Theory of Mechanism Design; Midterm Examination, 2024

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All questions assume independent private values model in the single object auction with values distributed for bidder i in $[0, a_i]$ with cdf F_i and positive density f_i. In the symmetric model, these distributions are the same, and we only consider symmetric equilibria (strictly increasing and differentiable strategies).

1. Consider a standard auction in the symmetric model. We have shown in the class that to show that a symmetric strategy is an equilibrium, we only need to show that imitation is not profitable. The objective of this question is to show that it is enough to show that only "local" imitation is not profitable. To remind, fix a symmetric strategy s, and denote by $Q(s(x); s)$ the interim allocation probability of winning the object for a bidder with value x by following s if others follow s. Similarly, denote by $P(s(x); s)$ the interim payment of bidder x by following s if others follow s. Imitation lemma shows that $(s, \ldots, s)$ is a Bayesian equilibrium if

$$Q(s(x); s)x - P(s(x); s) \geq Q(s(y); s)x - P(s(y); s) \qquad \forall x, y \in [0, a] \tag{1}$$

If constraint (1) holds for bidder type x not imitating to y, we denote it as $x \to y$.

(a) Pick x_1, x_2, x_3 such that x_2 lies between x_1 and x_3 ($x_1 > x_3$ and $x_1 < x_3$ are both possible). Suppose $x_1 \to x_2$ and $x_2 \to x_3$. Then, show that $x_1 \to x_3$. (3 marks)

Answer. The IC constraints $x_1 \to x_2$ and $x_2 \to x_3$ means

$$Q(s(x_1); s)x_1 - P(s(x_1); s) \geq Q(s(x_2); s)x_1 - P(s(x_2); s)$$
$$Q(s(x_2); s)x_2 - P(s(x_2); s) \geq Q(s(x_3); s)x_2 - P(s(x_3); s)$$

Adding them gives

$$\begin{aligned} Q(s(x_1); s)x_1 - P(s(x_1); s) &\geq Q(s(x_2); s)x_1 + Q(s(x_3); s)x_2 - Q(s(x_2); s)x_2 - P(s(x_3); s) \\ &= G(x_2)x_1 + \big(G(x_3) - G(x_2)\big)x_2 - P(s(x_3); s) \end{aligned}$$

where we used the fact that in the standard auction symmetric strategy, $G(x) \equiv Q(s(x); s)$ due to efficiency. Hence,

$$Q(s(x_1); s)x_1 - P(s(x_1); s) \geq G(x_2)x_1 + \big(G(x_3) - G(x_2)\big)x_2 - P(s(x_3); s)$$

If $x_1 > x_3$, then $G(x_3) < G(x_1)$ and this gives

$$\begin{aligned} Q(s(x_1); s)x_1 - P(s(x_1); s) &\geq G(x_2)x_1 + \big(G(x_3) - G(x_2)\big)x_2 - P(s(x_3); s) \\ &\geq G(x_2)x_1 + \big(G(x_3) - G(x_2)\big)x_1 - P(s(x_3); s) \\ &= G(x_3)x_1 - P(s(x_3); s) \\ &= Q(s(x_3); s)x_1 - P(s(x_3); s) \end{aligned}$$

which is the desired $x_1 \to x_3$ IC constraint.

If $x_1 < x_3$, then $x_1 < x_2$ and $G(x_2) < G(x_3)$ implies

$$\begin{aligned} Q(s(x_1); s)x_1 - P(s(x_1); s) &\geq G(x_2)x_1 + \big(G(x_3) - G(x_2)\big)x_2 - P(s(x_3); s) \\ &= G(x_3)x_2 + (x_1 - x_2)G(x_2) - P(s(x_3); s) \\ &\geq G(x_3)x_2 + (x_1 - x_2)G(x_3) - P(s(x_3); s) \\ &= G(x_3)x_1 - P(s(x_3); s) \\ &= Q(s(x_3); s)x_1 - P(s(x_3); s) \end{aligned}$$

which is the desired $x_1 \to x_3$ IC constraint.

(b) Suppose the space of valuation $[0, a]$ is split into intervals of ϵ, where $a > \epsilon > 0$: $I_1 = [0, \epsilon], I_2 = [\epsilon, 2\epsilon], \ldots, I_k = [(k-1)\epsilon, k\epsilon = a]$. Suppose for every interval I_ℓ and every $x, y \in I_\ell$, $x \to y$ and $y \to x$ holds. Then show that $(s, \ldots, s)$ is a Bayesian equilibrium. (6 MARKS)

Answer. This follows immediately from (a). We can do this using induction on k. If $k = 2$, this follows from (a) as follows. Pick $x_1, x_3 \in [0, a]$, and note that $[0, a] = I_1 \cup I_2$. If x_1, x_3 are interior of I_1 or I_2, then we are done. Else, we can choose x_2 to be the meeting point of I_1 and I_2, which is ϵ. Since $x_1 \to x_2$ and $x_2 \to x_3$ holds, by (a), $x_1 \to x_3$ holds.

Now, assume that the claim holds for any interval partitioned into $(k-1)$ subintervals. Then, let $[0,a]$ be partitioned into k intervals $I_1, \dots, I_k$. Pick $x_1, x_3 \in [0,a]$. If x_1, x_3 are in the interior of $I_1 \cup \dots \cup I_{k-1}$ or I_k, we are done by our induction hypothesis. Else, $x_1 \in I_1 \cup \dots \cup I_{k-1}$ and $x_3 \in I_k$. Then, we can choose, x_2 to be the meeting point of I_{k-1} and I_k, which is $(k-1)\epsilon$. Applying (a), we are done again.

(c) Can you give a formal definition of a "local Bayesian equilibrium" in a first-price auction using the above notion of locality? Discuss how it relates to the usual definition of Bayesian equilibrium and if this notion of equilibrium makes any sense (in your opinion). (3 MARKS)

Answer. For any strategy profile to be an equilibrium, we first need to ensure that imitation lemma holds. This is true because of standard auction property: if a bidder bids $s(a)$ and others follow s he wins for sure because of standard auction (highest bidder wins), and any bid more than $s(a)$ weakly less profitable. Hence, deviating to bid more than $s(a)$ is same as deviating to bid $s(a)$, establishing imitation lemmma.

Hence, given (a), we can say a strategy profile $(s, \dots, s)$ is a symmetric local Bayesian equilibrium if there exists an $\epsilon > 0$ such that for every $x \in [0,a]$ and for all $y \in [\max(0, x-\epsilon), \min(a, x+\epsilon)]$

$$Q(s(x); s)x - P(s(x); s) \geq Q(s(y); s)x - P(s(y); s)$$

Local Bayesian equilibrium appears weaker than Bayesian equilibrium, but it is equivalent to Bayesian equilibrium as shown in (a).

Local Bayesian equilibrium seems like an assumption on the behavior of agents: they would like to deviate within ϵ neighborhood of truth. Since we can characterize precise Bayesian equilibrium of standard auctions, (a) says that even with such limiting assumptions on behavior of agents, the equilibrium conditions remain the same.

2. A seller can sell its good in two markets using a first-price auction: (a) MARKET 1 consists of n symmetric CRRA bidders with $\alpha \in (0,1)$ being the coefficient of relative

risk aversion; (b) MARKET 2 consists of n symmetric risk neutal bidders (the bidders in both markets are different and cannot move across markets). The bidders in both the markets are ex-ante identical: they all draw values from a common distribution F with positive density f and support $[0, a]$.

(a) Which market will generate more expected revenue for the seller? (2 MARKS)

Answer. Risk averse buyers bid more aggressively and generate more expected revenue than risk neutral buyers in first-price auctions. So, MARKET 1 will generate more expected revenue.

(b) Suppose the seller has the option to **upgrade** the good before selling in one of the markets (but not in the other): an upgrade results in shifting the value distribution from F to another distribution F' such that F' first-order stochastically dominates F. Is there an upgrade possible such that the expected revenue in the market where the upgraded good is sold equals the expected revenue in the market where the standard good is sold? Which market should see the upgraded good? (4 MARKS)

Answer. We know that the expected revenue in MARKET 1 is the expected revenue of a risk-neutral economy where value of each buyer is drawn from a distribution with cdf $F'(x) \equiv [F(x)]^{\frac{1}{\alpha}}$, where $\alpha \in (0, 1)$ is the coefficient of risk aversion. Clearly, F' first-order stochastically dominates F. Hence, if the good in MARKET 2 is upgraded such that buyers draw their values from F', then the expected revenue in both the markets will be the same.

3. Consider a seller selling n ex-ante identical buyers with value distribution F, positive density f and support $[0, a]$. The seller is using a second-price auction with a reserve price.

(a) A regulator puts an upper bound $\bar{r} \in [0, a]$ on the reserve price that the seller can use. How will the optimal reserve price of the seller change with this upper bound? (3 MARKS)

Answer. Let $r^\star$ be the optimal reserve price without any constraint. If $\bar{r} \geq r^\star$, then $r^\star$ continues to be the optimal reserve price. However, if $r^\star > \bar{r}$, then the answer depends. To recall, the $r^\star$ solves

$$\max_r \int_r^a \psi(x)G(x)f(x)dx$$

If ψ satisfies single crossing, then $r^\star$ is the point where ψ crosses x-axis. If $\bar{r} < r^\star$, then clearly, it is optimal to choose the reserve price as $\bar{r}$. Hence, under ψ satisfying single-crossing, the optimal reserve price is $\min(\bar{r}, r^\star)$.

(b) A regulator puts an upper bound $\bar{r}$ and a lower bound $\underline{r}$ with $\underline{r} < \bar{r}$ on the reserve price that the seller can use. How will the optimal reserve price of the seller change with these bounds? (3 MARKS)

Answer. The answer is similar. If $r^\star \in [\underline{r}, \bar{r}]$, then $r^\star$ continues to be the optimal reserve price. Under ψ satisfying single crossing, if $r^\star < \underline{r}$, we should have $\underline{r}$ as the optimal reserve price. If $r^\star > \bar{r}$, we should have $\bar{r}$ as the optimal reserve price. Hence, the optimal reserve price is $\max(\underline{r}, \min(\bar{r}, r^\star))$.

(c) Suppose the seller releases more information about the product and this results in the value distribution "becoming better" in a stochastic dominance sense (with same support). What happens to the optimal reserve price and optimal (expected) revenue? Feel free to use any appropriate notion of stochastic dominance to derive your result. (4 MARKS)

How will your answers change to the above questions if the seller was using a first-price auction with a reserve price? ((2 MARKS)

Answer. Suppose the distribution moves from F to F'. Then the virtual value moves from $\psi(x) = x - \frac{1-F(x)}{f(x)}$ to $\psi'(x) = x - \frac{1-F'(x)}{f'(x)}$. So, if $\frac{1-F'(x)}{f'(x)} \geq \frac{1-F(x)}{f(x)}$ or F' hazard rate dominates F, then $\psi'(x) \leq \psi(x)$. Since the reserve price in both the auctions (under single crossing of ψ and ψ') is the point where the virtual value crosses zero, the reserve price under F' is higher than under F.[1]

Further, in any standard auction the expected payment of a bidder is increasing in value. Since expected revenue is $n \int P(x)f(x)dx$, if the distribution improves in the first-order stochastic dominance sense the expected revenue improves: here, we use

[1] We do not really need single crossing of ψ'. It is enough to note that ψ' will cross zero later than ψ.

the characterization of first-order stochastic dominance that F' first-order stochastic dominates F if and only if for all increasing functions P, we have $\int P(x)f'(x)dx \geq \int P(x)f(x)dx$.