JOURNAL OF ECONOMIC THEORY 59, 451–465 (1993)

The Possibility of Efficient Mechanisms for Trading an Indivisible Object*

LOUIS MAKOWSKI

Department of Economics, University of California, Davis, California 95616-8578

AND

CLAUDIO MEZZETTI

Department of Economics, University of North Carolina, Chapel Hill, North Carolina 27599-3305

Received February 15, 1991; revised January 26, 1992

We study a trading problem in which the seller of an indivisible object faces at least two potential buyers. Each trader's valuation is privately known and regarded by the others as an independent random variable. We assume that the highest possible buyers' valuation is greater, by an arbitrarily small amount, than the seller's highest possible valuation. We show that for a family of distributions forming an open set in the space of probability distributions, there exist individually rational, ex post efficient, Bayesian mechanisms for trading the object. In addition, we show that any such mechanism is equivalent to a simple bidding game. *Journal of Economic Literature* Classification Numbers: D82, C72. © 1993 Academic Press, Inc.

1. INTRODUCTION

Consider a seller of an indivisible object who faces a small number of potential buyers. Each individual privately knows his own valuation and regards the others' valuations as random variables independently drawn from commonly known distributions.[1] Is it possible to design an individually rational, ex post efficient, Bayesian mechanism for trading the object?

We characterize conditions under which the answer is "yes." In addition

* We thank the Associate Editor and two anonymous referees for useful comments on an earlier version. Louis Makowski's research was supported by a grant from the NSF.

[1] We should stress that, although it is commonly made in the literature, the assumption of independent private values is an important limitation of the analysis.

0022-0531/93 $5.00

we show that the family of distributions satisfying these conditions forms an open set in the space of probability distributions. Furthermore, we present a class of trading rules that implement the ex post efficient allocation whenever this is possible. These mechanisms are obtained by adding carefully chosen subsidies to a Vickrey's second price auction.

Our possibility result rests on two main assumptions. First, there are at least two potential buyers. Second, the type of buyer with the highest valuation values the good more than the type of seller with the highest valuation; formally, the support of the seller's distribution $[a_0, b_0]$ has a lower upper bound than the support of the buyers' distribution $[a, b]$.

The role of our first assumption is to ensure at least a modicum of competition among buyers.[2] This reduces the "bribe" one must offer buyers to induce truthful revelation of their valuations. When there is just one potential buyer, and the interiors of the intervals over which the buyer's and seller's valuations are distributed have a non-empty intersection, Myerson and Satterthwaite [16] proved that there is no individually rational, incentive compatible, ex post efficient trading mechanism. Their result was extended by Gresik and Satterthwaite [5] to the case of many buyers and/or sellers, under the assumption that everybody's valuation is independently drawn from the same distribution. The second assumption introduces some ex ante asymmetry, beyond owning the good, between buyers and seller. It guarantees that strictly positive gains from trade exist with positive probability even for the seller with valuation b_0. This makes it possible to design a game which imposes a positive lump sum charge on the seller, without violating his individual rationality constraint. The money thus obtained can be used to bribe buyers to truthfully reveal their valuations.

Our possibility result is somewhat surprising. Impossibility theorems predominate in the literature (e.g., Laffont and Maskin [10], Makowski and Ostroy [12]). One exception is the case of a large number of agents (Holt [8], Harris and Raviv [7], Gresik and Satterthwaite [6], and Satterthwaite and Williams [18]). Another exception is analyzed by Cramton, Gibbons, and Klemperer [2].[3]

2. The Model

Consider a trading problem where a seller indexed by $i=0$ owns an indivisible object and faces n potential buyers. Each of the $n+1$ players'

[2] Basically the same model, along with analogues of all the results, could be formulated for the case of one buyer facing n sellers, i.e., competition on the seller's side instead of on the buyer's side.

[3] They show that ex post efficient trading mechanisms exist when an indivisible asset is initially jointly owned, with no player having too large a share.

valuations v_i $(i=0, 1, ..., n)$ is privately known at the time of trading. But it is common knowledge that the valuations are independent random variables. Specifically, the seller's valuation v_0 is drawn from a distribution F_0 with support $[a_0, b_0]$ and positive continuous density f_0; while each buyer's valuation is independently drawn from a distribution F with support $[a, b]$ and positive continuous density f. To avoid trivial cases, we shall assume throughout that the seller's and buyers' supports have an interior intersection, i.e., $a < b_0$ and $b \geqslant a_0$.

In a *direct trading mechanism* all the players simultaneously report their valuations to a coordinator, who then determines whether the object is transferred to some buyer and the money payments between players. Thus, such a mechanism is described by a pair of outcome functions $\langle p, t \rangle$ on $[a_0, b_0] \times [a, b]^n$, where $p(v) = \{p_i(v)\}$ are the probabilities that the object will be transferred to each player i and $t(v) = \{t_i(v)\}$ are the money transfers to each player i if $v = (v_0, v_1, ..., v_n)$ is the vector of reported valuations. We assume throughout that the mechanism must be *feasible* (budget balancing), $\sum_{i=0}^{n} t_i(v) = 0$ for all v.

Let $v_{-i} = (v_0, ..., v_{i-1}, v_{i+1}, ..., v_n)$, and let $E_{-i}(\cdot)$ be the expectation operator with respect to v_{-i}. Then $P_i(v_i) = E_{-i}(p_i(v))$ is i's expected probability of receiving the object, and $T_i(v_i) = E_{-i}(t_i(v))$ is i's expected money transfer when he announces v_i. Consequently, the expected payoffs to type v_0 of the seller and type v_j of buyer j $(j = 1, ..., n)$, when everyone truthfully reports, are given by

$$U_0(v_0) = T_0(v_0) - [1 - P_0(v_0)]\, v_0$$
$$U_j(v_j) = T_j(v_j) + P_j(v_j)\, v_j.$$

A trading mechanism is Bayesian *incentive compatible* if each type of each player wants to report his valuation truthfully when others report truthfully:

$$U_0(v_0) \geqslant T_0(\hat{v}_0) - [1 - P_0(\hat{v}_0)]\, v_0 \qquad \forall v_0, \hat{v}_0 \in [a_0, b_0]$$
$$U_j(v_j) \geqslant T_j(\hat{v}_j) + P_j(\hat{v}_j)\, v_j \qquad \forall j, \forall v_j, \hat{v}_j \in [a, b].$$

By the Revelation Principle (e.g., Myerson [14] or Dasgupta, Hammond, and Maskin [3]), there is no loss of generality in restricting our attention to incentive compatible direct mechanisms.

To guarantee everybody's participation, the mechanism must be interim *individually rational*, i.e., each type of each player must obtain a non-negative expected payoff:

$$U_0(v_0) \geqslant 0 \qquad \forall v_0 \in [a_0, b_0]$$
$$U_j(v_j) \geqslant 0 \qquad \forall j, \forall v_j \in [a, b].$$

Note, we normalize preferences so that no trade yields zero utility. The following lemma characterizes individually rational, incentive compatible trading mechanisms. The proof is in the Appendix.

LEMMA 1. *For any incentive compatible trading mechanism,*

$$U_0(b_0)+\sum_{j=1}^{n} U_j(a)=\sum_{j=1}^{n} aP_j(a)-b_0[1-P_0(b_0)]$$
$$+\sum_{j=1}^{n}\int_a^b [1-F(u)]\,u\,dP_j(u)-\int_{a_0}^{b_0} F_0(u)\,u\,dP_0(u).$$

Furthermore, for any $p(v)$ such that $P_i(v_i)$ is weakly increasing in v_i for all $i\in\{0, 1, ..., n\}$, there exists a transfer function $t(v)$ such that the trading mechanism $\langle p, t\rangle$ is individually rational and incentive compatible if and only if

$$\sum_{j=1}^{n} aP_j(a)-b_0[1-P_0(b_0)]+\sum_{j=1}^{n}\int_a^b [1-F(u)]\,u\,dP_j(u)$$
$$-\int_{a_0}^{b_0} F_0(u)\,u\,dP_0(u)\geqslant 0.$$

3. THE POSSIBILITY OF EX POST EFFICIENT MECHANISMS

Ex post efficiency requires that the object be given to the trader with the highest valuation. Thus, the mechanism $\langle p, t\rangle$ is *ex post efficient* if and only if $p_i(v)=1$ whenever $v_i>v_k$ for all $k\neq i$, and $p_i(v)=0$ whenever $v_i<v_k$ for some k. When two or more players have the same valuations, then the object can be given with non-negative probability to any of them. Theorem 1 gives a necessary and sufficient condition for the existence of an individually rational, incentive compatible, ex post efficient mechanism.[4] The following conventions are used: $F(u)=0$ if $u<a$, $F(u)=1$ if $u>b$, $F_0(u)=0$ if $u<a_0$, $F_0(u)=1$ if $u>b_0$.

THEOREM 1. *A mechanism $\langle p, t\rangle$ is individually rational, incentive compatible, and ex post efficient if and only if $P_0(v_0)=F^n(v_0)$, $P_j(v_j)=F_0(v_j)\,F^{n-1}(v_j)$ for $j=1, ..., n$, and*

$$-n\int_a^{b_0} F_0(u)\,F^{n-1}(u)[1-F(u)]\,du-n\int_{b_0}^{b} F^{n-1}(u)[1-F(u)]\,du$$
$$+\int_{b_0}^{b}[1-F^n(u)]\,du\geqslant 0. \qquad \text{(E)}$$

[4] Theorem 1, along with Lemma 1, could be straightforwardly extended to the case of asymmetric buyers, i.e., buyers whose valuations are drawn from different distributions.

Proof. Ex post efficiency implies $P_i(v_i) = \text{Prob}\{v_i > \max_{k \neq i}(v_k)\}$ for all i. By independence of traders' valuations, the distribution function of the random variable $x_i = \max_{k \neq i}(v_k)$ is $F^n(x_0)$ if $i = 0$ and $F_0(x_j) F^{n-1}(x_j)$ if $i = j \neq 0$. By substituting into the inequality of Lemma 1 and integrating by parts, the theorem follows. ∎

The expression on the l.h.s. of (E) equals the sum of all traders' expected payoffs when they are of the worst-off types, i.e., $U_0(b_0) + \sum_{i=1}^{n} U_i(a)$. Hence, to satisfy individual rationality, it must be non-negative. We will provide a better understanding of this expression in Section 5. Observe that when $n = 1$ the last two terms on the l.h.s. of (E) cancel out and (E) reduces to the expression in Myerson and Satterthwaite [16, p. 272] which is clearly always negative. Also notice that if $b \leqslant b_0$, then the l.h.s. of (E) is again negative, since $F(u) = 1$ for all $u \geqslant b$ and hence the last two integrals in (E) equal zero. This proves that both our two assumptions ($n > 1$ and $b > b_0$) are necessary to avoid an impossibility result.

In the next theorem, we show how to construct families of distributions F_0, F that satisfy (E) and demonstrate that the class of distributions satisfying (E), i.e., such that an individually rational, incentive compatible, ex post efficient trading mechanism can be designed, forms an open set in the space of probability distributions. We need to introduce a topology on the set of possible distributions. Let I be any compact interval and let $\mathscr{F}[I]$ be the set of distributions G on I with positive continuous density g. Endow $\mathscr{F}[I]$ with the metrizable C^1 uniform convergence (relative) topology. Under this topology G_q converges to G if and only if both $G_q - G$ and $g_q - g$ converge uniformly to zero.[5]

THEOREM 2. (i) *For any distribution F_0 with positive continuous density over $[a_0, b_0]$ and all a, b with $b_0 < b$, there exists an open set of distributions $\mathscr{F}^1 \subset \mathscr{F}[a, b]$ such that* (E) *is satisfied for any pair (F_0, F) with $F \in \mathscr{F}^1$. Conversely*:

(ii) *For any distribution F with positive continuous density over $[a, b]$ and all a, b with $b_0 < b$, there exists an open set of distributions $\mathscr{F}^0 \subset \mathscr{F}[a_0, b_0]$ such that* (E) *is satisfied for any pair (F_0, F) with $F_0 \in \mathscr{F}^0$.*

Proof. (i) Take $F = F(\gamma) \equiv [(u-a)/(b-a)]^{\gamma}$ where γ is a positive parameter. Notice that as γ tends to $+\infty$ the integrands in the first and second integral on the l.h.s. of (E) tend to zero, whereas the integrand in the third integral tends to $b - b_0 > 0$. Thus, there is a sufficiently large γ, say γ^*, such that when $F = F(\gamma^*)$, (E) is satisfied with strict inequality. Observe next that the l.h.s. of (E) is a continuous function of F. Hence,

[5] See Mas-Colell [13, Chap. 1, K.1].

there exists an open neighborhood of $F(\gamma^*)$, say $\mathscr{F}^1 \subset \mathscr{F}[a, b]$, such that (E) is satisfied for all $F \in \mathscr{F}^1$.

(ii) Take $F_0 = F_0(\gamma) \equiv [(u - a_0)/(b_0 - a_0)]^\gamma$ where γ is a positive parameter. Rewrite (E) as

$$-n \int_a^{b_0} F_0(u) F^{n-1}(u)[1 - F(u)]\, du$$
$$+ \int_{b_0}^{b} [1 - nF^{n-1}(u) + (n-1) F^n(u)]\, du \geqslant 0. \qquad \text{(E}'\text{)}$$

Observe that the integrand in the first integral on the l.h.s. of (E′) tends to zero as γ tends to $+\infty$, while the integrand in the second integral is equal to $1 - F^n(u) - nF^{n-1}(u)[1 - F(u)]$ which is always positive for $u \neq b$. Thus, there is a γ, say γ^*, such that $F_0(\gamma^*)$ satisfies (E′) with strict inequality. Further, since the l.h.s. of (E′) varies continuously with F_0, there is an open neighborhood of $F_0(\gamma^*)$, say $\mathscr{F}^0 \subset \mathscr{F}[a_0, b_0]$, such that (E′) is satisfied for all $F_0 \in \mathscr{F}^0$. ∎

The proof is based on the observation that (E) is more likely to be satisfied the more the buyers' and seller's distributions are concentrated at the tops of their supports. Notice that while we need $b > b_0$, Theorem 2 applies even if $b - b_0$ is made arbitrarily small. Notice also that, since for any $u \neq b$, $nF^{n-1}(u)$ and $F^n(u)$ tend to zero as n tends to $+\infty$, the first two integrals in (E) tend to zero while the third integral tends to $b - b_0$ as n tends to infinity. Thus, for any distributions F_0, F there exists a finite number n^* of buyers such that if $n > n^*$ an individually rational, incentive compatible, ex post efficient mechanism exists. This accords with our intuition that increasing the number of buyers reduces the effect of incomplete information.

As an application of our possibility result, consider a variation of the example in Myerson and Satterthwaite [16, pp. 273–274]. Let the buyers' valuations v_j be uniformly distributed on $[0, 1]$ and the seller's valuation v_0 be uniformly distributed on $[0, c]$ with $c \leqslant 1$. Then (E) simplifies to

$$\phi(c, n) \equiv \frac{n-1}{n+1} - c + \frac{c^n}{n+1} + \frac{2c^{n+1}}{(n+1)(n+2)} \geqslant 0.$$

Observe that $\phi(c, n)$ is convex in c for $c \in [0, 1]$, with $\phi(0, n) > 0$ and $\phi(1, n) < 0$. Thus, by the continuity of ϕ and Bolzano's theorem, given any n, there exists a $c^*(n)$ such that $\phi(c, n) \geqslant 0$ if and only if $c \leqslant c^*(n)$. In other words, the uniform case displays a "cut-off property." Given the number of buyers n with valuations uniformly distributed on $[0, 1]$, there exists a cut-off point $c^*(n)$ such that if the seller's valuation is uniformly distributed on

$[0, c]$ then an individually rational, incentive compatible, ex post efficient mechanism exists if and only if $c \leqslant c^*(n)$. For $n=2$, $c^*(2) \approx 0.3959$. Further, $c^*(n)$ is increasing in n and tends to 1 as n tends to $+\infty$. However, the number of buyers does not have to be very large in order to obtain a cut-off point reasonably close to 1: $c^*(n)$ increase from 0.5510 to 0.7003 to 0.8353 as n increase from 3 to 5 to 10.

4. A Bidding Game That Implements the ex Post Efficient Outcome

Given any vector of valuations v, let $j(v)$ be the buyer with the highest valuation, and let $\pi_1(v)$ be the largest and $\pi_2(v)$ be the second largest valuation among buyers. Let $\pi_f(v) = \max(\pi_1(v), v_0)$, $\pi_s(v) = \max(\pi_2(v), v_0)$, and let $d(v)$ be their difference, i.e., the deficit in a second price auction with reserve price v_0, $d(v) = \pi_f(v) - \pi_s(v)$. Define the family of *second price auctions with seller* (SPAWS) as the set of bidding games in which the highest bidder gets the object and the total payments that each buyer j makes and the seller receives are given by

$$-t_j(v) = \begin{cases} \pi_s(v) + h_j(v) & \text{if } v_j = \pi_f(v) > v_0 \\ h_j(v) & \text{otherwise} \end{cases}$$
$$t_0(v) = \begin{cases} \pi_f(v) - h_0(v) & \text{if } \pi_f(v) > v_0 \\ -h_0(v) & \text{otherwise} \end{cases} \tag{S}$$

$$\sum_{i=0}^{n} h_i(v) = d(v) \qquad \text{for each} \quad v. \tag{F}$$

$$E_{-i}(h_i(v)) \equiv H_i(v_i) = H_i, \qquad \text{a constant, for each } i \text{ and } v_i.^6 \tag{L}$$

By construction, any SPAWS mechanism is incentive compatible and ex post efficient. Further, from Step 2 in the proof of Lemma 1, we know that in any individually rational SPAWS mechanism $H_j \leqslant 0$ for each buyer j. That is, in expectations the seller must pay for the entire deficit. A simple SPAWS game that satisfies this additional requirement is constructed below.

Let D be the ex ante expected deficit, $D = E(d(v))$, and $D_j(v_j)$ be the expected deficit conditional on j being of type v_j and being the highest bidding buyer, $D_j(v_j) = E_{-j}[d(v) \mid v_j = \pi_1(v)]$. In the *basic SPAWS mechanism*,

$$h_j(v) = h_j^*(v) \equiv \begin{cases} d(v) - D_j(v_j) & \text{if } v_j = \pi_1(v) \\ 0 & \text{otherwise} \end{cases}$$
$$h_0(v) = h_0^*(v) \equiv D_{j(v)}(v_{j(v)}). \tag{SB}$$

[6] The functions $h_i(v)$ are transfers to be paid or received by each player. Capitalizing h_i and d indicates expected values of the functions.

Notice that the extra charges $h_i^*(\cdot)$ on seller and buyers are expected lump sums, as required, $H_0^*(v_0)=D$, $H_j^*(v_j)=0$ for each v_j and each $j=1, ..., n$. Notice also that no buyer apart from the highest bidding one pays anything to the seller. The highest bidding buyer pays the entire deficit in v minus a "bonus." The bonus, equal to $D_j(v_j)$, is returned to the highest bidding buyer even if there are no gains from trade, i.e., even if $\pi_1(v)<v_0$ and the seller keeps the object.

Since $H_j^*=0$ for each j, the basic SPAWS game is always individually rational for buyers. Our next result shows that it will also be individually rational for the seller, for any F_0, F which satisfy inequality (E) in Theorem 1.

THEOREM 3. *The basic SPAWS mechanism is an individually rational, incentive compatible, ex post efficient mechanism for any distributions F_0, F that satisfy* (E).

Proof. By construction, the basic SPAWS game is incentive compatible, ex post efficient, and individually rational for buyers. To show that it is also individually rational for the seller we must check that $U_0(b_0)\geqslant 0$. Let

$$A\equiv n\int_{b_0}^{b} uf(u)\,F^{n-1}(u)\,du=b-b_0F^n(b_0)-\int_{b_0}^{b} F^n(u)\,du,$$

and

$$B\equiv b_0-b_0\,F^n(b_0).$$

From the point of view of a seller of type b_0, A is the expectation of π_f and B is the expected opportunity cost, or loss of value, of participating in the mechanism. In the basic SPAWS game the expected transfer received by a seller of type b_0 is equal to $A-D$; thus, $U_0(b_0)=A-D-B$. The ex ante expected deficit D is given by

$$\begin{aligned}D\equiv{}& n\int_a^b uF_0(u)f(u)\,F^{n-1}(u)\,du-n\int_a^b uf_0(u)\,F^{n-1}(u)[1-F(u)]\,du\\&-n(n-1)\int_a^b uF_0(u)f(u)\,F^{n-2}(u)[1-F(u)]\,du\\={}& n\int_a^b F_0(u)\,F^{n-1}(u)\,du-n\int_a^b F_0(u)\,F^n(u)\,du,\end{aligned}$$

where the equality is obtained integrating the second integral on the l.h.s. by parts, with $dF_0(u)$ being one of the parts. It is straightforward to verify that $A-D-B$ equals the l.h.s. of (E). Hence the basic SPAWS game is individually rational if and only if (E) is satisfied. ∎

If F_0 and F satisfy (E) with strict inequality, the basic SPAWS mechanism is not the only individually rational SPAWS mechanism. Our next result shows that given any individually rational, incentive compatible, ex post efficient mechanism, there exists an equivalent SPAWS mechanism.

THEOREM 4. *For any individually rational, incentive compatible, ex post efficient mechanism $\langle p, t \rangle$, there exists an individually rational SPAWS mechanism with precisely the same outcome and expected transfers $T_i(v_i)$ for each i and v_i.*

Proof. Let $T_i^*(\cdot)$ be the expected transfers to individual i in the basic SPAWS game, which, recall, is incentive compatible. By Step 1(ii) in the proof of Lemma 1, for all $i = 0, ..., n$ and all $v_i, \hat{v}_i$

$$T_i(v_i) - T_i^*(v_i) = T_i(\hat{v}_i) - T_i^*(\hat{v}_i) = C_i, \qquad \text{a constant.}$$

Consider the bidding game in which the winning bidder always gets the object, and which satisfies (S) with $h_i(v) = h_i^*(v) - C_i$ for all i. By construction, this mechanism is expected utility equivalent to $\langle p, t \rangle$. To show it is in the SPAWS family we need only verify (F) and (L). By feasibility, $\sum_{i=0}^n t_i(v) = 0$ for all v. Therefore,

$$\int_{a_0}^{b_0} T_0(v_0)\, dF_0(v_0) + \sum_{j=1}^{n} \int_a^b T_j(v_j)\, dF(v_j) = 0$$

and similarly for $\{T_i^*(\cdot)\}$. Hence, $\sum_{i=0}^n C_i = 0$. Consequently, for each v, $\sum_{i=0}^n h_i(v) = \sum_{i=0}^n h_i^*(v) - \sum_{i=0}^n C_i = \sum_{i=0}^n h_i^*(v) = d(v)$. By construction, for each i and v_i, $H_i(v_i) = H_i^*(v_i) - C_i = H_i^* - C_i$, a lump sum. So (F) and (L) are verified. ∎

All SPAWS mechanisms are incentive compatible and ex post efficient. They differ, in expectations, only in the lump sum terms $\{H_i\}$. Further, in any SPAWS game $\sum_{i=0}^n H_i = D$. Thus, since individual rationality for buyers requires $H_j \leqslant 0$ for each $j = 1, ..., n$, in an individually rational SPAWS game $H_0 \geqslant D$. Hence, the best one for the seller has $H_0 = D$, which is precisely the seller's lump sum charge in the basic SPAWS game. Let $H(F_0, F)$ be the value of the l.h.s. of (E); if $H(F_0, F) > 0$ many different individually rational SPAWS mechanisms exist. The best from the point of view of buyers, the one that maximizes the sum of their expected payoffs, is the one with $\sum_{j=1}^n H_j = H(F_0, F)$.[7] While the basic SPAWS game is optimal from the point of view of the seller among all ex post efficient games, the seller would not choose to use it in the absence of some regula-

[7] See Makowski and Mezzetti [11, Theorem 5] for a simple proof.

tion. It is well known (e.g., Myerson [15], Riley and Samuelson [17]), that the individually rational, incentive compatible mechanism which maximizes the seller's expected payoff is a second price auction with a reserve price exceeding the seller's valuation v_0.

5. Intuition behind the Main Result

By Theorem 4, there is no loss of generality in restricting one's attention to games in the SPAWS family. Notice that in any SPAWS game the seller receives the first price but the winning buyer only pays the second price. Consequently, $\pi_f - \pi_s$ may be regarded as a bribe (a discount relative to the first price) offered to buyers in order to induce them to truthfully reveal. How is this bribe to be financed? Since the expected difference between π_f and π_s equals D, D is the minimum amount one must collect (lump sum) from the seller to be able to bribe buyers. Can this be done without violating the seller's individual rationality constraint? From the proof of Theorem 3, we know that (E) can be written as $A - B \geqslant D$, where A is the expected first price conditional on the seller being of type b_0, i.e., the worst type seller's (non-lump sum) revenue from participating in any SPAWS game, and B is the worst type seller's expected loss of value or opportunity cost of participating in the game. Observe that $A - B$ is the maximum expected charge which can be levied on the seller without violating his individual rationality constraint. Thus Theorem 1 can be reformulated as saying that an individually rational, incentive compatible, ex post efficient mechanism exists if and only if it is possible to devise a SPAWS game in which the maximum expected charge one can impose on the seller, without violating his individual rationality constraint, exceeds the minimum amount one must collect from him in order to successfully bribe buyers to tell the truth.

It is now clear why when $b \leqslant b_0$ an individually rational, incentive compatible, ex post efficient mechanism never exists. If $b \leqslant b_0$, then $A = 0$, i.e., one cannot collect anything from the seller without violating his individual rationality constraint. It is also clear why more than one buyer is needed for a possibility result. When $n = 1$, the ex ante expected gains from trade coincide with the ex ante expected deficit D, the minimum we must collect from the seller to be able to bribe buyers. But $A - B$, the interim expected gains from participating for a seller of the worst type b_0, is always less than the ex ante expected gains from trade. Thus, when $n = 1$, $A - B < D$ and (E) cannot be satisfied. With only one buyer the maximum lump sum charge one can make the seller pay is never sufficient to bribe the buyer to truthfully report his valuation.

APPENDIX: PROOF OF LEMMA 1[8]

STEP 1. *$\langle p, t\rangle$ is incentive compatible if and only if $\forall i, \forall v_i, \hat{v}_i$:*

(i) *$P_i(v_i)$ is weakly increasing, and*

(ii) $T_i(\hat{v}_i) - T_i(v_i) = \int_{\hat{v}_i}^{v_i} u\, dP_i(u)$.

Proof of Step 1 (Only if). If $\langle p, t\rangle$ is incentive compatible then

$$U_0(\hat{v}_0) \geqslant U_0(v_0) - (\hat{v}_0 - v_0)[1 - P_0(v_0)] \qquad \forall v_0, \hat{v}_0 \in [a_0, b_0],$$

and

$$U_j(\hat{v}_j) \geqslant U_j(v_j) + (\hat{v}_j - v_j)\, P_j(v_j) \qquad \forall j, \forall v_j, \hat{v}_j \in [a, b].$$

The above inequalities imply

$$(\hat{v}_0 - v_0)[1 - P_0(v_0)] \geqslant U_0(v_0) - U_0(\hat{v}_0) \geqslant (\hat{v}_0 - v_0)[1 - P_0(\hat{v}_0)]$$

and

$$(v_j - \hat{v}_j)\, P_j(v_j) \geqslant U_j(v_j) - U_j(\hat{v}_j) \geqslant (v_j - \hat{v}_j)\, P_j(\hat{v}_j).$$

Thus, $\forall i \in \{0, 1, ..., n\}$ if $v_i > \hat{v}_i$, it must be $P_i(v_i) \geqslant P_i(\hat{v}_i)$, i.e., $P_i(v_i)$ is weakly increasing. Since $P_i(v_i)$ is increasing, it is also Riemann integrable, and thus $U_0'(v_0) = P_0(v_0) - 1$ and $U_j'(v_j) = P_j(v_j)$ almost everywhere. Integrating gives

$$U_0(v_0) - U_0(\hat{v}_0) = \int_{\hat{v}_0}^{v_0} [P_0(u) - 1]\, du$$

and

$$U_j(v_j) - U_j(\hat{v}_j) = \int_{\hat{v}_j}^{v_j} P_j(u)\, du.$$

Integration by parts yields the expression in Step 1.

(If) Adding the identity $v_i[P_i(v_i) - P_i(\hat{v}_i)] \equiv v_i \int_{\hat{v}_i}^{v_i} dP_i(u)$ to the expression in Step 1 and rearranging yields

$$U_0(v_0) \geqslant T_0(\hat{v}_0) - v_0[1 - P_0(\hat{v}_0)]$$

[8] The proof of Lemma 1 makes substantial use of the analysis contained in the papers of Myerson and Satterthwaite [16, Theorem 1] and Cramton, Gibbons, and Klemperer [2, Lemmata 1–4].

and

$$U_j(v_j) \geqslant T_j(\hat{v}_j) + v_j P_j(\hat{v}_j),$$

which is incentive compatibility. ∎

STEP 2. *An incentive compatible mechanism* $\langle p, t \rangle$ *is individually rational if and only if*:

(i) $T_0(b_0) \geqslant b_0[1 - P_0(b_0)]$ *and*

(ii) $T_j(a) \geqslant -aP_j(a)$ *for all buyers* j.

Proof of Step 2. By Step 1, $P_i(v_i)$ is increasing $\forall i$, $U_0'(v_0) = P_0(v_0) - 1$ and $U_j'(v_j) = P_j(v_j)$ almost everywhere. Thus, $v_0 = b_0$ is the seller's type and $v_j = a$ the type of buyer j who are worst-off in expected utility terms. Hence, $U_0(b_0) \geqslant 0$ and $U_j(a) \geqslant 0\ \forall j$ are necessary and sufficient for individual rationality. ∎

STEP 3 (Conclusion of the proof of Lemma 1). *Suppose* $\langle p, t \rangle$ *is incentive compatible. Then from Step* 1,

$$T_0(v_0) = T_0(b_0) + \int_{v_0}^{b_0} u\, dP_0(u)$$

and

$$T_j(v_j) = T_j(a) - \int_a^{v_j} u\, dP_j(u) \qquad \textit{for each } j.$$

Integrating over $[a_0, b_0]$ *and* $[a, b]$ *yields*

$$E(t_0(v)) = E_0(T_0(v_0)) = T_0(b_0) + \int_{a_0}^{b_0} \int_{v_0}^{b_0} u\, dP_0(u)\, dF_0(v_0)$$

and

$$E(t_j(v)) = E_j(T_j(v_j)) = T_j(a) - \int_a^b \int_a^{v_j} u\, dP_j(u)\, dF(v_j).$$

Integrating by parts and rearranging yields

$$E_0(T_0(v_0)) = T_0(b_0) + \int_{a_0}^{b_0} uF_0(u)\, dP_0(u)$$

and

$$E_j(T_j(v_j)) = T_j(a) - \int_a^b u[1 - F(u)]\, dP_j(u).$$

By linearity of the expectation operator, $\sum_{i=0}^{n} E(t_i(v)) = E(\sum_{i=0}^{n} t_i(v))$. By feasibility $\sum_{i=0}^{n} t_i(v) = 0$. Thus

$$T_0(b_0) + \sum_{j=1}^{n} T_j(a) = \sum_{j=1}^{n} \int_a^b u[1 - F(u)]\, dP_j(u) - \int_{a_0}^{b_0} uF_0(u)\, dP_0(u).$$

Adding $\sum_{j=1}^{n} aP_j(a) - b_0[1 - P_0(b_0)]$ to both sides of this equality yields the first statement in Lemma 1. Further, by Step 2 and individual rationality,

$$T_0(b_0) + \sum_{j=1}^{n} T_j(a) \geqslant b_0[1 - P_0(b_0)] - \sum_{j=1}^{n} aP_j(a).$$

Substituting the last equality into this inequality yields the "only if" part of the second statement in Lemma 1.

The proof of the "if" part is by construction. Suppose $P_i(v_i)$ is increasing and the inequality in Lemma 1 is satisfied. We need to find a $\{t_i(v)\}$ such that $\langle p, t\rangle$ is an individually rational, incentive compatible trading mechanism. Define $t_i(v)$ as $t_i(v) = c_i - \int_{a_i}^{v_i} u\, dP_i(u) + (1/n) \sum_{k \neq i} \int_{a_k}^{v_k} u\, dP_k(u)$, with $\sum_{i=0}^{n} c_i = 0$, $a_k = a_0$ for $k = 0$, and $a_k = a$ for $k \neq 0$. Notice, $\sum_{i=0}^{n} c_i = 0$ implies $\sum_{i=0}^{n} t_i(v) = 0$, i.e., $\langle p, t\rangle$ is feasible. Taking the expectation over v_{-i} and integrating by parts yields

$$E_{-i}(t_i(v)) \equiv T_i(v_i) = c_i - \int_{a_i}^{v_i} u\, dP_i(u) + \frac{1}{n} \sum_{k \neq i} \int_{a_k}^{b_k} u[1 - F_k(u)]\, dP_k(u)$$

with $b_k = b_0$, $F_k = F_0$ for $k = 0$, and $b_k = b$, $F_k = F$ for $k \neq 0$. Thus

$$T_i(\hat{v}_i) - T_i(v_i) = \int_{\hat{v}_i}^{v_i} u\, dP_i(u)$$

and, by Step 1, $\langle p, t\rangle$ is incentive compatible. Observe that the defined $t(v)$ is such that

$$\begin{aligned} C &\equiv T_0(b_0) + \sum_{j=1}^{n} T_j(a) + \sum_{j=1}^{n} aP_j(a) - b_0[1 - P_0(b_0)] \\ &= \sum_{j=1}^{n} aP_j(a) - b_0[1 - P_0(b_0)] \\ &\quad + \sum_{j=1}^{n} \int_a^b u[1 - F(u)]\, dP_j(u) - \int_{a_0}^{b_0} uF_0(u)\, dP_0(u). \end{aligned}$$

By the inequality in Lemma 1, the r.h.s. of the above equation is non-negative, hence $C \geqslant 0$. Now, we can choose

$$c_0 = \frac{C}{n+1} + b_0[1 - P_0(b_0)] + \int_{a_0}^{b_0} u \, dP_0(u) - \frac{1}{n} \sum_{j=1}^{n} \int_{a}^{b} u[1 - F(u)] \, dP_j(u)$$

and

$$c_j = \frac{C}{n+1} - aP_j(a) - \frac{1}{n} \sum_{k \neq j} \int_{a_k}^{b_k} u[1 - F_k(u)] \, dP_k(u).$$

Notice that $\sum_{i=0}^{n} c_i = 0$. The choice of c_i's guarantees that $T_0 - b_0[1 - P_0(b_0)] \geqslant 0$ and, for each buyer j, $T_j(a) + aP_j(a) \geqslant 0$. Thus, by Step 2, $\langle p, t \rangle$ is individually rational. ∎

References

1. K. Chatterjee and W. Samuelson, Bargaining under incomplete information, *Oper. Res.* **31** (1983), 835–851.
2. P. Cramton, R. Gibbons, and P. Klemperer, Dissolving a partnership efficiently, *Econometrica* **55** (1987), 615–632.
3. P. Dasgupta, P. Hammond, and E. Maskin, The implementation of social choice rules: Some general results on incentive compatibility, *Rev. Econ. Stud.* **46** (1979), 185–216.
4. C. D'Aspremont and L. A. Gerard-Varet, Incentives and incomplete information, *J. Public Econ.* **11** (1979), 25–45.
5. T. Gresik and M. Satterthwaite, "The Number of Traders Required to Make a Market Competitive: The Beginnings of a Theory," CMSEMS D.P. No. 551, Northwestern University, 1983.
6. T. Gresik and M. Satterthwaite, The rate at which a simple market converges to efficiency as the number of traders increases: An asymptotic result for optimal trading mechanisms, *J. Econ. Theory* **48** (1989), 304–332.
7. M. Harris and A. Raviv, Allocation mechanisms and the design of auctions, *Econometrica* **49** (1981), 1477–1499.
8. C. Holt, Uncertainty and the bidding for incentive contracts, *Amer. Econ. Rev.* **69** (1979), 697–705.
9. B. Holmström and R. Myerson, Efficient and durable decision rules with incomplete information, *Econometrica* **51** (1983), 1799–1820.
10. J. J. Laffont and E. Maskin, A differential approach to expected utility maximizing mechanisms, *in* "Aggregation and Revelation of Preferences" (J. J. Laffont, Ed.), pp. 289–308, North-Holland, Amsterdam, 1979.
11. L. Makowski and C. Mezzetti, "The Possibility of Efficient Mechanisms for Trading and Indivisible Object," W. P. No. 344, University of California at Davis, 1989.
12. L. Makowski and J. Ostroy, Efficient and individually rational Bayesian mechanisms only exist on perfectly competitive environments, mimeo, 1989.
13. A. Mas-Colell, "The Theory of General Economic Equilibrium: A Differentiable Approach," Cambridge Univ. Press, Cambridge, 1985.
14. R. Myerson, Incentive compatibility and the bargaining problem, *Econometrica* **47** (1979), 61–73.

15. R. MYERSON, Optimal auction design, *Math. Oper. Res.* **6** (1981), 58–73.
16. R. MYERSON AND M. SATTERTHWAITE, Efficient mechanisms for bilateral trading, *J. Econ. Theory* **29** (1983), 265–281.
17. J. RILEY AND W. SAMUELSON, Optimal auctions, *Amer. Econ. Rev.* **71** (1981), 381–392.
18. M. SATTERTHWAITE AND S. WILLIAMS, Rate of convergence to efficiency in the buyers bid double auction as the market becomes large, *Rev. Econ. Stud.* **56** (1989), 477–498.
19. W. VICKREY, Counterspeculation, auctions, and competitive sealed tenders, *J. Finance* **16** (1961), 8–37.