferent between any two members of the opposite side. In practice, however, ties in preferences arise naturally and are widespread. Market design needs to handle ties carefully, because in the presence of indifferences, stability no longer implies Pareto efficiency, and the deferred acceptance algorithm cannot be applied to produce a Pareto efficient or a worker-optimal stable matching.

We allow ties in preference rankings and show that the Pareto dominance relation on stable matchings can be captured by two simple operations which involve rematching of workers and firms via *cycles* or *chains*. Likewise, the Pareto relation defined via workers' welfare can also be broken down to two similar procedures which preserve stability. Using these structural results we design *fast* algorithms to compute a Pareto efficient and stable matching, and a worker-optimal stable matching.

JEL classification: C78; D47; D02

Keywords: Two-sided matching; Matching with ties; Matching with indifferences; Efficient and stable matching

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1. Introduction

Much of the matching market design literature relies on all preference (and priority) rankings being strict. However, in many settings, ties in preferences and priorities are widespread. For example, agents on one side of the market might treat some alternatives as identical (e.g., entry level posts in a firm or seats in a school) even when those alternatives have different preferences or criteria in the way they rank potential matches.[1] Even when potential matching partners are different, some agents may well be indifferent between some and be willing to reveal so when asked.[2] Faced with a large number of alternatives to consider, the task of evaluating and ranking them all can be prohibitive, and the agents might instead use simple scoring systems (such as high, medium, low) which lead to coarse rankings. In allocating public resources (as in school admissions), large numbers of recipients are given equal priority ranking for legal reasons. In various practical matching platforms, instead of observing all possible alternatives, agents express simple criteria to describe their preferences. For example admission criteria to courses, software compatibility between a programmer and a task, or expressing availability on a scheduling platform lead to preferences with large indifference classes.[3]

We argue in this paper that the way such ties are handled has important consequences, not only from a distributional perspective, but also in terms of overall welfare. We show why matching mechanisms should be cautious about treating ties arbitrarily, and we propose novel designs to deal with indifferences carefully and efficiently.

The notion of stability in a two-sided matching market rests on the premise that each agent should prefer their match to staying unmatched (*individual rationality*), and that a matching should be robust to two unmatched agents' temptation to match because both prefer each other over their current partners (*no blocking pairs*).[4] Clearly, if either agent in a blocking pair is indifferent between the new prospect and their current match, stability of the matching is not

1 When firms [schools] care about the match-specific quality or the overall composition of workers [students] due to diversity concerns, firms' preferences [schools' priorities] over workers [students] depend on the specific post [seat], even though workers [students] will perceive posts [seats] at a given firm [school] as identical. See, e.g, Kominers and Sönmez (2016) for a detailed analysis of such preferences [priorities].

2 In their study of school choice in Amsterdam, when de Haan et al. (2016) asked parents to assign numerical values to schools, some assigned equal values to some schools even though they had the option of rating schools with a fine metric. Scottish Foundation Allocation Scheme (which matches about 750 student doctors to 50 training programmes every year) allows hospitals to express ties in preferences. Irving (2008) reports that most hospitals do express ties: some rank doctors in three indifference classes, whereas some have many more indifference classes with several doctors in each class. This suggests that hospitals maintain ties in their preferences even after substantial deliberation over the alternatives.

3 Harvard University Freshman Seminars Program encourages faculty members to accept all students who meet their criteria for admission. If a faculty member wishes, they can rank order applicants allowing for ties. For details, see http://freshmanseminars.college.harvard.edu/how-review-and-rank-applications. Another example with widespread ties on both sides of the market is the Harvard Business School Health Care Initiative's matching of alumni to students for mentoring, where coarse rankings appear to originate from simple criteria that guide preferences. See http://www.hbs.edu/healthcare/for-alumni/Pages/default.aspx.

4 Roth (2002) finds strong correlation between a clearinghouse being successful and its delivering stable matchings. Various regional markets for new physicians and surgeons in the UK provide field data on this, and the lab experiments by Kagel and Roth (2000) confirm this prediction in a controlled environment. In the context of school choice (Abdulkadiroğlu and Sönmez, 2003) the very same notion captures the idea of *respecting priorities*. Accordingly stability has been a central property of many centralized matching schemes.

threatened.[5] While weak preferences abound in various real life settings, in practice most centralized matching mechanisms either force agents to reveal strict rankings or break ties as part of the mechanism. Tie-breaking introduces new potential “blocking pairs”, and therefore new stability constraints. Thanks to Gale and Shapley (1962) we know that a stable matching exists in the strict preferences environment derived after tie-breaking. Of course this matching is stable with respect to the original preferences, because after all whichever pair that blocks in the absence of ties would continue to be a blocking pair after ties are broken. However, any form of tie-breaking will typically “destabilize” some matchings that were stable in the original problem with indifferences. This might have serious welfare consequences, as we illustrate in the following example. Consider two firms, A and B, with one position each, and two workers i and j. Suppose firm B prefers i over j, denoted $i \succ_B j$; whereas worker j prefers firm A over B, denoted $A \succ_j B$. Firm A and worker i are indifferent between who they match with, denoted $i \sim_A j$ and $A \sim_i B$, respectively:

Actual preferences with ties

$\succsim_A$	$\succsim_B$
$i \sim_A j$	i
	j

$\succsim_i$	$\succsim_j$
$A \sim_i B$	A
	B

If everyone breaks ties alphabetically when asked to reveal strict rankings, we obtain the strict preferences, denoted $\succsim'$, below:

Preferences derived via tie-breaking and the resulting stable matching

$\succsim'_A$	$\succsim'_B$
$\boxed{i}$	i
j	$\boxed{j}$

$\succsim'_i$	$\succsim'_j$
$\boxed{A}$	A
B	$\boxed{B}$

For these strict rankings $\succsim'$, there is a unique stable matching $\begin{pmatrix} i & j \\ A & B \end{pmatrix}$. However, if the workers swap their jobs, both worker j and firm B are better off, while worker i and firm A are as happy as before. Moreover, this new matching is stable with respect to the actual preferences $\succsim$. The alphabetical tie-breaking turns $\{i, A\}$ into a blocking pair for the unique efficient matching $\begin{pmatrix} i & j \\ B & A \end{pmatrix}$, and thus *de-stabilizes* it. Consequently, $\begin{pmatrix} i & j \\ A & B \end{pmatrix}$ survives as the only stable matching, leaving both sides of the market worse off compared with the stable and efficient matching. Moreover, applying the *workers-proposing deferred acceptance algorithm* after the above tie-breaking does not return a *worker-optimal stable matching*, invalidating one of Gale and Shapley’s key results.

The extent of the inefficiency due to not properly handling ties will, of course, depend on the realization of preferences. To get a sense of the potential scope of inefficiency, consider a market consisting of $n \geq 2$ workers and an equal number of firms each having one position to fill. Every agent finds those on the other side acceptable. Firms are indifferent between any two workers. Each worker w_i has a strict preference top ranking firm f_i and bottom ranking firm f_{i-1} (mod n) for $i = 0, 1, \ldots, n-1$. Both the matching ν which assigns w_i to f_i and the matching μ which assigns each w_i to f_{i-1} (mod n) are stable. The size of the Pareto inefficiency in the

[5] It is not plausible that an agent would be part of a blocking pair unless it was worthwhile to do so. Therefore the standard definition of a blocking pair requires that both parties should be strictly better off from getting together.

stable matching μ is n in terms of the number of affected agents, and $n(n-1)$ in terms of the total steps up the preference lists of the agents.[6]

Despite the simplicity of our motivating example and the wide range of contexts where indifferences are common, the literature has been missing an effective way to compute an efficient and stable matching or a worker-optimal stable matching. Our findings regarding the structure of the set of stable matchings allow us to resolve these issues via economically intuitive and *computationally fast*[7] algorithms.

Since our model and terminology are closely related to those of Erdil and Ergin (2008, henceforth EE'08) it is worth highlighting how the current work differs in its conceptual framework and technical novelty. EE'08 is motivated by improving efficiency in school choice, and critically, only students' preferences constitute the welfare criteria in their analysis. In contrast, we begin with a two-sided efficiency concept. The difference between the one-sided and the two-sided perspectives is stark. First, in the absence of ties, there isn't even a question of efficiency when both sides' preferences matter, because *all* stable matchings are efficient. In a one-sided analysis, however, there is usually a conflict between stability and one-sided efficiency. When respecting exogenously imposed priorities is indispensable, the welfare benchmark for the one-sided analysis is constrained efficiency (i.e., student-optimal stability). Gale and Shapley (1962) solve this stability constrained efficiency problem when there are no ties. EE'08 note that this solution fails if school priorities have ties (as is often the case in practice), and suggest a novel class of mechanisms (*stable improvement cycles*) to achieve student-optimal stable allocations. Abdulkadiroğlu et al. (2009) find strong empirical evidence that the way ties are resolved in school priorities matter significantly for student's welfare.[8] However the one-sided approaches of EE'08 and APR'09 have *no* implications for two-sided efficiency whether there are ties or not. In particular *stable improvement cycles* improve students' welfare always at the expense of schools' welfare. Our two-sided perspective in the current work sheds light on a separate and wider class of applications where agents on *both* sides of the market feature in the efficiency analysis. Secondly, instead of focusing on a careful treatment of ties in *exogenously fixed priorities* for allocation problems, we are motivated by the prevalence of ties in *preferences* on *both* sides of matching problems. Given the remarkable growth of matching platforms with increasing numbers of participants, the possibility of expressing coarse preferences is becoming an attractive design feature. A key insight for these markets follows from our findings: allowing, even encouraging, agents to express indifferences when ranking alternatives not only simplifies preference

6 Diebold and Bichler (2017) present simulations to compare the performance of a number of matching algorithms. Their efficiency notion is one-sided, but they do include *Efficient and Stable Matching Algorithm (ESMA)* which we introduce in Section 3. It is worth noting that even though ESMA insists on improving both workers and firms, Diebold and Bichler find that it improves the average match rank for workers by about 10%.

7 The computer scientific benchmark for what makes an algorithm fast it its so-called *computationally complexity* which is roughly the number of "steps" an algorithm requires to complete as a function of the size of the input of the algorithm. While this function being a polynomial is sufficient to call an algorithm fast in the asymptotic sense, whether it performs fast in practice is an empirical question which depends on the size of the data set. Both our simulations and those of other authors (e.g., Diebold and Bichler, 2017) who employ our algorithms (e.g., Diebold and Bichler, 2017) confirm that these algorithms are indeed fast enough with real life size data sets even to be able to run thousands of simulations on standard computers.

8 In particular, they find that of the 73115 students matched by deferred acceptance (DA) following arbitrary tie-breaking, stable improve cycles would improve 1488 students. If the improvements are weighted to incorporate how many schools these students go up in their preferences, the total size of the improvement is 3600 steps up the preference lists.

revelation and market participation, but also improves efficiency.[9] The extent of such efficiency gains can be large, and our *computationally fast* algorithms enable such welfare improvements. In Section 4 we turn to *worker-optimality*, bringing our framework closer to EE'08 in spirit, but with a more general model incorporating indifferences in workers' preferences, which allows a wider range of applications. This generalization is far from a straightforward technical exercise, and at a conceptual level, it requires the notion of *stable improvement chains* which could not have existed in the environment of EE'08. In contrast with *cycles* which rotate workers between firms (while preserving stability), these *chains* allow the size of the matching to grow by assigning an unemployed worker in place of an employed one, shifting employed workers to the places of other employed ones, and finally filling an empty post with a previously employed worker (while preserving stability). In particular, such worker-improvement chains can change the size of the matching (not possible with cycles), and might even improve firms' welfare (again never possible with worker-improvement cycles). We defer the discussion of technical differences to the relevant sections, and summarize below our main results.

A brief summary of results

Our first structural result (Theorem 1) establishes the nature of the Pareto dominance relation on the set of stable matchings. Namely, a stable matching μ can be Pareto dominated if and only if the workers can form a trading cycle or a trading chain, where every worker and firm involved gets weakly better off, with at least one of them getting strictly better off. Therefore, it is sufficient to search for these Pareto improvement (PI) cycles and chains to check whether a stable matching is Pareto efficient or not. Using the fact that Pareto improvements preserve stability (Lemma 1), we conclude that by successively searching for PI-cycles and chains and carrying them out whenever found, we can reach a Pareto efficient and stable matching. This procedure, which we call the Efficient and Stable Matching Algorithm, has polynomial time complexity, and it is remarkably fast in real-life size datasets.

In some applications, the policy maker compares matchings according to one side of the market. Perhaps the best known example is school choice. Stability matters, because it captures the notion of *respecting priorities*. Beyond that, the welfare considerations involve students' preferences only. Therefore, the concern is to find a student-optimal stable matching. While Gale and Shapley's (1962) deferred acceptance algorithm yields one when both school and student rankings are strict, it fails to do so when there are ties. Our second main result, Theorem 2, provides a clear solution. If a stable matching is not worker-optimal stable, then it admits a trading cycle or a trading chain, where workers get weakly better off, stability is preserved, and at least one worker gets strictly better off. Our Worker-Optimal Stable Matching Algorithm is a successive search for such cycles and chains until we exhaust them, and hence reach a worker optimal stable outcome. In contrast with carrying out an exhaustive search (which would be of exponential time complexity and prohibitively slow in any reasonable size of real life data), our algorithm is polynomial time and performs fast in practice.

Our main message is that centralized matching mechanisms can make use of existing ties in preferences and priorities. When agents are forced to reveal strict rankings over alternatives they are indifferent about, mechanisms can result in efficiency loss. Our algorithms offer a practical solution to the problem of finding stable and efficient matchings when there are ties. Instead

[9] Irving's (2008) report on the Scottish doctor-hospital matching scheme echoes the idea of permitting, even encouraging, ties as a policy idea in the light of the efficiency gains we study.

of requiring agents to submit strict rank orders, the policy maker should allow and encourage revelation of ties in rankings when preferences involve indifferences.

Related literature

While the concept of indifferences or ties in preferences is widely acknowledged in practice, the implications for matching markets have been studied relatively little. We briefly note here some of the earlier work closest to our paper.

Crawford and Knoer (1981) incorporate the deferred acceptance algorithm into a model with discrete salaries. (See also Kelso and Crawford, 1982.) They note that ties invalidate Gale and Shapley's result of convergence to a worker-optimal stable matching. Monetary transfers ensures that the *strict core* is non-empty, while it can be empty in our environment. The difference between the core and the strict core in their model resembles the difference between inefficient stable and efficient stable matchings in our model. Their computation of a strict core allocation is not in polynomial time. Jones (1983) modifies the Crawford–Knoer algorithm to get polynomial time convergence in a market with discrete valuations and salaries. His algorithm is different in nature from ours, and neither of the procedures can be translated into the other's environment to solve the same problem.

Dur et al. (2017), in their study of the Boston School Choice System, identify a widespread phenomenon of ties in student preferences. A school in Boston can rank students differently depending on which *slot* it is considering. These slots are identical from the perspective of the student, but the admissions system treats them differently in order to have some control over diversity. The mechanism used in Boston specifies an order (called a *precedence order*) over slots to fill these slots. Effectively, this is using a tie-breaking rule: each student's indifference over the slots at a school are resolved according to the precedence order. Their empirical results show that the distributional consequences of different precedence orders (i.e., different tie-breaking rules) are huge. Hence the effect of ties in students' preferences is not a mere theoretical curiosity. Kominers and Sönmez (2016) study slot specific priorities in a much more general set up which allows contracts. In these papers, the notion of indifference is widespread, but it is implicit and is restricted to indifferences over slots in the same school (or at the same branch, respectively). It is worth noting, however, that these models do not capture the observed ties in various matching platforms in practice such as those we mention in Footnotes 2 and 3.

Another special case of our model is Shapley and Scarf's (1974) housing market model. The top trading cycles mechanism is well known to be efficient, but this result relies on all preferences being strict. Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) allow the agents to have ties in their preferences, and design strategy-proof and Pareto efficient mechanisms.

Bogomolnaia and Moulin (2004) restrict preferences to be dichotomous: each agent views those on the other side of the market as either acceptable or unacceptable. By allowing randomized mechanisms, which can be interpreted as time sharing, they obtain efficient and strategy-proof mechanisms. Their results and techniques, while powerful in the dichotomous preference domain, do not generalize to our environment.

2. The model

Let W and F denote disjoint finite sets of workers and firms, respectively. Let $A = W \cup F$ stand for the set of all agents. Let $q = (q_f)_{f \in F}$ where $q_f \geq 1$ denotes the number of positions that firm f has. A **preference profile** is a vector of weak orders (complete and transitive relations)

$\succsim = (\succsim_a)_{a \in A}$ where $\succsim_w$ denotes the preference of worker w over $F \cup \{\varnothing\}$ and $\succsim_f$ denotes the preference of firm f over $W \cup \{\varnothing\}$. For a worker, $\varnothing$ represents being unemployed. For a firm, it stands for an unfilled position. Let $\succ_a$ and $\sim_a$ denote the antisymmetric and symmetric parts of $\succsim_a$, respectively. Throughout, we will assume that there is no worker w and firm f such that $w \sim_f \varnothing$ or $f \sim_w \varnothing$. We will call this the **no indifference to unemployment/vacancy (NI$\varnothing$)** assumption. A worker w is said to be **acceptable** to firm f if $w \succ_f \varnothing$; similarly a firm f is acceptable to worker w if $f \succ_w \varnothing$. A preference profile $\succsim = (\succsim_a)_{a \in A}$ is **strict** if $\succsim_a$ is anti-symmetric for each $a \in A$.

A strict $\succsim'$ is called a **tie-breaking** of $\succsim$ if $x \succsim_a y$ implies $x \succsim'_a y$ for all $x, y, a \in A$. This general formulation allows different agents to break ties differently. If, on the other hand, every agent a breaks ties in $\succsim_a$ according to the same linear order, then we obtain a *single tie-breaking*. Formally speaking, $\succsim'$ is called a **single tie-breaking** of $\succsim$, if there exist bijections $\phi^W : W \to \{1, \ldots, |W|\}$ and $\phi^F : F \to \{1, \ldots, |F|\}$ such that:

$$f \sim_w g \implies [f \succ'_w g \Leftrightarrow \phi^F(f) < \phi^F(g)],$$

and

$$w \sim_f v \implies [w \succ'_f v \Leftrightarrow \phi^W(w) < \phi^W(v)],$$

for all $w, v \in W$ and $f, g \in F$.

Note that our model specifies firms' preferences only over workers. This preference information is enough to check for stability of a given matching. However in order to conduct welfare analysis, we need to know more about how firms rank *sets* of workers. Let 2^W stand for the set of subsets of W. We will extend[10] the preference $\succsim_f$ over $W \cup \{\varnothing\}$, to a reflexive and transitive (but typically incomplete) preference $\succsim_f$ over 2^W. A preference relation $\tilde{\succsim}_f$ over 2^W is called **responsive** if it is complete, transitive, and for any $S, T, K \subseteq W$ where $S \cap K = T \cap K = \varnothing$ and $|S|, |T| \leq 1$:

$$(S \cup K) \tilde{\succsim}_f (T \cup K) \iff S \tilde{\succsim}_f T.$$

Now, for any two subsets I and J of W, we will define $I \succsim_f J$ if and only if $I \tilde{\succsim}_f J$ for every responsive extension $\tilde{\succsim}_f$ of $\succsim_f$. This is a "minimal responsive extension" in the following sense. Given f's preferences over individual workers, and the fact that f's preferences over sets of workers are responsive, we will conclude that f weakly prefers I to J if and only if $I \succsim_f J$. A useful observation[11] is that $I \succsim_f J$ if and only if the sets I and J can be indexed as $I : i_1, \ldots, i_n$ and $J : j_1, \ldots, j_n$, where for each worker short of n, a copy of $\varnothing$ is written and $i_t \succsim_f j_t$ for each $t \in \{1, \ldots, n\}$.

A **matching** is a function $\mu : W \to F \cup \{\varnothing\}$ such that $|\mu^{-1}(f)| \leq q_f$ for each $f \in F$. A matching μ is **individually rational** if $\mu(w) \succsim_w \varnothing$ for each worker w; and $v \succsim_f \varnothing$ for each $v \in \mu^{-1}(f)$ and firm f. Given a matching μ, a worker firm pair (w, f) is said to form a **blocking pair** if (i) $f \succ_w \mu(w)$, and (ii) $w \succ_f v$ for some $v \in \mu^{-1}(f)$, or $|\mu^{-1}(f)| < q_f$ and $w \succ_f \varnothing$. A matching μ is **stable** if it is individually rational and if there is no blocking pair.[12]

[10] An extension is naturally defined as: the preference $\tilde{\succsim}_f$ is an **extension** of $\succsim_f$ if (i) for any $w, v \in W$: $\{w\} \tilde{\succsim}_f \{v\}$ if and only if $w \succsim_f v$ and (ii) for any $w \in W$, $\{w\} \tilde{\succsim}_f \varnothing$ if and only if $w \succsim_f \varnothing$.

[11] The proof of this observation is included in the Appendix for completeness.

[12] Relaxing this condition to allow one side to be indifferent in a blocking pair would give us the notion of a *strongly stable* matching, which may fail to exist. For example, when there is a single firm with one position, and two workers who both find the firm acceptable, if the firm is indifferent between the workers, then no matching is strongly stable.

We define the partial orders $\succsim_W$, $\succsim_F$ and $\succsim_A$ on the set of matchings as follows. Let $\mu \succsim_W \nu$, if $\mu(w) \succsim_w \nu(w)$ for each $w \in W$; let $\mu \succsim_F \nu$, if $\mu^{-1}(f) \succsim_f \nu^{-1}(f)$ for each $f \in F$; and let $\mu \succsim_A \nu$ if $\mu \succsim_W \nu$ and $\mu \succsim_F \nu$. Let $\sim_W$, $\sim_F$, and $\sim_A$ denote the symmetric parts, whereas $\succ_W$, $\succ_F$, and $\succ_A$ denote the asymmetric parts of these relations. A matching μ **Pareto dominates** ν if $\mu \succ_A \nu$. This is equivalent to the requirement that all workers and firms weakly prefer μ to ν, and at least one worker or a firm strictly prefers μ to ν. A matching is **Pareto efficient** if it is not Pareto dominated by any other matching. A stable matching μ is called **W-optimal stable** if there is no stable matching ν such that $\nu \succ_W \mu$.

Gale and Shapley (1962) described an algorithm, which is polynomial-time in the number of workers and firms, that yields a stable matching for a strict preference profile $\succsim$. This is known as the worker proposing **deferred acceptance (DA)** algorithm:

> At the first step, every worker applies to her favorite acceptable firm. For each firm f, q_f most preferred acceptable applicants (or all if there are fewer than q_f) are placed on the waiting list of f, and the others are rejected.
>
> At the kth step, those applicants who were rejected at step $k-1$ apply to their next best acceptable firms. For each firm f, the most preferred acceptable q_f workers among the new applicants and those in the waiting list are placed on the new waiting list and the rest are rejected.

The algorithm terminates when every worker is either on a waiting list or has been rejected by every firm that is acceptable to her. After this procedure ends, firms admit workers on their waiting lists which yields the desired matching. When $\succsim$ is strict, $DA_W(\succsim)$ denotes the outcome of the worker proposing DA algorithm.

Theorem. *(Gale and Shapley, 1962) When preferences are strict, the worker proposing deferred acceptance algorithm returns the unique worker-optimal stable matching.*

The DA algorithm is not well-defined when the preference profile $\succsim$ is not strict. When there are indifferences in preferences, the above algorithm is employed after the ties are exogenously broken. Since a matching that is stable with respect to a tie-breaking $\succsim'$ of $\succsim$ is also stable with respect to $\succsim$, an immediate corollary of the above theorem is that there always exists a stable matching in our model.[13] However, as we have illustrated in the introduction, using the DA after arbitrary tie-breaking does not result in an efficient nor worker-optimal stable matching. Are there intuitive and computationally efficient algorithms to find Pareto efficient and/or worker-optimal stable matchings? Before we design such algorithms, we proceed to develop a better understanding of the structure of the set of stable matchings.

3. Pareto efficient and stable matchings

We start by noting that a matching must be stable if every agent weakly prefers it to some other stable matching.

Lemma 1. *If $\nu \succsim_A \mu$ for some stable matching μ, then ν is also stable.*

[13] To every stable matching, corresponds at least one tie-breaking at which it is stable. On the other hand it is far from clear which tie-breaking rules should be used in order to find a Pareto efficient, or a worker-optimal stable matching.

Proof. Since μ is individually rational and every agent is weakly better-off at ν, ν is also individually rational. Next consider any worker firm pair (w, f). There exist enumerations $\nu^{-1}(f) : i_1, \ldots, i_{q_f}$ and $\mu^{-1}(f) : j_1, \ldots, j_{q_f}$ such that for each worker short of q_f a copy of $\varnothing$ is inserted and $i_t \succsim_f j_t$ for $t \in \{1, \ldots, q_f\}$. If (w, f) is a blocking pair for ν, then $f \succ_w \nu(w)$ and $w \succ_f i_t$ for some $t \in \{1, \ldots, q_f\}$. But then $f \succ_w \nu(w) \succsim_w \mu(w)$ and $w \succ_f i_t \succsim_f j_t$ implying that (w, f) is a blocking pair for μ, a contradiction. □

Lemma 1 implies that a stable matching that is not Pareto efficient is Pareto dominated by a stable matching. Therefore, starting from an arbitrary stable matching, it is possible to reach a Pareto efficient and stable matching through a finite sequence of Pareto improving stable matchings.

Corollary 1. *There exists a stable and Pareto efficient matching.*

The argument behind Corollary 1 suggests a constructive method to find a Pareto efficient and stable matching. However the argument does not explicitly specify (1) how to check whether a given stable matching μ is Pareto efficient, and (2) if not, how to find a matching that Pareto dominates it. Since the model is finite, one can imagine answering these questions by comparing μ exhaustively to every other matching. However such an approach is computationally infeasible, since the number of matchings grows exponentially in $\min\{|W|, |F|\}$. In order to produce a stable and Pareto efficient matching in a real-life centralized matching market, it is therefore necessary to provide polynomial time methods to answer these questions.

Given a preference profile $\succsim$ and a matching μ, we will next introduce and discuss two tests: the existence of *Pareto improvement cycles* and the existence of *Pareto improvement chains*. The existence of these cycles or chains will immediately imply that μ is not Pareto efficient. Conversely we will prove, in Theorem 1, that if such cycles or chains do not exist for a stable matching μ, then μ is Pareto efficient. We will then use these findings to describe a polynomial time method for producing a stable and efficient matching.

Definition 1. A **Pareto improvement (PI) cycle** consists of distinct workers $w_1, \ldots, w_n \equiv w_0$ ($n \geq 2$) such that:

(i) Each w_t is matched to some firm,
(ii) $\mu(w_{t+1}) \succsim_{w_t} \mu(w_t)$ and $w_t \succsim_{\mu(w_{t+1})} w_{t+1}$ for $t \in \{0, 1, \ldots, n-1\}$,
(iii) At least one of the preference relations in (ii) is strict for some $t \in \{0, 1, \ldots, n-1\}$.

Each worker w_t in a PI-cycle *weakly desires* the position of the following worker w_{t+1}, and the employer $\mu(w_{t+1})$ of the latter would not mind replacing w_{t+1} with w_t. Moreover, at least one worker strictly *envies* the following worker or at least one firm $\mu(w_{t+1})$ prefers w_t to w_{t+1}. If there is a PI-cycle, then the matching μ can be Pareto improved, where the Pareto dominating matching μ' is obtained by letting each worker move into the firm of the next worker:

$$\mu'(w) = \begin{cases} \mu(w_{t+1}) & \text{if } w = w_t \text{ for some } t \in \{0, \ldots, n-1\}, \\ \mu(w) & \text{otherwise.} \end{cases}$$

Definition 2. A **Pareto improvement (PI) chain** consists of distinct workers $w_1, \ldots, w_n$ ($n \geq 2$) and a firm f with an empty position such that:

(i) a. w_1 is unmatched,
    b. w_t is matched with some firm for $t \in \{2, \dots n\}$,

(ii) a. $\mu(w_{t+1}) \succsim_{w_t} \mu(w_t)$ and $w_t \succsim_{\mu(w_{t+1})} w_{t+1}$ for $t \in \{1, \dots, n-1\}$.
    b. $f \succsim_{w_n} \mu(w_n)$ and $w_n \succsim_f \varnothing$.

Each worker w_t in a PI-chain except w_n, weakly envies the following worker w_{t+1}, and as in a PI-cycle, the employer $\mu(w_{t+1})$ of the latter would not mind replacing w_{t+1} with w_t. The last worker w_n weakly desires the empty position of f and is acceptable to f. Note that in the definition of a PI-chain, we do not need to require that at least some of the preferences in (ii) is strict, because the NI$\varnothing$ assumption guarantees that $\mu(w_2) \succ_{w_1} \varnothing = \mu(w_1)$ and $w_n \succ_f \varnothing$.[14] Moreover the requirement that w_1 is not matched is crucial for μ' to Pareto dominate μ, because otherwise w_1's employer could be worse-off at μ'. If there is a PI-chain, then the matching μ can be Pareto improved, where the Pareto dominating matching μ' is obtained by letting each worker other than w_n move into the firm of the next worker and letting w_n move to f:

$$\mu'(w) = \begin{cases} \mu(w_{t+1}) & \text{if } w = w_t \text{ for some } t \in \{1, \dots, n-1\}, \\ f & \text{if } w = w_n, \\ \mu(w) & \text{otherwise.} \end{cases}$$

By **carrying out a PI-cycle or a PI-chain**, we mean constructing the new matching μ' which Pareto dominates μ as in above. Our next theorem proves a converse to the above observations: if μ is stable and there are no PI-cycles nor PI-chains, then we can conclude that μ is Pareto efficient.[15]

A directed graph $G = (V, E)$ consists of a set V of vertices and a set E of directed edges, where a directed edge is an ordered pair of vertices, i.e., an element of the cartesian product $V \times V$. The word 'directed' will be omitted throughout the text. We will write an edge (x, y) as $x \to y$ as we will visualize the vertices as nodes, and the edges as arrows between these nodes. A **directed cycle** in G consists of distinct vertices $x_0, \dots, x_{n-1}$ $(n \geq 2)$ such that $x_0 \to x_1 \to \dots \to x_{n-1} \to x_n \equiv x_0$.[16] We will simply refer to these as 'cycles' for the rest of the text unless we prefer to emphasize the directed structure. Note that, given a directed graph, if each vertex of this graph has exactly one arriving and one leaving edge, then each edge of the graph is part of a cycle.

Theorem 1. *A stable matching is Pareto efficient if and only if it does not admit PI-cycles nor PI-chains.*

[14] Note also that if the matching μ is stable, then in part (ii) of the definition of a PI-cycle and part (ii.a) of the definition of a PI-chain, at least one of the preferences should be an indifference for each t. Similarly in part (ii.b) of the definition of a PI-chain, we must have $f \sim_{w_n} \mu(w_n)$.

[15] It is possible to embed the definition of a PI-chain into that of a PI-cycle as follows: introduce sufficiently many "ghost workers" and "ghost firms" into the model such that each ghost worker is indifferent between all firms including the ghost firms, each actual firm is indifferent between all ghost workers, finds them acceptable, and ranks them below all actual workers it finds acceptable; each ghost firm is indifferent between all workers including ghost workers; and each actual worker is indifferent between all ghost firms, finds them acceptable, and ranks them below all actual firms she finds acceptable. Now, w_1 in the PI-chain is matched with a ghost firm, $\varnothing^F$, and the vacancy at f is occupied by a ghost worker $\varnothing^W$. By letting $\varnothing^W$ move to $\varnothing^F$, we can interpret the PI-chain of the original model as a PI-cycle of the modified model.

[16] Our terminology allows for self pointing edges $x \to x$, but we do not call them cycles since our definition of a cycle involves at least two distinct vertices.

Proof. It only remains to prove the "if" part. Assume that μ is stable but not Pareto efficient and let ν be a matching that Pareto dominates μ. Then by NI$\varnothing$, every worker matched at μ is matched at ν, and each firm is matched with at least as many workers at ν as it is matched at μ. Let $W' = \{w \in W \mid \mu(w) \neq \nu(w)\}$ and note that by NI$\varnothing$, each worker in W' is matched to a firm at ν. For each firm f fix enumerations $\nu^{-1}(f) : i_1^f, \dots, i_{q_f}^f$ and $\mu^{-1}(f) : j_1^f, \dots, j_{q_f}^f$ such that (1) for each worker short of q_f a copy of $\varnothing$ is inserted, (2) $\nu(j_t^f) = f \Rightarrow i_t^f = j_t^f$, and (3) $i_t^f \succsim_f j_t^f$, for $t \in \{1, \dots, q_f\}$. Construct a directed graph G with the vertex set W' as follows. For any $w \in W'$, consider the unique t such that $w = i_t^{\nu(w)}$, and let $w \rightarrow j_t^{\nu(w)}$ if $j_t^{\nu(w)} \neq \varnothing$. Note that if $w \rightarrow v$ then $\mu(v) \succsim_w \mu(w)$, and $w \succsim_{\mu(v)} v$. Call an edge of G *strict* if one of these preferences is strict and denote a strict edge by $w \twoheadrightarrow v$.

If there is no extra worker matched at ν, each firm must be matched with the same number of workers in μ and ν. In particular each vertex in G has exactly one leaving edge and one arriving edge. Therefore each edge in this directed graph must be part of a cycle. Since ν Pareto dominates μ, G must have a strict edge. In particular each strict edge is part of a cycle, leading to a PI-cycle.

If there is a worker w_1 who is matched at ν but not at μ, then by NI$\varnothing$ and stability of μ, $\nu(w_1)$ cannot have an empty position at μ. Therefore there exists a worker w_2 such that $w_1 \twoheadrightarrow w_2$. Then either w_2 moved to a firm with an empty position at μ or there is a worker w_3 such that $w_2 \rightarrow w_3$. In the first case, w_1, w_2, and $\nu(w_2)$ form a PI-chain. In the second case, w_3 must have moved to a firm which had an empty position at μ, or there is a worker w_4 such that $w_3 \rightarrow w_4$. In the first case, w_1, w_2, w_3, and $\nu(w_3)$ form a PI-chain. Proceeding analogously, we find a PI-chain in at most $|W'|$ steps. □

The above theorem naturally suggests an algorithm which returns a stable and Pareto efficient matching: First obtain a stable matching by applying the DA algorithm to a tie-breaking. So long as the matching is not Pareto efficient, by Theorem 1, there will be a PI-cycle or a PI-chain. If so, find one and carry it out to obtain a Pareto improving matching. Since the original matching is stable, the new matching continues to be stable by Lemma 1. Repeat this as long as the obtained matching has a PI-cycle or a PI-chain. A more precise description can be found in Appendix A.3.

By the finiteness of our model one cannot keep Pareto improving indefinitely, hence the procedure will stop after finitely many steps and yield a Pareto efficient matching. The fact that we started with a stable matching guarantees that each matching along the procedure, and in particular the final matching, is stable. We call this procedure the **Efficient and Stable Matching Algorithm (ESMA)**. We show in Proposition 1 in Appendix A.3, that the ESMA is polynomial in the number of workers and the total number of positions.[17] Computational efficiency of the algorithm is thanks to its ability to discover PI-cycles and PI-chains quickly. It is worth noting that, unlike the *stable improvement cycles algorithm* of Erdil and Ergin (2008), a straightforward depth-first-search for cycles in a directed graph is not sufficient in this setting. Instead we rely on Tarjan's (1972) algorithm to explore a graph's strongly connected components. See Appendix A.3 for the details.

[17] To be more accurate, the procedure we describe above corresponds to a family of algorithms because of the way we order the agents and the firms in various steps will affect the outcome. By specifying a precise a selection rule, we can completely describe a deterministic mechanism. We present one such rule in the Appendix.

In the domain of strict preferences, it is well known (Roth, 1982) that when both sides of the market are strategic actors, no stable mechanism is strategy-proof.[18] Since our domain with weak preferences generalize that of strict preferences, the impossibility result automatically extends.

4. Worker-optimal stable matchings

We next turn to the question of how to compute W-optimal stable matchings. Let μ be a stable matching for some fixed $\succsim$. We will say that a worker w **weakly [strictly] desires** firm f if $\mu(w) \neq f$ and she weakly [strictly] prefers f to her match at μ, that is $f \succsim_w \mu(w)$ [$f \succ_w \mu(w)$]. Let D_f^μ denote the set of workers who weakly desire f and are acceptable to f, such that there is no other worker who strictly desires f and is ranked strictly higher in $\succsim_f$. Clearly D_f^μ depends on $\succsim$, too, but for notational simplicity we suppress the dependence of D_f^μ on the preference profile.

Definition 3. A **stable worker improvement (SWI) cycle** consists of distinct workers $w_1, \ldots, w_n \equiv w_0$ $(n \geq 2)$ such that:

(i) Each w_t is matched to some firm,
(ii) $w_t \in D^\mu_{\mu(w_{t+1})}$ for each $t \in \{0, \ldots, n-1\}$,
(iii) $\mu(w_{t+1}) \succ_{w_t} \mu(w_t)$ for some $t \in \{0, 1, \ldots, n-1\}$.

Each worker w_t in an SWI-cycle weakly desires the employer of the following worker w_{t+1}. The employer of w_{t+1} (that is, firm $\mu(w_{t+1})$) finds w_t acceptable.[19] And there is no other worker who strictly desires $\mu(w_{t+1})$, and is ranked strictly higher than w_t by $\mu(w_{t+1})$.

If μ is a stable matching which admits an SWI-cycle, then it can be improved from the workers' perspective, to another matching μ', obtained by letting each worker move into the firm of the next worker:

$$\mu'(w) = \begin{cases} \mu(w_{t+1}) & \text{if } w = w_t \text{ for some } t \in \{0, \ldots, n-1\}, \\ \mu(w) & \text{otherwise.} \end{cases}$$

By **carrying out an SWI-cycle**, we mean constructing the new stable matching μ' which improves μ from the workers' perspective, as done above. Note that although workers improve from μ to μ', firms may become worse-off in the transition, because unlike in a PI-cycle, in an SWI-cycle we do not require that $w_t \succsim_{\mu(w_{t+1})} w_{t+1}$. Therefore we cannot rely on Lemma 1 to conclude that μ' is stable. Instead, condition (ii) is key in guaranteeing that the new matching μ' continues to be stable.

Lemma 2. *Let μ be a stable matching which admits an SWI-cycle. Then the matching μ' obtained by carrying out the SWI-cycle is stable as well.*

[18] Perhaps surprisingly, however, if agents' preferences are dichotomous in the sense that each agent is indifferent between all partners they find acceptable, Bogomolnaia and Moulin (2004) establish stable, efficient and strategy-proof mechanisms.

[19] The definition does not rule out the possibility that some workers in an SWI-cycle are matched with the same firm. However, note that no two consecutive workers in an SWI-cycle are matched with the same firm, because for a worker w_t to desire $\mu(w_{t+1})$, it must be that $\mu(w_t) \neq \mu(w_{t+1})$.

Proof. μ is individually rational, because it is stable to begin with. Since μ' makes all workers weakly better off, it is individually rational from workers' perspective. Secondly, the definition of D_f^μ and condition (ii) ensure that no firm admits an unacceptable worker as part of an SWI-cycle, therefore μ' is individually rational for firms as well. Hence, suppose, for a contradiction, that (w, f) is a blocking pair for μ'. That is, $f \succ_w \mu'(w)$ and $w \succ_f v$ for some v with $\mu'(v) = f$. All workers weakly prefer μ' to μ, and in particular $\mu'(w) \succsim_w \mu(w)$, therefore we must have $f \succ_w \mu(w)$.

If f is not part of the SWI-cycle, then it has the same set of workers under both matchings μ and μ'. In particular $\mu(v) = f$. Since $f \succ_w \mu(w)$, matching μ is blocked by (w, f), contradicting stability of μ. Thus f is part of the SWI-cycle. Say $f = \mu(w_t)$ for some t.

We know that w desires f at μ. Stability of μ implies that according to $\succsim_f$ every worker matched with f under μ is ranked at least as high as w. Condition (ii), on the other hand, implies that whoever moved to f via the SWI-cycle generating μ' from μ is weakly preferred to w by f. Thus, every worker matched with f under μ' is ranked at least as high as w, contradicting with (w, f) being a blocking pair for μ'. $\square$

Definition 4. A **stable worker improvement (SWI) chain** consists of distinct workers $w_1, \ldots, w_n$ $(n \geq 2)$ and a firm f with an empty position such that:

(i) a. If w_1 is matched to a firm, then no worker in W strictly desires and is acceptable to $\mu(w_1)$,
   b. w_t is matched to some firm for each $t \in \{2, \ldots, n\}$,
(ii) $w_t \in D^\mu_{\mu(w_{t+1})}$ for each $t \in \{1, \ldots, n-1\}$, and $w_n \in D_f^\mu$,
(iii) $\mu(w_{t+1}) \succ_{w_t} \mu(w_t)$ for some $t \in \{1, \ldots, n-1\}$.

Each worker w_t in an SWI-chain except w_n, weakly desires the firm of the following worker w_{t+1}, and as in an SWI-cycle, the employer $\mu(w_{t+1})$ of the latter finds w_t acceptable. Also, there is no other worker who strictly desires $\mu(w_{t+1})$ and is ranked strictly higher than w_t by $\mu(w_{t+1})$. The last worker w_n weakly desires and is acceptable to f.[20]

If there is an SWI-chain, then the matching μ can be improved from the workers' perspective, to a new stable matching μ' obtained by letting each worker other than w_n move into the firm of the next worker and letting w_n move to f:

$$\mu'(w) = \begin{cases} \mu(w_{t+1}) & \text{if } w = w_t \text{ for some } t \in \{1, \ldots, n-1\}, \\ f & \text{if } w = w_n, \\ \mu(w) & \text{otherwise.} \end{cases}$$

This construction of the new stable matching μ' which improves μ from the workers' perspective is what we call **carrying out an SWI-chain**. As in an SWI-cycle, although workers improve from μ to μ', firms may become worse-off in the transition and we can again not make use of Lemma 1 to conclude that μ' is stable. Like in the proof of Lemma 2, condition (ii) plays the analogous key role in guaranteeing that μ' is stable.

We are now ready to state our second structural theorem which states that the converse of the above observations holds. Namely, if μ is stable and admits no SWI-cycles nor SWI-chains, then μ must be W-optimal stable. The proof can be found in Appendix A.2.

[20] Similar to Footnote 15, the definition of an SWI-chain can be embedded into that of an SWI-cycle.

Theorem 2. *A stable matching μ is W-optimal stable if and only if there are no SWI-cycles nor SWI-chains.*

The theorem suggests a procedure to find a W-optimal stable matching: First obtain a stable matching by applying the DA algorithm to a tie-breaking. So long as the stable matching is not W-optimal stable, by Theorem 2, there will be an SWI-cycle or an SWI-chain. If that is the case, find an SWI-cycle or an SWI-chain and carry it out to obtain a new stable matching that improves the original one from the workers' perspective. Repeat this as long as the obtained stable matching has an SWI-cycle or SWI-chain. A precise description of this procedure is in Appendix A.3.

By finiteness of our model, the procedure will stop after finitely many steps and yield a W-optimal stable matching. We call this procedure the **Worker-Optimal Stable Matching Algorithm (WOSMA)**. We show in Proposition 2 in Appendix A.3, that the WOSMA is polynomial in the number of workers and the number of firms.

Suppose that a stable matching μ admits an SWI-chain $w_1, \ldots, w_n$ and f. Then at μ, f has an empty position and w_n weakly desires f (in particular w_n is not matched to f). The worker w_n must be indifferent between $\mu(w_n)$ and f for otherwise (w_n, f) would form a blocking pair for μ. In particular if μ is stable and if the workers have strict preferences, then μ does not admit any SWI-chains. Thus, when workers have strict preferences, a stable matching is worker-optimal stable if and only if there are no stable worker improvement cycles. Assuming strict preferences for workers brings us back to the framework of Erdil and Ergin (2008), where students have strict preferences, whereas schools' priority rankings over students have ties. In that setup, if μ and ν are stable matchings such that $\nu \succ_W \mu$, then there exist stable matchings $\mu_1, \ldots, \mu_n$ such that $\mu = \mu_1$, $\nu = \mu_n$, and μ_{t+1} is obtained by carrying out an SWI-cycle at μ_t, for $t \in \{1, \ldots, n-1\}$. That is, if workers prefer a stable matching ν to another stable matching μ, then ν can be reached from μ by a sequence of SWI-cycles. An analogue of this result does not hold in our framework where both sides have weak preferences. Even if $\nu \succ_W \mu$, where μ and ν are both stable matchings we do not necessarily have a sequence of SWI-cycles and SWI-chains that take us from μ to ν. This is because there might be moves from one matching to the other that do not affect any worker's welfare, and therefore would be impossible to execute by improving chains or cycles.

Given preferences, WOSMA finds a worker-optimal stable matching. However, there is one final round of improvements possible, and that's from the firms' perspective. By employing ESMA, or to be more precise, by finding Pareto improving cycles and carrying them out, we can improve over the WOSMA outcome according to the firms' preferences.[21]

When all preferences are strict, Dubins and Freedman (1981) and Roth (1982) show that the worker-proposing DA (which yields the unique worker-optimal stable matching) is strategy-proof from workers' perspective. In contrast, ties in firms' preferences imply that even when only workers are strategic, there is no strategy-proof mechanism which achieves worker-optimal stability (Erdil and Ergin, 2008). In particular WOSMA is not strategy-proof even if only workers are strategic.

[21] If we apply the model to a school choice problem where school priority rankings over students are interpreted as exogenous constraints, then Kesten's (2010) relaxation of the stability notion (by allowing students to consent to their priorities to be violated when they are not hurt by such violations) yields outcomes which Pareto dominate WOSMA from students' perspective at the expense of respecting priorities and "schools' welfare."

5. Single tie-breaking and Pareto efficiency

While our understanding of the effects of different tie breaking rules in matching markets has been limited, the conventional wisdom in the literature suggests that efficiency loss arising from tie-breaking increases when different agents break their ties differently, which is likely to be the case if tie-breaking is de-centralized. In the example we discussed in the introduction, all agents used a single tie-breaking rule (the alphabetical one), which resulted in efficiency loss. However, if they used another single tie-breaking, the reverse alphabetical, then we would reach the unique efficient stable matching in that example. Is it possible to reach all stable and Pareto efficient matchings by focusing on single tie-breaking rules? The following example shows that it is not.[22]

Example 1. Suppose there are two workers, w and v, and two firms, f and g. Both firms have two positions each, that is, $q_f = q_g = 2$. If the preferences are given as

$\succsim_w$	$\succsim_v$
f, g	f, g

$\succsim_f$	$\succsim_g$
w	w
v	v

,

then the efficient and stable matching $\mu = (wf, vg)$ is not stable at any single tie-breaking, because both workers would end up with the firm which is ranked higher by the tie-breaking rule. Hence there may be Pareto efficient (also W-optimal) stable matchings which cannot be reached by using the (worker proposing) DA algorithm after all possible ways of single tie-breaking. $\diamond$

If, however, the workers have strict preferences, single tie-breaking never leads to an efficiency loss as shown below.

Theorem 3. *Let $\succsim$ be a preference profile where workers have strict preferences. If $\succsim'$ is a single tie-breaking of $\succsim$ and if μ is stable with respect to $\succsim'$, then μ is Pareto efficient at $\succsim$.*

Proof. Let $\phi^W : W \to \{1, \ldots, |W|\}$ be a bijection that induces the tie-breaking $\succsim'_F$. Suppose for a contradiction that μ is Pareto dominated by a matching ν at $\succsim$. Since the workers have strict preferences, their being indifferent between μ and ν would imply that $\mu = \nu$, therefore some worker(s) must strictly prefer ν to μ at $\succsim$. We also know by Lemma 1 that ν is stable, therefore μ is not W-optimal stable at $\succsim$.

By Theorem 1 in Erdil and Ergin (2008), there exist stable matchings $\mu_1, \ldots, \mu_n$ such that $\mu = \mu_1$, $\nu = \mu_n$, and μ_{t+1} is obtained by carrying out an SWI-cycle at μ_t, for $t \in \{1, \ldots, n-1\}$. Note that $\mu_t \succsim_F \mu_{t+1}$, for otherwise if the tth SWI-cycle rematches a firm f to a worker w such that $w \succ_f w'$ for some $w' \in \mu_t^{-1}(f)$, then (w, f) would block μ_t, a contradiction. Hence $\mu = \mu_1 \succsim_F \mu_2 \succsim_F \cdots \succsim_F \mu_n = \nu$. We also have that $\nu \succsim_F \mu$ since ν Pareto dominates μ, therefore $\mu = \mu_1 \sim_F \mu_2 \sim_F \cdots \sim_F \mu_n = \nu$.

Let $w_1, \ldots, w_n$ be the SWI-cycle at $\mu = \mu_1$ above. Then $\mu(w_{t+1}) \succ_{w_t} \mu(w_t)$ and $w_t \sim_{\mu(w_t)} w_{t+1}$ for $t \in \{0, 1, \ldots, n-1\}$, which, by the definition of the single tie-breaking $\succsim'_F$, implies that $\phi^W(w_0) < \phi^W(w_1) < \cdots < \phi^W(w_{n-1}) < \phi^W(w_n) = \phi^W(w_0)$, a contradiction. $\square$

[22] It is an open question whether there is always a *single tie-breaking* at which there is a stable and Pareto efficient matching. Theorem 4 below answers this question affirmatively when each firm has one position only.

A directed graph G is **acyclic** if it has no cycles. A **topological ordering** of a directed graph is a bijection $\phi\colon X \to \{1, \dots, |X|\}$ such that $x \to y$ implies that $\phi(x) \geq \phi(y)$. It is not hard to see that a directed graph is acyclic if and only if it is topologically ordered.

Theorem 4. *If each firm has one position and μ is stable and Pareto efficient at $\succsim$, then there exists a* single tie-breaking $\succsim'$ *such that μ is stable with respect to $\succsim'$.*

Proof. Assume that μ is Pareto efficient and stable at $\succsim$. We will construct the *single tie breaking* $\succsim'$ in two steps, by first breaking the ties in firms' preferences and then those in workers' preferences.

Consider a directed graph G with vertex set W, where $w \to v$ if v is matched to a firm, $\mu(v) \succ_w \mu(w)$, and $w \sim_{\mu(v)} v$. Such an edge means that w strictly envies v, and the firm $\mu(v)$ would not mind replacing v with w. Pareto efficiency of μ implies that this graph is acyclic: If the graph has a cycle $w_0 \to w_1 \to \cdots \to w_{n-1} \to w_n \equiv w_0$, then the new matching obtained by rematching each worker w_t in the cycle to $\mu(w_{t+1})$ for $t \in \{0, \dots, n-1\}$, would Pareto dominate μ. Let $\phi^W\colon W \to \{1, \dots, |W|\}$ be a bijection inducing a topological ordering of G. Let $\succsim'_F$ denote the single tie-breaking of $\succsim_F$ induced by ϕ^W.

By NI$\varnothing$ and individual rationality of μ before the tie-breaking, μ continues to be individually rational after the tie-breaking. Suppose that (w, f) blocks μ at $(\succsim_W, \succsim'_F)$, i.e. $f \succ_w \mu(w)$ and $w \succ'_f \mu^{-1}(f)$. Stability of μ at $\succsim$ implies that $w \sim_f \mu^{-1}(f)$, in particular by NI$\varnothing$, f is matched to a worker v. Note that $w \to v$ since at $\succsim$, w strictly envies v, and the firm $f = \mu(v)$ would not mind replacing v with w. Hence $\phi^W(w) \geq \phi^W(v)$, a contradiction to $w \sim_f v$ and $w \succ'_f v$. We conclude that μ is stable at $(\succsim_W, \succsim'_F)$.

Next consider an analogous directed graph G' with vertex set F, where $f \to g$ if g is matched to a worker, $\mu^{-1}(g) \succ'_f \mu^{-1}(f)$, and $f \sim_{\mu^{-1}(g)} g$. Suppose that there is a cycle $f_0 \to f_1 \to \cdots \to f_{n-1} \to f_n \equiv f_0$. Consider the new matching μ' obtained by rematching each firm f_t in the cycle to $\mu^{-1}(f_{t+1})$ for $t \in \{0, \dots, n-1\}$. At $(\succsim_W, \succsim'_F)$, all workers, as well as the firms not involved in the cycle are indifferent between μ and μ', whereas all the firms involved in the cycle strictly prefer μ' to μ. No firm strictly prefers μ' to μ at $\succsim$, since otherwise μ' would Pareto dominate μ at $\succsim$. Since $\mu^{-1}(f_{t+1}) P \succ'_{f_t} \mu^{-1}(w_t)$ and $\mu^{-1}(f_{t+1}) \sim_{f_t} \mu^{-1}(w_t)$ for $t \in \{0, \dots, n-1\}$, by the definition of the single tie-breaking $\succsim'_F$, we have $\phi^W(\mu^{-1}(f_0)) = \phi^W(\mu^{-1}(f_n)) < \phi^W(\mu^{-1}(f_{n-1})) < \cdots < \phi^W(\mu^{-1}(f_1)) < \phi^W(\mu^{-1}(f_0))$, a contradiction. Therefore G' is acyclic, let $\phi^F : F \to \{1, \dots, |F|\}$ be a bijection inducing a topological ordering of G'. Let $\succsim'_W$ be the single tie-breaking of $\succsim_W$ induced by ϕ^F. By the same argument as in the above paragraph switching the roles of firms and workers, we conclude that μ is stable with respect to $(\succsim'_W, \succsim'_F)$. $\square$

One implication of this theorem is that when the matching environment is one-to-one, all stable and Pareto efficient matchings can be discovered by restricting to single tie-breakings only. While such a restriction is a significant reduction in the number of tie-breakings to consider, and exhaustive exploration through them all would still be computationally prohibitive in most applications.

Finally, by combining Theorems 3 and 4, we get the following equivalence result.

Corollary 2. *Assume that each firm has one position and one side has strict preferences at $\succsim$. Then μ is stable and Pareto efficient at $\succsim$ if and only if there exists a single tie-breaking $\succsim'$ such that μ is stable at $\succsim'$.*

6. Conclusion

We took as our motivation a number of simple observations: when indifferences are incorporated into standard models of two-sided matching, stability no longer implies Pareto efficiency, and the celebrated deferred acceptance algorithm is not guaranteed to produce a Pareto efficient or a worker-optimal stable matching. We have explored the structure of the set of stable matchings to characterize the nature of Pareto relations between stable matchings. By using the simple operations that capture these relations, we have designed new algorithms, ESMA and WOSMA, to find an efficient and stable matching, and a worker-optimal stable matching, respectively.

Ties in preference rankings are increasingly common. With the surge of matching platforms, online and offline, participants frequently need to evaluate large numbers of possible partners. Faced with the task of comparing hundreds of alternatives, it is a lot easier to mark them on a discrete scale of 1 to 5, than forming a strict ranking. Instead of going through candidates' profiles, an employer might rather submit criteria to a matchmaker, who, in turn, would derive the employer's ranking from such criteria. It is important for these platforms to help participants reach *efficient* outcomes *quickly*. In order to fulfill that role successfully, the matchmaker has to handle ties carefully, and our algorithms, by their intuitive and fast nature, answer the need for quick and efficient matchmaking.

Appendix A

A.1. Characterization of a minimal responsive extension

We now prove the observation from Section 2: $I \succsim_f J$ if and only if the sets I and J can be indexed as $I : i_1, \dots, i_n$ and $J : j_1, \dots, j_n$, where for each worker short of n a copy of $\varnothing$ is written and $i_t \succsim_f j_t$ for each $t \in \{1, \dots, n\}$

In order to see the "if" part, note that if such indexing of I and J exist, it is straightforward to verify that $I \tilde{\succsim} J$ for every responsive extension of $\succsim$.

For the "only if part," given sets $I \succsim_f J$, index the elements in I and J in decreasing preference order, that is, $i_1 \succsim i_2 \succsim_f \cdots \succsim_f i_n$ and $j_1 \succsim j_2 \succsim_f \cdots \succsim_f j_n$, respectively.

We are claiming that $i_t \succsim_f j_t$ for all $t = 1, \dots, n$. Suppose, for a contradiction, the contrary, and let k be the smallest index for which $j_k \succ_f i_k$.

Let $v : W \to \mathbb{R}$ be a function capturing the value of each worker to a firm whose valuation of sets of workers is additive. Set these values so that $v(j_k) > v(\ell) + 1$ for all workers i such that $j_k \succ_f \ell$, whereas $v(\ell) - v(j_k) < \epsilon$ for all workers ℓ with $\ell \succ_f j_k$. Since $i_1 \succsim_f i_2 \succsim_f \cdots \succsim_f i_{k-1} \succsim_f j_{k-1} \succsim_f j_k$, the above assumptions on valuations imply

$$v(i_1) + \cdots + v(i_{k-1}) < v(j_1) + \cdots + v(j_{k-1}) + (k-1)\epsilon.$$

Setting $\epsilon < 1/(k-1)$, we obtain

$$v(i_1) + \cdots + v(i_{k-1}) + v(i_k) < v(j_1) + \cdots + v(j_{k-1}) + v(j_k). \tag{$\star$}$$

Now, consider additive preferences over sets of workers in the sense that a set S of workers is of value $\sum_{\ell \in S} v(\ell)$, and higher value corresponds to higher preference. It is trivially verified that such additive preferences constitute a responsive extension of $\succsim_f$ according to which J must be preferred to I, because $v(I) < v(J)$ as seen in ($\star$). But remember the definition of a minimal responsive extension: we write $I \succsim_f J$ only if $I \tilde{\succsim}_f J$ for every responsive extension of $\succsim_f$. This yields the contradiction with the above additive preferences ranking J strictly higher than I. □

A.2. Proof of Theorem 2

Apart from SWI-chains and SWI-cycles, it will be useful for the purposes of the proof to consider chains and cycles that do not change any worker's welfare.

Definition 5. Given two matchings μ and ν, **a reversible cycle from μ to ν** consists of distinct workers $w_1, \dots, w_n \equiv w_0$ $(n \geq 2)$ such that:

(i) Each w_t is matched to some firm both at μ and ν,
(ii) $\nu(w_t) = \mu(w_{t+1}) \neq \mu(w_t)$ for $t \in \{0, 1, \dots, n-1\}$,
(iii) $\mu(w_t) \sim_{w_t} \nu(w_t)$ for $t \in \{1, \dots, n\}$.

In a reversible cycle, consecutive workers are matched with distinct firms. Each worker w_t moves to the position held by worker w_{t+1}, but is indifferent between her position under μ, and the position she is moving to.

Definition 6. Given two matchings μ and ν, **a reversible chain from μ to ν** consists of distinct workers $w_1, \dots, w_n$ $(n \geq 1)$ and a firm f with an empty position at μ such that:

(i) a. Each w_t is matched to some firm both at μ and ν,
b. $\mu(w_1)$ has an empty position at ν,
(ii) $\nu(w_n) = f$ and $\nu(w_t) = \mu(w_{t+1}) \neq \mu(w_t)$ for $t \in \{1, \dots, n-1\}$,
(iii) $\mu(w_t) \sim_{w_t} \nu(w_t)$ for $t \in \{1, \dots, n\}$.

If there is a reversible cycle [chain] from μ to ν, to **reverse** such a cycle [chain] will mean replacing ν with ν' by simply reassigning the workers who are involved in the cycle [chain] back to their firms at μ, i.e.,

$$\nu'(w) = \begin{cases} \mu(w) & \text{if } w \text{ is involved in the reversible cycle [chain]} \\ \nu(w) & \text{otherwise.} \end{cases}$$

Clearly, the reversing process does not effect the welfare of the workers.

Lemma 3. *Assume that μ and ν are stable matchings such that $\nu \succsim_W \mu$. If ν' is obtained by reversing a reversible cycle or chain from μ to ν, then ν' is also stable.*

Proof. Let μ, ν, and ν' be as in above. Take any firm f and worker w such that $f = \nu'(w)$. Then by the definition of ν', $f = \nu(w)$ or $f = \mu(w)$. Since both μ and ν are individually rational, i.e., $f \succsim_w \varnothing$ and $w \succsim_f \varnothing$, ν' is individually rational, too.

Suppose for a contradiction that (w, f) is a blocking pair for ν'. Then (i) $f \succ_w \nu'(w)$ and (ii.a) $w \succ_f v$ for some $v \in \nu'^{-1}(f)$, or (ii.b) $w \succ_f \varnothing$ and f has an empty position at ν'. Since $\nu' \sim_W \nu \succsim_W \mu$, we have (i.$\nu$) $f \succ_w \nu(w)$ and (i.μ) $f \succ_w \mu(w)$. In case (ii.a), f is matched to v at ν or μ, which along with (i.ν) and (i.μ) imply that ν or μ is unstable, a contradiction. In case (ii.b), f has an empty position at ν or μ, which along with (i.ν) and (i.μ) imply that ν or μ is unstable, again a contradiction. □

If μ is a stable matching that is not W-optimal stable, then there exists a stable matching ν^0 such that $\nu^0 \succ_W \mu$. If there are any reversible cycles or chains from μ to ν^0, by Lemma 3, we can arbitrarily select one and reverse it to obtain a new stable matching ν^1 such that $\nu^1 \sim_W \nu^0 \succ_W \mu$. If there exist any reversible chains or cycles from μ to ν^1, by Lemma 3, we can again arbitrarily select one and reverse it to obtain a yet another stable matching ν^2 such that $\nu^2 \sim_W \nu^1 \sim_W \nu^0 \succ_W \mu$. Proceeding analogously, we will eventually obtain a stable matching ν such that $\nu \succ_W \mu$ and there are no reversible cycles or chains from μ to ν. We summarize this observation in the following Lemma.

Lemma 4. *If μ is a stable matching that is not W-optimal stable, then there exists a stable matching ν such that $\nu \succ_W \mu$ and there are no reversible cycles nor chains from μ to ν.*

Lemma 5. *Let μ be a stable matching and ν be an individually rational matching such that $\nu \succ_W \mu$. Assume that μ does not admit an SWI-cycle nor an SWI-chain, and that there are no reversible cycles nor chains from μ to ν. Then each firm f is matched to at least as many workers at μ as at ν.*

Proof. Let $W' = \{w \in W : \mu(w) \neq \nu(w)\}$. For each firm f, if there exists a worker $u \in W$ who strictly desires f at μ and is acceptable to f, then fix u_f to be a highest ranked such u with respect to $\succsim_f$. Otherwise we will say that "u_f does not exist." If u_f does not exist and there exists $v \in W'$ such that $\nu(v) = f$, then fix v_f to be any such v. Otherwise, i.e., if u_f exists *or* if there is no $v \in W'$ such that $\nu(v) = f$, we will say that "v_f does not exist." By definition u_f and v_f cannot co-exist. If u_f exists then $f \succ_{u_f} \mu(u_f)$ and $u_f \in D^{\mu}_{f}$. If v_f exists, then $f = \nu(v_f) \neq \mu(v_f)$, $f = \nu(v_f) \sim_{v_f} \mu(v_f)$, and $v_f \in D^{\mu}_{f}$.[23]

A finite sequence $(w_1, \ldots, w_n)$ of $n \geq 1$ workers is of **Type I** if (i) they are all distinct, (ii) each one is matched to some firm both at μ and ν, (iii) $\nu(w_1)$ has an empty position at μ, (iv.a) $w_1 = v_{\nu(w_1)}$, and (iv.b) $w_{t+1} = v_{\mu(w_t)}$ for $t \in \{1, \ldots, n-1\}$. Note that in a Type I sequence, $\nu(w_t) \sim_{w_t} \mu(w_t)$ and $w_t \in D^{\mu}_{\nu(w_t)}$ for each $t \in \{1, \ldots, n\}$.

A finite sequence $(w_1, \ldots, w_n)$ of $n \geq 2$ workers is of **Type II** if (i) they are all distinct, (ii) there exists a $k \leq n-1$ such that: (ii.a) $(w_1, \ldots, w_k)$ is of Type I, (ii.b) each one of $w_{k+1}, \ldots, w_n$ is matched to some firm at μ, and (ii.c) $w_{t+1} = u_{\mu(w_t)}$ for $t \in \{k, \ldots, n-1\}$. Note that in a Type II sequence, $\nu(w_t) \sim_{w_t} \mu(w_t)$ and $w_t \in D^{\mu}_{\nu(w_t)}$ for each $t \in \{1, \ldots, k\}$; and $\mu(w_{t-1}) \succ_{w_t} \mu(w_t)$ and $w_t \in D^{\mu}_{\mu(w_{t-1})}$ for each $t \in \{k+1, \ldots, n\}$.

We will show in step 1 below that, if there exists a Type I sequence of length $n \geq 1$, then there exists a Type I or Type II sequence of length $n+1$. We will prove in step 2 that, if there exists a Type II sequence of length $n \geq 2$, then there exists a Type II sequence of length $n+1$. The two steps imply that there cannot be any Type I sequence of length one, otherwise it is possible to generate an arbitrarily large sequence of distinct workers, contradicting finiteness of W. To see that this is enough to prove the lemma, suppose that there exists a firm f who is matched to less

[23] This last inclusion uses individual rationality of ν. v_f is matched to f at ν, so she must be acceptable to f. But since there are no workers that strictly desire f and are acceptable to f (since u_f does not exist), v_f does not strictly desire f (and since $\nu \succ_W \mu$, v_f must be indifferent between $f = \nu(v_f)$ and $\mu(f)$).

workers at μ than at ν. Then f must have an empty position at μ. By stability of μ and NI$\varnothing$, u_f does not exist. Since f is matched to more workers at ν, v_f exists. Since $f = \nu(v_f) \sim_{v_f} \mu(v_f)$, by NI$\varnothing$, v_f is matched to a firm at μ. Hence (v_f) constitutes a Type I sequence of length one, a contradiction. It remains to prove steps 1 and 2.

Step 1: Let $(w_1, \ldots, w_n)$ be a Type I sequence. Then $\mu(w_n)$ does not have an empty position at ν, since otherwise $w_n, \ldots, w_1$ (yes, in the reverse order) and $\nu(w_1)$ would constitute a reversible chain from μ to ν. Since $\nu(w_n) \neq \mu(w_n)$ and the positions of $\mu(w_n)$ are full at ν, there exists a worker in W' matched to $\mu(w_n)$ at ν. Hence either $u_{\mu(w_n)}$ or $v_{\mu(w_n)}$ exists.

If $u_{\mu(w_n)}$ exists, let $w_{n+1} = u_{\mu(w_n)}$. Since $\mu(w_n) \neq \mu(u_{\mu(w_n)})$, $w_{n+1} \neq w_n$. Also w_{n+1} is distinct from $w_1, \ldots, w_{n-1}$, because otherwise if $w_{n+1} = w_k$ for some $k \leq n-1$, then $w_{n+1}, w_n, w_{n-1}, \ldots, w_{k+1}$ (yes, in this order) would constitute an SWI-cycle. Moreover, w_{n+1} must be matched to a firm at μ, since otherwise $w_{n+1}, w_n, w_{n-1}, \ldots, w_1$ and $\nu(w_1)$ would constitute an SWI-chain. Hence in this case $(w_1, \ldots, w_n, w_{n+1})$ is a Type II sequence of length $n+1$.

If $v_{\mu(w_n)}$ exists, let $w_{n+1} = v_{\mu(w_n)}$. Since $\mu(w_n) \neq \mu(v_{\mu(w_n)})$, $w_{n+1} \neq w_n$. Also w_{n+1} is distinct from $w_1, \ldots, w_{n-1}$, for otherwise if $w_{n+1} = w_k$ for some $k \leq n-1$, then $w_{n+1}, w_n, w_{n-1}, \ldots, w_{k+1}$ would constitute a reversible cycle from μ to ν. Moreover, w_{n+1} must be matched to a firm at μ, because of the NI$\varnothing$ assumptions, her indifference between μ and ν, and her being matched with $\mu(w_n)$ at ν. Thus, in this case $(w_1, \ldots, w_n, w_{n+1})$ is a Type I sequence of length $n+1$.

Step 2: Let $w_1, \ldots, w_n$ ($n \geq 2$) be a Type II sequence where k is as in part (ii) of the definition of a Type II sequence. There exists a worker who strictly desires $\mu(w_n)$ at μ and is acceptable to $\mu(w_n)$, because otherwise $w_n, \ldots, w_1$ and $\nu(w_1)$ would constitute an SWI-chain. Hence $u_{\mu(w_n)}$ exists.

Let $w_{n+1} = u_{\mu(w_n)}$. Since $\mu(w_n) \neq \mu(u_{\mu(w_n)})$, $w_{n+1} \neq w_n$. Also w_{n+1} is distinct from $w_1, \ldots, w_{n-1}$, because otherwise if $w_{n+1} = w_k$ for some $k \leq n-1$, then $w_{n+1}, w_n, w_{n-1}, \ldots, w_{k+1}$ would constitute an SWI-cycle. Moreover, w_{n+1} must be matched to a firm at μ, for otherwise $w_{n+1}, w_n, w_{n-1}, \ldots, w_1$ and $\nu(w_1)$ would constitute an SWI-chain. Hence in this case $(w_1, \ldots, w_n, w_{n+1})$ is a Type II sequence of length $n+1$. □

Lemma 6. *Let μ be a stable matching and ν be an individually rational matching such that $\nu \succ_W \mu$. Assume that μ does not admit an SWI-cycle nor an SWI-chain and that there are no reversible cycles nor chains from μ to ν. Let $W' = \{w \in W : \mu(w) \neq \nu(w)\}$ and $F' = \mu(W')$. Then:*

(i) *For each firm f, the number of workers in W' who are matched to firm f is the same at μ and ν. In particular, $F' = \nu(W')$.*
(ii) *Each worker in W' is matched to a firm in both μ and ν.*

Proof. By $\nu \succ_W \mu$, individual rationality of μ, and NI$\varnothing$, each worker in W' is matched to a firm at ν. To see part (i), note that Lemma 5 implies that $|W' \cap \mu^{-1}(f)| \geq |W' \cap \nu^{-1}(f)|$ for any firm f. Suppose that the inequality $|W' \cap \mu^{-1}(f)| \geq |W' \cap \nu^{-1}(f)|$ holds strictly for some firm f^*. Summing across all firms we have:

$$\sum_{f \in F} |W' \cap \mu^{-1}(f)| > \sum_{f \in F} |W' \cap \nu^{-1}(f)|.$$

That is, the number of workers in W' matched to some firm at μ is more than the number of workers in W' matched to some firm at ν. This implies that there exists a worker in W' who is unmatched at ν, a contradiction. Part (ii) follows from part (i) and the fact that each worker in W' is matched to a firm at ν. $\square$

Proof of Theorem 2. It only remains to prove the "if" part. Assume that μ is stable but not W-optimal stable. By Lemma 4, there exists a stable matching ν such that $\nu \succ_W \mu$ and there are no reversible cycles nor chains from μ to ν. Suppose for a contradiction that μ admits no SWI-cycle nor SWI-chain. Let W' and F' be as in Lemma 6. That is, W' is the set of workers for whom the match has changed from μ to ν, and F' is the set of firms with whom these workers are matched at μ.

For any $f \in F'$, let W'_f denote the set of workers in W' who weakly desire f at μ, and are acceptable to f, such that there is no other worker in W' who strictly desires f and is ranked strictly higher in $\succsim_f$. By Lemma 6, $f \in F' = \nu(W')$, hence there exist workers in W' who are matched to f at ν. Those workers who have left the firm they were matched with at μ, and are now matched with f under ν must have weakly improved since $\nu \succ_W \mu$. Hence, the set of workers W' who weakly desire f at μ, and are acceptable to f is nonempty. In particular, W'_f is nonempty.

If there is any worker in W'_f who is matched to f at ν, fix w_f to be such a worker who is ranked highest with respect to $\succsim_f$. If not, all those workers in W' who are matched to f at ν are ranked lower than some worker $w' \in W'_f$ who strictly desires f at μ; and w_f to be one such worker. Note that if $u \in W'$ is matched to f at ν, then $w_f \succsim_f u$. Also note that $\mu(w_f) \in F'$ and $\mu(w_f) \neq f$.

We next show that $w_f \in D^{\mu}_f$. Suppose not, then there is a worker $v \notin W'$ who strictly desires f and is strictly higher in $\succsim_f$ than w_f. Since $v \notin W'$, $\nu(v) = \mu(v)$, therefore $f \succ_v \nu(v)$. Let u be a worker in W' who is matched to f at μ, then $v \succ_f w_f \succsim_f u$, a contradiction to the stability of ν.

Now, consider a directed graph G with vertex set F', where for each firm f there is a unique incoming edge given by $\mu(w_f) \to f$. Since each firm in F' is pointed to by a different firm in F', there exists a cycle $f_1, \ldots, f_n = f_0$ in F'.[24] Let $w_t = w_{f_{t+1}}$ for $t \in \{0, \ldots, n-1\}$. Since $f_t \to f_{t+1}$ and $w_t = w_{f_{t+1}}$, we have $\mu(w_t) = f_t$. In particular $w_1, \ldots, w_n$ are distinct and each one is matched to some firm at μ. By construction $w_t \in D^{\mu}_{\mu(w_{t+1})}$, and if $\mu(w_{t+1}) \sim_{w_t} \mu(w_t)$, then $\nu(w_t) = \mu(w_{t+1})$, for $t \in \{0, 1, \ldots, n-1\}$. Hence $w_1, \ldots, w_n$ constitute either an SWI-cycle or a reversible cycle from μ to ν, a contradiction. $\square$

A.3. The algorithms and their time complexity analysis

In this section we give precise descriptions of the algorithms announced earlier. In doing so we introduce the notion of a 2-labeled graph and a strict cycle, and then establish an upper bound on the time complexity of strict cycle search on a 2-labeled graph in Lemma 7. In Propositions 1 and 2, we use this result to establish that the algorithms introduced are polynomial time.

An algorithm being of polynomial time means that the time required in order for it to return its outcome or halt is a polynomial in the size of its input. This property becomes especially important as the size of the input grows. In the problems studied in this paper, even with 100

[24] Remember that our definition of a cycle in a graph requires that the vertices are distinct and $n \geq 2$.

agents, there are 100! (more than 10^{145}) different ways of single tie-breaking, and many more arbitrary tie breaking rules. Therefore methods of exhaustion are not computationally feasible. On the other hand, *polynomial time* is a theoretical benchmark for 'algorithmically efficient' computation. Time complexity is expressed via the big-Oh notation, as the theory is concerned with the asymptotic behavior of the number of steps it takes to complete the algorithm as a function of the size of the problem. However, such notation can hide arbitrarily large constants, and may not always give a realistic sense of what the actual running times in practice could be. Partly to address this issue, we conducted simulations for our earlier paper, Erdil and Ergin (2008), where the indifference classes had several hundred agents. We confirmed that on an average desktop computer, with such data set as the input, the actual running time was always at most a few minutes.

Given a directed graph $G = (V, E)$, a **path** from a vertex x to a vertex y is a sequence of distinct vertices $x_1, \dots, x_n$ such that $x = x_1 \to x_2 \to \cdots \to x_n = y$. A directed graph is called **strongly connected** if for every pair of vertices x and y there is a path from x to y and a path from y to x. The strongly connected components of a directed graph are its maximal strongly connected subgraphs. These form a partition of the graph.

A **2-labeled graph**[25] is a graph $G = (V, E)$ and a function $\ell : E \to \{0, 1\}$. That is, each edge is assigned one of the two labels.

We will denote the edges labeled 0 with $x \to y$, and those labeled 1 with $x \twoheadrightarrow y$. The edges labeled 1 will be called **strict edges of** G. A cycle of G with at least one strict edge on is called a **strict cycle**.

Lemma 7. *Strict cycle search on a 2-labeled graph $G = (V, E)$ is $\mathrm{O}\big(|V| + |E|\big)$.*

Proof. Note that G has a strict cycle if and only if a strongly connected component includes a strict edge. Identifying strongly connected components of G is $\mathrm{O}\big(|V| + |E|\big)$ by Tarjan (1972), checking for strict edges is $\mathrm{O}\big(|E|\big)$, and finding a cycle that includes a specific strict edge is $\mathrm{O}\big(|V|\big)$.[26] Hence strict cycle search is $\mathrm{O}\big(|V| + |E|\big)$. $\square$

Efficient and Stable Matching Algorithm (ESMA)

Given a preference profile $\succsim$, to a stable matching μ we will associate a 2-labeled graph Γ^μ with the vertex set $W \cup \{\varnothing\}$, and the edges and their labels specified as follows:

(i) $w \to v$ if $\mu(v)$ is a firm such that $\mu(v) \succsim_w \mu(w)$ and $w \succsim_{\mu(v)} v$.
(ii) $w \to \varnothing$ if there is a firm f with an empty position such that $f \succsim_w \mu(w)$ and $w \succsim_f \varnothing$.
(iii) $\varnothing \to w$ if $\mu(w) = \varnothing$.

Label the strict edges as follows:

(iv) $w \twoheadrightarrow v$ if $w \to v$ and one of the preferences in (i) is strict.
(v) $w \twoheadrightarrow \varnothing$ for each $w \to \varnothing$.[27]

[25] We restrict our attention to edge labeled graphs and assume that vertices are not labeled. It is worth noting that the notion of a *labeled graph* is different from that of *graph labeling*.

[26] If $x \twoheadrightarrow y$ is such an edge, we need to explore each vertex only once as we employ a depth-first search starting from y only checking whether there is an edge from the explored vertex back to x.

[27] Since the second preference in (ii) is always strict by NI$\varnothing$.

Note that in Γ^μ, a strict cycle with [without] $\varnothing$ as one of its vertices, corresponds to a PI-chain [PI-cycle] at μ. Conversely any PI-chain or PI-cycle at μ corresponds to a strict cycle of Γ^μ.

In what follows, let us write Γ^k instead of Γ^{μ^k} for notational simplicity. Then the ESMA is described as:

Step 0:
Select a strict preference profile $\succsim'$ from $\mathcal{T}(\succsim)$. Run the DA algorithm and obtain a temporary matching μ^0.

Step t $\geq$ 1:
$(t.a)$ Given μ^{t-1}, construct the associated 2-labeled graph Γ^{t-1}.
$(t.b)$ Find a strict cycle in Γ^{t-1} if there exists any, let the corresponding PI-cycle or the PI-chain take place to obtain μ^t, and go to step $(t+1.a)$. If there is no strict cycle, then return μ^{t-1} as the output of the algorithm.

Note that without specifying the choices involved in the algorithm, the above description really corresponds to a family of mechanisms. The following selection rule for tie-breaking, cycles and chains completes the description of the algorithm as a deterministic mechanism: (1) fix an enumeration of firms and workers, and add $\varnothing$ as the very last worker, (2) break all ties according to this order before running the DA in Step 0, and (3) run Tarjan's algorithm to identify the strongly connected components (where the enumeration fixed in (1) is the order in which Tarjan's search is implemented), (4) look for strict cycles beginning with exploring the lowest indexed strict edge, and the lowest indexed edge at every step afterwards, where the edges $i \to j$ are ranked in the lexicographic order for pairs (i, j).

Proposition 1. *The ESMA terminates in* $\mathrm{O}\big(|W|^3 \cdot Q\big)$ *time where* $Q = \sum_{f \in F} q_f$.

Proof. Each step t of the ESMA involves a strict cycle search in Γ^t which is $\mathrm{O}\big(|W \cup \{\varnothing\} + |E|\big)$, where E is the set of edges, by Lemma 7.

The DA algorithm which is conducted initially is $\mathrm{O}\big(|W| \cdot |F|\big)$, hence also $\mathrm{O}\big(|W|^3 \cdot Q\big)$ since $|F| \leq Q$. From the above paragraph, each subsequent step of the ESMA is $\mathrm{O}\big(|W|^2\big)$ since $|E| \leq (|W|+1)^2$. At each step, at least a worker or a firm improves, so these steps can be repeated at most $|W| \cdot |F|$ times in workers' favor and $|W| \cdot Q$ times in firms' favor. Hence the algorithm terminates in $\mathrm{O}\big(|W|^3 \cdot Q\big)$ time. $\square$

Worker-Optimal Stable Matching Algorithm (WOSMA)

Given a preference profile $\succsim$, to a stable matching μ, let us associate a 2-labeled graph G^μ with the vertex set $W \cup \{\varnothing\}$, and the edges and their labels specified as follows:

(i) $w \to v$ if $\mu(v)$ is a firm such that $w \in D^\mu_{\mu(v)}$.
(ii) $w \to \varnothing$ if there is a firm f with an empty position such that $w \in D^\mu_f$.
(iii) $\varnothing \to w$ if $\mu(w) = \varnothing$ or there is no worker who strictly desires and is acceptable to $\mu(w)$.

Label the strict edges as follows:

(iv) $w \twoheadrightarrow v$ if $w \to v$ and $\mu(v) \succ_w \mu(w)$.

In G^μ a strict cycle with [without] $\varnothing$ as one of its vertices, corresponds to an SWI-chain [SWI-cycle]. Conversely any SWI-chain or SWI-cycle corresponds to a strict cycle of G^μ.

Let us write G^k instead of G^{μ^k} in what follows, for notational simplicity.

Step 0:
Select a strict preference profile $\succsim'$ from $\mathcal{T}(\succsim)$. Run the DA algorithm and obtain a temporary matching μ^0.

Step t ≥ 1:
$(t.a)$ Given μ^{t-1}, let G^{t-1} be the associated 2-labeled graph as constructed above.
$(t.b)$ Find a strict cycle in G^{t-1}, if there exists any, let the corresponding SWI-cycle or the SWI-chain take place to obtain μ^t, and go to step $(t+1.a)$. If there is no strict cycle, then return μ^{t-1} as the output of the algorithm.

Proposition 2. *The WOSMA terminates in* $\mathrm{O}\big(|W|^3 \cdot |F|\big)$ *time.*

Proof. Each step t of the WOSMA involves a strict cycle search in G^t which is $\mathrm{O}\big(|E| + |W \cup \{\varnothing\}|\big)$ by Lemma 7.

The DA algorithm which is conducted initially is $\mathrm{O}\big(|W| \cdot |F|\big)$, hence also $\mathrm{O}\big(|W|^3 \cdot |F|\big)$. From the above paragraph, each subsequent step of the WOSMA is $\mathrm{O}\big(|W|^2\big)$ since $|E| \leq (|W|+1)^2$. At each step, at least a worker improves, so these steps can be repeated at most $|W| \cdot |F|$ times. Hence the algorithm terminates in $\mathrm{O}\big(|W|^3 \cdot |F|\big)$ time. □

References

Abdulkadiroğlu, A., Pathak, P., Roth, A.E., 2009. Strategyproofness versus efficiency: redesigning the NYC high school match. Am. Econ. Rev. 99 (5), 1954–1978.

Abdulkadiroğlu, A., Sönmez, T., 2003. School choice: a mechanism design approach. Am. Econ. Rev. 93 (3), 729–747.

Alcalde-Unzu, J., Molis, E., 2011. Exchange of indivisible goods and indifferences: the Top Trading Absorbing Sets mechanisms. Games Econ. Behav. 73 (1), 1–16.

Bogomolnaia, A., Moulin, H., 2004. Random matching under dichotomous preferences. Econometrica 72 (1), 257–279.

Crawford, V.P., Knoer, E.M., 1981. Job matching with heterogeneous firms and workers. Econometrica 49 (2), 437–450.

de Haan, M., Gautier, P.A., Oosterbeek, H., van der Klaauw, B., 2016. The performance of school assignment mechanisms in practice. University of Amsterdam. Mimeo.

Diebold, F., Bichler, M., 2017. Matching with indifferences: a comparison of algorithms in the context of course allocation. Eur. J. Oper. Res. 260 (1), 268–282.

Dubins, L., Freedman, D., 1981. Machiavelli and the Gale–Shapley algorithm. Am. Math. Mon. 88 (7), 485–494.

Dur, U., Kominers, S.D., Pathak, P., Sönmez, T., 2017. Reserve design: unintended consequences and the demise of Boston's walk zones. J. Polit. Econ. Forthcoming.

Erdil, A., Ergin, H., 2008. What's the matter with tie-breaking? Improving efficiency in school choice. Am. Econ. Rev. 98 (3), 669–689.

Gale, D., Shapley, L., 1962. College admissions and the stability of marriage. Am. Math. Mon. 69 (1), 9–15.

Irving, R., 2008. SFAS: February 2008 – a brief report. Computing Science Department, University of Glasgow. Mimeo.

Jaramillo, P., Manjunath, V., 2012. The difference indifference makes in strategy-proof allocation of objects. J. Econ. Theory 147 (5), 1913–1946.

Jones, P.C., 1983. A polynomial time market mechanism. J. Inf. Optim. Sci. 4 (2), 193–203.

Kagel, J., Roth, A.E., 2000. The dynamics of reorganization in matching markets: a laboratory experiment motivated by a natural experiment. Q. J. Econ. 115 (1), 201–235.

Kelso, A.S., Crawford, V.P., 1982. Job matching, coalition formation, and gross substitutes. Econometrica 50 (6), 1483–1504.

Kesten, O., 2010. School choice with consent. Q. J. Econ. 125 (3), 1297–1348.

Kominers, S.D., Sönmez, T., 2016. Matching with slot-specific priorities: theory. Theor. Econ. 11 (2), 683–710.

Roth, A.E., 1982. The economics of matching: stability and incentives. Math. Oper. Res. 7 (4), 617–628.

Roth, A.E., 2002. The economist as engineer: game theory, experimentation, and computation as tools for design economics. Econometrica 70 (4), 1341–1378.
Shapley, L., Scarf, H., 1974. On cores and indivisibility. J. Math. Econ. 1 (1), 23–28.
Tarjan, R.E., 1972. Depth-first search and linear graph algorithms. SIAM J. Comput. 1 (2), 146–160.