Mid-term exam; Sept. 2017 solutions

Q1

a

Consider 2 values $x > y$.
If $y < r$, then the bid amount is 0.
So, the increasingness is obvious.

So, assume $x > y \geq r$.

Let the eqm. bid function be denoted as $b : [0, \beta] \to \mathbb{R}_+$.

$$b(x) - b(y) = (x-y) - \frac{1}{G(x)} \int_r^x G(z)\,dz + \frac{1}{G(y)} \int_r^y G(z)\,dz$$

$$\geq (x-y) - \frac{1}{G(x)} \int_r^x G(z)\,dz + \frac{1}{G(x)} \int_r^y G(z)\,dt$$

$(\because G(x) > G(y))$

$$= (x-y) - \frac{1}{G(x)} \int_y^x G(z)\,dz$$

$$\geq (x-y) - \frac{1}{G(x)} G(x)\cdot(x-y)$$

$(\because G \text{ is increasing})$

$$= 0$$

This proves b is increasing.

b// Direct revelation mechanism

- Every agent reports his value.

- Agent with the highest value wins if his value is $\geq r$.
- If the value of winner is x, he pays

$$b(x) = x - \frac{1}{G(x)} \int_r^x G(z)\,dz.$$

This is the direct revelation mechanism since (both points are important)

1) Increasing bid function in the auction implies whenever the object is allocated it is given to ~~the~~ buyer with highest value.

2) If any buyer has a value $\geq r$, the object is allocated. This follows since

$$x - \frac{1}{G(x)} \int_r^x G(z)\,dz \geq x - \frac{1}{G(x)} G(x)(x-r)$$

$$= r.$$

This means if highest value bidder has value $\geq r$, he wins. Payment is $b(x)$.

c. Consider a Vickrey auction with reserve price r.

Note that the direct revelation mechanism of first-price auction showed that first-price auction with reserve price r allocates the object to highest valued buyer whenever his value $\geq r$. Else it does not allocate the object.

This allocation rule is same as Vickrey with reserve price r.

The expected payment of a buyer with value 0 is zero in both the auctions.

By revenue equivalence, both the auctions generate the same revenue.

d. The optimal mechanism with symmetric bidders is Vickrey with an optimally chosen reserve price.

By (c), the result follows.

2. a) f is non-decreasing. Hence, it is implementable

b) Using revenue equivalence formula,

for every i, v_i, $v_j \geq v_k$

$$P_i(v_i, v_j, v_k) = -\frac{1}{3}v_k + v_i \cdot f_i(v_i, v_j, v_k) - \int_0^{v_i} f_i(x, v_j, v_k)\,dx$$

For generic $v_1 > v_2 > v_3$

$$f_1(v_1, v_2, v_3) = \frac{2}{3} + \frac{1}{3}\frac{v_3}{v_2}$$

So, $$P_1(v_1, v_2, v_3) = -\frac{1}{3}v_3 + v_1\left(\frac{2}{3} + \frac{1}{3}\frac{v_3}{v_2}\right) - (v_1 - v_2)\left(\frac{2}{3} + \frac{1}{3}\frac{v_3}{v_2}\right)$$

$$= \cancel{-\frac{1}{3}v_3} + \frac{2}{3}v_2 + \cancel{\frac{1}{3}v_3}$$

$$= \frac{2}{3}v_2$$

$$f_2(v_1, v_2, v_3) = f_3(v_1, v_2, v_3) = 0$$

$\therefore$ $P_2(v_1, v_2, v_3) = P_2(v_1, 0, v_3) = -\frac{1}{3}v_3$

$P_3(v_1, v_2, v_3) = P_3(v_1, v_2, 0) = -\frac{1}{3}v_2$

c// NO. Since generically $(v_1 > v_2 > v_3)$

$$P_1(v_1, v_2, v_3) + P_2(v_1, v_2, v_3) + P_3(v_1, v_2, v_3)$$

$$= \frac{1}{3}(v_2 - v_3) > 0$$

d// Welfare in this mechanism:

$$v_1\left(\frac{2}{3} + \frac{1}{3}\frac{v_3}{v_2}\right) - \frac{2}{3}v_2 + \frac{1}{3}v_3 + \frac{1}{3}v_2$$

$$= \frac{2}{3}(v_1 - v_2) + \frac{1}{3}\left[\frac{v_1 v_3}{v_2} + v_2 + v_3\right]$$

Welfare in GL (it is BB)

$$\frac{2}{3}v_1 + \frac{1}{3}v_2$$

Welfare in Vickrey

$$v_1 - v_2$$

Welfare in GL and Vickrey <u>not</u> comparable.
Welfare in this mecha and GL is also <u>not</u> comparable.
Diff in Welfare in this mech and Vickrey:

$\frac{1}{3}(v_2 - v_1) + \frac{1}{3}\left[\frac{v_1 v_3}{v_2} + v_2 + v_3\right]$, which is again not comparable, check $v_3 = 0$, $v_1 = 2v_2$

e// Ex-ante comparasion possible.

d-AGV is eff + BB + BIC
$\Rightarrow$ Ex-ante welfare is the highest of all BB + BIC mechanisms (including the one suggested)

Vickrey auction ex-ante welfare and this mechanism ex-ante welfare comparision will <u>depend</u> on the distn of <u>values</u>.

3//

a//
$$R = \max_{f} \int_0^1 \underbrace{\left[x - \frac{1 - G(x)}{g(x)}\right]}_{\text{Virtual value}} g(x)\, f(x)\, dx$$

b//
$$R = \max_{f} \int_0^1 \left[x\, g(x) + G(x) - 1\right] f(x)\, dx$$

Since $x\,G(x)$ is strictly convex, $x\,g(x) + G(x)$ is strictly increasing. Hence, R is max. if $f(x) = 1$ when $x\,g(x) + G(x) > 1$

Else, set $f(x) = 0$.

This is the standard optimal sol^n.

Since $x g(x) + G(x) - 1$ is strictly increasing from -1 to $g(1) > 0$, there is a unique point x^* where

$$x^* g(x^*) + G(x^*) - 1 = 0$$

Or $$x^* - \frac{1 - G(x^*)}{g(x^*)} = 0$$