Notes on Auction Theory

August 22, 2022

1 Introduction

Auctions are widely used methods of allocating resources. A typical auction involves two types of agents: (a) *sellers*, those who own the resources and want to sell them; (b) *buyers*, those who do not own the resources but want to buy them. Depending on the number of buyers and the number of sellers, one can roughly classify the settings as follows.

- The most commonly studied auctions are those where there is a single seller, who is interested in selling a single object (or multiple objects) to a set of buyers. We will be mainly interested in analyzing such auctions. Examples of such auction setting include: Govt selling rights to mine to various companies; Google selling advertisement slots on search pages; Used cars sold on various websites (`cars24.com`) using auction. In these examples, there is a single seller (for instance, the owner of the used car) who is selling the object she owns to a set of buyers (those who logged in to the website to buy the car). In some of these cases, the seller need not be the auctioneer, but an intermediary agent conducts the auction.

- An analogous theory of auctions can also be developed for settings where there is a single buyer who is interested to buy an object and there are many sellers who can sell or supply the object. These are *procurement auctions*. Procurement auctions are mainly used by firms to procure raw materials for manufacturing. They are also used by Governments and other organizations to procure vaccines and other medical supplies. The analysis of auctions for single seller and multiple buyers can be straightforwardly adapted to the procurement settings. However, there are other concerns in a procurement auction, which separates it from normal auction setting. For instance, consider procurement of a raw material (say, spare parts of a car) by a firm (a car manufacturer). Several *suppliers (sellers)* can supply the raw material. The firm is interested in two dimensions of the raw material: (a) *price* and (b) *quality*. Each supplier can supply the raw material at different (price, quality) pairs. The buyer (firm) has to choose a supplier by considering *offers* of suppliers in both the dimensions. A standard method to aggregate these offers is through *scoring* rule, where a weight is given to each dimension and the aggregated score of each supplier is used to select the final supplier.

- There are settings where multiple sellers simultaneously sell their objects to a set of buyers. These are *double* auctions, where sellers post *ask* prices and buyers put *offers/bids*, and a market-clearing mechanism matches buyers and sellers. While used in many settings, we will not cover such auctions. Double auctions usually have the additional requirement that trade has to be *budget-balanced*: payments received by the sellers must equal the buyers' payments.

Unless stated explicitly, we will only be discussing settings with a single seller and multiple buyers, and that too in a single object model.

Why auction? The most prominent procedure for selling products is the posted-price mechanism. The posted-price mechanism is an excellent procedure when (a) the seller has a good idea about the willingness to pay of buyers; (b) the buyers cannot come together to an auction. If the seller does not have a good idea of the willingness to pay of buyers, then the seller can potentially get low revenues from posted-price mechanism: too low a posted price generates low revenue and too high a posted price reduces the probability of winning. On the other hand, auction allows us a *discovery* of willingness to pay.

Tools for analysis. The analysis of auctions is based on game theory. The willingness to pay information is private to individual buyers. Hence, an auction setting induces a Bayesian game of incomplete information.

2 Standard auction formats

We see various auction formats in practice (for selling a single object). Broadly, these auctions can be classified into two categories:

(a) *sealed-bid* auctions; These are auctions where bidders submit a one-time bid and winner and payments are decided based on these bids.

(b) *open-cry* auctions; These are auctions where prices are announced iteratively and demands of bidders at these prices are elicited. The auction ends when demands of bidders equal supply.

Under sealed-bid auctions, there are many variants. The two most common variants are (a) first-price auction and (b) second-price auction. In both the auctions, bidders (buyers) place bids and the bidder with the highest bid wins the auction. In both the auctions, a bidder pays only if she wins the object. The auctions differ in their payment rule: (a) in the first-price auction, the winner pays her own bid; (b) in the second-price auction, the winner pays the second-highest bid. While these are two popular sealed-bid auctions, there are other sealed-bid auctions which are studied in the auction theory literature. One such auction format is called the *all-pay auction*. As the name suggests, in an all-pay auction, the highest bidder wins the object but every bidder (including losers) pay their bid. Such auction are used to model contests, where the effort level works as a proxy for bid, which is paid by every bidder.

In open-cry auctions, there are two popular auction formats: (a) ascending price auction (English auction) and (b) descending price auction (Dutch auction). While various implementations of these auctions are present, it is convenient to think of the *continuous clock* implementation. In this implementation, the seller keeps a continuous price clock. In the ascending price auction, this price clock starts at a low (zero) price and the price keeps increasing continuously. Bidders can decide to exit the auction at any time during the auction. Once a bidder exits the auction, she may not come back. The price clock stops as soon as there is exactly one bidder remaining in the auction.[1] At that point, the only other bidder remaining wins the auction and pays the price in the auction clock.

There are practical benefits of each auction. For instance, price-based auctions, like the English and the Dutch auctions are transparent procedures with a lot of privacy preserving features. On the other hand, they require presence of bidders when auction takes place and can become complex in terms of communication. The sealed-bid auctions can allow bidders to send bids by communicating them beforehand. A sealed-bid auction is a centralized algorithm where inputs are processed centrally by the seller. On the other hand, ascending and descending price auctions are decentralized iterative communication procedures. We will see that there are differences in theoretical properties of these auction formats.

[1] There is an implicit tie-breaking used here. If two bidders exit the auction at the same time, the auction may order the bidders and allow them to exit one after the other. In particular, if the last exit results in no bidder in the auction, the auction picks one of the bidders at random and allows everyone else to exit.

3 Modeling auctions

The willingness to pay for the object of a bidder determines her strategy in any auction. The willingness to pay of a bidder is the *maximum amount* a bidder is willing to pay such that she is indifferent between buying the object and not buying. This is referred to as the **valuation** of the bidder. Models of auctions differ in the way they model valuations of the bidders.

3.1 An example: auction in 19th century Gujarat guilds

To understand models of auctions, let us consider an example of auction conducted in the guilds of Gujarat in the 19th century. These auctions are studied in Sen and Swamy (2004). The guilds of Gujarat were trading associations involving traders doing similar trades. Like any professional association, such guilds needed money to do various community activities and provide public goods. They had a unique procedure to raise funds for the guild. Sen and Swamy (2004) quote the following from the Gazetter of the city of Surat:

> *A favorite device for raising money is for men of the craft or trade to agree, on a certain day, to shut all their shops but one. The right to keep open this one shop is then put up to auction, and the amount bid is credited to the guild fund.*

While it is not easy to analyze such auctions because the winning bid in this auction is used by guild (bidders themselves), let us make the simplifying assumption that winning bid is used to provide a public good, which does not change the payoff of the bidders. For instance, the public good is provided irrespective of which bidder wins, but the winning bidder gets the additional benefit of keeping its shop open. So, the valuation of a bidder is its valuation for keeping the shop open.

What is the valuation of keeping a shop open? This valuation will depend on the demand on the shop on the day. We consider three models with n bidders.

(i) Suppose all shops trade the same good (medicine). Then, by keeping its shop open, a bidder captures the aggregate demand of all the shops in the guild. If we write d_i as the demand (no of customers) to shop i, then the valuation of a shop is a function of $\sum_i d_i$. If the prices are the same across all the shops, then it is reasonable to assume that the valuation is some function $v : \mathbb{R}_+ \to \mathbb{R}_+$, where $v(\sum_i d_i)$ is the valuation of *any* bidder which keeps the shop open. We observe that the valuation is the same for all the bidders in this model. However, each bidder i only observes his own demand d_i. So, even though the bidders know that everyone has the same valuation, they do not this value ex-ante.

Such a model of valuation is called the *common values model*. In common values model, each bidder receives a signal (demand for the shop) which is her private information, and the signals of all bidders determine a common valuation for the object. Common value models were first analyzed theoretically in Wilson (1967, 1969). Common value models are used to analyze sale of oil tracts, sale of goods in the resale market (for instance, most car buyers in the used car market are dealers who resell the car).

(ii) Suppose all shops trade in different goods. Then, by keeping its shop open, a bidder captures the demand of her own shop. Since she know the demand of her shop, she knows the valuation. However, this demand (and hence, valuation) information is private to her – each shop only knows its own demand. Such a model of valuation is called the *the private values model*. Private values models are used to analyze sale of art, procurement auctions, sale of real estate by auction. The first study of auctions in private values model is Vickrey (1961).

(iii) In reality, most practical models of auction are somewhere between the private values and the common values. To understand this, suppose half the shops in the guild trade medicines and the other half trade books. Each shop only observes her own demand but cares about demand of shops which trade the same good as hers. So, the valuation of a medicine shop will depend on the aggregate demand of all medicine shops, but it will not depend on the the demand of book shops.

Such a model of valuation is called the *interdependent values model*. Interdependent values model is general enough to capture the common values and the private values as special cases. These models were first studied in Milgrom and Weber (1982).

4 Objectives of an auction

There are two reasons to analyze auctions. First, we would like to understand the behavior of bidders. For this, we will adopt an appropriate notion of equilibrium and analyze equilibrium behavior of bidders. Second, we would like to compare auction formats in terms of their equilibrium outcomes. We will carry out these exercises in all the models we will study: (a) private values model and (b) interdependent values model. The private values model is a special case of the interdependent values model, but it is analyzed separately because it is more tractable and simpler than the general interdependent values model.

When comparing auction formats, we usually use two parameters: (a) *expected revenue* to seller; (b) *efficiency*. Efficiency is the standard notion of Pareto efficiency here and boils down to the following simple notion: an auction is efficient if the bidder with highest valuation of the object wins the object. This is an ex-post notion of efficiency. Expected revenue reflects an ex-ante objective of the seller to maximize expected revenue across auction formats. Under reasonable conditions, we will be able to rank standard auctions in terms of expected revenue and efficiency.[2]

5 Private values model and prior-free auctions

We now formally define a private values model. There is a single object for sale by a seller. There are n bidders and the set of bidders is denoted by $N = \{1, \ldots, n\}$. The valuation of each bidder is a random variable denoted by V_i, and its realization is denoted by v_i. Each bidder privately observes the realization of her valuation before entering the auction: this is the private values model.

We assume that the support of the distribution of this random variable is a set T_i, which we refer to as the *type set* of bidder i. The utility of not winning the object is normalized to zero. If the probability of winning the object with a payment p_i is q_i, utility from this outcome is given by $q_i(v_i - p_i)$. This form of utility function is consistent with a *risk neutral* bidder, and we will study extensions to other forms of utility functions later.

[2]Most popular criticism of an auction is by looking at the revenue of *one* instance. But, revenue to a seller is a random variable, and observed revenue is just one realization of that random variable. Hence, the right criticism of an auction format should be based on *expected revenue* it can generate.

5.1 Second-price auction

In a second-price auction, the strategy of a bidder is a map $s_i : T_i \to \mathbb{R}_+$. If bidders bid $b \equiv (b_1, \ldots, b_n)$, let $q_i(b)$ denote the winning probability of bidder i and $p_i(b)$ denote the payment of bidder i. Note that

$$q_i(b) = \begin{cases} 1 & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

Further, $p_i(b) = 0$ if $q_i(b) = 0$ and $p_i(b) = \max_{j \neq i} b_j$ otherwise.

Definition 1 *Bidding strategy s_i of bidder i is weakly dominant if for every (v_i, v_{-i}) and for every s_{-i},*

$$q_i(s_i(v_i), s_{-i}(v_{-i}))\Big[v_i - p_i(s_i(v_i), s_{-i}(v_{-i}))\Big] \geq q_i(b_i, s_{-i}(v_{-i}))\Big[v_i - p_i(b_i, s_{-i}(v_{-i}))\Big] \qquad \forall\ b_i.$$

Strategy s_i is **truthful** for bidder i if $s_i(v_i) = v_i$ for all $v_i \in T_i$.

Theorem 1 (**Vickrey** (**1961**)) *In the Vickrey auction, truthful strategy is a weakly dominant strategy for every bidder.*

Proof: Fix a profile of valuations $v \equiv (v_i, v_{-i})$. Fix a buyer i and suppose each of the other bidder $j \neq i$ bids b_j – so, we have fixed an arbitrary profile of bids of other bidders $\{b_j\}_{j \neq i}$. This profile of bids is generated due to some arbitrary strategy profile of other bidders. We will argue whatever this bid profile may be, bidder i weakly prefers to bid v_i to every other bid.

Before proceeding with the proof, consider Figure 1. It plots the payoff of a buyer i along the Y-axis and bid of the buyer i along the X-axis. The payoff of the buyer i is zero if it bids below $\max\limits_{j \neq i} b_j$. Otherwise (if he bids above $\max\limits_{j \neq i} b_j$),

- if the value of the buyer i is above $\max\limits_{j \neq i} b_j$, then its payoff of the buyer is given by the blue line (line above Y-axis),
- if the value of the buyer i is below $\max\limits_{j \neq i} b_j$, then its payoff of the buyer is given by the red line (line below Y-axis).

Hence, each bidder i, independent of its value can partition its strategies into two sets: (i) below $\max_{j \neq i} b_j$ and (ii) above $\max_{j \neq i} b_j$. It gets the same payoff by bidding anything in each of these sets. A buyer whose value is above $\max_{j \neq i} b_j$ prefers the blue part to the orange part in Figure 1, but a buyer whose value is below $\max_{j \neq i} b_j$ prefers the orange part to the blue part in Figure 1.

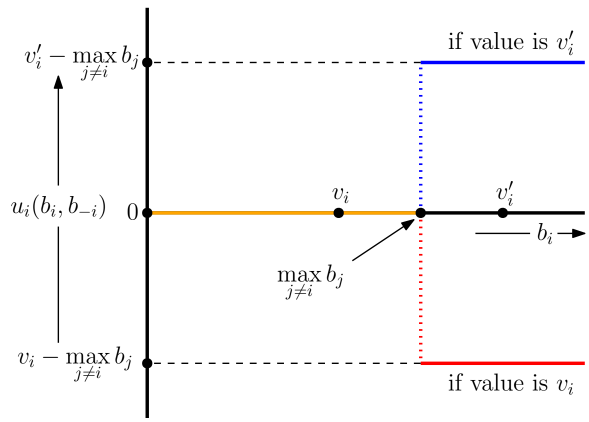

Figure 1: Weakly dominant strategy in Vickrey auction

Figure 1 gives an idea on why bidding value maximizes payoff of any buyer. Below, we formally show that it is indeed a weakly dominant strategy. Suppose buyer i has value v_i. We consider two cases.

Case 1. $v_i > \max_{j \neq i} b_j$. In this case, the payoff of buyer i from bidding v_i is $v_i - \max_{j \neq i} b_j > 0$. As long as he bids more than $\max_{j \neq i} b_j$, buyer i's payoff remains the same: she still wins the object and pays the same. By bidding strictly less than $\max_{j \neq i} b_j$ she does not win the object and gets a payoff of zero. By bidding equal to $\max_{j \neq i} b_j$, she gets the object but with some probability $q \leq 1$ and pays $\max_{j \neq i} b_j$. Hence, her payoff is $q(v_i - \max_{j \neq i} b_j)$, which is not more than what she was getting by bidding v_i.

Case 2. $v_i \leq \max_{j \neq i} b_j$. In this case, the payoff of buyer i from bidding v_i is zero. This is because either she is not getting the object (in which case his payoff is zero) or she is sharing the object in which case she is paying $\max_{j \neq i} b_j = v_i$. This is the case for all bids strictly less than $\max_{j \neq i} b_j$. If she bids greater than or equal to $\max_{j \neq i} b_j$, she wins (with some probability) but pays $\max_{j \neq i} b_j \geq v_i$. Hence, her payoff is non-positive. Hence, bidding

v_i is at least as good as bidding anything else.[3] ■

The weak dominance is a very strong strategic requirement. It states that the truthful strategy is better than every other strategy (a) in every state of the world and (b) for any strategy of other players. Thus, it is independent of the distributional assumptions. The English auction shares similar properties.

5.2 Ascending price auction

The ascending price auction induces an extensive form game. Strategy in an extensive form game is more complicated. Remember, we modelled the ascending price auction using a continuous price clock. At every price p, denote the history at p as h^p. This will include all the bidders who have dropped out and at what prices they have dropped out. Let $\mathcal{H}$ be the set of all possible histories. A strategy of bidder i is a map

$$s_i : T_i \times \mathbb{R}_+ \times \mathcal{H} \to [0, 1]$$

with the requirement that $s_i(\cdot, p, \cdot) = 0$ implies $s_i(\cdot, p', \cdot) = 0$ for all $p' > p$ (i.e., once you exit an auction, you cannot come back). So, $s_i(v_i, p, h^p) = 1$ denotes that bidder i with value v_i stays in auction at price p with history h^p. With this definition of strategy, Definition 1 also works for ascending price auction to define a weakly dominant strategy.

Strategy s_i is **truthful** for bidder i in ascending price auction if $s_i(v_i, p, h^p) = 1$ for all $p \leq v_i$ and for all h^p, and $s_i(v_i, p, h^p) = 0$ otherwise.

THEOREM **2** *In the ascending price auction, truthful strategy is a weakly dominant strategy for each bidder.*

Proof: The proof is quite simple and does not require any notation. Fix the strategies of other players, and consider any other strategy in which bidder i is not truthful. Consider an arbitrary valuation profile. Then, there are couple of cases to consider.

[3] To show that bidding v_i is weakly dominant, we must also show that v_i is strictly better than any other bid for *some* bid vector of other players. For this, fix v_i and some strategy $b_i \neq v_i$. As we saw from the two cases, if $b_i > v_i$, then when $v_i < \max_{j \neq i} b_i < b_i$, it is strictly better for buyer i to bid v_i. Similarly, if $b_i < v_i$, then when $b_i < \max_{j \neq i} b_i < v_i$, it is strictly better for buyer i to bid v_i.

Case 1. Suppose bidder i wins the auction at price p_i by being truthful. In that case, the only action performed by bidder i is 1 at each price $p \leq p_i$. By following any other strategy if bidder i wins then also the only action performed by bidder i is 1 at each price in the auction. Since other bidders are following the same strategy, the history in the auction remains the same. As a result, the auction again ends at price p_i.

By not being truthful, if she does not win the auction, then she gets zero payoff, which is weakly worse than following the truthful strategy and winning.

Case 2. Suppose bidder i does not win the auction by being truthful. By not being truthful, if she still does not win the auction, then her payoff remains the same.

By not being truthful, if she wins the auction, then the actions taken by other bidders remain the same till price hits v_i. In that case, the auction must end at a price $\geq v_i$. So, she will win the auction at price above v_i, which gives lower payoff than zero. ■

Though the truthful strategy is weakly dominant in both second-price and ascending price auctions, the definition of truthful strategy is different in both the auction formats. Ascending price auction is a more complex extensive form game, and we established that truthful strategy is a weakly dominant strategy. A recent paper by Li (2017) studies a stronger equilibrium concept than weakly dominant strategies in extensive games called *obviously strategy-proof*, and shows that the ascending price auction is obviously strategy-proof but the second-price auction is not.

Since truthful strategy is weakly dominant, the payment of winning bidder is equal to the second highest value of all bidders. This is exactly the payment of the winning bidder in the second-price auction. Further, both the auctions are *efficient*, i.e., the bidder with the highest value wins the object. Thus, we have established the following corollary.

Corollary **1** *The outcome of the second-price auction and the ascending price auction are identical and efficient in weakly dominant strategies.*

6 Symmetric Bayesian equilibria in first-price auctions

Let $T_i = [0, a]$ for all $i \in N$. We will assume that the values of bidders are *independently and identically distributed*. The cummulative distribution function of values will be denoted by F. We will assume F is differentiable with a positive density f. Hence, bidders are symmetric ex-ante.

In any sealed-bid auction, a strategy of bidder i is a map: $s_i : [0, a] \to \mathbb{R}_+$. A strategy profile $\mathbf{s} \equiv (s_1, \ldots, s_n)$ is **symmetric** if $s_1 = \ldots, s_n$. In that case we will denote the strategy of each bidder as s. A strategy s is **monotone** if $s(x) > s(y)$ for each $x, y \in [0, a]$ with $x > y$.

Given a monotone strategy s, the value $s(x)$ denotes the bid amount of *any* bidder with valuation $x \in [0, a]$. In first-price auction, given a bid b of bidder i and given all the other bidders are following symmetric monotone strategy s, bidder i wins if $b > s(v_j)$ for all $j \neq i$. The probability of this event is $[F(s^{-1}(b))]^{n-1}$, where we use $s^{-1}(b) = a$ if $b > s(a)$ and $s^{-1}(b) = 0$ if $b < s(0)$. Denote this as

$$Q(b; s) := [F(s^{-1}(b))]^{n-1}$$

.

We formally define a Bayesian equilibrium using this notation. The definition accounts for the fact that if a bidder does not win the object she pays zero and her payoff is zero.

Definition **2** *A symmetric strategy profile* $\mathbf{s} \equiv (s, \ldots, s)$ *is a Bayesian equilibrium of first-price auction if for every bidder* i*, for every value* $v_i \in [0, a]$

$$Q(s(v_i); s)\big[v_i - s(v_i)\big] \geq Q(b; s)\big[v_i - b\big] \qquad \forall\ b \in \mathbb{R}_+ \tag{1}$$

The following lemma shows that only a particular kind of incentive constraints must hold for a symmetric strategy profile to be a Bayesian equilibrium.

Lemma **1 (Imitation lemma)** *A symmetric strategy profile* $\mathbf{s} \equiv (s, \ldots, s)$*, where* s *is monotone, is a Bayesian equilibrium of first-price auction if and only if for every bidder* *i, for every value* $v_i \in [0, a]$

$$Q(s(v_i); s)\big[v_i - s(v_i)\big] \geq Q(s(v_i'); s)\big[v_i - s(v_i')\big] \qquad \forall\ v_i' \in [0, a] \tag{2}$$

Proof: Constraints in (1) clearly imply (2). For the other direction, suppose for every bidder i, for every value $v_i \in [0, a]$, (2) holds. Note that $Q(s(0); s) = 0$. This is because if other bidders follow s, whenever one other bidder has value $x > 0$, she bids $s(x) > s(0)$ and the bidder bidding $s(0)$ does not win. As a result, the probability of winning by bidding $s(0)$ is the probability that all the other bidders have value zero, which is zero.

Now, for every $\epsilon > 0$, we have $Q(s(\epsilon); s) > 0$. This is because if other bidders have value less than ϵ, then they will bid less than $s(\epsilon)$, and the bidder bidding $s(\epsilon)$ wins. The probability that $(n-1)$ bidders have value less than ϵ is positive since density f is positive.

Using (2) with $v_i = \epsilon$ and $v_i' = 0$ implies that $Q(s(\epsilon); s)(\epsilon - s(\epsilon)) \geq 0$ or $\epsilon \geq s(\epsilon)$. Using monotonicity of s, we get $\epsilon > s(0)$. Hence, $s(0) < \epsilon$ for all ϵ, which means $s(0) = 0$.

Now, pick some $b \in \mathbb{R}_+$. Since other bidders follows s, they never bid more that $s(a)$. Hence, by bidding $b > s(a)$, bidder i always wins. Hence, if $b > s(a)$, then $Q(b; s) = 1$. But $Q(s(a); s) = 1$ too. This is because the only event when bidder i does not win with probability 1 is when one of the other bidders have value equal to a. This has zero probability.

Hence, we have for every $v_i \in [0, a]$

$$Q(s(v_i); s)\big[v_i - s(v_i)\big] \geq Q(s(a); s)\big[v_i - s(a)\big] \geq Q(b; s)\big[v_i - b\big],$$

where the first inequality follows from (2). Hence, (1) holds for all $b > s(a)$.

Now, using $s(0) = 0$, we only need to show (1) holds for any $b \in [s(0), s(a)]$. Since s is strictly increasing, for every $b \in [s(0), s(a)]$, there exists a unique $v_i' \in [0, a]$ such that $s(v_i') = b$. Then, (2) implies (1). ■

THEOREM **3** *Suppose* $\mathbf{s} \equiv (s, \ldots, s)$ *is a symmetric strategy profile, where* s *is a monotone and differentiable strategy in the first-price auction. Then, the following are equivalent.*

1. $(s, \ldots, s)$ *is a Bayesian equilibrium.*

2. s satisfies

$$s(x) = x - \frac{1}{[F(x)]^{n-1}} \int_0^x [F(y)]^{n-1} dy \qquad \forall\, x \in [0, a] \tag{3}$$

Proof: For every $x \in [0, a]$, let $u(x) = Q(s(x); s)\big[x - s(x)\big]$. Since s is monotone and highest bidder wins, $Q(s(x); s) = [F(x)]^{n-1}$, and we write $G(x) \equiv Q(s(x); s)$. Notice that G is the cdf of highest $(n-1)$ draws using F. Let g denote the density of this random variable: $g(x) = (n-1)[F(x)]^{n-2} f(x)$ for each $x \in [0, a]$. Hence, $u(x) = G(x)(x - s(x))$. Note that if s is differentiable, u is differentiable. By Lemma 2, we know that s is a Bayesian equilibrium if and only if

$$u(x) \geq u(y) + G(y)(x - y) \qquad \forall\, x, y \in [0, a] \tag{4}$$

Necessity. Suppose s is a Bayesian equilibrium. Then, fix some $x, x + \delta \in [0, a]$, where $\delta > 0$. Using (4) we get

$$\begin{aligned} u(x+\delta) &\geq u(x) + \delta G(x) \\ u(x) &\geq u(x+\delta) - \delta G(x+\delta) \end{aligned}$$

Hence, we get

$$\delta G(x+\delta) \geq u(x+\delta) - u(x) \geq \delta G(x)$$

By continuity of G, we thus get that

$$\frac{d[u(x)]}{dx} = G(x) \qquad \forall\, x \in [0, a] \tag{5}$$

Since $u(0) = 0$, (11) and the fundamental theorem of calculus implies that

$$u(x) = \int_0^x G(y)dy$$

$$\Rightarrow G(x)(x - s(x)) = \int_0^x G(y)dy$$

$$\Rightarrow s(x) = x - \frac{1}{[F(x)]^{n-1}} \int_0^x [F(y)]^{n-1} dy$$

Sufficiency. Suppose s is as defined in (9). Then, for every $x \in [0, a]$, we have

$$u(x) = G(x)(x - s(x)) = \int_0^x G(y)dy$$

Hence, for any $x, y \in [0, a]$, we have

$$u(x) - u(y) = \int_y^x G(z)dz,$$

If $x > y$, then since G is increasing, $G(z) > G(y)$ for all $z > y$. Hence, $\int_y^x G(z)dz > (x-y)G(y)$. If $x < y$, then $G(z) < G(y)$ and this means $\int_y^x G(z)dz = -\int_x^y G(z) > (x-y)G(y)$. Thus, (4) holds, and we are done. ■

Remark. Theorem 3 shows that there is a unique symmetric equilibrium in monotone and differentiable strategies. Focusing on symmetric equilibrium is natural in an environment where bidders draw their value independently and identically. However, one may ask if there are *asymmetric* and *non-monotone* equilibria in this environment. Maskin and Riley (2003) show that the equilibrium identified in Theorem 3 is unique under reasonable conditions.

By Theorem 3, in a symmetric equilibrium (with monotone and differentiable) strategies, a bidder with value x bids according to (4). Since this is a symmetric strategy profile with monotone strategies, for any two bidders with values x, y we see that $s(x) > s(y)$ if and only if $x > y$. Hence, the symmetric equilibrium identified in Theorem 3 is efficient: the highest *valued* bidder makes the highest bid and wins. Hence, the probability of winning of a bidder with value x is $G(x) = \big[F(x)\big]^{n-1}$. Hence, using Theorem 3, the expected payment of a bidder with value x in this symmetric equilibrium is given by

$$G(x)s(x) = xG(x) - \int_0^x G(y)dy = \int_0^x yg(y)dy \tag{6}$$

The last expression $\int_0^x yg(y)dy$ is the expected value of the random variable highest of $(n-1)$ values given that it is less than x. Hence, the expected payment of a bidder with value x is the expected value of the second highest *valuation* in the region she wins.

In a second-price auction, bidders have a weakly dominant strategy to bid their value. In this equilibrium, a bidder with value x pays zero if she does not win but pays the highest of $(n-1)$ other bidders' values if she wins. Hence, her expected payment is the expected value of the random variable highest of $(n-1)$ values given that it is less than x, which is exactly (6).

Since all the bidders are symmetric, the expected revenue of a seller in the first-price and the second-price auction (in the equilibrium described) is given by

$$\begin{aligned} n\int_0^a G(x)s(x)f(x)dx &= n\int_0^a \Big(\int_0^x yg(y)dy\Big)f(x)dx \\ &= n\int_0^a \Big(\int_x^a f(y)dy\Big)xg(x)dx \\ &= n\int_0^a x(1-F(x))g(x)dx \\ &= n(n-1)\int_0^a x(1-F(x))\big[F(x)\big]^{n-2}f(x)dx \end{aligned}$$

Thus, we have come to an important result in auction theory.

Theorem 4 **(Revenue equivalence, Vickrey (1961))** *The expected payment of each bidder in the symmetric equilibrium of the first-price auction and the weakly dominant strategy of the second-price auction are the same. The expected revenue of the seller is identical across these two auctions:*

$$n(n-1)\int_{0}^{a} x(1-F(x))\big[F(x)\big]^{n-2} f(x)dx.$$

While this is a striking result, let us remember the assumptions we have made:

- Values are private.
- Bidders are ex-ante identical: values are *independently* and *identically* distributed.
- Bidders are risk neutral.

But this benchmark result will serve as a template for the rest of the course. We will relax various assumptions in this result and compare auction formats.

6.0.1 Descending price and first-price auctions

The equivalence between first-price and second-price auction formats automatically induce equivalence with the ascending price auction (1). We explore the equivalence with the descending price auction. A strategy in a descending price auction can be defined similar to an ascending price auction. The history in a descending auction does not change till the auction ends: the action to show interest in the object, ends the auction. Hence, strategy in a descending auction is just a function $s_i : [0,a] \times [0,\bar{P}] \to \{0,1\}$, where $\bar{P}$ is the highest possible price in the auction. Here, $s_i(v_i,p) = 0$ indicates that the bidder is *not* interested in the object and $s_i(v_i,p) = 1$ indicates the bidder is *interested* in the object. Hence, $s_i(v_i,p) = 1$ implies $s_i(v_i,p') = 1$ for all $p' < p$. Thus, there is a cutoff price p^* such that $s_i(v_i,p) = 0$ for all $p > p^*$ and $s_i(p) = 1$ for all $p \le p^*$. Thus, a strategy of a bidder is to figure out for each v_i, a cut-off price p^* such that the bidder is interested in the object below that price. Note that if the bidder wins the auction she pays p^* in this case.

The decision in a first-price auction is similar for a bidder. Given the value v_i of bidder i, she has to decide how much to bid. This bid is the amount she pays if she wins. This bid is exactly similar to p^*. Hence, the first-price auction and the descending price auction are equivalent strategically. Thus, the equivalence in Theorem 4 extends to all standard auctions. We formalize this below.

6.1 Symmetric equilibrium in standard auctions

There are many sealed-bid auction formats that one can think of: first-price, second-price, third-price etc. A sealed-bid auction is a *standard auction* if

(a) highest bidder wins;

(b) winner pays non-negative amount;

(c) losers payment is zero;

(d) payment is *non-decreasing*: higher bid does not lead to lower payment to a bidder.

Each of these assumptions are trivially satisfied by first-price, second-price, and third-price auctions. We will see the analysis of the first-price auction extends to any standard auction.

Let s be a strategy in a standard auction. If all the other bidders follow strategy s, then the probability of winning by bidding b is given similarly: bidder i with bid b wins if $b > \max_{j \neq i} s(v_j)$ and probability of this event is $[F(s^{-1}(b))]^{n-1}$. As before, we will denote this as $Q(b; s)$. Note that if bidder i also follows the strategy s, then probability of winning is $[F(s^{-1}(s(v_i)))]^{n-1} = [F(v_i)]^{n-1}$, which we will denote as $G(v_i)$.

If all bidders play s, then let $P(b; s)$ be the expected payment of a bidder by bidding b. In the first-price auction, this expected payment is $bQ(b; s)$. In the second-price auction, this expected payment is the expected value of $\mathbf{E}_{v_{-i}:\max(s(v_{-i}))<b} \max(s(v_{-i}))$.[4]

It is without loss of generality to denote the standard auction by (Q, P): these are the only things that will be required for analysis (given a strategy). Using this notation a symmetric strategy profile $\mathbf{s} \equiv (s, \ldots, s)$ is a Bayesian equilibrium in a standard auction if for every $i \in N$, for every $v_i \in [0, a]$

$$Q(s(v_i); s)v_i - P(s(v_i); s) \geq Q(b; s)v_i - P(b; s) \qquad \forall\, b \in \mathbb{R}_+ \tag{7}$$

[4] We know that in the second-price auction $s(v_i) = v_i$ is a weakly dominant strategy but we do not use this specific s here.

Lemma **2 (Imitation lemma)** *Let s be a monotone strategy in a standard auction (Q, P). Strategy s is a Bayesian equilibrium of a standard auction (Q, P) if and only if for every bidder i, for every value $v_i \in [0, 1]$*

$$Q(s(v_i); s)v_i - P(s(v_i); s) \geq Q(s(v_i'); s)v_i - P(s(v_i'); s) \qquad \forall\ v_i' \in [0, a] \tag{8}$$

Proof: Constraints in (7) clearly imply (8). For the other direction, note that by bidding less than $s(0)$, a bidder always loses and pays zero. Hence, for any $b < s(0)$, $Q(b; s) = 0$ and $P(b; s) = 0$. Similarly, by bidding $b = s(0)$, a bidder wins with positive probability only if all other bidders have value 0 (in which case they bid $s(0)$), and this event happens with zero probability. Hence, $Q(s(0); s) = P(s(0); s) = 0$. Using this for all $b < s(0)$, we see that for every $v_i \in [0, a]$, (8) implies

$$Q(s(v_i); s)v_i - P(s(v_i); s) \geq Q(s(0); s)v_i - P(s(0); s) = 0 = Q(b; s)v_i - P(b; s)$$

Similarly, for all $b > s(a)$, $Q(b; s) = Q(s(a); s) = 1$ and $P(b; s) \geq P(s(a); s)$ by non-decreasing payment. Hence, (8) implies that for every $b > s(a)$ and for every $v_i \in [0, a]$,

$$Q(s(v_i); s)v_i - P(s(v_i); s) \geq Q(s(a); s)v_i - P(s(a); s) \geq Q(b; s)v_i - P(b; s)$$

Now, consider $b \in [s(0), s(a)]$. Since s is monotone, there exists v_i' such that $s(v_i') = b$. Hence, (8) implies (7) holds.

This exhausts all cases, and hence, (8) implies that $\mathbf{s} \equiv (s, \ldots, s)$ is a Bayesian equilibrium. ■

Once the imitation lemma is done, the equilibrium characterization proof is similar. Note that we do not need differentiable strategy now.

Theorem **5** *Suppose $\mathbf{s} \equiv (s, \ldots, s)$ is a symmetric strategy profile, where s is a monotone strategy in a standard auction (Q, P). Then, the following are equivalent.*

1. *$(s, \ldots, s)$ is a Bayesian equilibrium.*

2. s satisfies

$$P(s(x);s) = x[F(x)]^{n-1} - \int_0^x [F(y)]^{n-1}dy \qquad \forall\, x \in [0,a] \tag{9}$$

Proof: For every $x \in [0,a]$, let $u(x) = Q(s(x);s)x - P(s(x);s)$. Since the highest bidder wins and s is monotone, $Q(s(x);s) = [F(x)]^{n-1}$, and we write $G(x) \equiv Q(s(x);s)$. Hence, $u(x) = xG(x) - P(s(x);s)$. Unlike the proof of Theorem 3, we cannot assume u is differentiable without loss of generality. However, the rest of the proof can be modified slightly. By Lemma 2, we know that s is a Bayesian equilibrium if and only if

$$u(x) \geq xG(y) - P(s(y);s) = u(y) + G(y)(x-y) \qquad \forall\, x,y \in [0,a] \tag{10}$$

Necessity. Suppose s is a Bayesian equilibrium. Then, (10) holds. Pick $x,y \in [0,a]$ and $\lambda \in [0,1]$ with $z = \lambda x + (1-\lambda)y$. Then, $u(x) \geq u(z) + G(z)(x-z)$ and $u(y) \geq u(z) + G(z)(y-z)$. Multiplying the first inequality by λ and the second by $(1-\lambda)$ gives $\lambda u(x) + (1-\lambda)u(y) \geq u(z)$. Hence, u is convex. A convex function is differentiable almost everywhere in the interior of $[0,a]$.

Then, fix some $x, x+\delta \in [0,a]$, where $\delta > 0$ and u is differentiable at x. Using (10) we get

$$\begin{aligned} u(x+\delta) &\geq u(x) + \delta G(x) \\ u(x) &\geq u(x+\delta) - \delta G(x+\delta) \end{aligned}$$

Hence, we get

$$\delta G(x+\delta) \geq u(x+\delta) - u(x) \geq \delta G(x)$$

By continuity of G, we thus get that

$$\frac{d[u(x)]}{dx} = G(x) \qquad \forall\, x \in [0,a] \tag{11}$$

where u is differentiable. Since $u(0) = 0$ and using the fact that u is differentiable almost everywhere in $[0,a]$, (11) and the fundamental theorem of calculus imply that for all $x \in [0,a]$

$$u(x) = \int_0^x G(y)dy$$

$$\Rightarrow G(x)x - P(s(x);s) = \int_0^x G(y)dy$$

$$\Rightarrow P(s(x);s) = x[F(x)]^{n-1} - \int_0^x [F(y)]^{n-1}dy$$

SUFFICIENCY. Suppose s is as defined in (9). Then, for every $x \in [0,a]$, we have

$$u(x) = G(x)x - P(s(x);s) = \int_0^x G(y)dy$$

Note that $G(x) = [F(x)]^{n-1}$, and hence, G is increasing. Hence, for any $x, y \in [0,a]$, we have

$$u(x) - u(y) = \int_y^x G(z)dz \geq (x-y)G(y),$$

where the inequality follows since G is increasing. Thus, (10) holds, and we are done. ■

Remark. Theorem 5 is a characterization of symmetric equilibrium in monotone strategies in a standard auction. Not all the four conditions in the definition of standard auction are used in both the direction. A careful look at the proof of the theorem shows that for (1) implies (2), we only need that (a) winner is the highest bidder; and (b) losers payment is zero (or $u(0) = 0$). For the other direction ((2) implies (1)), we need all the four conditions of a standard auction, which is primarily used to reduce the set of incentive constraints in the imitation lemma. Usually, the direction (1) implies (2) is referred to as the revenue equivalence theorem in auction theory.

6.2 Ranking distributions of revenue

The revenue of any auction is a random variable for the seller. We denote the random variable corresponding to the revenue of the first-price auction as R_1 and that of the second-price auctions as R_2. Let H_i be the cdf and h_i be the density function of random variable R_i for each $i \in \{1, 2\}$. The revenue equivalence theorem showed that

$$\int_0^a r h_1(r) dr = \int_0^a r h_2(r) dr$$

This compares the two random variables R_1 and R_2 based on the mean. But there are other ways to compare two random variables.

Definition **3** *Random variable R_1 concave dominates random variable R_2 if for all concave functions $u : \mathbb{R}_+ \to \mathbb{R}$,*

$$\int_0^a u(r) h_1(r) dr \geq \int_0^a u(r) h_2(r) dr$$

Concave dominance is equivalent to the well known second order stochastic dominance.

Theorem **6** *The first price auction revenue concave dominates the second-price auction revenue. Hence, a risk averse seller prefers the first-price auction over the second-price auction.*

Proof: Let s be the unique symmetric equilibrium strategy of the first-price auction. The random variable R_2 is the second-highest of n draws using F. To be precise we denote the k-th highest of n draws using F as $X^n_{(k)}$. The conditional expectations are related in the following way:

$$E[X^n_{(2)} | X^n_{(1)} = x] = E[X^{n-1}_{(1)} | X^{n-1}_{(1)} < x]$$

where the inequality follows from the fact that taking expectation of $X^n_{(2)}$ when $X^n_{(1)} = x$ is the same as taking expectation of $X^{n-1}_{(1)}$ when its value is less than x. But the equilibrium bid in the first price auction is

$$s(x) = \frac{1}{G(x)} \int_0^x y g(y) dy = E[X_{(1)}^{n-1} | X_{(1)}^{n-1} < x]$$

Hence,

$$E[R_2 | R_1 = r] = E[X_{(1)}^{n-1} | X_{(1)}^{n-1} < s^{-1}(r)] = s(s^{-1}(r)) = r$$

■

7 Reserve Prices

Reserve price is commonly used in many auction formats. In a sealed-bid auction (first-price or second-price), with reserve price, a bid is *eligible* if it exceeds the reserve price. The payment of the winning bidder in the first-price auction is still her bid, but a bidder wins only if she bids the highest *and* the bid exceeds the reserve price. The payment of the winning bidder in the second-price auction is the maximum of the second highest bid and the reserve price.

A consequence of reserve price is that an object is not sold at some profiles of bids. This seems like a wasted opportunity to raise some revenue. So, why do sellers post reserve prices in sealed-bid auctions? The simple intuition for this is that even though the seller loses revenue by not selling some times, she raises more revenue when the object is sold. To see this, consider a second-price auction with two bidders whose values are uniformly distributed in $[0, 1]$. The expected revenue in a second-price auction without a reserve price is the expected value of the lowest of two values, which is $\frac{1}{3}$. Now, suppose we conduct a second-price auction with a reserve price of $\frac{1}{2}$. Then, the object is sold only when one of the bidder bids more than $\frac{1}{2}$. But note that even when the losing bidder bids less than $\frac{1}{2}$, the winning bidder pays $\frac{1}{2}$. Indeed, as we will show next, bidding your value is still a weakly dominant strategy in second-price auction with reserve price. Hence, the expected revenue in this auction can be calculated as follows: the object is sold if at least one bidder has value $\frac{1}{2}$ and this probability is $\frac{3}{4}$. When the object is sold, the price paid by the winning bidder is at least $\frac{1}{2}$. Hence, the second-price auction with a reserve price of $\frac{1}{2}$ collects at least $\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$ expected revenue. Since $\frac{3}{8} > \frac{1}{3}$, setting this particular reserve price improves revenue. Of course, setting too high a reserve price means the object is not sold often and the expected revenue will be low. Hence, there is some *optimal* reserve price which maximizes expected revenue.

7.1 Reserve price in second-price auction

The second-price auction with a reserve price r is defined as follows. At every profile of bids, if the highest bid is less than r, the object is not sold. Else, the highest bidder wins and pays an amount equal to the maximum of r and the second highest bid. If there are multiple highest bidders with bid more than r, then each of them becomes the winning bidder with equal probability (and pay the maximum of r and the second highest bid with equal probability).

A simple way to interpret this auction is as if the seller (a non-strategic bidder) places a bid of r. Clearly, the incentives in the standard second-price auction works for any r (refer to Theorem 1). As a result, we have the following.

THEOREM **7** *In the second-price auction with a reserve price, truthful strategy is a weakly dominant strategy.*

What is the expected payment of a bidder with value x in a second-price auction with reserve price r? If $x \leq r$, she does not win the auction and pays zero. If $x > r$, she pays r if the maximum of other bidders' values is less than r and pays the maximum of other bidders' values if maximum of other bidders' values is between r and x. Let G be the cummulative distribution function of maximum of $(n-1)$ draws of values using F and let g be the density function. Then, the probability that maximum of $(n-1)$ values is less than r is $G(r)$. Hence, the expected payment of bidder with value $x > r$ is:

$$rG(r) + \int_r^x yg(y)dy = rG(r) + [yG(y)]_r^x - \int_r^x G(y)dy = xG(x) - \int_r^x G(y)dy \qquad (12)$$

Hence, the expected payment from a bidder is (noting that bidder with value less than r

pays zero)

$$\int_r^a xG(x)f(x)dx - \int_r^a \Big(\int_r^x G(y)dy\Big)f(x)dx = \int_r^a xG(x)f(x)dx - \int_r^a \Big(\int_x^a f(y)dy\Big)G(x)dx$$
$$= \int_r^a xG(x)f(x)dx - \int_r^a (1-F(x))G(x)dx$$
$$= \int_r^a \Big[x - \frac{1-F(x)}{f(x)}\Big]G(x)f(x)dx$$

Hence, the expected revenue from a second-price auction with reserve price r is (using that all n bidders are ex-ante identical):

$$\text{Rev}^2(r) = n\int_r^a \Big[x - \frac{1-F(x)}{f(x)}\Big]G(x)f(x)dx \tag{13}$$

The term $x - \frac{1-F(x)}{f(x)}$ is called the **virtual value** of bidder with value x. We denote the virtual value function as ψ

$$\psi(x) = x - \frac{1-F(x)}{f(x)} \qquad \forall\ x \in [0,a]$$

Note that the virtual value function depends on the distribution. In order to find the optimal reserve price, we use the following assumption on distributions.

Definition 4 *The virtual value function satisfies* **single crossing** *if there exists $v^* \in [0,a]$ such that $\psi(x) < 0$ for all $x < v^*$ and $\psi(x) > 0$ for all $x \geq v^*$.*

Since ψ is continuous, single crossing also implies that $\psi(v^*) = 0$. Since

$$\text{Rev}^2(r) = n\int_r^a \psi(x)G(x)f(x)dx \tag{14}$$

it is clear that $r = v^*$ is an optimal reserve price if virtual value satisfies single crossing.

A distribution F satisfies **monotone hazard rate (MHR)** if $\frac{f(x)}{1-F(x)}$ is non-decreasing in x. MHR implies that $x - \frac{1-F(x)}{f(x)}$ is strictly increasing in x. Since $x - \frac{1-F(x)}{f(x)}$ is negative at

$x = 0$ and positive at $x = a$, MHR implies ψ is increasing and crosses zero at most once.

If $r < v^*$, where $\psi(v^*) = 0$, then the expected revenue (14) can be improved by raising r a little bit because that gets rid of negative terms in the expressions. If $r > v^*$, then the expected revenue (14) can be improved by lowering r a little bit because that adds some positive terms in the expression. Hence, the optimal reserver price is v^* under the single crossing condition. This leads to the main theorem of this section.

THEOREM **8** *Suppose the distribution of values of bidders is such that the virtual value function satisfies single crossing. Then, the optimal (expected revenue maximizing) reserve price in a second-price auction is the unique solution to the equation* $r - \frac{1-F(r)}{f(r)} = 0$.

Uniform distribution satisfies MHR (and hence, single crossing condition): $f(x) = \frac{1}{a}$ and $F(x) = \frac{x}{a}$. Then, $\frac{f(x)}{1-F(x)} = \frac{1}{a-x}$, which is increasing in x. Hence, the solution to $r - \frac{1-F(r)}{f(r)} = r - (a - r) = 2r - a = 0$ or $r^* = \frac{a}{2}$.

7.2 Reserve price in first-price auction

The first price auction with a reserve price r works as follows. Bidders submit bids and the highest bidder wins the object (with ties broken in some way) if her bid is more than r. Else, the object is not sold. The winning bidder pays her bid.

In the first-price auction placing any bid less than or equal to r has the same effect as placing a bid of r: in either case, the bidder does not win the object and pays zero. So, we will assume that bidders only use strategies where they bid at least r.

DEFINITION **5** *A strategy* $s : [0, a] \to [r, \infty)$ *is* r-**monotone** *if there exists a cutoff* v^* *such that* $s(x) = r$ *for all* $x \leq v^*$ *and* $s(x) > s(y)$ *for all* $x > y \geq v^*$.

THEOREM **9 (Riley and Samuelson (1981))** *Let* s *be a* r*-monotone strategy which is differentiable in* (r, a)*. Then, the following are equivalent.*

1. $(s, \ldots, s)$ *is a Bayesian equilibrium of the first-price auction with reserve price* r*.*

2. *For every* $x \in [0, a]$,

$$s(x) = \begin{cases} r & \text{if } x \leq r \\ x - \frac{1}{G(x)} \int_r^x G(y)dy & \text{if } x > r. \end{cases}$$

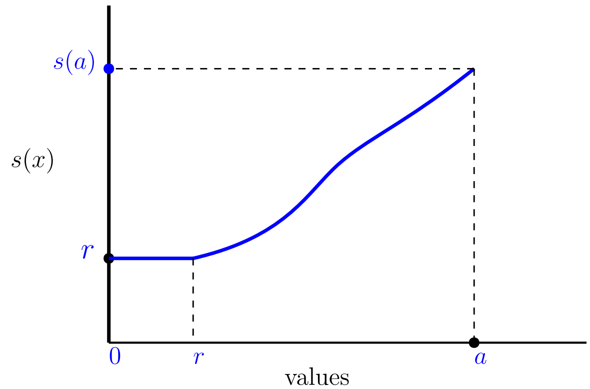

Figure 2: A r-monotone strategy

Proof: Let s be a r-monotone strategy and $(s, \ldots, s)$ is a Bayesian equilibrium. We first argue that $v^* = r$. If $v^* > r$, for sufficiently small $\epsilon > 0$, consider the value $x = r + \epsilon$. When other bidders have value less than x, they place a bid of r. In that the bidder can place a bid of $r + \frac{\epsilon}{2}$ and win the object to get a payoff $\frac{\epsilon}{2}$. This happens with positive probability (since the probability that others have value $r + \epsilon$ is positive). Hence, by bidding $r + \frac{\epsilon}{2}$, the bidder gets positive payoff. On the other hand, by following s, the bidder would have bid r and not won the object. This contradicts that $(s, \ldots, s)$ is a Bayesian equilibrium. Hence, $v^* \leq r$.

Next, suppose $v^* < r$. By continuity of s, there exists a value $x > v^*$ but arbitrarily close to v^* with $x < r$ such that $s(x) > r$. When other bidders have value less than x, which happens with positive probability, this bidder wins (as others bid less than $s(x)$ by r-monotonicity of s). By winning, the bidder pays $s(x) > r > x$, and hence, gets negative payoff. By bidding r, she gets zero payoff. This contradicts that $(s, \ldots, s)$ is a Bayesian equilibrium. Hence, we conclude that $v^* = r$. The strategy is shown in Figure 2.

Now, fix a bidder i and suppose other bidders follow s. Suppose bidder i has a value $x \geq r$. The payoff she gets by following s is $u(x) := G(x)(x - s(x))$, where $G(x)$ is the probability that other bidders have value less than x, which is also the probability with which bidder i wins. Suppose bidder i bids $b \in [r, s(a)]$. By r-monotonicity, there is a value $y \in [r, a]$ such that $s(y) = b$. Hence, Bayesian equilibrium implies that

$$u(x) \geq G(y)(x - s(y)) = u(y) + (x - y)G(y) \tag{15}$$

Inequality (15) holds for all $x, y \in [r, a]$. Using an argument analogous to the proof of Theorem 3, we conclude that $\frac{d[u(x)]}{dx} = G(x)$ for all $x \in [r, a]$. Hence, by fundamental theorem of calculus,

$$u(x) = u(r) + \int_r^x G(y)dy = \int_r^x G(y)dy \qquad \forall\, x \in [r, a],$$

where we use the fact that $u(r) = G(r)(r - s(r)) = 0$ since $s(r) = r$. But $u(x) = G(x)(x - s(x)$ implies that for all $x \in [r, a]$, we must have

$$s(x) = x - \frac{1}{G(x)} \int_r^x G(y)dy \tag{16}$$

For the converse, we want to show that $(s, \ldots, s)$, where s is defined as in (16), is a Bayesian equilibrium. For this, suppose all the bidders exccept i follow s. Then the maximum bid by others is $s(a)$. If i bids $b > s(a)$ she wins for sure with payoff equal to $x - b$, where x is her payoff. By bidding $s(a)$ also i wins with probability 1 (the probability that others bid less than $s(a)$ is 1) with payment $s(a) < b$. Hence, as long as we can show that i cannot manipulate to $s(a)$, we can also ensure that she cannot manipulate to $b > s(a)$. Similarly, bidding less than r gives a payoff of zero and following s ensures non-negative payoff. Finally, if value of i is less than or equal to r, she gets a payoff of zero by using s. Any bid $b > r$ implies i wins with non-zero probability and pays $b > r$. This means her expected payoff is negative. So, all types with value less than or equal to r must follow s.

So, to show $(s, \ldots, s)$ is a Bayesian equilibrium, we need to ensure that if bidder i has value $x > r$, she should bid $s(x)$ and cannot be better off by bidding $b \in [r, s(a)]$. By r-monotonicity, by bidding $b \in [r, s(a)]$ is equivalent to bidding $s(y) \equiv b$ where $y \in [r, a]$. So, we need to show that

$$\begin{aligned} u(x) &\geq G(y)(x - s(y)) = u(y) + (x - y)G(y) \\ \Leftrightarrow u(x) - u(y) &\geq (x - y)G(y). \end{aligned}$$

But $u(x) - u(y) = \int_r^x G(z)dz - \int_r^y G(z)dz = \int_y^x G(z)dz \geq (x - y)G(y)$, where the inequality follows from the fact that G is increasing. ■

By Theorem 9, the expected payment of a bidder with value $x > r$ in the first-price auction with reserve price r is

$$xG(x) - \int_{r}^{x} G(y)dy,$$

and the expected payment of a bidder with value $x \leq r$ is zero (as such a bidder never wins). This is identical to the second-price auction with reserve price r (see (12)). Hence, expected revenue in the first-price auction with reserve price r is equal to the expected revenue in the second-price auction with reserve price r. Further, the optimal reserve price in the first-price auction is the same as that in the second-price auction. We summarize these discussions below using Theorem 8.

THEOREM **10** *The expected revenue from a first-price auction with reserve price r and a second-price auction with reserve price r is the same. Under MHR, the optimal reserve price is the unique solution to the equation*

$$r - \frac{1 - F(r)}{f(r)} = 0.$$

The optimal reserve prices depend on the MHR assumption. Without the MHR assumption, the determination of an optimal reserve price is tricky. Kotowski (2018) shows that dividing the bidders into two groups and setting different reserve prices for them improves revenue over a single reserve price for all the bidders.

Again, it is important to remind ourselves of the assumptions that drive these results: (a) private values (b) independent and identical bidders; (c) risk neutral bidders. Also, an ascending price auction where the clock starts at price r is strategically equivalent to a second-price auction with reserve price r. Similarly, a descending price auction where the clock stops at price r is equivalent to a first-price auction with reserve price r. Hence, Theorem 10 extends to all standard auctions with appropriate implementation of reserve price.

In practice, reserve prices have other uses besides revenue maximization. For instance, a reserve price may just indicate the cost of producing a good (which is normalized to zero here). A reserve price may be used to stop bidders from colluding – collusion is a strategic behaviour among groups of bidders (*bidding rings*) where all group members place lower bids.

8 Risk Averse Bidders

We are going to assume that bidders are *risk averse*. So, each bidder has a utility function $\pi : \mathbb{R} \to \mathbb{R}$, which is strictly increasing, concave and differentiable. The interpretation of π is the following. If a bidder i with value v_i receives the object and pays p for it, her utility from that is

$$\pi(v_i - p)$$

We are going to normalize and assume that $\pi(0) = 0$.

In the risk neutral case, this utility was just $\pi(v_i - p) = v_i - p$. An important feature of this assumption is the following. Suppose a bidder with value v_i faces a lottery $\frac{1}{3}$ probability of receiving the object at price p_1 and $\frac{2}{3}$ probability of receiving the object at price p_2, then according to a bidder with π, she evaluates the lottery as

$$\frac{1}{3}\pi(v_i - p_1) + \frac{2}{3}\pi(v_i - p_2) < \pi\Big(\frac{1}{3}(v_i - p_1) + \frac{2}{3}(v_i - p_2)\Big)$$

In the case of risk neutral bidder, the above expression would be an equality. Notice that the preference of the bidder over eventual alternatives `(winning/not winning, payment)` is still uniquely determined by a single parameter: her value v_i. It is just that how she evaluates a lottery changes. So, the interim preference of bidders which is over such lotteries of the eventual ex-post outcome will be shaped by the π function. This π function is assumed to be known in the model.

How does risk aversion change bidding behavior in first-price and second-price auctions?[5]

Theorem **11** *In a second-price auction with risk averse bidders, it is weakly dominant strategy for each bidder to bid her value.*

[5] The equivalence of first-price with descending price auction and second-price with the ascending price auction remains even with risk averse bidders.

Proof: The proof is identical to the case where bidders were risk neutral. The basic idea of the earlier proof carries over: your bid does not determine your payment in case you win. To give an idea, suppose bidder i is deciding to bid with value v_i and others bid b_{-i}. If she bids v_i and wins, she pays $b^* \equiv \max(b_{-i})$, and her utility is $\pi(v_i - b^*)$. Can she do better? As long as she wins, her utility remains the same as her payment still remains b^*. Obviously, bidding something to lose is not profitable. Similarly, if she bids v_i and loses, her $v_i < b^*$. In that case, the only way to win is to bid more than b^*, in which case her utility is $\pi(v_i - b^*) < \pi(0)$ since $v_i < b^*$ and π is increasing. ■

Notice that the proof does not even require that π is concave. The robustness of second-price (and ascending price) auction to preferences over lotteries makes it complelling in its own rights. This stems from the fact that the second-price auction gives payoffs to agents in an *ex-post* sense. On the other hand, first-price auction gives payoffs to agents at an *interim* stage.

Theorem **12** (**Holt Jr** (**1980**)) *Let $(s, \ldots, s)$ be the unique symmetric monotone equilibrium of the first-price auction with risk-neutral bidders. Let $(\bar{s}, \ldots, \bar{s})$ be a symmetric monotone equilibrium of first-price auction with risk-averse bidders. Then, for almost all $x \in [0, a]$,*

$$\bar{s}(x) > s(x)$$

Hence, the expected revenue in a first-price auction is greater than the expected revenue in a second-price auction with risk-averse bidders.

Proof: The proof does not derive an expression for equilibrium in a first-price auction (as was done in Theorem 3). It starts from the premise that a symmetric monotone equilibrium $(\bar{s}, \ldots, \bar{s})$ exist. Any such equilibrium has the following feature. Consider bidder i. If other bidders follow $\bar{s}$, for this to be equilibrium, i must bid $\bar{s}(x)$ for each $x \in [0, a]$. In particular, she should not be able to *imitate* to a type y when her true type is x. What is her probability of winning if she bids $\bar{s}(y)$ when others follow $\bar{s}$? Well, for this others have to bid less than $\bar{s}(y)$, which in turn means the highest of $(n-1)$ values have to be less than y. Hence, the probability of winning by bidding $\bar{s}(y)$ remains $G(y)$.

So, following $\bar{s}$ gives bidder i with value x a payoff equal to

$$u(x) = G(x)\pi(x - \bar{s}(x))$$

Imitating to y gives a payoff equal to

$$G(y)\pi(x - \bar{s}(y))$$

Note that equilibrium requires that the maximum of the above expression must occur at $y = x$. A necessary condition for that is the first-order condition needs to be satisfied at $y = x$.

$$G(y)\pi'(x - \bar{s}(y))\bar{s}'(y) = g(y)\pi(x - \bar{s}(y)),$$

where π' and $\bar{s}'$ denotes the derivatives of the respective functions.

Since this must hold at $y = x$, we get

$$\frac{G(x)}{g(x)} = \frac{\pi(x - \bar{s}(x))}{\pi'(x - \bar{s}(x))} \frac{1}{\bar{s}'(x)} \tag{17}$$

Now, in case of risk-neutral bidders, Theorem 3 showed that for all $x \in [0, a]$,

$$G(x)s(x) = xG(x) - \int_0^x G(y)dy$$

$$\Rightarrow G(x)s'(x) + g(x)s(x) = xg(x)$$

$$\Rightarrow \frac{G(x)}{g(x)} = (x - s(x))\frac{1}{s'(x)}$$

Using this with Equation (17), we get

$$\frac{\pi(x - \bar{s}(x))}{\pi'(x - \bar{s}(x))} \frac{1}{\bar{s}'(x)} = (x - s(x))\frac{1}{s'(x)}$$

Hence, we must have for all $x \in [0, a]$,

$$\frac{\bar{s}'(x)}{s'(x)} = \frac{\pi(x - \bar{s}(x))}{\pi'(x - \bar{s}(x))} \frac{1}{x - s(x)} \tag{18}$$

Since π is a concave and increasing function: $\pi(z) = \int_0^z \pi'(y)dy > z\pi'(z)$ for all $z > 0$. Hence, we can conclude from Equation (18), that for all $x \in (0, a]$,

$$\frac{\bar{s}'(x)}{s'(x)} > \frac{x - \bar{s}(x)}{x - s(x)} \tag{19}$$

We will now argue that $\bar{s}(x) > s(x)$ for almost all $x \in [0, a]$. Note that a feature of the equilibrium, with risk averse and risk-neutral bidders is that $\bar{s}(0) = s(0) = 0$. First, a small claim.

Claim **1** *For every $x \in (0, a]$, if $s(x) \geq \bar{s}(x)$, then $\bar{s}'(x) > s'(x)$.*

Proof: If $s(x) \geq \bar{s}(x)$, then (19) implies that

$$\frac{\bar{s}'(x)}{s'(x)} > \frac{x - \bar{s}(x)}{x - s(x)} \geq 1.$$

Hence, $\bar{s}'(x) > s'(x)$. ■

Now, we argue that there cannot be an interval $[\tilde{x}, \tilde{x} + h]$, where $h > 0$ such that $s(\tilde{x}) = \bar{s}(\tilde{x})$ and $s(x) \geq \bar{s}(x)$ for all $x \in [\tilde{x}, \tilde{x} + h]$. Such an interval is shown in Figure 3.

If such an interval exists, then by Claim 1, for all $x \in [\tilde{x}, \tilde{x} + h]$, we have $\bar{s}'(x) > s'(x)$. As a result, for any $x \in (\tilde{x}, \tilde{x} + h]$,

$$\bar{s}(x) = \bar{s}(\tilde{x}) + \int_{\tilde{x}}^{x} \bar{s}'(y)dy > s(\tilde{x}) + \int_{\tilde{x}}^{x} s'(y)dy = s(x),$$

which is a contradiction.

We can now complete our argument that $\bar{s}(x) > s(x)$ for almost all $x \in [0, a)$. If not, there must exist an interval $[\tilde{x}, \tilde{x} + h]$ with $h > 0$ such that $s(\tilde{x}) = \bar{s}(\tilde{x})$ and $s(x) \geq \bar{s}(x)$ for all x in the interval. But we just showed that such an interval cannot exist. This proves the first part of the theorem.

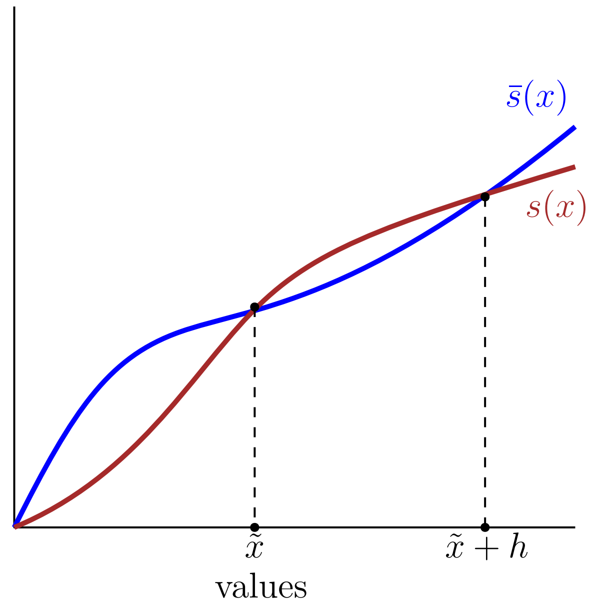

Figure 3: An interval where $s(x) \geq \bar{s}(x)$

For the second part, we observe that the expected payment of a bidder with value x in second-price auction remains the same (due to Theorem 11) in the case of risk-neutral and risk-averse bidders. Hence, a bidder with value x makes an expected payment equal to her expected payment in a first-price auction with risk neutral bidder (Theorem 4): $G(x)s(x)$. But the expected payment of a bidder with value x in a first-price auction with risk averse bidders is $G(x)\bar{s}(x)$. Hence, we have

$$G(x)s(x) < G(x)\bar{s}(x).$$

So, with risk averse bidders, a bidder with value x makes higher expected payment in the first-price auction than in a second-price auction. Thus, the expected revenue in a first-price auction is higher than the second-price auction with risk averse bidders. ■

So, the usual revenue equivalence between first-price and second-price auction breaks down with risk-averse bidders. A corollary of this result is also that the descending price auction (equivalent to the first-price auction) generates more expected revenue than an ascending price auction (equivalent to the second-price auction) with risk-averse bidders.

Why does risk aversion lead to aggressive bidding in first-price auction? The basic intuition is that an increase in bid leads to two outcomes: (a) *an increase* in probability of winning and (b) *decrease* in ex-post payoff. With risk aversion, a bidder cares more about increasing the probability of winning. We now look at two specific form of risk aversion and see how bidding of such bidders change.

8.1 CRRA Bidders

A bidder is called a **CRRA bidder**, if her coefficient of relative risk aversion $\frac{-z\pi''(z)}{\pi'(z)}$ is constant. With a CRRA bidder, the utility function takes the following specific form:

$$\pi(z) = z^{\alpha},$$

where $0 < \alpha < 1$ and the coefficient of relative risk aversion becomes

$$-\frac{z\pi''(z)}{\pi'(z)} = (1 - \alpha)$$

We can describe the functional form of symmetric equilibrium for CRRA bidders.

THEOREM **13** *Let $(\bar{s}, \ldots, \bar{s})$ be a symmetric and monotone strategy profile. Then, the following are equivalent.*

1. *$(\bar{s}, \ldots, \bar{s})$ is a Bayesian equilibrium.*
2. *For every $x \in [0, a]$,*

$$\bar{s}(x) = x - \frac{1}{G_{\alpha}(x)} \int_{0}^{x} G_{\alpha}(y)dy$$

where $G_{\alpha}(y) = [G(x)]^{\frac{1}{\alpha}}$.

Proof: $1 \Rightarrow 2$. Suppose $(n - 1)$ bidders follow the equilibrium strategy $\bar{s}$. A bidder with value x by imitating a bidder of type y

- wins the auction with probability $G(y)$
- and gets a payoff $(x - \bar{s}(y))^{\alpha}$

Expected utility:= $G(y)(x - \bar{s}(y))^\alpha$. In equilibrium, this expected utility must be maximized at $y = x$. First order condition gives

$$g(y)(x - \bar{s}(y))^\alpha - G(y)\alpha\bar{s}'(y)(x - \bar{s}(y))^{\alpha-1} = 0$$
$$\iff g(y)(x - \bar{s}(y)) - G(y)\alpha\bar{s}'(y) = 0$$

$$\alpha\frac{G(x)}{g(x)} = \frac{x - \bar{s}(x)}{\bar{s}'(x)}$$

This is similar to risk-neutral case except the α multiplier.

Let $G_\alpha(x) = [G(x)]^{\frac{1}{\alpha}}$ for all $x \in [0, a]$. Note G_α is a probability distribution. Let its pdf be g_α. For every $x \in [0, a]$,

$$\frac{G_\alpha(x)}{g_\alpha(x)} = \frac{[G(x)]^{\frac{1}{\alpha}}}{g(x)\frac{1}{\alpha}[G(x)]^{\frac{1}{\alpha}-1}} = \alpha\frac{G(x)}{g(x)}$$

So, first order condition reduces to

$$\alpha\frac{G(x)}{g(x)} = \frac{G_\alpha(x)}{g_\alpha(x)} = \frac{x - \bar{s}(x)}{\bar{s}'(x)}$$

So, in any symmetric and monotone equilibrium with CRRA bidders,

$$\frac{G_\alpha(x)}{g_\alpha(x)} = \frac{x - \bar{s}(x)}{\bar{s}'(x)}$$

When $\alpha = 1$, we get the same condition as risk-neutral bidders.

Hence, with CRRA bidders, equilibrium involves bidding like in risk-neutral case but as if value is drawn from $F_\alpha \equiv F^{\frac{1}{\alpha}}$. Solving similar to risk-neutral case, we get

$$\bar{s}(x) = x - \frac{1}{G_\alpha(x)}\int_0^x G_\alpha(y)dy$$

2 $\Rightarrow$ 1. For this, we only show that a bidder with type x cannot gain by *imitating* a bidder with type y by bidding $\bar{s}(y)$, given that other bidders follow $\bar{s}$.

Expected utility by bidding $\bar{s}(y)$ is

$$G(y)\Big[x-\bar{s}(y)\Big]^{\alpha} = [G_{\alpha}(y)]^{\alpha}\Big[x-\bar{s}(y)\Big]^{\alpha} = \Big[G_{\alpha}(y)\big(x-\bar{s}(y)\big)\Big]^{\alpha}$$

If all bidders were risk-neutral and bidders drew their values using $[F]^{\frac{1}{\alpha}}$, then the equilibrium is $\bar{s}$. Hence,

$$G_{\alpha}(y)\big(x-\bar{s}(y)\big) \leq G_{\alpha}(x)\big(x-\bar{s}(x)\big)$$

Putting all together,

$$\begin{aligned} G(y)\Big[x-\bar{s}(y)\Big]^{\alpha} &\leq \Big[G_{\alpha}(x)\big(x-\bar{s}(x)\big)\Big]^{\alpha} \\ &= [G_{\alpha}(x)]^{\alpha}\Big[x-\bar{s}(x)\Big]^{\alpha} \\ &= G(x)\Big[x-\bar{s}(x)\Big]^{\alpha}, \end{aligned}$$

which is the required incentive constraint. ■

Hence, a CRRA bidder with coeffficient of risk-aversion α, bids as if the highest of $(n-1)$ values is drawn from G_{α}. Since G_{α} first-order stochastic dominates G, the expected revenue with risk-averse bidders is higher. In particular,

$$\begin{aligned} \frac{1}{G_{\alpha}(x)}\int_0^x G_{\alpha}(y)dy &= \int_0^x \Big[\frac{G(y)}{G(x)}\Big]^{\frac{1}{\alpha}} dy \\ &\leq \int_0^x \Big[\frac{G(y)}{G(x)}\Big] dy = \frac{1}{G(x)}\int_0^x G(y)dy \end{aligned}$$

where we use $\alpha < 1$. This implies that

for every $x \in [0, a]$,

$$\bar{s}(x) = x - \frac{1}{G_{\alpha}(x)}\int_0^x G_{\alpha}(y)dy \geq x - \frac{1}{G(x)}\int_0^x G(y)dy = s(x)$$

Finally, using the fact that the probability of winning in both cases is $G(x)$ for a bidder with value x implies that the expected payment of a bidder with type x satisfies $G(x)\bar{s}(x) > G(x)s(x)$. Hence, expected revenue is higher with risk-averse bidders.

8.2 CARA Bidders

A bidder is called a **CARA bidder**, if her coefficient of absolute risk aversion $\frac{-\pi''(z)}{\pi'(z)}$ is constant. With a CARA bidder, the utility function has the following specific form:

$$\pi(z) = 1 - \exp(-\alpha z),$$

where $\alpha > 0$ is the coefficient of absolute risk aversion.

Now, consider the uncertainty over *prices* faced by a CARA bidder in a second-price auction. Since the bidder pays the highest of $(n-1)$ values when she is the winner, her expected utility conditional on winning when value is x and bid is z is given by

$$\mathbf{E}_{Y_1}\left[\pi(x - Y_1) : Y_1 < z\right],$$

where Y_1 is the random variable of highest $(n-1)$ values. Let the certainty equivalent of this gamble be $\rho(x,z)$. Formally,

$$\pi(x - \rho(x,z)) = \mathbf{E}_{Y_1}\left[\pi(x - Y_1) : Y_1 < z\right]$$

Using the expression for π, we see that

$$\begin{aligned}
1 - \exp(-\alpha(x - \rho(x,z)) &= \frac{1}{G(z)} \int_0^z \big(1 - \exp(-\alpha(x-y))\big) g(y)dy \\
&= \frac{1}{G(z)}\left[G(z) - \int_0^z \exp(-\alpha(x-y))g(y)dy\right] \\
&= 1 - \frac{1}{G(z)} \int_0^z \exp(-\alpha(x-y))g(y)dy
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\exp(-\alpha(x-\rho(x,z)) &= \frac{1}{G(z)}\int_0^z \exp(-\alpha(x-y))g(y)dy \\
\iff \frac{\exp(\alpha\rho(x,z))}{\exp(\alpha x)} &= \frac{1}{G(z)}\int_0^z \frac{\exp(\alpha y)}{\exp(\alpha x)}g(y)dy \\
\iff \exp(\alpha\rho(x,z)) &= \frac{1}{G(z)}\int_0^z \exp(\alpha y)g(y)dy
\end{aligned}$$

Notice that the RHS is independent of x. Hence, ρ is independent of x, and we simply write $\rho(x,z) \equiv \rho(z)$, and for every z, $\rho(z)$ solves

$$\exp(\alpha\rho(z)) = \frac{1}{G(z)}\int_0^z \exp(\alpha y)g(y)dy$$

Hence, we write

$$\pi(x-\rho(z)) = \mathbf{E}_{Y_1}\left[\pi(x-Y_1) : Y_1 < z\right] \tag{20}$$

We now argue that ρ is the *unique* symmetric and monotone equilibrium with CARA bidders. Further, bidders are indifferent between first-price and second-price auctions. So, even though the seller prefers the first-price auction (Theorem 12), CARA bidders are indifferent between auction formats.

Theorem **14** (**Matthews** (**1987**)) *There is a unique symmetric and monotone equilibrium* $(\rho,\ldots,\rho)$ *in the first-price auction with CARA bidders:*

$$\exp(\alpha\rho(x)) = \frac{1}{G(x)}\int_0^x \exp(\alpha y)g(y)dy \qquad \forall\ x \in [0,a] \tag{21}$$

Further, the expected utility of every bidder is the same in the first-price and the second-price auction.

Proof: First, by Theorem 11, truthful bidding is a Bayesian equilibrium (weakly dominant) in the second-price auction. In a second-price auction, by bidding z, a bidder with value x gets an expected utility equal to

$$G(z)\mathbf{E}_{Y_1}\left[\pi(x - Y_1) : Y_1 < z\right]$$

Since truthful strategy is a Bayesian equilibrium, this expression is maximized at $z = x$.

$$x \in \arg\max_z \left[G(z)\mathbf{E}_{Y_1}\left[\pi(x - Y_1) : Y_1 < z\right]\right]$$

But Equation (20) implies that

$$x \in \arg\max_z \left[G(z)\pi(x - \rho(z))\right]$$

Conversely, this equilibrium must be unique. This is because if there is some equilibrium $(\bar{s}, \ldots, \bar{s})$, then it must be the case that

$$x \in \arg\max_z \left[G(z)\pi(x - \bar{s}(z))\right]$$

We know that the certainty equivalent of the "price gamble" in the second-price auction is given by a solution to the Equation (20). Hence, $\bar{s} \equiv \rho$, and ρ is *uniquely* determined. This shows uniqueness.

Finally,

$$G(x)\pi(x - \rho(x)) = G(x)\mathbf{E}_{Y_1}\left[\pi(x - Y_1) : Y_1 < x\right]$$

implies the expected payoff of a bidder with value x is the same in both the first-price and the second-price auction for a CARA bidder. ■

9 Asymmetric auctions: two bidders

While symmetry is a plausible assumption in some settings, it is violated in many settings: bidders come from heterogeneous backgrounds and there is no reason to believe that their values will be distributed similarly. For instance, two teams bidding for a player in a cricket league auction will most likely draw their values from different distributions (value of a player will depend on the players the teams they already have, which may be different across teams).

In an asymmetric environment, the strategies of agents become asymmetric – remember, strategy of a player is a map from set of types to real numbers, and the set of types are potentially different across players. So, we allow for asymmetric equilibria in first-price auction. The analysis of equilibrium in first-price auction with asymmetric bidders is quite complex – seminal papers are Lebrun (1999); Maskin and Riley (2000b). Of course, truthful bidding remains a weakly dominant strategy in the second-price auction. So, the focus of this section is on the analysis of first-price auction.

9.1 Two examples

We present two examples to illustrate the effect of asymmetry. In both the examples, there are two bidders: $\{1, 2\}$

- In this example, bidder 1 draws her value uniformly from $[0, 1]$ and bidder 2 draws her value from $[2, 3]$. The expected revenue in a second-price auction is the expected value of bidder 1: $\frac{1}{2}$. The following is an asymmetric equilibrium of the first-price auction: bidder 1 bids her value and bidder 2 bids 1. To see this, if bidder 2 bids 1, then it is optimal for bidder 2 to bid her value. If bidder 1 bids her value, consider a bid b of bidder 2 such that $b \leq 1$. The expected payoff of bidder 2 by bidding b is $b(v_2 - b)$, where v_2 is the value of bidder 2. Differentiating, $v_2 - 2b \geq 0$ since $v_2 \geq 2$ and $b \leq 1$, we see that the expected payoff is maximized at $b = 1$. Bidding more than 1 is not optimal because bidder 1 never bids more than 1. This shows that the given strategies constitute a Bayesian equilibrium. The expected revenue in the first-price auction is 1 – bidder 2 always wins and pays 1. Hence, there exists an equilibrium in the first-price auction where the expected revenue of the first-price auction is higher than in the second-price auction.

- The second example is somewhat special. It has a finite type space and we will not compute an equilibrium (a mixed strategy equilibrium will exist). The type space is as follows. Bidder 1 has a value of 2 with probability 1 but bidder 2 has a value of 0 with probability $\frac{1}{2}$ and a value of 2 with probability $\frac{1}{2}$. Bidder 2 of type 0 must bid 0 in any equilibrium. Hence, bidder 1 can always bid arbitrarily close to 0 and win the auction whenever bidder 2 has type 0. This happens with probability $\frac{1}{2}$. Hence, bidder 1 can guarantee herself a payoff of $\frac{1}{2}(2-0) = 1$. Hence, she will never bid more than 1 in equilibrium (by doing so, her payoff is less than $2 - 1 = 1$). So, bidder 2 of type 2 can always bid slightly more (but arbitrarily close to) 1 to win the auction, and ensure a payoff of $2 - 1 = 1$. This happens with probability $\frac{1}{2}$. Hence, bidder 2 can guarantee an expected payoff of $\frac{1}{2}$ in any equilibrium. So, total expected payoffs of bidders is at least $\frac{3}{2}$ in any equilibrium. Notice that the winner is always a bidder with value not more than 2. Hence, the total expected surplus is not more than 2. Since expected revenue is expected surplus minus expected payoff of bidders, we conclude that the expected revenue in the first-price auction is not more than $\frac{1}{2}$ in any equilibrium.

The expected revenue in a second-price auction is $\frac{1}{2} \times 2 = 1$ (i.e., second higest value is 0 with probability $\frac{1}{2}$ and 2 with probability $\frac{1}{2}$). Thus, the second-price auction generates more expected revenue than the first-price auction in any equilibrium.

9.2 First-price auction: two bidders

We analyze properties of equilibria in a two bidder setting. Suppose bidder 1 draws her value from $[0, a_1]$ using distribution F_1 and bidder 2 draws her value from $[0, a_2]$ using distribution F_2. We will **assume** that distribution of bidder 2 dominates the distribution of bidder 1 in terms of *reverse hazard rate*:

$$a_2 \geq a_1$$
$$\frac{f_2(x)}{F_2(x)} > \frac{f_1(x)}{F_1(x)} \qquad \forall\, x \in (0, a_1)$$

Reverse hazard rate ordering of random variables is stronger than usual first-order stochastic dominance of random variables (Shaked and Shanthikumar, 2007). We will refer to bidder 1 as the *weak bidder* and bidder 2 as the *strong bidder*. A class of distribution that can be ordered in terms of reverse hazard rate dominance is: for all $x \in [0, 1]$, we have $F(x) = x^{\alpha}$ for some $\alpha \in (0, 1]$. These distributions have support $[0, 1]$ and for $\alpha > \hat{\alpha}$, we have two

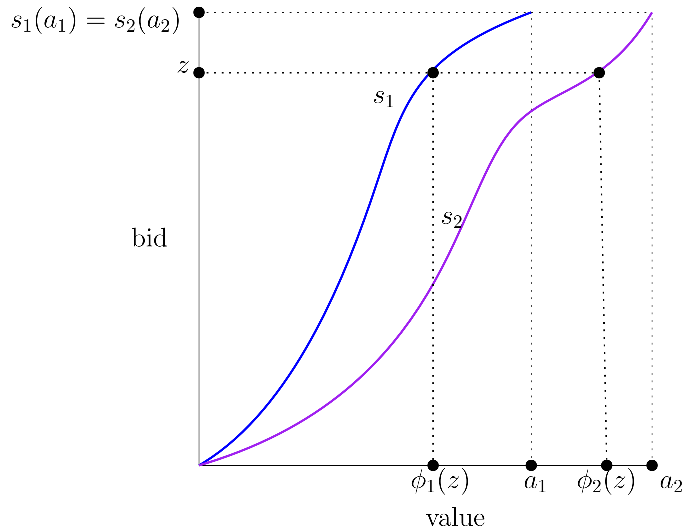

Figure 4: Asymmetric bidding in first-price auction

distributions F and $\hat{F}$ such that

$$\frac{f(x)}{F(x)} = \frac{\alpha}{x} > \frac{\hat{\alpha}}{x} = \frac{\hat{f}(x)}{\hat{F}(x)} \qquad \forall\ x \in (0,1)$$

A strategy for bidder $i \in \{1,2\}$ is a map $s_i : [0, a_i] \to \mathbb{R}_+$. We assume that s_i is *strictly increasing and differentiable*. The main result of the section is the following.

Theorem **15** (**Maskin and Riley** (**2000a**)) *Suppose* (s_1, s_2) *is a Bayesian equilibrium in the first-price auction. Then, the weak bidder bids more aggressively than the strong bidder in equilibrium:*

$$s_1(x) > s_2(x) \qquad \forall\ x \in [0, a_1]$$

Proof: In any equilibrium (s_1, s_2), it must be that $s_1(0) = s_2(0) = 0$. Further, $s_1(a_1) = s_2(a_2)$. If $s_1(a_1) > s_2(a_2)$, then bidder 1 can do better by lowering her bid when her type is a_1. A similar argument works if $s_2(a_2) > s_1(a_1)$. Hence, we assume that $s_1(a_1) = s_2(a_2)$. We let $\bar{b} = s_1(a_1) = s_2(a_2)$. The two bid functions are shown in Figure 4.

For every bidder $i \in \{1,2\}$, define $\phi_i(z) := s_i^{-1}(z)\ \forall\ z \in [0, \bar{b}]$. This is the inverse bidding function of each bidder. Note that since $s_1(a_1) = s_2(a_2)$, the domain of the inverse bidding function is the same for both the bidders. Figure 4 illustrates this.

Suppose bidder j follows s_j. Then, bidder $i \neq j$ does not deviate by bidding $b \in [0, \bar{b}]$. But by bidding b, bidder i wins if $s_j(v_2) < b$ or $v_2 < \phi_j(b)$. The probability of this event is $F_j(\phi_j(b))$. Hence, expected payoff of bidder i with value x when she bids b is

$$F_j(\phi_j(b))\big[x - b\big]$$

The first order condition is $f_j(\phi_j(b))\phi_j'(b)(x - b) = F_j(\phi_j(b))$. This must hold for all $x \in (0, a_i)$. Since in equilibrium $b = s_i(x)$, we can write $x = \phi_i(b)$. So, one way to write the first order condition is for all $z \in [0, \bar{b}]$, we must have

$$f_j(\phi_j(z))\phi_j'(z)(\phi_i(z) - z) = F_j(\phi_j(z)) \tag{22}$$

$$\iff \frac{f_j(\phi_j(z))}{F_j(\phi_j(z))} = \frac{1}{\phi_j'(z)(\phi_i(z) - z)} \tag{23}$$

Note that this implies that $\phi_i(z) > z$ if z is in the interior.

Suppose $\phi_1(z) = \phi_2(z)$ for some z. Then, using this condition and the fact that F_2 reverse harzard rate dominates F_1, we get

$$\frac{1}{\phi_2'(z)(\phi_1(z) - z)} = \frac{f_2(\phi_2(z))}{F_2(\phi_2(z))} = \frac{f_2(\phi_1(z))}{F_2(\phi_1(z))} > \frac{f_1(\phi_1(z))}{F_1(\phi_1(z))} = \frac{1}{\phi_1'(z)(\phi_2(z) - z)} = \frac{1}{\phi_1'(z)(\phi_1(z) - z)}$$

Using $\phi_1(z) > z$, we get that $\phi_1'(z) > \phi_2'(z)$ whenever $\phi_1(z) = \phi_2(z)$. An implication of this is that whenever ϕ_1 curve meets ϕ_2 curve, they cross each other – if they only tangentially touched each other, then $\phi_1'(z)$ must equal $\phi_2'(z)$.

We next show that if $\phi_1(z) > \phi_2(z)$ for some $z \in (0, \bar{b})$, then $\phi_1(\hat{z}) > \phi_2(\hat{z})$ for all $\hat{z} \in [z, \bar{b})$. Suppose not. Then, by continuity, for some $\hat{z}$, we have $\phi_1(\hat{z}) = \phi_2(\hat{z})$. Hence, $\phi_1'(\hat{z}) > \phi_2'(\hat{z})$. Then, for some interval $[z, \hat{z})$ we have $\phi_1(y) > \phi_2(y)$ and $\phi_1'(y) > \phi_2'(y)$ for all $y \in [z, \hat{z})$. Then, $\phi_1(\hat{z}) = \phi_1(z) + \int_z^{\hat{z}} \phi_1'(y) > \phi_2(z) + \int_z^{\hat{z}} \phi_2'(y) = \phi_2(\hat{z})$, a contradiction. Now, we consider two cases.

Case 1. Suppose $a_1 < a_2$. In that case $\phi_1(\bar{b}) = a_1 < a_2 = \phi_2(\bar{b})$. Hence, there is some point $\hat{z}$ in interior $(0, \bar{b})$ but sufficiently close to $\bar{b}$ such that $\phi_1(\hat{z}) < \phi_2(\hat{z})$. But then, there cannot be a $z \in (0, \bar{b})$ such that $\phi_1(z) > \phi_2(z)$.

CASE 2. Suppose $a_1 = a_2$. Then, assume for contradiction that there is a $z \in (0, \bar{b})$ such that $\phi_1(z) > \phi_2(z)$. We know that this implies for any $\hat{z}$ arbitrarily close to $\bar{b}$, we have $\phi_1(b) > \phi_2(b)$. Since $\phi_1(\bar{b}) = a_1 = a_2 = \phi_2(\bar{b})$ and b is arbitrarily close to $\bar{b}$, we get that $F_1(\phi_1(b)) > F_2(\phi_2(b))$ and the derivatives of F_1 and F_2 at these points must have the opposite relation: $\phi_1'(b)f_1(\phi_1(b)) < \phi_2'(b)f_2(\phi_2(b))$. Using (23), we see that

$$\phi_1(b) = b + \frac{F_2(\phi_2(z))}{\phi_2'(b)f_2(\phi_2(b))} < b + \frac{F_1(\phi_1(z))}{\phi_1'(b)f_1(\phi_1(b))} = \phi_2(b),$$

which is a contradiction.

Hence, we have shown that there cannot be a $z \in (0, \bar{b})$ such that $\phi_1(z) > \phi_2(z)$. But there cannot be any interval where $(\hat{z}, \hat{z} + h)$ such that $\phi_1(z) = \phi_2(z)$ for all z in this interval. This is because, by our earlier claim, we will have $\phi_1'(z) > \phi_2'(z)$ for all z in this interval, which contradicts the fact that $\phi_1(z) = \phi_2(z)$ in the interval. We have already argued that ϕ_1 and ϕ_2 cannot tangentially touch each other. This shows that $\phi_1(z) < \phi_2(z)$ for all $z \in (0, \bar{b})$, which is same as $s_1(x) > s_2(x)$ for all $x \in [0, a_1]$. ■

Theorem 15 has efficiency consequences. Because at a type profile (x, x), where both the bidders have same value $x \in (0, a_1)$, bidder 1 bids more than bidder 2. By continuity, there is a profile $(x - \epsilon, x)$, where $\epsilon > 0$ but sufficiently small, such that $s_1(x - \epsilon) > s_2(x)$. That is, bidder 1 wins even though she has a lower value. Hence, the first-price auction is *not efficient* with asymmetric bidders.

The other comment about the proof of Theorem 15 is that the first order conditions along with the relevant boundary conditions define a unique Bayesian equilibrium of the first-price auction. This and some conditions on F_1 and F_2 under which revenue in two auction formats can be compared are discussed in Maskin and Riley (2000a).

10 Optimal auction design

In this section, we discuss the design of optimal auctions for selling a single object. Optimal auction refers to an auction that maximizes expected revenue over all possible auctions, where a Bayesian equilibrium exists. However, we consider an even larger class of *mechanisms* which need not be an auction, e.g. a posted-price mechanism, where a price is announced and the first buyer to express willingness to pay buys at the announced price.

Auction design is slightly different from analyzing auctions. Typically, when we theoretically analyze auctions, we try to look for its equilibria. In design of auctions, we only consider auctions which have an equilibrium. We do not worry about characterizing the equilibria. Rather, we try to see what outcomes can be achieved in some equilibrium.

To understand design of optimal auctions, we first have to formally reduce the same of *mechanisms* that we need to consider. For this, we first define the notion of a *social choice function (SCF)*. There are n agents and let the type space of each agent i be $\mathcal{D}_i$. Let $\mathcal{D} = \mathcal{D}_1 \times \ldots \times \mathcal{D}_n$ be the set of type profiles. We associate with every agent i a utility function: $u_i : [0, 1] \times \mathbb{R} \times \mathcal{D} \to \mathbb{R}$, i.e., for every allocation probability, payment, and type, it specifies a utility. So, $u_i(q_i, p_i; v_i)$ denotes the utility from winning the object with probability q_i and paying p_i when type is v_i. A special form of this utility function is $q_i v_i - p_i$, the quasilinear utility function. Potentially, this can also be a risk averse utility function.

A **social choice function (SCF)** is a pair of maps (q_i, p_i) for each $i \in N$ such that $q_i : \mathcal{D} \to [0, 1]$ is the allocation function of agent i and $p_i : \mathcal{D} \to [0, 1]$ is the payment function of agent i. There is no restriction on the value (positive, negative, zero) of p_i. But q_is need to satisfy feasibility: $\sum_{i=1}^{n} q_i(v) \leq 1$ for each $v \in \mathcal{D}$. We will denote such an SCF by simply (q, p).

An SCF reflects designer's goal from a mechanism, i.e., if the designer knew the types of the agents, how she would set the outcomes. A mechanism is a more complicated object than an SCF. The main objective of a mechanism is to set up rules of interaction between agents. These rules are often designed with the objective of realizing the outcomes of a social choice function. The basic ingredient in a mechanism is a **message**. A message is a communication between an agent and the mechanism designer. You can think of it as an action chosen in various contingencies of a Bayesian game - these messages will form the actions for various contingencies of agents in a Bayesian game that the designer will set up.

A mechanism must specify the **message space** for each agent. A message space has to specify various contingencies that may arise in a mechanism and available actions at each of the contingencies. This in turn induces a Bayesian game with messages playing the role of actions. Given a message profile, the mechanism chooses an outcome.

Definition 6 *A* **mechanism** *is a collection of message spaces and a decision rule:* $\mathcal{M} \equiv (M_1, \ldots, M_n, (\phi, \pi))$, *where*

- *for every* $i \in N$, M_i *is the message space of agent* i *and*
- $\phi : M_1 \times \ldots \times M_n \to [0,1]^n$ *is the allocation decision and* $\pi : M_1 \times \ldots \times M_n \to \mathbb{R}^n$ *is the payment decision.*

A mechanism is a **direct mechanism** *if* $M_i = \mathcal{D}_i$ *for every* $i \in N$.

In a mechanism $((\mathcal{M}_1, \ldots, \mathcal{M}_n), (\phi, \pi))$, if a message profile $(m_1, \ldots, m_n)$ is sent by agents, then agent i gets the object with probability $\phi_i(m_1, \ldots, m_n)$ and pays $\pi_i(m_1, \ldots, m_n)$.

In a direct mechanism, every agent communicates a type from his type space to the mechanism designer. Hence, if (q, p) is an scf, then $((\mathcal{D}_1, \ldots, \mathcal{D}_n), (q, p))$ is a direct mechanism - for simplicity, we will just refer to (q, p) as a (direct) mechanism.

The message space of a mechanism can be quite complicated. Consider the sale of a single object by a "price-based" procedure. The mechanism designer announces a price and asks every buyer to communicate if he wants to buy the object at the announced price. The price is raised if more than one buyer expresses interest in buying the object, and the procedure is repeated till exactly one buyer shows interest. The message space in such a mechanism is quite complicated. Here, a message must specify the communication of the buyer (given his type) for every contingent price.

10.1 Dominant Strategy Incentive Compatibility

We now introduce the notion of incentive compatibility. The idea of a mechanism and incentive compatibility is often attributed to the works of Hurwicz - see (Hurwicz, 1960). The goal of mechanism design is to design the message space and decision rules in a way such that when agents participate in the mechanism they have (best) actions (messages) that they can choose as a function of their private types such that the desired outcome is achieved. The most fundamental, though somewhat demanding, notion of incentive compatibility in mechanism design is the notion of dominant strategies.

A strategy is a map $s_i : \mathcal{D}_i \to M_i$, which specifies the message each agent i will choose for every realization of her type. A strategy s_i is a **dominant strategy** for agent i in mechanism $(M_1, \ldots, M_n, (\phi, \pi))$, if for every $v_i \in \mathcal{D}_i$ we have

$$u_i(\phi_i(s_i(v_i), m_{-i}), \pi_i(s_i(v_i), m_{-i}); v_i) \geq u_i(\phi_i(m'_i, m_{-i}), \pi_i(m'_i, m_{-i}); v_i) \qquad \forall\, m'_i,\ \forall\, m_{-i}$$

Definition **7** *A social choice function (q, p) is* **implemented** *in dominant strategy equilibrium by a mechanism $(M_1, \ldots, M_n, (\phi, \pi))$ if there exists strategies $(s_1, \ldots, s_n)$ such that*

1. *$(s_1, \ldots, s_n)$ is a dominant strategy equilibrium of $(M_1, \ldots, M_n, (\phi, \pi))$, and*
2. *$\phi_i(s_1(v_1), \ldots, s_n(v_n)) = q_i(v_1, \ldots, v_n)$ and $\pi_i(s_1(v_1), \ldots, s_n(v_n)) = p_i(v_1, \ldots, v_n)$ for all $i \in N$ and for all $(v_1, \ldots, v_n) \in \mathcal{D}$.*

For direct mechanisms, we will look at equilibria where everyone tells the truth.

Definition **8** *A direct mechanism is* **strategy-proof** *or* **dominant strategy incentive compatible (DSIC)** *if for every agent $i \in N$ and every $v_i \in \mathcal{D}_i$, the* **truth-telling** *strategy $s_i(v_i) = v_i$ for all $v_i \in \mathcal{D}_i$ is a dominant strategy.*

So, to verify whether a social choice function is implementable or not, we need to search over infinite number of mechanisms whether any of them implements this SCF. A fundamental result in mechanism design says that one can restrict attention to the direct mechanisms.

Proposition **1 (Revelation Principle, Myerson (1979))** *If a mechanism implements a social choice function (q, p) in dominant strategy equilibrium, then the direct mechanism (q, p) is strategy-proof.*

Proof: Suppose mechanism $(M_1, \ldots, M_n, (\phi, \pi))$ implements (q, p) in dominant strategies. Let $s_i : \mathcal{D}_i \to M_i$ be the dominant strategy of each agent i.

Fix an agent $i \in N$. Consider two types $v_i, v'_i \in \mathcal{D}_i$. Consider v_{-i} to be the report of other agents in the direct mechanism. Let $s_i(v_i) = m_i$ and $s_{-i}(v_{-i}) = m_{-i}$. Similarly, let $s_i(v'_i) = m'_i$. Then, using the fact that (q, p) is implemented by our mechanism in dominant strategies, we get

$$u_i(\phi_i(m_i, m_{-i}), \pi_i(m_i, m_{-i}); v_i) \geq u_i(\phi_i(m'_i, m_{-i}), \pi_i(m'_i, m_{-i}); v_i)$$

But $q_i(v_i, v_{-i}) = \phi_i(m_i, m_{-i})$, $q_i(v'_i, v_{-i}) = \phi_i(m'_i, m_{-i})$, $p_i(v_i, v_{-i}) = \pi_i(m_i, m_{-i})$ and $p_i(v'_i, v_{-i}) = \pi_i(m'_i, m_{-i})$. Then: $u_i(q_i(v_i, v_{-i}), p_i(v_i, v_{-i}); v_i) \geq u_i(q_i(v'_i, v_{-i}), p_i(v'_i, v_{-i}); v_i)$, which establishes that (q, p) is strategy-proof. ■

Thus, a social choice function (q, p) is implementable in dominant strategies if and only if the direct mechanism (q, p) is strategy-proof. Revelation principle is a central result in mechanism design. One of its implications is that if we wish to find out what social choice functions can be implemented in dominant strategies, we can restrict attention to direct mechanisms. This is because, if some non-direct mechanism implements a social choice function in dominant strategies, revelation principle says that the corresponding direct mechanism is also strategy-proof. For instance, if we know that the equilibrium in the ascending price auction implements the second-price auction outcome, then it is without loss of generality to focus attention on the direct mechanism, which is the second-price auction.

10.2 Bayesian Incentive Compatibility

While dominant strategy incentive compatibility required the equilibrium strategy to be the best strategy under all possible strategies of opponents, Bayesian incentive compatibility requires this to hold in *expectation*. This means that in Bayesian incentive compatibility, an equilibrium strategy must give the highest expected utility to the agent, where we take expectation over types of other agents. To be able to take expectation, agents must have information about the probability distributions from which types of other agents are drawn. Hence, Bayesian incentive compatibility is informationally demanding. In dominant strategy incentive compatibility the mechanism designer needed information on the type space of agents, and every agent required no prior information of other agents to compute his equilibrium. In Bayesian incentive compatibility, every agent needs to know the distribution from which agents' types are drawn.

Since we need to compute expectations, we will assume that values of agents $v \equiv (v_1, \ldots, v_n)$ are jointly drawn using a distribution F. Hence, we are being more general than the models of auctions we studied by allowing for correlation of values of agents. Given agent i has value v_i, we denote by $F_{-i}(\cdot|v_i)$ the conditional distribution of values of agents in $N \setminus \{i\}$

To understand Bayesian incentive compatibility, fix a mechanism $(M_1, \ldots, M_n, \phi, \pi)$. A strategy of agent $i \in N$ for such a mechanism is a mapping $s_i : \mathcal{D}_i \to M_i$. A strategy profile $(s_1, \ldots, s_n)$ is a **Bayesian equilibrium** if for all $i \in N$, for all $v_i \in \mathcal{D}_i$ we have

$$\begin{aligned}
&\int_{v_{-i}} u_i(\phi_i(s_i(v_i), s_{-i}(v_{-i})), \pi_i(s_i(v_i), s_{-i}(v_{-i})); v_i) dF_{-i}(v_{-i}|v_i) \\
&\geq \int_{v_{-i}} u_i(\phi_i(m_i, s_{-i}(v_{-i})), \pi_i(m_i, s_{-i}(v_{-i})); v_i) dF_{-i}(v_{-i}|v_i) \qquad \forall\, m_i \in M_i.
\end{aligned}$$

A direct mechanism (social choice function) (q, p) is **Bayesian incentive compatible** if $s_i(v_i) = v_i$ for all $i \in N$ and for all $v_i \in \mathcal{D}_i$ is a Bayesian equilibrium, i.e., for all $i \in N$ and for all $v_i, v_i' \in \mathcal{D}_i$ we have

$$\begin{aligned}
&\int_{v_{-i}} u_i(q_i(v_i, v_{-i}), p_i(v_i, v_{-i}); v_i) dF_{-i}(v_{-i}|v_i) \\
&\geq \int_{v_{-i}} u_i(q_i(v_i', v_{-i}), p_i(v_i', v_{-i}); v_i) dF_{-i}(v_{-i}|v_i).
\end{aligned}$$

A dominant strategy incentive compatible mechanism is Bayesian incentive compatible. A mechanism $(M_1, \ldots, M_n, \phi, \pi)$ **implements** a social choice function (q, p) in Bayesian equilibrium if there exists strategies $s_i : \mathcal{D}_i \to M_i$ for each $i \in N$ such that

1. $(s_1, \ldots, s_n)$ is a Bayesian equilibrium of $(M_1, \ldots, M_n, \phi, \pi)$ and

2. $\phi_i(s_1(v_1), \ldots, s_n(v_n)) = q_i(v_1, \ldots, v_n)$, $\pi_i(s_1(v_1), \ldots, s_n(v_n)) = p_i(v_1, \ldots, v_n)$ for all $i \in N$ and for all $(v_1, \ldots, v_n) \in \mathcal{D}$.

Analogous to the revelation principle for dominant strategy incentive compatibility, we also have a revelation principle for Bayesian incentive compatibility.

Proposition **2 (Revelation Principle)** *If a mechanism implements a social choice function (q, p) in Bayesian equilibrium, then the direct mechanism (q, p) is Bayesian incentive compatible.*

Proof: Suppose $(M_1, \ldots, M_n, \phi, \pi)$ implements (q, p). Let $(s_1, \ldots, s_n)$ be the Bayesian equilibrium strategies of this mechanism which implements (q, p). Fix agent i and $v_i, v_i' \in \mathcal{D}_i$.

Now,

$$
\begin{aligned}
&\int_{v_{-i}} u_i(q_i(v_i, v_{-i}), p_i(v_i, v_{-i}); v_i) dF_{-i}(v_{-i}|v_i) \\
&= \int_{v_{-i}} u_i(\phi_i(s_i(v_i), s_{-i}(v_{-i})), \pi_i(s_i(v_i), s_{-i}(v_{-i})); v_i) dF_{-i}(v_{-i}|v_i) \\
&\geq \int_{v_{-i}} u_i(\phi_i(s_i(v'_i), s_{-i}(v_{-i})), \pi_i(s_i(v'_i), s_{-i}(v_{-i})); v_i) dF_{-i}(v_{-i}|v_i) \\
&= \int_{v_{-i}} u_i(q_i(v'_i, v_{-i}), p_i(v'_i, v_{-i}); v_i) dF_{-i}(v_{-i}|v_i),
\end{aligned}
$$

where the equalities come from the fact that the mechanism implements (q, p) and the inequality comes from the fact that $(s_1, \ldots, s_n)$ is a Bayesian equilibrium of the mechanism. ■

Like the revelation principle of dominant strategy incentive compatibility, the revelation principle for Bayesian incentive compatibility is not immune to criticisms for multiplicity of equilibria.

An example of (symmetric) first-price auction.

Consider a symmetric environment and first-price auction. In particular, suppose there are two bidders whose values are drawn independently from $[0, 1]$ using uniform distribution. We know that a unique symmetric Bayesian equilibrium of this first-price auction is that each buyer i bids $\frac{1}{2}v_i$.

What does the revelation principle say here? Since there is *an* equilibrium of this mechanism, the revelation principle says that there is a direct mechanism with a truth-telling Bayesian equilibrium. Such a direct mechanism is easy to construct here.

1. Ask buyers to submit their values (v_1, v_2).

2. The buyer i with the highest value wins but pays $\frac{1}{2}v_i$.

Notice that the first-price auction implements the outcome of this direct mechanism. Since the first-price auction had this outcome in Bayesian equilibrium, this direct mechanism is Bayesian incentive compatible.

10.3 Characterization of Bayesian incentive compatibility

Due to the revelation principle, we can focus our analysis to direct mechanisms. Our first step is to characterize, i.e., give an alternate description, of the set of direct mechanisms. The characterization is an important step to simplify our goal of describing an *optimal auction*.

From this section onwards, we will assume *independent* distribution of values. So, we will assume that value of bidder i is distributed in $[0, a_i]$ using cdf F_i and positive density f_i. Note that even though values are independent, we allow the distributions to be different.

Take any Bayesian incentive compatible (BIC) mechanism (q, p). Consider any agent $i \in N$ who has value v_i. Her expected payoff from reporting v_i' when her type is v_i is (given that others are truthfully reporting types)

$$v_i \int\limits_{v_{-i}} q_i(v_i', v_{-i}) f_{-i}(v_{-i}) dv_{-i} - \int\limits_{v_{-i}} p_i(v_i', v_{-i}) f_{-i}(v_{-i}) dv_{-i}$$

To make this notationally simple, we will introduce two notations,

$$Q_i(v_i') = \int\limits_{v_{-i}} q_i(v_i', v_{-i}) f_{-i}(v_{-i}) dv_{-i}$$
$$P_i(v_i') = \int\limits_{v_{-i}} p_i(v_i', v_{-i}) f_{-i}(v_{-i}) dv_{-i}$$

So, $Q_i(v_i')$ is the *interim* allocation probability of winning the object for agent i when she reports v_i'. Similarly, $P_i(v_i')$ is the *interim* payment made by agent i when she reports v_i'. This is calculated by integrating out (taking expectation over) v_{-i} of the *ex-post* allocation probability and payment terms.

Hence, the BIC constraints, can be written succinctly as

$$v_i Q_i(v_i) - P_i(v_i) \geq v_i Q_i(v_i') - P_i(v_i')$$

In other words, a mechanism (q, p) *is Bayesian incentive compatible* if for every $i \in N$ and every $v_i, v_i' \in \mathcal{D}_i$ we have

$$v_i Q_i(v_i) - P_i(v_i) \geq v_i Q_i(v_i') - P_i(v_i')$$

Given a mechanism (q,p), we can define the *interim utility* of each agent i from the mechanism by a function $u_i : \mathcal{D}_i \to \mathbb{R}$ as follows:

$$u_i(v_i) = v_i Q_i(v_i) - P_i(v_i) \qquad \forall\ v_i \in \mathcal{D}_i$$

Note that for any v_i, v_i',

$$v_i Q_i(v_i') - P_i(v_i') = (v_i - v_i')Q_i(v_i') + v_i' Q_i(v_i') - P_i(v_i') = u_i(v_i') + (v_i - v_i')Q_i(v_i')$$

Hence, a mechanism (q,p) *is Bayesian incentive compatible* if for every $i \in N$ and every $v_i, v_i' \in \mathcal{D}_i$ we have

$$u_i(v_i) \geq u_i(v_i') + (v_i - v_i')Q_i(v_i')$$

Theorem **16 (Myerson (1981))** *A mechanism (q,p) is Bayesian incentive compatible if and only if for each $i \in N$*

1. *Q_i is monotone, i.e., $Q_i(v_i) \geq Q_i(v_i')$ for all $v_i > v_i'$*
2. *$u_i(v_i) = u_i(0) + \int_0^{v_i} Q_i(x)dx$ for all $v_i \in [0, a_i]$*

Before proceeding with the proof, we point out that $u_i(v_i) = v_i Q_i(v_i) - P_i(v_i)$ for any v_i. Hence, a simple substitution reveals that (2) in the theorem can be alternatively written as

$$P_i(v_i) = P_i(0) + v_i Q_i(v_i) - \int_0^{v_i} Q_i(x)dx \qquad \forall\ v_i \in [0, a_i] \tag{24}$$

Proof: Suppose (q,p) is Bayesian incentive compatible. Then, for any $v_i > v_i'$, the two IC constraints (one where v_i type does not manipulate to v_i' and the other where v_i' type does not manipulate to v_i) must hold:

$$u_i(v_i) \geq u_i(v_i') + (v_i - v_i')Q_i(v_i')$$
$$u_i(v_i') \geq u_i(v_i) + (v_i' - v_i)Q_i(v_i)$$

Adding these two IC constraints give us $(v_i - v_i')(Q_i(v_i) - Q_i(v_i')) \geq 0$. Since $v_i > v_i'$, we get

$Q_i(v_i) \geq Q_i(v_i')$. This proves necessity of (1).

For necessity of (2), we first show that u_i is a convex function. Take any $v_i, v_i' \in [0, a_i]$ and suppose $v_i'' = \lambda v_i + (1-\lambda)v_i'$ where $\lambda \in (0,1)$. Then, IC constraints $v_i \to v_i''$ (i.e., v_i type not reporting v_i'') and $v_i' \to v_i''$ give us:

$$\lambda u_i(v_i) \geq \lambda u_i(v_i'') + \lambda(v_i - v_i'')Q_i(v_i'')$$
$$(1-\lambda)u_i(v_i') \geq (1-\lambda)u_i(v_i'') + (1-\lambda)(v_i' - v_i'')Q_i(v_i'')$$

Adding gives the necessary convexity constraint:

$$\lambda u_i(v_i) + (1-\lambda)u_i(v_i') \geq u_i(v_i'')$$

A convex function need not be differentiable everywhere (for instance, a convex function consisting of two line segments with different slopes will not be differentiable at the point of intersection of the line segments), but it is differentiable almost everywhere. That is, the set of points where a convex function is not differentiable has zero measure.

Indeed if u_i is differentiable at v_i in the interior of $[0, a_i]$, then we can pick h arbitrarily close to zero and write the IC constraint $v_i + h \to v_i$:

$$u_i(v_i + h) \geq u_i(v_i) + hQ_i(v_i)$$

Hence, we have

$$\frac{u_i(v_i + h) - u_i(v_i)}{h} \geq Q_i(v_i)$$

Taking $h \to 0$, we get

$$\frac{du_i(v_i)}{dv_i} \geq Q_i(v_i) \tag{25}$$

Next, consider the IC constraint $(v_i - h) \to v_i$:

$$u_i(v_i - h) \geq u_i(v_i) - hQ_i(v_i)$$

Hence, we have

$$\frac{u_i(v_i) - u_i(v_i - h)}{h} \le Q_i(v_i)$$

Taking $h \to 0$, we get

$$\frac{du_i(v_i)}{dv_i} \le Q_i(v_i) \tag{26}$$

Combining (25) and (26), we get for almost all $v_i \in (0, a_i)$, we have

$$\frac{du_i(v_i)}{dv_i} = Q_i(v_i)$$

By the fundamental theorem of calculus,

$$u_i(v_i) = u_i(0) + \int_0^{v_i} Q_i(x)dx \qquad \forall\ v_i \in [0, a_i]$$

This completes one direction of the proof.

For the other direction, suppose (q, p) satisfies (1) and (2). Take $v_i, v_i' \in [0, a_i]$ and note that

$$u_i(v_i) - u_i(v_i') = \int_{v_i'}^{v_i} Q_i(x)dx \ge (v_i - v_i')Q_i(v_i'),$$

where the equality follows from (2) and inequality follows from (1), i.e., monotonicity of Q_i. Hence, every IC constraint $v_i \to v_i'$ holds. So, (q, p) is Bayesian incentive compatible. ■

Implications. One crucial implication of Theorem 16 is the usual revenue equivalence result in auction theory (Theorem 5). Take any two auction formats which in equilibrium satisfy the following two conditions:

1. They allocate the object to the *highest valued bidder*. This will happen if the two auctions have symmetric equilibria – in that case highest bidder is also the highest valued bidder. Hence, these two auctions have the same Q_i function for each bidder i. As an example, if bidders are symmetric (i.e., draw values from same distribution),

then the allocation rules in the first-price and the second-price auction are the same.

2. Further, if two auction formats are such that the utility of the lowest type (zero type) is the same. This also happens in all the standard auctions. This assumption is not true if there are two auction formats, and one of them charges entry fee and the other one does not.

If these two assumptions hold, then Theorem 16 (in particular, (24)) says that the expected payment of every bidder in these two auction formats are the same. This result generalizes Theorem 5 since it holds for mechanisms where equilibrium need not allocate efficiently. This also explains why in asymmetric environment, first-price and second-price are not revenue equivalent. The first-price auction need not allocate the object efficiently in equilibrium (Theorem 15) but the second-price auction continues to allocate the object efficiently. This leads to different Q_i functions in the two auction formats. As a result, the expected revenue is different.

10.4 Characterization of dominant strategy incentive compatibility

A characterization similar to Theorem 16 is possible for dominant strategy incentive compatible (DSIC) direct mechanisms. The only difference is we will have ex-post version (instead of interim version) of monotonicity and payoff equivalence.

Note that the DSIC constraints, can be written succinctly as: for all i, for all v_{-i}, and for all v_i, v_i'

$$v_i q_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i q_i(v_i', v_{-i}) - p_i(v_i', v_{-i})$$

Given a mechanism (q, p), we can define the *ex-post utility* of each agent i from the mechanism by a function $U_i : \mathcal{D} \to \mathbb{R}$ as follows:

$$U_i(v_i, v_{-i}) = v_i q_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \qquad \forall\ (v_i, v_{-i}) \in \mathcal{D}$$

Note that for any v_i, v_i',

$$v_i q_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) = U_i(v_i, v_{-i}) \geq v_i q_i(v_i', v_{-i}) - p_i(v_i', v_{-i}) = U_i(v_i', v_{-i}) + (v_i - v_i') q_i(v_i', v_{-i})$$

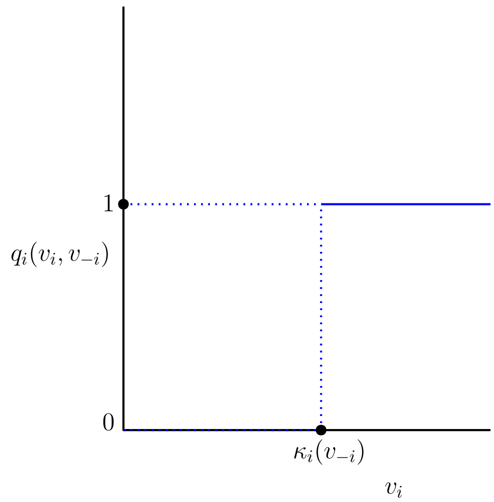

Figure 5: Step function

Hence, a mechanism (q, p) *is DSIC* if for every $i \in N$, for every v_{-i} and every $v_i, v_i' \in \mathcal{D}_i$ we have

$$U_i(v_i, v_{-i}) \geq U_i(v_i', v_{-i}) + (v_i - v_i')q_i(v_i', v_{-i})$$

So, the analogue of Theorem 16 is as follows – we skip the proof, which is almost identical to Theorem 17.

THEOREM **17 (Myerson (1981))** *A mechanism (q, p) is dominant strategy incentive compatible if and only if for each $i \in N$*

1. *q_i is monotone, i.e., $q_i(v_i, v_{-i}) \geq q_i(v_i', v_{-i})$ for all $v_i > v_i'$ and for all v_{-i}*
2. *$U_i(v_i, v_{-i}) = U_i(0, v_{-i}) + \int_0^{v_i} q_i(x, v_{-i})dx$ for all $v_i \in [0, a_i]$ and for all v_{-i}*

10.4.1 Deterministic mechanisms

A mechanism (q, p) is *deterministic* if $q_i(v) \in \{0, 1\}$ for all v. First-price and second-price auctions (with deterministic tie-breaking) are deterministic. If q is deterministic, monotonicity means it has to be a *step function* (as in Figure 5).

For every i and every v_{-i}, $q_i(\cdot, v_{-i})$ is zero below some *cutoff* $\kappa_i(v_{-i})$ and 1 above it.

Suppose $p_i(0, v_{-i}) = 0$ in a mechanism (zero type always pays zero) – this may not hold if there is an *entry fee*. Using DSIC characterization

$$p_i(v_i, v_{-i}) = p_i(0, v_{-i}) + v_i q_i(v_i, v_{-i}) - \int_0^{v_i} q_i(x, v_{-i}) dx$$

If $q_i(v_i, v_{-i}) = 0$, then $q_i(x, v_{-i}) = 0$ for all $x < v_i$ (by monotonicity). So, $p_i(v_i, v_{-i}) = 0$. If $q_i(v_i, v_{-i}) = 1$, then $q_i(x, v_{-i}) = 1$ for all $x \in (\kappa_i(v_{-i}, v_i)$ and $q_i(x, v_{-i}) = 0$ for all $x < \kappa_i(v_{-i})$. Hence, if $q_i(v_i, v_{-i}) = 1$, then

$$p_i(v_i, v_{-i}) = v_i - \int_0^{v_i} q_i(x, v_{-i}) dx = v_i - (v_i - \kappa_i(v_{-i})) = \kappa_i(v_{-i})$$

So, we can write

$$p_i(v_i, v_{-i}) = q_i(v_i, v_{-i}) \kappa_i(v_{-i})$$

This leads to a cleaner result for deterministic mechanisms.

Theorem **18** (**Myerson** (**1981**)) *A mechanism (q, p) is deterministic dominant strategy incentive compatible if and only if for each $i \in N$*

1. *q_i is a step function, i.e., for all v_{-i}, there is a cutoff $\kappa_i(v_{-i})$ such that $q_i(v_i, v_{-i}) = 0$ if $v_i < \kappa_i(v_{-i})$ and $q_i(v_i, v_{-i}) = 1$ if $v_i > \kappa_i(v_{-i})$*

2. *$P_i(v_i, v_{-i}) = P_i(0, v_{-i}) + q_i(v_i, v_{-i}) \kappa_i(v_{-i})$ for all $v_i \in [0, a_i]$ and for all v_{-i}*

Given v_{-i}, what is $\kappa_i(v_{-i})$? The amount $\kappa_i(v_{-i})$ is the minimum i needs to bid to have $q_i(\cdot, v_{-i}) = 1$. Now, consider the second-price auction. If bidder i wins in a second-price auction than $v_i \geq \max_{j \neq i} v_j$. What is the mininum bidder i needs to bid to win? This is clearly $\max_{j \neq i} v_j$: *highest of other bidders' values* or the *second highest value*. If bidder i wins, she pays this: second-highest value/bid.

We could think of many deterministic DSIC auctions using Theorem 18. Consider the following direct mechanism with two bidders whose values are in $[0, 1]$. if bidder 1 reports

v_1 and bidder 2 reports v_2, the object is allocated as follows. Bidder 1 wins if $(v_1)^2$ is more than v_2. Else, bidder 2 wins.

We use Theorem 18 to compute payments. First, given v_2, the bidder 1 wins as long as her report v_1 is more than $\sqrt{v_2}$. So, this is a step function as required in (1) of Theorem 18. Similarly, for bidder 2: given v_1, bidder 2 wins as long as her report is more than $(v_1)^2$, again a step function.

For payment, at any type profile (v_1, v_2), we consider two cases.

1. Bidder 1 wins: $v_1 > \sqrt{v_2}$. In that case, $\kappa_1(v_2) = \sqrt{v_2}$ – this is the minimum value bidder 1 needs to have to win against bidder 2 with value v_2. So bidder 1 pays $\sqrt{v_2}$.

2. Bidder 2 wins: $v_1 \leq \sqrt{v_2}$. In that case, $\kappa_2(v_1) = (v_1)^2$ – this is the minimum value bidder 2 needs to have to win against bidder 1 with value v_2. So bidder 2 pays $(v_1)^2$.

Using Theorem 18, this is a DSIC mechanism.

10.5 Participation constraints

If we are designing a mechanism, we must ensure that there is incentive for bidders to participate in the auction. The incentive to participate will depend on the *outside option* of the bidders. Here, we assume that bidders get zero utility if they do not participate in the auction. Thus, to ensure participation, bidders should be given non-negative utility (in equilibrium) in the auction. There are again two stages where such utility comparisions can be done.

1. Ex-post. The final payoff in the mechanism is non-negative. This means for every type profile, the payoff of each bidder must be non-negative in the direct mechanism. Since the direct mechanism has truth-telling equilibrium, this boils down to the following definition of participation constraint.

   Definition 9 *A mechanism (q, p) is* **ex-post individually rational (EIR)** *if for every bidder i and every type profile (v_i, v_{-i}), we have*

$$v_i q_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq 0$$

Equivalently, this says that $u_i(v_i, v_{-i}) \geq 0$ for all i and all (v_i, v_{-i}).

2. Interim. This participation cares about the interim payoff, i.e., the expected payoff from the direct mechanism (in truth-telling equilibrium) given the type of the agent.

   Definition **10** *A mechanism (q,p) is* **interim individually rational (IIR)** *if for every bidder i and every type v_i, we have*

   $$v_i Q_i(v_i) - P_i(v_i) \geq 0$$

   Equivalently, this says that $U_i(v_i) \geq 0$ for all i and all v_i

Clearly, if a mechanism is EIR, it is also IIR. The following lemma characterizes EIR and IIR through simpler constraints.

Lemma **3** *Suppose (q,p) is a DSIC mechanism. Then, it is EIR if and only if $p_i(0, v_{-i}) \leq 0$ for all $i \in N$ and for all v_{-i}.*

Suppose (q,p) is a BIC mechanism. Then, it is IIR if and only if $P_i(0) \leq 0$.

Proof: Suppose (q,p) is a BIC mechanism. Then, by Theorem 16, we know that for every i and every v_i, we have $u_i(v_i) = u_i(0) + \int_0^{v_i} Q_i(x)dx \geq u_i(0)$. Hence, if $u_i(0) \geq 0$ ensures $u_i(v_i) \geq 0$ for all v_i. Of course, $u_i(v_i) \geq 0$ for all v_i implies $u_i(0) \geq 0$. Hence, IIR is equivalent to requiring for all $i \in N$, we have $u_i(0) \geq 0$. But $u_i(0) = -P_i(0)$ implies IIR is equivalent to requiring for all $i \in N$, we have $P_i(0) \leq 0$.

The proof for the DSIC mechanism and EIR is similar (using Theorem 17). ■

10.6 Optimal auction design

In this section, we will be concerned with designing a direct mechanism which maximizes the expected revenue of the seller under incentive and participation constraints. More precisely, this is how we define an optimal mechanism.

Definition 11 *A BIC and IIR mechanism (q,p) is* **optimal** *if for every other BIC and IIR mechanism (q',p'),*

$$\int_v \Big[\sum_{i=1}^n p_i(v)\Big] f(v)dv \geq \int_v \Big[\sum_{i=1}^n p'_i(v)\Big] f(v)dv$$

An equivalent way of writing down the expected revenue expression is through interim payments:

$$\sum_{i=1}^n \Big[\int_0^{a_i} P_i(v_i)f_i(v_i)dv_i\Big] \geq \sum_{i=1}^n \Big[\int_0^{a_i} P'_i(v_i)f_i(v_i)dv_i\Big]$$

Our first result says that every optimal mechanism must maximize the *expected virtual* values of agents. The virtual value of agent i with value x is defined as:

$$\psi_i(x) = x - \frac{1-F_i(x)}{f_i(x)}$$

The virtual value is useful in deriving a simple expression for expected revenue. For an arbitrary BIC and IIR mechanism $(\hat{q},\hat{p})$, we use the characterization in Theorem 16) to write an expression for expected revenue. The expected payment of bidder i of type x is

$$\hat{P}_i(x) = \hat{P}_i(0) + x\hat{Q}_i(x) - \int_0^x \hat{Q}_i(z)dz$$

Hence, expected payment of bidder i to the seller is

$$\begin{aligned}
\int_0^{a_i} \hat{P}_i(x)f_i(x)dx &= \hat{P}_i(0) + \int_0^{a_i} x\hat{Q}_i(x)f_i(x)dx - \int_0^{a_i}\Big[\int_0^x \hat{Q}_i(z)dz\Big]f_i(x)dx \\
&= \hat{P}_i(0) + \int_0^{a_i} x\hat{Q}_i(x)f_i(x)dx - \int_0^{a_i}\int_x^{a_i} f_i(z)dz\hat{Q}_i(x)dx \\
&= \hat{P}_i(0) + \int_0^{a_i} x\hat{Q}_i(x)f_i(x)dx - \int_0^{a_i}(1-F_i(x))\hat{Q}_i(x)dx \\
&= \hat{P}_i(0) + \int_0^{a_i}\Big[x - \frac{1-F_i(x)}{f_i(x)}\Big]\hat{Q}_i(x)f_i(x)dx \\
&= \hat{P}_i(0) + \int_0^{a_i} \psi_i(x)\hat{Q}_i(x)f_i(x)dx,
\end{aligned}$$

where the second equality follows by changing the order of integration. Hence, the expected revenue of any BIC and IIR mechanism is (sum of expected payments of all bidders):

$$\begin{aligned}
&\sum_{i=1}^{n}\hat{P}_i(0)+\sum_{i=1}^{n}\int_0^{a_i}\psi_i(x)\hat{Q}_i(x)f_i(x)dx\\
&=\sum_{i=1}^{n}\hat{P}_i(0)+\sum_{i=1}^{n}\int_0^{a_i}\psi_i(x)\Big(\int_{v_{-i}}\hat{q}_i(x,v_{-i})f_{-i}(v_{-i})\Big)f_i(x)dx\\
&=\sum_{i=1}^{n}\hat{P}_i(0)+\sum_{i=1}^{n}\int_0^{a_i}\int_{v_{-i}}\psi_i(v_i)\hat{q}_i(v_i,v_{-i})f_{-i}(v_{-i})f_i(v_i)dv_i\\
&=\sum_{i=1}^{n}\hat{P}_i(0)+\sum_{i=1}^{n}\int_v\psi_i(v_i)\hat{q}_i(v)f(v)dv\\
&=\sum_{i=1}^{n}\hat{P}_i(0)+\int_v\sum_{i=1}^{n}\Big[\psi_i(v_i)\hat{q}_i(v)\Big]f(v)dv && (27)
\end{aligned}$$

Since this mechanism is IIR, by Lemma 3, $\hat{P}_i(0)\leq 0$ for all $i\in N$.

THEOREM **19** *Suppose (q,p) is a BIC and IIR mechanism. Then, (q,p) is an optimal mechanism if and only if for every BIC and IIR mechanism (q',p'), we have*

$$\int_v\sum_{i=1}^{n}\Big[\psi_i(v_i)q_i(v)\Big]f(v)dv\geq\int_v\sum_{i=1}^{n}\Big[\psi_i(v_i)q'_i(v)\Big]f(v)dv \quad (28)$$

$$P_i(0)=0 \qquad \forall\ i\in N \quad (29)$$

Proof: *Necessary direction.* If (q,p) is an optimal mechanism, we must have $P_i(0)=0$ for all $i\in N$ – if not, we can construct another BIC and IIR mechanism with (q,p') (same allocation rule but different p') such that $P'_i(0)=0$ and $P'_i(v_i)$ is given by the revenue equivalence formula (by Theorem 16 such a mechanism is BIC and IIR). By Equation (27), this mechanism generates more revenue since $P_i(0)\leq 0$ for all i. This means that the expected revenue of the optimal mechanism (q,p) is

$$\int_v\sum_{i=1}^{n}\Big[\psi_i(v_i)q_i(v)\Big]f(v)dv$$

Since (q,p) is optimal, its expected revenue is greater than the expected revenue of any BIC and IIR mechanism (q',p') where $P'_i(0)=0$. The expected revenue from such a mechanism is

$$\int_v \sum_{i=1}^{n} \Big[\psi_i(v_i)q_i'(v)\Big] f(v)dv.$$

By optimality of (q,p), we get

$$\int_v \sum_{i=1}^{n} \Big[\psi_i(v_i)q_i(v)\Big] f(v)dv \geq \int_v \sum_{i=1}^{n} \Big[\psi_i(v_i)q_i'(v)\Big] f(v)dv$$

Sufficient direction. Suppose (q,p) is a BIC and IIR mechanism satisfying (28) and (29). The expected revenue from any BIC and IIR mechanism (q',p') is

$$\begin{aligned}\sum_{i=1}^{n} P_i'(0) + \int_v \sum_{i=1}^{n} \Big[\psi_i(v_i)q_i'(v)\Big] f(v)dv &\leq \int_v \sum_{i=1}^{n} \Big[\psi_i(v_i)q_i'(v)\Big] f(v)dv \\ &\leq \int_v \sum_{i=1}^{n} \Big[\psi_i(v_i)q_i(v)\Big] f(v)dv \\ &= \sum_{i=1}^{n} P_i(0) + \int_v \sum_{i=1}^{n} \Big[\psi_i(v_i)q_i(v)\Big] f(v)dv\end{aligned}$$

where the first inequality is due to the fact that (q',p') is IIR and $P_i'(0) \leq 0$ for all i and the last inequality and equality follow from (28) and (29) respectively. So, (q,p) generates more expected revenue. ■

We make the following assumption on virtual values (assumed in Theorem 8).

Definition 12 *A distribution F_i of bidder i is* **regular** *if the virtual value function is strictly increasing, i.e., for all $v_i' > v_i$, we have $\psi_i(v_i') > \psi_i(v_i)$.*

Note that if $\frac{f_i(x)}{1-F_i(x)}$ is increasing, then ψ is strictly increasing. Further, $\frac{f_i(x)}{1-F_i(x)}$ is called the *hazard rate* of distribution F_i at x. Hence, hazard rate increasingness implies regularity. Several well-known distributions satisfy hazard rate monotonicity: uniform, exponential. For uniform, $F_i(x) = \frac{x}{a_i}$, and hence, hazard rate is $\frac{1}{a_i - x}$, which is clearly increasing in x.

The main result of this section is the following.

Theorem **20** (**Myerson** (**1981**)) *Suppose distributions of all bidders are regular. Then,* *there is an optimal mechanism* (q,p) *such that for all* v *and for all* $i \in N$

$$q_i(v) = \begin{cases} 1 & \text{if } \psi_i(v_i) > \max_{j \neq i} \psi_j(v_j) \text{ and } \psi_i(v_i) \geq 0 \\ 0 & \text{if } \psi_i(v_i) < \max_{j \neq i} \psi_j(v_j) \text{ or } \psi_i(v_i) < 0 \end{cases}$$

In words, theorem is saying that an optimal mechanism must allocate the object to the bidder with the highest non-negative virtual value – in case of ties, it can be allocated to any highest non-negative virtual value agent. The proof follows immediately from Theorem 19.

Proof: From Theorem 19, an optimal mechanism must maximize the expression

$$\int_v \sum_{i=1}^n \Big[\psi_i(v_i)q_i(v)\Big] f(v)dv$$

over all BIC and IIR mechanisms (q,p). If we forget the fact that we maximize over BIC and IIR mechanisms, and just maximize the expression $\int_v \sum_{i=1}^n \Big[\psi_i(v_i)q_i(v)\Big] f(v)dv$, then we can do so by *point-wise* maximizing it. That is, for each v, we maximize the expression $\sum_{i=1}^n \Big[\psi_i(v_i)q_i(v)\Big]$. This can be maximized by choosing a (q,p) such that $q_i(v)$ is 1 whenever $\psi_i(v) \geq 0$ and $\psi_i(v_i) \geq \max_{j \neq i} \psi_j(v_j)$ and zero otherwise.

This defines an optimal solution to $\int_v \sum_{i=1}^n \Big[\psi_i(v_i)q_i(v)\Big] f(v)dv$. But are the ignored BIC and IIR constraints satisfied by this mechanism? We invoke Theorem 16. For this, we check monotonicity of q_i. For this fix v_{-i}, and $v_i' > v_i$. If $q_i(v_i, v_{-i}) = 1$, then $\psi_i(v_i) \geq \max(\max_{j \neq i} \psi_j(v_j), 0)$. By regularity, $\psi_i(v_i') > \psi_i(v_i) \geq \max(\max_{j \neq i} \psi_j(v_j), 0)$. Hence, $q_i(v_i', v_{-i}) = 1$ by the definition of q. So, q_i is monotone in the sense of Theorem 17. Indeed, this is a step function. To satisfy DSIC characterization of Theorem 17, we choose $p_i(0, v_{-i}) = 0$ (satisfies EIR) and choose payment according to $\kappa_i(v_{-i})$: minimum value at which virtual value crosses zero and exceeds the virtual value of others.

Hence, the chosen mechanism is a **deterministic DSIC mechanism**. satisfying EIR. Thus, there is an optimal mechanism satisfying the claim of the theorem. ■

This completes the description of the optimal mechanism: it is a DSIC mechanism which allocates the object to the agent with highest non-negative virtual value. In particular, if there is no agent with non-negative virtual value, the object is unassigned.

10.6.1 Symmetric bidders

If there are symmetric bidders, then they have identical type space: $[0, a]$ and values are distributed identically: $\hat{F}$ with density $\hat{f}$. In that case, all the bidders have identical virtual value *functions*, i.e., for all $x \in [0, a]$, virtual value of any bidder with type x is

$$\psi(x) = x - \frac{1 - \hat{F}(x)}{\hat{f}(x)}$$

Suppose the distribution is regular. At a type profile $v \equiv (v_1, \ldots, v_n)$ if $v_i > v_j$ then

$$\psi(v_i) = v_i - \frac{1 - \hat{F}(v_i)}{\hat{f}(v_i)} > v_j - \frac{1 - \hat{F}(v_j)}{\hat{f}(v_j)} = \psi(v_j)$$

where the inequality follows from regularity (virtual value function is strictly increasing). Hence, with symmetric distribution, we see that $v_i > v_j$ if and only if $\psi(v_i) > \psi(v_j)$. Using Theorem 20, the object goes to the *highest valued bidder* who has non-negative virtual value. When does a bidder have non-negative virtual value, i.e., for what x does $\psi(x) = 0$. Since ψ is strictly increasing, there is a unique value $\psi^{-1}(0)$ at which virtual value becomes zero. This means if the highest bidder has value more than $\psi^{-1}(0)$, she gets the object; else the object is not sold. The loser pays zero and the winner pays the cutoff type when she starts winning: this will be highest of $\psi^{-1}(0)$ and max of others values. Hence, with symmetric type, the optimal auction is the second-price auction with a reserve price $\psi^{-1}(0)$ – this is also the optimal reserve price in a second-price auction (Theorem 8).

THEOREM **21** *Suppose values of bidders are indepdendently and identically distributed using a regular distribution. Then, the optimal mechanism is a second-price auction with an optimally chosen reserve price.*

For instance, if values of bidders are uniformly distributed in $[0, 1]$, then $\psi^{-1}(0) = \frac{1}{2}$ (the optimal reserve price in a second-price auction does not depend on the number of bidders). Hence, the optimal auction (with any number of bidders n) is a second-price auction with reserve price $\frac{1}{2}$.

10.6.2 Examples

It is instructive to look at some particular examples where values are differently distributed. For instance, suppose $n = 2$ and bidder 1 draws its value uniformly from $[0, 1]$ and bidder 2 draws its value uniformly from $[0, 2]$.

For uniform distribution with support $[0, a]$, the virtual value function is $x - \frac{1-\frac{x}{a}}{\frac{1}{a}} = x - (a - x) = 2x - a$. Hence, $\psi_1(v_1) = 2v_1 - 1$ and $\psi_2(v_2) = 2v_2 - 2$. This means, bidder 1 will have negative virtual value below $\frac{1}{2}$ and cannot win in the optimal auction if her value is less than $\frac{1}{2}$. Similarly, bidder 2 cannot win in the optimal auction if her value is less than 1. Consider the following value profiles.

- $v_1 = 0.3, v_2 = 1.2$. Virtual value of bidder 1 is negative and bidder 2 is positive. Hence, bidder 2 wins and pays the cutoff price: the minimum she needs to bid to win if $v_1 = 0.3$ is 1.

- $v_1 = 0.9, v_2 = 1.1$. In this case $\psi_1(0.9) = 2\times0.9-1 = 0.8$ and $\psi_2(1.1) = 2\times1.1-2 = 0.2$. So, both bidders have positive virtual value but bidder 1 has a higher virtual value (even though she has a lower value than bidder 2). So, bidder 1 wins. To determine cutoff price, note that bidder 1 has to beat the virtual value of bidder 2 and have non-negative virtual value. When bidder 1 has a value of 0.6 she has a virtual value of 0.2, equal to the virtual value of bidder 2. Hence, her payment is 0.6.

- $v_1 = 0.4, v_2 = 0.8$. In this case, the virtual values of both the bidders are negative and the object is not sold.

These examples show two sources of inefficiency in the optimal auction (a) inefficiency due to the fact that the object may be not sold (even though there are bidders with positive value) and (b) inefficiency due to the fact that the object is sold to the lower valued bidder. The latter inefficiency does not arise with symmetric bidders, but may occur with asymmetric bidders.

10.6.3 The must-sell case

In many settings, the object must be sold in an auction. This is not a feature of the optimal auction: optimal auction necessarily does not sell the object if virtual values are negative.

If the object must be sold, then similar arguments to Theorem 20 reveals that allocating the object to the bidder with the highest virtual value (even if this bidder's virtual value is negative) is optimal (under regularity). Note that the bidder with the highest virtual value need not be the bidder with the highest value: if bidders are symmetric, highest virtual value is also highest value. Hence, even in the must-sell case, the optimal auction need not be an efficient auction (like a second-price auction). If bidders are symmetric, then the second-price auction is also an optimal auction.

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