Theory of mechanism design

Midterm Examination

February 27, 2019; Duration: 3 hours; Total marks: **40**

All the questions assume quasilinear utility and independent private values model.

Wherever not specified, assume that an agent's outside option gives zero utility.

Write your answers clearly without unnecessary arguments.

1. Suppose there is a single buyer for a single object. Suppose the value of the buyer (type space) lies in $[0,1]$. Consider two possible distributions of values of the buyer: F_1 and F_2 with positive densities f_1 and f_2 respectively. We say F_1 **first order stochastic dominates (FOSD)** F_2 if for every non-decreasing function $\phi : [0,1] \to \mathbb{R}$, we have

$$\int_0^1 \phi(x) f_1(x) dx \geq \int_0^1 \phi(x) f_2(x).$$

   (a) Show that if F_1 FOSD F_2 then $F_1(x) \leq F_2(x)$ for all $x \in [0,1]$. **(2 marks)**

   Answer. Take any $x \in [0,1]$ and choose $\phi(y) = 1$ if $y \geq x$ and $\phi(y) = 0$ otherwise. We get from the definition of FOSD, $1 - F_1(x) \geq 1 - F_2(x)$, which is $F_1(x) \leq F_2(x)$.

   (b) Show that the revenue from an optimal mechanism according to F_1 is no less than the revenue from an optimal mechanism according to F_2. **(2 marks)**

   Answer. We know that the optimal mechanism is a posted price mechanism whose expressions for revenue according to F_1 and F_2 are:

$$\max_p p\Big(1 - F_1(p)\Big)$$
$$\max_p p\Big(1 - F_2(p)\Big)$$

   Since F_1 FOSD F_2, the optimal revenue using F_1 is larger than that using F_2.

   (c) Consider an arbitrary dominant strategy incentive compatible mechanism (q,p), where q is the allocation rule and p is the payment rule. Show that the expected revenue from the mechanism (q,p) according to F_1 is no less than the revenue according to F_2. **(4 marks)**

   Answer. Take $v > v'$ and incentive compatibility gives $v' f(v') - p(v') \geq v' f(v) - p(v)$. This implies that $p(v) \geq v'\big[f(v) - f(v')\big] + p(v')$. Since $f(v) \geq f(v')$, we have $p(v) \geq p(v')$.

So, p is a non-decreasing function. By our earlier part, $\int_0^1 p(v)f_1(v)dv \geq \int_0^1 p(v)f_2(v)dv$, i.e., the expected revenue according to F_1 is larger than that from F_2.

2. A seller is selling a single object to a single buyer whose value is distributed in $[0,1]$. Consider a mechanism (q,p), where q is the allocation rule and p is the payment rule. The mechanism (q,p) is **locally incentive compatible** if there exists some $\epsilon > 0$ such that for all types $v, v' \in [0,1]$ with $|v - v'| \leq \epsilon$, the following holds:

$$q(v)v - p(v) \geq q(v')v - p(v').$$

Show that if (q,p) is locally incentive compatible, it is dominant strategy incentive compatible. **(10 marks)**

Answer. This is a simple application of Myersonian characterization of incentive compatibility. First, we divide the type space as: into $[0,\epsilon], [\epsilon, 2\epsilon], \ldots, [K\epsilon, 1]$, where in each part $T^k \equiv [k\epsilon, (k+1)\epsilon]$ the mechanism (q,p) restricted to T^k is incentive compatible. But, then $q(v) \geq q(u)$ for each $u > v$ with $u, v \in T^k$. Applying it for all T^k, we immediately get that q is non-decreasing. To see this rigorously, take any $v \in T^k$ and $u \in T^{k+\ell}$ with $u > v$. Applying non-decreasingness of q in each part, we have $q(v) \leq q((k+1)\epsilon) \leq q((k+2)\epsilon) \leq \ldots \leq q((k+\ell)\epsilon) \leq q(u)$.

Next, we argue that p satisfies revenue equivalence. For this, again, we know that revenue equivalence formula holds in each T^k. Denoting the net utility function of this mechanism as $\mathcal{U}$, we can write,

$$\begin{aligned}
\mathcal{U}(\epsilon) &= \mathcal{U}(0) + \int_0^{\epsilon} q(x)dx \\
\mathcal{U}(2\epsilon) &= \mathcal{U}(\epsilon) + \int_{\epsilon}^{2\epsilon} q(x)dx = \mathcal{U}(0) + \int_0^{2\epsilon} q(x)dx \\
\ldots &= \ldots \\
\ldots &= \ldots \\
\mathcal{U}(K\epsilon) &= \mathcal{U}(0) + \int_0^{K\epsilon} q(x)dx
\end{aligned}$$

Now, pick any $v \in T^k$, and use the revenue equivalence in T^k to conclude $\mathcal{U}(v) = \mathcal{U}(k\epsilon) + \int_{k\epsilon}^{v} q(x)dx$. Then, use the above sequence of revenue equivalence formula to conclude $\mathcal{U}(v) = \mathcal{U}(0) + \int_0^v q(x)dx$.

Hence, (q, p) satisfies the fact f is monotone and revenue equivalence formula holds. Then, by Myerson, (q, p) is incentive compatible.

3. There are two agents $N = \{1, 2\}$. There is a single divisible object to be allocated to the agents. If an agent gets $q \in [0, 1]$ share of the object and receives a transfer of t, then her payoff is $qv + t$, where v is the per unit value of the agent.

   Suppose values of both the agents are drawn from $[0, 1]$. Consider the following allocation rule $q \equiv (q_1, q_2)$: at every profile (v_1, v_2),

$$q_1(v_1, v_2) = \frac{1}{2} + \frac{1}{2}(v_1 - v_2)$$
$$q_2(v_1, v_2) = 1 - q_1(v_1, v_2).$$

   (a) Argue that q is dominant strategy implementable, i.e., there exists transfer rules $t \equiv (t_1, t_2)$ such that (q, t) is dominant strategy incentive compatible. **(2 marks)**

   Hint. To show the above, you may want to consider the type profile where both agents have the same value and use symmetry and revenue equivalence at that type profile.

   Answer. Since q is increasing, it is implementable.

   (b) Call a transfer rule $t \equiv (t_1, t_2)$ **symmetric** if $t_1(v_1, v_2) = t_2(v_2, v_1)$ for all $v_1, v_2 \in [0, 1]$. Either show that there exists a symmetric and budget-balanced transfer rule $t \equiv (t_1, t_2)$ such that (q, t) is dominant strategy incentive compatible or conclude that such a mechanism cannot exist. **(10 marks)**

   Answer. I give a *sketch* of the proof. There is a transfer rule (t_1, t_2) such that (q, t_1, t_2) is DSIC, symmetric and budget-balanced. We can explicitly construct such a transfer rule. First, because the rule is symmetric $t_1(v, v) = t_2(v, v)$ for any v. Also, $q_1(v, v) = q_2(v, v) = \frac{1}{2}$. By budget-balance, $vq_1(v, v) + t_1(v, v) + vq_2(v, v) + t_2(v, v) = v$. Denote by $\mathcal{U}_1$ and $\mathcal{U}_2$, the net utility functions of the two agents. Note that since q_1 and q_2 are symmetric and t_1 and t_2 are symmetric, $\mathcal{U}_1$ and $\mathcal{U}_2$ are also symmetric.

Then, revenue equivalence formula gives $\mathcal{U}_1(v,v) = \mathcal{U}_1(0,v) + \int_0^v q_1(x,v)dx$. Similarly, $\mathcal{U}_2(v,v) = \mathcal{U}_2(v,0) + \int_0^v q_2(v,x)dx = \mathcal{U}_1(0,v) + \int_0^v q_1(x,v)dx$, where the last equality uses symmetry. Adding them and using that the LHS sum is v, we get $v = 2\mathcal{U}_1(0,v) + 2\int_0^v q_1(x,v)dx$. The integral can be easily computed from the expression of q_1 and this pins down $\mathcal{U}_1(0,v)$ for **every** v. But once this is determined, we can compute $\mathcal{U}_1(u,v)$ for each u, v. By symmetry $\mathcal{U}_2(u,v)$ can be computed. What remains to be verified is that $\mathcal{U}_1(u,v) + \mathcal{U}_2(u,v)$ will be equal to $uq_1(u,v) + vq_2(u,v)$, which will imply that budget-balance will hold. But this can be easily verified. Revenue equivalnce and symmetry holds by construction.

4. A seller is selling K units of a single good to n buyers. The value (type) of a buyer i is denoted by $v_i : \{0, 1, \ldots, K\} \to \mathbb{R}_+$ with $v_i(0) = 0$. Assume that the value for every buyer i satisfies **increasing marginal values**: $v_i(k+1) - v_i(k) \geq v_i(k) - v_i(k-1)$ for each $k \in \{1, \ldots, K-1\}$.

   (a) The seller considers the following mechanism. It bundles all K units as one good and does a Vickrey auction of all K units. In other words, it chooses a buyer in $\arg\max_i v_i(K)$ and allocates **all** K units to this buyer. It asks the winning buyer (say i) to pay the second highest value on K units, i.e., $\max_{j \neq i} v_j(K)$. Is this mechanism DSIC and ex-post individually rational? **(4 marks)**

   Answer. Since the information of all units less than K is ignored, this is like a single object Vickrey auction. We know that the Vickrey auction is DSIC and ex-post IR. Hence, this mechanism is also DSIC and ex-post IR.

   (b) Suppose the seller uses the VCG mechanism for K units. How will that compare to the Vickrey auction of K units described above (compare the allocation and payment rules)? **(6 marks)**.

   Answer. For this problem, the VCG mechanism is identical to bundling all K units and doing a Vickrey auction. We will argue this by arguing that efficiency implies giving **all** K units to one buyer is efficient. Once this is argued, the VCG mechanism just collapses to the Vickrey auction of the bundle of K units.

   Suppose not. This means in all efficient allocations at a valuation profile v, two buyers, say i and j, get allocated x_i and x_j such that $1 \leq x_i < K$ and $1 \leq x_j < K$.

But if this is efficient then

$$
\begin{aligned}
v_i(x_i+1) - v_i(x_i) &\leq v_j(x_j) - v_j(x_j-1) && \text{(due to efficiency)} \\
&\leq v_j(x_j+1) - v_j(x_j) && \text{(due to increasing marginal values)} \\
&\leq v_i(x_i) - v_i(x_i-1) && \text{(due to efficiency)},
\end{aligned}
$$

where either the first inequality or the last inequality must hold strictly. Then, $v_i(x_i+1) - v_i(x_i) < v_i(x_i) - v_i(x_i-1)$, which contradicts increasing marginal values.