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EQUILIBRIUM BID FUNCTIONS FOR AUCTIONS WITH AN UNCERTAIN NUMBER OF BIDDERS *

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The number of rivals may be unknown when a bidding strategy is formulated in an auction. In a symmetric model with risk-neutral bidders holding independent information, we obtain explicit equilibrium bidding functions for first-price and second-price auctions with uncertainty about the number of rivals. Five auctions are revenue-equivalent: first-price and second-price auctions, each with the number of bidders known or uncertain, and English auctions.

1. Introduction

Most auctions models assume the number of bidders is fixed and common knowledge. Often, though, bidders do not know when formulating strategies how many rivals will compete, particularly in markets organized with sealed bidding. Recent papers by Matthews (1987) and McAfee and McMillan (1987) have made the number of bidders random, with a known distribution. Both papers adopt the independent-private-values model [Vickrey (1961)], and analyze effects of bidder risk aversion upon seller's and bidders' preferences across auction institutions. Both implicitly assume that the seller knows the number of bidders (in advance), and the key institutional choice is whether the seller reveals this information. All their analysis is indirect; equilibrium bid functions when the number of rivals is uncertain are not presented.

Clearly, an equilibrium bid facing this 'numbers uncertainty' will be weighted average of the bids that would have been chosen for each number of rivals. What are the weights? We obtain explicit symmetric equilibrium bid functions for risk-neutral bidders in a more general independent model which allows for asset value uncertainty. Weights differ for first-price and second-price auctions. Five auctions are revenue-equivalent: first-price and second-price auctions, each with the number of bidders known or uncertain, and English auctions.

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2. The model

An indivisible asset is for sale at auction. All $n < \infty$ potential bidders are risk neutral. The number who actually bid is a, which may be unknown when formulating strategies. A known symmetric distribution governs a, established as follows. Each subset A of $N = \{1, \ldots, n\}$ is the set of bidders with probability π_A; sets with the same cardinality have the same chances. So the seller's ex ante probability of a bidders is

$$s_a := \sum_{\substack{|A|=a \\ A \subset N}} \pi_A, \quad a = 1, \ldots, n. \tag{1}$$

Let $\underline{a} := \min\{a \in N \mid s_a > 0\}$. The corresponding probability for a bidder is influenced by his becoming an actual bidder. Using symmetry, it is

$$p_a := \sum_{\substack{|A|=a \\ 1 \in A \subset N}} \frac{\pi_A}{\sum_{1 \in A \subset N} \pi_A} = \frac{a s_a}{\sum_{i=1}^{n} i s_i}. \tag{2}$$

Thus, relative to the seller, bidders rationally place lower probability on events with fewer bidders.

Each actual bidder i observes a real-valued signal X_i, an independent draw from distribution F with support $[\underline{x}, \bar{x}]$ and density $f(x) > 0$ on $(\underline{x}, \bar{x})$. The asset value to actual bidder i depends, in general, on his signal and those of actual rivals, and is represented by a symmetric function

$$V_{ia} = V_{ia}(X_1, \ldots, X_a) = V_a\left(X_i, \{X_j\}_{i \neq j \leq a}\right), \tag{3}$$

which is assumed non-negative, non-decreasing, continuous, and bounded in expectation for all $a \in N$. Let $(Y_1^a, \ldots, Y_{a-1}^a)$ be $(X_2, \ldots, X_a)$ arrayed descendingly. Define

$$v_a(x) := \mathrm{E}\left[V_{1a} \mid X_1 = x = Y_1^a\right], \quad a \in N, \tag{4}$$

which is the expected asset value conditional on the two highest of a signals being x.[1] We assume zero reserve prices and entry fees throughout.

3. First-price auctions

The symmetric equilibrium bid function when bidders are informed that the actual number of rivals is a ('contingent bidding'; see section 6) is defined by the differential equation

$$b_a'(x) = \left[v_a(x) - b_a(x)\right](a-1)(f(x)/F(x)), \tag{5}$$

with initial condition $b_a(\underline{x}) = v_a(\underline{x})$ [from Milgrom and Weber (1982)].

[1] The independent-private-values model is a special case. Common-value auctions [Wilson (1977)] are inconsistent with independence of signals, but the characteristic that the event of winning is informative about asset value can be incorporated, e.g. $V_{1a}(X_1, \ldots, X_a) = \frac{1}{3}[X_1 + \max\{X_2, \ldots, X_a\} + \min\{X_2, \ldots, X_a\}]$.

Suppose $b(x)$ with $b'(x) > 0$ is a symmetric equilibrium of a first-price auction with numbers uncertainty. [2] It must solve:

$$\max_b \sum_a p_a \mathrm{E}\big[(V_{1a} - b) 1_{(b(Y_1^a) < b)} \mid X_1 = x,\ a\big], \tag{6}$$

where $1_{(z)}$ takes value 1 in event (z), 0 otherwise. Equation (6) yields:

$$b'(x) = \Big(\sum_a [v_a(x) - b(x)](a-1) w_a(x)\Big) \frac{f(x)}{F(x)}, \tag{7}$$

with initial condition $b(\underline{x}) = v_{\underline{a}}(\underline{x})$, where the weights

$$w_a(x) = \frac{F^{a-1}(x) p_a}{\sum_i F^{i-1}(x) p_i} \geq 0 \tag{8}$$

are the probabilities of a bidders conditional on x being the highest signal observed [derivation follows Milgrom and Weber (1982, pp. 1106–1107)].

Theorem 1. In first-price auctions, the unique symmetric equilibrium bid function facing numbers uncertainty is precisely the expected payment conditional upon winning under contingent bidding:

$$b(x) = \sum_a w_a(x) b_a(x). \tag{9}$$

Proof. Applying L'Hôpital's rule to (8) at $\underline{x}$, $w_a(\underline{x}) = 0$ for $a > \underline{a}$, to satisfy the initial condition. Non-differentiable symmetric equilibria can be ruled out as in Maskin and Riley (1984, Theorem 2). Differentiating (9):

$$b'(x) = \sum_a w_a(x) b_a'(x) + (f(x)/F(x)) \Big(\sum_a w_a(x) b_a(x)(a-1) - \sum_a \Big[w_a(x) b_a(x) \sum_i (i-1) w_i(x) \Big] \Big).$$

Removing $b_a'(x)$ via (5) and inserting $b(x)$ via (9) yields

$$\begin{aligned} b'(x) &= (f(x)/F(x)) \Big\{ \sum_a w_a(x)([v_a(x) - b_a(x)](a-1) + b_a(x)(a-1)) - b(x) \sum_i (i-1) w_i(x) \Big\} \\ &= (f(x)/F(x)) \sum_a [v_a(x) - b(x)](a-1) w_a(x), \end{aligned}$$

the last by switching the index i to a. □

[2] The assumption that $b'(x)$ is positive is seldom justified [cf. McAfee and McMillan (1987)], but can be shown when $v_a(x)$ is non-increasing with a (see footnote 3).

Thus, in first-price auctions, the equilibrium weights bidders attach to differing numbers of rivals are the probabilities that will be correct in the event of winning.

4. Second-price auctions

The unique symmetric equilibrium contingent bidding function for second-price auctions is $v_a(x)$ in (4) [cf. Matthews (1977), Levin and Harstad (1986)]. We state without proof:

Proposition 1. Under contingent bidding, first-price and second-price auctions have the same expected payment upon winning, and the same expected revenue.

Let the uncertain-number analogue to (4) be $B(x)$, expected asset value given that the two highest of an uncertain number of signals are both x. This is

$$B(x) := \sum_a W_a(x) v_a(x), \tag{10}$$

where

$$W_a(x) := \left((a-1)F^{a-2}(x)p_a\right) \Big/ \left(\sum_i (i-1)F^{i-2}(x)p_i\right). \tag{11}$$

Theorem 2. In second-price auctions with numbers uncertainty,
(A) If $B(x)$ in (10) is non-decreasing, then $(B,\ldots,B)$ is a symmetric equilibrium.
(B) If a symmetric equilibrium exists, then $B(x)$ is non-decreasing, and $(B,\ldots,B)$ is the unique symmetric equilibrium. [3]
(C) Expected payment conditional on winning and expected revenue are the same as under contingent bidding.

Proof (A). Focus on bidder 1. Let a random variable M denote the highest rival bid when all actual rivals use $B(\cdot)$. When $X_1 = x$, 1's choice between $B(x)$ and any $\bar{b} > B(x)$ is solely decided by the positive probability event (ignoring ties for brevity):

$$\theta := \{B(x) < M < \bar{b}\}.$$

[3] Suppose, as is intuitive, that (4) falls as a rises. Then (10) can be used to show that (9) is increasing, as asserted. Let $d(x)$ be the denominator of (8) divided by the denominator of (11). Substituting into (7):

$$b'(x) = \sum_a [v_a(x) - b_a(x)] W_a(x) d(x) f(x) = d(x) f(x) \left[B(x) - b(x) \sum_a W_a(x) \right]$$

$$= [B(x) - b(x)] d(x) f(x) > 0, \quad \text{since}$$

$$B(x) \geq \sum_a \mathrm{E}[V_{1a} \mid X_1 = x \geq Y_{1a}] W_a(x) \geq \sum_a \mathrm{E}[V_{1a} \mid X_1 = x \geq Y_{1a}] w_a(x) > b(x),$$

where the first inequality follows from (4), (10) and (3) being non-decreasing, the second from (8) weighting cases of fewer bidders proportionately higher than (11), the last from expected profit being positive.

(Thus, Σ_z will refer to $\Sigma_{z=2}^{n}$.) Define $\underline{X}$ by $M = B(\underline{X})$. In event θ: $\underline{X} > x$; bidding $B(x)$ earns zero profit; bidding $\underline{b}$ obtains an asset of expected value

$$\sum_a \mathrm{E}[V_{1a} \mid X_1 = x, \theta] \Pr[a \mid X_1 = x, \theta]$$

$$\leq \sum_a \mathrm{E}[V_{1a} \mid X_1 = x, Y_{a1} = \underline{X}] \Pr[a \mid X_1 = x, \theta]$$

$$= \sum_a \mathrm{E}[V_{1a} \mid X_1 = x, Y_{a1} = \underline{X}] \frac{(a-1) F^{a-2}(\underline{X}) f(x) f(\underline{X}) p_a}{\sum_i (i-1) F^{i-2}(\underline{X}) f(x) f(\underline{X}) p_i}$$

$$= \sum_a \mathrm{E}[V_{1a} \mid X_1 = x, Y_{a1} = \underline{X}] W_a(X)$$

$$\leq \sum_a \mathrm{E}[V_{1a} \mid X_1 = Y_{a1} = \underline{X}] W_a(\underline{X}) = B(\underline{X}),$$

which is the price. The first inequality results from substituting a more optimistic event (a rival signal above $\underline{X}$ is inconsistent with θ); the second is strict except when V_{1a} is degenerate. The argument for any $\underline{b} < B(x)$ is parallel.

(B). B modifies A in ways parallel to Levin and Harstad (1986), details are omitted for brevity.

(C). Let $S(y \mid x)$ represent the density of the second-highest signal being y given that the highest signal is x. Let $\overline{W}_a(x, y)$ be the probabilities of a bidders, given that x is the highest and y the second-highest signal; these are the weights used to calculate expected payment upon winning with contingent bidding. These weights are

$$\overline{W}_a(x, y) = \frac{(a-1) f(x) f(y) F^{a-2}(y) p_a}{\sum_i (i-1) f(x) f(y) F^{i-2}(y) p_i} = W_a(y). \tag{12}$$

Multiplying both sides of (12) by $v_a(y) S(y \mid x)\, \mathrm{d}y$ and integrating both over $[\underline{x}, x]$ yields the conclusion. □

Thus, the weights in second-price auctions are the probabilities of a bidders conditional on the event that a bidder is indifferent over winning in equilibrium.

5. English auctions

Given independence of signals, information learned about rivals' signals from their drop-out prices in English auctions may affect behavior, but it does not affect expected payment.

Proposition 2. The second-highest bidder drops out of an English auction, on average, at his second-price (contingent) equilibrium bid.

Proof. Rewriting (4):

$$v_a(x) = \int_{\underline{x}}^{x}\int_{\underline{x}}^{x}\cdots\int_{\underline{x}}^{x} V_a(x, x, x_3, x_4,\dots, x_a)\prod_{i=3}^{a}[f(x_i)\,\mathrm{d}x_i]$$

$$= (a-2)!\int_{\underline{x}}^{x_{a-1}}\int_{\underline{x}}^{x_{a-2}}\int_{\underline{x}}^{x_3}\int_{\underline{x}}^{x} V_a(x, x, x_3, x_4,\dots, x_a) f(x_3)\,\mathrm{d}x_3\dots f(x_a)\,\mathrm{d}x_a. \tag{13}$$

The right-hand side of (13) is the expected drop-out point of the second-highest bidder in an English auction, rewriting eq. (6) in Milgrom and Weber (1982). □

This result combines with Proposition 1 and Theorems 1 and 2(C) to yield expected payment and expected revenue equivalence of all five auction forms discussed.

6. Contingent bidding

McAfee and McMillan (1987, p. 2) suggest that 'knowledge about the set of bidders... matters, because if different bidders have different (albeit Bayesian consistent) expectations over the set of bidders, then if the set of bidders is not known, the optimal auction cannot be implemented using a first-price sealed-bid auction...' This paper has shown that bidders unsure of the number of rivals are always led to select bids on the basis of different expectations about the number of bidders. However, the quoted concern that the seller may not be able to resolve this uncertainty (if, for example, congregating bidders is expensive) is unnecessary. The seller can introduce contingent bidding. That is, each potential bidder submits a list of bids, one for each number ('I bid $42 if there are 3 bidders, $47 if 4, $48 if...'). The seller is precommitted to using only the contingent bid corresponding to the number of submissions received. In essence, each bidder selects his bid effectively knowing the number of rivals he faces.

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