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Ordinal Bayesian incentive compatibility in restricted domains [☆]

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Abstract

We study deterministic voting mechanisms by considering an ordinal notion of Bayesian incentive compatibility (OBIC). If the beliefs of agents are independent and generic, we show that a mechanism is OBIC and satisfies an additional condition called *elementary monotonicity* if and only if it is a dominant strategy incentive compatible mechanism. Our result works in a large class of preference domains (that include the unrestricted domain, the single-peaked domain, the single-dipped domain, and some single-crossing domains). We can significantly weaken elementary monotonicity in our result in the single-peaked domain if we assume unanimity and in a large class of domains if we assume unanimity and tops-onlyness.

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1. Introduction

In standard models of voting, dominant strategy incentive compatibility (DSIC) is usually too demanding. This is illustrated by the Gibbard–Satterthwaite theorem (Gibbard, 1973;

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Satterthwaite, 1975), which shows that the only DSIC and unanimous deterministic voting mechanism in the unrestricted domain is a dictatorship.¹ This motivates the study of weaker solution concepts in these models. In this paper, we consider ordinal Bayesian incentive compatibility (OBIC) introduced by d'Aspremont and Peleg (1988). A voting mechanism is OBIC if for every agent, his interim/expected outcome probability vector from truth-telling first-order stochastic-dominates any interim outcome probability vector obtained by deviating. In the unrestricted domain of preferences, Majumdar and Sen (2004) show that OBIC with independent and generic priors is equivalent to DSIC under unanimity.² We investigate the robustness of this result to the unrestricted domain assumption.

We construct a *non-DSIC*, unanimous, and *anonymous* mechanism that is OBIC with respect to some generic priors when the domain of preferences is restricted to be the single-peaked domain. However, our main results suggest that the equivalence between OBIC and DSIC voting mechanisms can be restored in various restricted domains under weak additional axioms. The main additional axioms that we use are *elementary monotonicity* and its weaker versions along with unanimity. Elementary monotonicity, which we formally define later, is a very mild form of Maskin monotonicity, and requires a mechanism to respond *positively* to changes in the preferences of agents. It is satisfied by a variety of mechanisms.

Our core result says that OBIC and elementary monotonicity are equivalent to DSIC in a large class of domains. In the single-peaked domain, the equivalence between OBIC and DSIC holds with a significantly weaker version of elementary monotonicity if we assume unanimity. If we assume unanimity and *tops-onlyness*, the weakened version of elementary monotonicity and OBIC are equivalent to DSIC in a large class of domains.

Our results provide a foundation for using dominant strategy voting mechanisms in various restricted domains if we use ordinal deterministic mechanisms.³ An implication of our results is that if we want to design Bayesian incentive compatible voting mechanisms, we must consider randomized and/or cardinal mechanisms. All our results hold even if we weaken OBIC to only prevent manipulations of each agent to his *adjacent* preferences – we call this requirement *locally* OBIC (LOBIC). Incentive compatibility with local incentive constraints were recently studied in Carroll (2012) and Sato (2013), who identified domains where local incentive constraints imply all incentive constraints. All our proofs use ideas from this literature. Thus, our results bring together two different ideas (OBIC and local incentive compatibility) in strategic voting literature.

Our results extend the result in Majumdar and Sen (2004) by identifying the precise connection between DSIC and OBIC mechanisms with (and without) unanimity in restricted domains. Also, they corroborate the different implications of OBIC with generic and uniform priors (a uniform prior requires that each preference in the domain is drawn with equal probability). This is because Majumdar and Sen (2004) had shown that every *neutral* mechanism satisfying elementary monotonicity is OBIC under uniform priors in the unrestricted domain – this covers many reasonable mechanisms. In contrast, our results show a very different implication of elementary monotonicity with generic priors.

¹ Throughout the paper, we only consider deterministic voting mechanisms.

² We define a generic prior formally later – it is a generic subset of the set of independent priors.

³ Restricting attention to ordinal and deterministic mechanisms in this setting is with loss of generality – see Borgors and Postl (2009), Borgors and Smith (2014), who discuss this issue in detail.

1.1. Relation to literature

Our results parallel recent contributions in the single object auction quasilinear utility models (and some of its extensions) by [Manelli and Vincent \(2010\)](#) and [Gershkov et al. \(2013\)](#), who establish a weaker version of equivalence between Bayesian incentive compatible and DSIC mechanisms. The equivalence in these papers is in terms of interim outcome probabilities. Besides, these papers look at cardinal mechanisms with quasilinearity, whereas we consider ordinal mechanisms without transfer.

One way to interpret our results is that we are replacing DSIC by weaker axioms – LOBIC and some additional axioms. In a recent paper, such an approach of decomposing the DSIC axiom is pursued in [Muto and Sato \(2014\)](#). They use three weaker axioms than DSIC and show that they are equivalent to DSIC in the *unrestricted* domain. Though they do not consider OBIC, some of their axioms (more precisely, some weakening of their axioms) are implied by OBIC with generic priors. Their result only applies to the unrestricted domains, while our results apply to many restricted domains.

Besides [Majumdar and Sen \(2004\)](#), [Bhargava et al. \(2015\)](#) study OBIC voting mechanisms with correlated priors in the unrestricted domain. They show how correlation allows one to escape the Gibbard–Satterthwaite impossibility result using OBIC. OBIC mechanisms have been studied in the context of matching problems in [Majumdar \(2003\)](#), [Ehlers and Masso \(2007, 2015\)](#). These papers study the implication of stability and OBIC in two-sided matching problems. Though some of our results extend to private good allocation problems, most of our results are specific to voting environment.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 formally states the unrestricted domain result and gives two counterexamples to show that the result breaks down in a restricted domain or if the number alternatives is two. Section 4 formally defines all the domains we use in the paper. Section 5 contains all the results and Section 5.5 gives specific examples of domains where our results can be applied. We end with some discussions in Section 6.

2. The model

We formally introduce our model in this section. Let A be a finite set of alternatives and \mathcal{P} be the set of all strict linear orders over A – \mathcal{P} will be referred to as the unrestricted domain of preferences. Let $\mathcal{D} \subseteq \mathcal{P}$ be some subset of strict linear orders. We will refer to \mathcal{D} as the **domain**. There are n agents. The set of agents is denoted by $N = \{1, \dots, n\}$. The private preference of each agent $i \in N$ is a strict linear order $P_i \in \mathcal{D}$.

A **social choice function** (scf) is a map $f : \mathcal{D}^n \rightarrow A$.

Definition 1. An scf $f : \mathcal{D}^n \rightarrow A$ is **dominant strategy incentive compatible (DSIC)** if for every $i \in N$, every P_{-i} , and every $P_i \in \mathcal{D}$, there exists no $P'_i \in \mathcal{D}$ such that

$$f(P'_i, P_{-i}) P_i f(P_i, P_{-i}).$$

We now introduce the notion of ordinal Bayesian incentive compatibility. We first introduce the assumptions that we make about the priors in our model. We will assume that there are common beliefs that each agent i **independently** draws his preference using a probability distribution $\mu_i : \mathcal{D} \rightarrow [0, 1]$. Hence, the belief of agent i that agents other than i have a preference profile P_{-i} will be denoted as $\mu(P_{-i}) \equiv \times_{j \neq i} \mu_j(P_j)$. We will refer to $\{\mu_i\}_{i \in N}$ as a profile of priors.

We make the following assumption about generic priors in the paper.

Definition 2. The profile of priors $\{\mu_i\}_{i \in N}$ is **generic** if for every $j \in N$ and for every $\mathcal{S}, \mathcal{T} \subseteq \mathcal{D}^{n-1}$ we have

$$\left[\sum_{P_{-j} \in \mathcal{S}} \mu(P_{-j}) = \sum_{P_{-j} \in \mathcal{T}} \mu(P_{-j}) \right] \Rightarrow [\mathcal{S} = \mathcal{T}].$$

Genericity requires that if we consider an agent j and consider two distinct subsets of profiles of preferences of agents in $N \setminus \{j\}$, then the probability that agents in $N \setminus \{j\}$ have preferences in these subsets cannot be the same. Mathematically, these priors are generic in a topological sense (Majumdar and Sen, 2004) – to be precise, Majumdar and Sen (2004) show this fact when $\mathcal{D} = \mathcal{P}$, but an identical proof works if $\mathcal{D} \subseteq \mathcal{P}$.⁴ One notable prior that is not generic is the uniform prior that assigns the same probability to all the preferences.

Given a social choice function f , we can compute the interim outcome probability of each agent from this scf using the priors. For this, consider a profile of priors $\{\mu_i\}_{i \in N}$. For each agent $i \in N$, define $\pi_i^f(a, P_i)$ as the interim outcome probability of the scf f choosing alternative a when agent i reports P_i as his preference (and other agents report truthfully):

$$\pi_i^f(a, P_i) = \sum_{P_{-i} \in \mathcal{D}^{n-1}: f(P_i, P_{-i})=a} \mu(P_{-i}).$$

Note that π_i^f depends on the priors, but we have suppressed it from the notation for simplicity.

For any alternative $a \in A$ and any $P_i \in \mathcal{D}$, let $B(a, P_i) := \{a\} \cup \{b \in A : b P_i a\}$.

Definition 3. (d'Aspremont and Peleg, 1988) An scf f is **ordinally Bayesian incentive compatible (OBIC)** with respect to profile of priors $\{\mu_i\}_{i \in N}$ if for every $i \in N$, for every $P_i, P'_i \in \mathcal{D}$, and for every $a \in A$, we have

$$\sum_{b \in B(a, P_i)} \pi_i^f(b, P_i) \geq \sum_{b \in B(a, P'_i)} \pi_i^f(b, P'_i).$$

An equivalent definition of OBIC is to require that for every $i \in N$, for every $P_i, P'_i \in \mathcal{D}$, for every utility function $u : A \rightarrow \mathbb{R}$ representing P_i , we have

$$\sum_{a \in A} u(a) \pi_i^f(a, P_i) \geq \sum_{a \in A} u(a) \pi_i^f(a, P'_i).$$

A well known fact to note is that if an scf is OBIC with respect to *all* beliefs, then it is DSIC. We also want to point out that OBIC is a solution concept for *ordinal* scfs. In principle, the private information (or *type*) of an agent is a vNM-utility function. But we restrict attention to scfs that only elicit ordinal preference over alternatives. In other words, OBIC requires robustness with respect to cardinal representation of ordinal preferences. Hence, the incentive constraints in OBIC have to make sure that an agent does not manipulate irrespective of which cardinal utility function represents his ordinal preferences.

⁴ Denote by Δ the unit simplex of dimension $|\mathcal{D}| - 1$. The set of common independent beliefs is the n -th order Cartesian product of unit simplices Δ , and is given by Δ^n . The set of priors ruled out by genericity are given by equations that define a finite set of hyperplanes in Δ^n , which has Lebesgue measure zero in Δ^n .

Our approach of modeling a type does not have any concrete decision-theoretic foundation. One way to think about it is that agents find it difficult to think about their preferences over lotteries, and the designer exogenously imposes a restriction on mechanisms where agents only state their ordinal preferences over alternatives.

Throughout, we also restrict attention to OBIC with respect to generic priors.

Definition 4. An scf f is **G-OBIC** if there exists a profile of generic priors $\{\mu_i\}_{i \in N}$ such that f is OBIC with respect to $\{\mu_i\}_{i \in N}$.

G-OBIC requires OBIC with respect to *some* profile of generic priors (it may be just one profile of generic priors), but need *not be all* profile of generic priors.

3. The unrestricted domain result

We now discuss the implication of G-OBIC along with unanimity in the unrestricted domain. At any preference ordering P , we will denote by $P(k)$ the k -th ranked alternative according to P .

Definition 5. An scf $f : \mathcal{D}^n \rightarrow A$ is **unanimous** if for every $\mathbf{P} \equiv (P_1, \dots, P_n)$ with $P_1(1) = \dots = P_n(1)$, we have $f(\mathbf{P}) = P_1(1)$.

The following result extends the Gibbard–Satterthwaite theorem using G-OBIC in the unrestricted domain.

Theorem 1. ([Majumdar and Sen, 2004](#)) *Let $|A| \geq 3$ and $f : \mathcal{P}^n \rightarrow A$ be a unanimous scf, where \mathcal{P} is the unrestricted domain. Then, f is G-OBIC if and only if it is DSIC.*

Instead of showing that every unanimous and G-OBIC scf is DSIC, and then using the Gibbard–Satterthwaite theorem to conclude dictatorship, [Majumdar and Sen \(2004\)](#) directly prove that G-OBIC and unanimity imply dictatorship. Hence, one does not obtain any intuition from their proof whether [Theorem 1](#) will hold in other domains.

3.1. A two alternatives example

We give an example with two alternatives to show how [Theorem 1](#) can break down. The example gives insights on why G-OBIC and non-DSIC mechanisms may exist when the assumptions of [Theorem 1](#) are relaxed.

Let $A = \{a, b\}$ and $N = \{1, 2, 3\}$. We define an scf \bar{f} as follows. If all the agents have the same top ranked alternative, then \bar{f} picks that alternative. Else, \bar{f} picks the alternative which is top ranked for less number of agents.

Clearly, \bar{f} is unanimous. We next argue that \bar{f} is G-OBIC. Since there are only two alternatives, we only need to show that $\pi_i^{\bar{f}}(P_i(1), P_i) \geq \pi_i^{\bar{f}}(P_i(1), P'_i)$ for all $i \in N$, for all P_i, P'_i . For simplicity, for every agent $i \in N$, we will denote the preference ordering where a is top ranked as P_i and the preference ordering where b is top ranked as P'_i . For every $i \in N$, let the probability that agent i has preference P_i be p_i and the probability that he has preference P'_i be $(1 - p_i)$. Now, we can compute the interim outcome probabilities for agent 1 as follows.

If agent 1 has preference P_1 (where a is top ranked), then the probability that b will be the outcome in \bar{f} is the probability that agents 2 and 3 have different preferences. This probability is

Table 1
Interim outcome probabilities of agent 1 in \bar{f} .

	P_1	P'_1
$\pi_1^{\bar{f}}(a, \cdot)$	$1 - p_2 - p_3 + 2p_2p_3$	$p_2 + p_3 - 2p_2p_3$
$\pi_1^{\bar{f}}(b, \cdot)$	$p_2 + p_3 - 2p_2p_3$	$1 - p_2 - p_3 + 2p_2p_3$

exactly $p_2(1 - p_3) + p_3(1 - p_2) = p_2 + p_3 - 2p_2p_3$. Hence, if agent 1 has preference P_1 , then the probability that a will be the outcome in \bar{f} is $1 - p_2 - p_3 + 2p_2p_3$. An analogous calculation can be used to compute interim outcome probabilities when agent 1 has preference P'_1 . This is summarized in Table 1.

The interim outcome probabilities of agents 2 and 3 can be computed in an analogous manner. It is easily seen from Table 1 that OBIC constraints can be satisfied for agent 1 if and only if $1 - p_2 - p_3 + 2p_2p_3 \geq p_2 + p_3 - 2p_2p_3$, which is equivalent to requiring that $(1 - 2p_2)(1 - 2p_3) \geq 0$.

Collecting the OBIC constraints for all the agents, we can then conclude that \bar{f} is OBIC if and only if priors satisfy

$$(1 - 2p_2)(1 - 2p_3) \geq 0$$

$$(1 - 2p_1)(1 - 2p_3) \geq 0$$

$$(1 - 2p_1)(1 - 2p_2) \geq 0$$

This is possible if and only if either $p_1, p_2, p_3 \in (0, 0.5)$ or $p_1, p_2, p_3 \in (0.5, 1)$. To see why p_1, p_2, p_3 can be picked such that the priors become generic, note that the set of priors satisfying either $p_1, p_2, p_3 \in (0, 0.5)$ or $p_1, p_2, p_3 \in (0.5, 1)$ is a subset of $(0, 1)^3$ with a non-empty interior. Hence, it will have a non-empty intersection with the set of generic priors.⁵

Intuitively, if agent 1 has a as top, then the only profiles where agent 1 gets a as outcome are those where the other two agents have the same preference. So, if the probability that any pair of agents have the same preference is high enough, then interim outcome probability of a will be high, and the scf will be OBIC.

3.2. An example in the single-peaked domain

We now extend the two alternatives example to the single-peaked domain. Single peaked domain is an important domain restriction in strategic voting literature with applications in political economy and other disciplines. It is a domain where existence of anonymous, unanimous, and DSIC scfs is guaranteed (Moulin, 1980), and thus, allows one to escape the negative consequences of the Gibbard–Satterthwaite theorem.⁶ In this domain, we give an example of an scf with three alternatives and three agents that is G-OBIC, unanimous, and *anonymous* but not DSIC.⁷

⁵ For instance, one can verify that $p_1 = 0.49, p_2 = 0.47, p_3 = 0.43$ results in generic priors.

⁶ Conversely, Chatterji et al. (2013) have shown that any domain that admits a “well-behaved” social choice function must be roughly single-peaked.

⁷ Informally, anonymity requires that if we permute the preferences of agents and consider the new profile of preferences, the outcome at the new profile must be the same as the old one.

Table 2
The single-peaked domain.

a	b	b	c
b	a	c	b
c	c	a	a

The single-peaked domain is defined as follows. Let \succ be a strict linear order of the set of alternatives A .

Definition 6. A preference ordering P is single-peaked with respect to \succ if for every $b, c \in A$ with $P(1) \succ b \succ c$ or $c \succ b \succ P(1)$, we have bPc .

We assume that $A = \{a, b, c\}$ and $N = \{1, 2, 3\}$. Suppose the preferences are single-peaked with respect to the strict linear order \succ given by $a \succ b \succ c$. The set of all single-peaked preference orderings with respect to \succ is denoted by \mathcal{S} . For this example, the domain \mathcal{S} is shown in Table 2, where each column is a preference in \mathcal{S} .

Our scf considers the number of agents who prefer a to b and the number of agents who prefer b to a . We say a is a **loser** in $\{a, b\}$ at \mathbf{P} if $|\{i \in N : aP_i b\}| < |\{i \in N : bP_i a\}|$. Else, we say b is a loser in $\{a, b\}$ at \mathbf{P} . Now, the scf f^* is defined as follows. For any preference profile $\mathbf{P} \in \mathcal{S}^3$,

1. if all the agents have the same top ranked alternative then f^* chooses that alternative,
2. if all the agents do not have the same top ranked alternative but every agent prefers b to a , then $f^*(\mathbf{P}) = b$,
3. if both the above conditions fail, then f^* chooses the loser alternative at \mathbf{P} .

The following proposition shows that f^* is G-OBIC.

Proposition 1. *The scf f^* is unanimous, anonymous, and G-OBIC, but not DSIC.*

The proof of Proposition 1 and all subsequent proofs are given in the Appendix. The intuition for why Proposition 1 works is similar to that of the two alternatives example. In contrast to the unrestricted domain, the set of preferences in the single-peaked domain is less. This allows us to extend the two alternatives example in a natural way to this domain. However, in the unrestricted domain, there are too many manipulations to take care of, and these rules fail to be G-OBIC.

4. The local domains

We will now formally define the restricted domains that we consider in this paper. For each of these restricted domains, we show equivalence of G-OBIC and DSIC under some additional conditions.

All our results work in a class of restricted domains that we call local domains. To define these restricted domains, we first introduce the notion of local incentive compatibility. Consider an agent i and two alternatives $a, b \in A$. Suppose P_i is a preference ordering such that $P_i(k) = a$ and $P_i(k+1) = b$. Now, consider P'_i such that $P'_i(k+1) = a$, $P'_i(k) = b$, and $P'_i(j) = P_i(j)$ for all $j \notin \{k, k+1\}$. In other words, a and b are consecutively ranked in P_i , and P'_i is constructed by swapping only their positions. In this case, we say that P'_i is an (a, b) -swap of P_i . Note that

if P'_i is an (a, b) -swap of P_i , then the position of b improves from P_i to P'_i . Hence, (a, b) -swap is different from (b, a) -swap.

Definition 7. An scf f is **locally dominant strategy incentive compatible (LDSIC)** if for every $i \in N$, every $P_{-i} \in \mathcal{D}^{n-1}$, and every $P_i \in \mathcal{D}$ there exists no $P'_i \in \mathcal{D}$ such that P'_i is an (a, b) -swap of P_i for some $a, b \in A$ and

$$f(P'_i, P_{-i})P_i f(P_i, P_{-i}).$$

Local DSIC only prevents manipulations across preferences which are *swaps* of each other. Using the notion of LDSIC, we now define a class of domains.

Definition 8. A domain \mathcal{D} is a **local domain** if every LDSIC $f : \mathcal{D}^n \rightarrow A$ in that domain is also DSIC.

Many interesting domains like the unrestricted domain, the single-peaked domain, and a *successive single-crossing* domain are known to be local domains (Carroll, 2012; Sato, 2013). We formally define and discuss some of these domains in Section 5.5.

We now introduce a subclass of local domains where we have stronger results. These are a class of local domains introduced in Sato (2013). Two preferences P and P' are **adjacent** if there is $x, y \in A$ such that P' is an (x, y) -swap of P .

Definition 9. A distinct sequence of preferences $(P^0, P^1, \dots, P^k, P^{k+1})$ in \mathcal{D} is **without restoration** if

- for every $j \in \{0, 1, \dots, k\}$, P^j and P^{j+1} are adjacent,
- there exists no distinct $j, j' \in \{0, 1, \dots, k\}$ and $x, y \in A$ such that P^{j+1} is a (x, y) -swap of P^j and $P^{j'+1}$ is a (y, x) -swap of $P^{j'}$.

A domain \mathcal{D} is **connected without restoration** if for every $P, P' \in \mathcal{D}$, there exists a sequence of distinct preferences $(P = P^0, P^1, \dots, P^k, P^{k+1} = P')$ in \mathcal{D} without restoration.

The without restoration property requires that no pair of alternatives is swapped more than once along the sequence. The connected without restoration property requires the existence of at least one such sequence. Sato (2013) shows that if a domain is connected without restoration, then it is a local domain. Further, he shows that the unrestricted domain and the single-peaked domain are connected without restoration.

We remark that besides the single-peaked domain, there are other interesting domains that are connected without restoration – see Sato (2013). We identify some specific domains that are connected without restoration in Section 5.5.

For some of our results, we will impose the following richness condition.

Definition 10. A domain \mathcal{D} is **rich** if for every alternative a , there exists a preference ordering $P \in \mathcal{D}$ such that $P(1) = a$.

5. The results

We will now present our results. Our results extend [Theorem 1](#) to various local domains under additional conditions. We define and discuss the additional conditions before stating the results. Though we use additional axioms, we use the following weaker notion of incentive compatibility than G-OBIC.

Definition 11. An scf f is **locally ordinal Bayesian incentive compatible (LOBIC)** with respect to profile of priors $\{\mu_i\}_{i \in N}$ if for every $i \in N$, for every $P_i, P'_i \in \mathcal{D}$ such that P'_i and P_i are adjacent, and for every $a \in A$, we have

$$\sum_{b \in B(a, P_i)} \pi_i^f(b, P_i) \geq \sum_{b \in B(a, P'_i)} \pi_i^f(b, P'_i).$$

An scf f is **G-LOBIC** if there exists some profile of generic priors $\{\mu_i\}_{i \in N}$ such that f is LOBIC with respect to $\{\mu_i\}_{i \in N}$.

Clearly, if f is OBIC, then it is also LOBIC. In general, LOBIC is a very weak incentive compatibility requirement since it requires only a small subset of incentive constraints to hold. We do not know if LOBIC implies OBIC in a local domain – by definition, LDSIC implies DSIC in a local domain.

5.1. Equivalence in local domains

Our first result uses one of the following two axioms.

Definition 12. An scf f satisfies **elementary monotonicity** if for every $i \in N$, every $P_{-i} \in \mathcal{D}^{n-1}$, and every $P_i, P'_i \in \mathcal{D}$ such that P'_i is an (a, b) -swap of P_i for some $a, b \in A$ and $f(P_i, P_{-i}) = b$, we have $f(P'_i, P_{-i}) = b$.

The next axiom is similar in spirit to elementary monotonicity.

Definition 13. An scf f is **positively responsive** if for every agent $i \in N$, for every preference profile P_{-i} , and for every P_i, P'_i , with $f(P_i, P_{-i}) = a$ and $P'_i(1) = a$, we have $f(P'_i, P_{-i}) = a$.

Both the axioms are one-agent axioms – we fix preferences of other agents at some P_{-i} and change the preference of agent i from P_i to P'_i . These axioms require that the scf responds in a positive manner if the change in preference lifts the outcome. In case of elementary monotonicity, the outcome is lifted from P_i to P'_i by one position in a local way. In case of positive responsiveness, the outcome is lifted to the top in P'_i , but the ranking among other alternatives are allowed to change. In both the cases, agent i is providing positive support to the current outcome by lifting its position. In case of positive responsiveness, the support is more natural because it is lifted to the top.

These are relatively weak axioms that are satisfied by many well known social choice functions. We refer to discussions on them in [Moulin \(1983\)](#), [Majumdar and Sen \(2004\)](#). We provide a brief discussion on them in Section 6. We are now ready to state our first main result.

Theorem 2. Let $f : \mathcal{D}^n \rightarrow A$ be an scf. Consider the following statements

1. f is DSIC.
2. f is G-LOBIC and elementary monotone.
3. f is G-LOBIC and positively responsive.

Statements (1) and (2) are equivalent if \mathcal{D} is a local domain. Statements (1), (2), and (3) are equivalent if \mathcal{D} is a rich connected domain without restoration.

Note that [Theorem 2](#) does not assume unanimity. It also does not require any condition on the range of the scf. An immediate corollary of [Theorem 2](#) gives an indirect characterization of local domains.

Corollary 1. Suppose \mathcal{D} is any domain. Every scf on \mathcal{D}^n satisfying elementary monotonicity and G-LOBIC is DSIC if and only if \mathcal{D} is a local domain.

Proof. [Theorem 2](#) already establishes one direction. For the other direction, fix a domain \mathcal{D} where every scf satisfying elementary monotonicity and G-LOBIC is DSIC. By [Lemma 4](#) (see proof of [Theorem 2](#) in the Appendix), every LDSIC scf satisfies elementary monotonicity and G-LOBIC. So, every LDSIC scf is also DSIC. Hence, the domain is a local domain. \square

5.2. Equivalence in the single-peaked domain

We will now investigate subdomains where unanimity allows us to weaken elementary monotonicity (or positive responsiveness). Our strongest result comes in the single-peaked domain, where we weaken elementary monotonicity as follows.

Fix a domain \mathcal{D} . A profile of preferences $\mathbf{P} \in \mathcal{D}^n$ is a **top-2 profile** if for every $i, j \in N$, $P_i(k) = P_j(k)$ for all $k > 2$. At a top-2 profile, agents differ in their ranking of alternatives only for the top two alternatives. Further, if (P_i, P_{-i}) and (P'_i, P_{-i}) are two top-2 profiles, then P'_i is a $(P_i(1), P_i(2))$ -swap of P_i . Let $\mathcal{D}^n(2)$ be the set of all top-2 profiles in \mathcal{D} .

Definition 14. An scf $f : \mathcal{D}^n \rightarrow A$ satisfies **weak elementary monotonicity** if f restricted to $\mathcal{D}^n(2)$ satisfies elementary monotonicity.

Note that weak elementary monotonicity is also a weakening of positive responsiveness property since in the subdomain $\mathcal{D}^n(2)$, elementary monotonicity and positive responsiveness is the same. Weak elementary monotonicity applies to very specific preference profiles. First, it requires that P'_i is a $(P_i(1), P_i(2))$ -swap of P_i . Second, it requires that every agent in $N \setminus \{i\}$ must have either P_i or P'_i as his preference. A typical pair of preference profiles where weak elementary monotonicity can be applied is shown in [Table 3](#). We assume $A = \{a, b, c, d\}$ and $n = 3$. Profiles (P_1, P_2, P_3) and (P'_1, P_2, P_3) in [Table 3](#) differ in agent 1's preference and P'_1 is a (a, b) -swap of P_1 . Also, notice that $P_2 = P_1$, $P_3 = P'_1$ (hence, $P_2 \neq P_3$). Weak elementary monotonicity applies to such profiles and requires that if $f(P_1, P_2, P_3) = b$, then $f(P'_1, P_2, P_3) = b$.

As before, we denote by \mathcal{S} the set of all single-peaked preferences with respect to the strict linear order $>$. We now state our result for the single-peaked domain.

Table 3
Weak elementary monotonicity.

P_1	P_2	P_3	P'_1	P_2	P_3
a	a	b	b	a	b
b	b	a	a	b	a
c	c	c	c	c	c
d	d	d	d	d	d

Theorem 3. Suppose $f : \mathcal{S}^n \rightarrow A$ is a unanimous scf, where \mathcal{S} is the single-peaked domain with respect to $>$. Then, the following statements are equivalent.

1. f is G-LOBIC and satisfies weak elementary monotonicity.
2. f is DSIC.

Since weak elementary monotonicity is a relatively weak condition, this result shows how little is required on top of unanimity to get the counterpart of [Theorem 1](#) in the single-peaked domain. In other words, though [Theorem 1](#) breaks down in the single-peaked domain, the nature of the break down is very nuanced.

The result in [Theorem 3](#) can also be extended to some extensions of single-peaked domain. For instance, [Demange \(1982\)](#) defines a notion of single-peakedness on a *tree graph*, which requires single-peakedness along paths of a tree graph whose vertices are alternatives. Our result can be easily extended to such a domain. Similarly, it can also be extended to *multiple single-peaked domain* discussed in [Reffgen \(2015\)](#).

5.3. Equivalence in a large class of local domains

We now provide a generalization of [Theorem 3](#) to a larger class of local domains with the help of an additional condition.

Definition 15. An scf f is **tops-only** if for every pair of profiles \mathbf{P} and \mathbf{P}' in its domain with $P_i(1) = P'_i(1)$ for all $i \in N$, we have $f(\mathbf{P}) = f(\mathbf{P}')$.

Tops-only property requires that the scf is only sensitive to the top ranked alternatives of each agent. This is a well-studied axiom in social choice theory. In many domains, DSIC and unanimity implies the tops-only property ([Weymark, 2008](#); [Saporiti, 2009](#); [Chatterji and Sen, 2011](#)). One of the motivations for using voting mechanisms which satisfy this axiom is computational – in models with a large number of alternatives, such scfs can only use information of the top alternatives of the agents. Tops-onlyness and unanimity allow us to weaken positive responsiveness in [Theorem 2](#).

Theorem 4. Suppose $f : \mathcal{C}^n \rightarrow A$ is a tops-only and unanimous scf, where \mathcal{C} is any connected domain without restoration. Then, the following statements are equivalent.

1. f is G-LOBIC and satisfies weak elementary monotonicity.
2. f is DSIC.

The proof of [Theorem 4](#) is given in the Appendix. We do not know if the tops-only property is redundant in this result.

5.4. Equivalence in the unrestricted domain

We now come to the final result of the paper, which is a strengthening of [Theorem 1](#).

Theorem 5. *Suppose $|A| \geq 3$ and $f : \mathcal{P}^n \rightarrow A$ is a G-LOBIC scf, where \mathcal{P} is the unrestricted domain. If f satisfies unanimity, then it satisfies elementary monotonicity. Hence, if f is G-LOBIC and unanimous, then it is DSIC.*

[Theorem 5](#) is a strengthening of [Theorem 1](#) since we use G-LOBIC instead of G-OBIC. However, one can still deduce the result of [Theorem 5](#) from existing results in the literature as follows. [Carroll \(2012\)](#) notes that his results also hold if he considers Bayesian incentive compatibility. Since he shows local incentive compatibility implies full incentive compatibility in the unrestricted domain, we can conclude that if f is LOBIC (with respect to some prior) then it is OBIC. We can then use [Theorem 1](#) to conclude that if f is G-LOBIC and unanimous, then it must be a dictatorship. Since the proof of [Theorem 1](#) in [Majumdar and Sen \(2004\)](#) directly establishes dictatorship (using induction on the number of agents), our proof provides an alternate and stronger version of their result.

5.5. Connected domains without restoration

In this section, we identify some domains that are connected without restoration to show that the results in [Theorems 2 and 4](#) apply to a large class of domains. As we noted earlier, [Sato \(2013\)](#) has already shown that the unrestricted domain and the single-peaked domain are connected without restoration. We present two more domains that are connected without restoration.

Definition 16. A preference ordering P is **single-dipped** with respect to an ordering \succ over alternatives if for every $a, b \in A$ with $a \succ b \succ P(|A|)$ or $P(|A|) \succ b \succ a$, we have aPb .

A domain is the single-dipped domain if it contains all the single-dipped preferences with respect to \succ .

The single-dipped domain is important in studying various practical problems including the problem of locating a public “bad”. It admits various interesting DSIC scfs ([Klaus et al., 1997](#); [Peremans and Storcken, 1999](#); [Barberà et al., 2012](#); [Manjunath, 2014](#)). Notice that there are only two alternatives (the maximal and the minimal alternatives in \succ) which can be top ranked in any single-dipped preference.

Lemma 1. *The single-dipped domain is a connected domain without restoration.*

Proof. Take any preference ordering P , and denote by \bar{P} the *reverse* of the preference ordering P , i.e., $\bar{P}(j) = P(|A| - j + 1)$ for all j . Note that P is single-dipped with respect to \succ if and only if \bar{P} is single-peaked with respect to \succ . Now, consider two single-dipped preference orderings P and P' . Since the single-peaked domain is connected without restoration, there is a sequence of distinct single-peaked preferences without restoration between \bar{P} and \bar{P}' . By taking

the reverse of each preference ordering in this sequence, we get the desired sequence without restoration between P and P' . \square

Another important class of domains that can be shown to be connected without restoration is the class of successive single-crossing domains introduced in [Carroll \(2012\)](#).

Definition 17. A set of preferences \mathcal{D} is a **single-crossing** domain if there exists a strict linear order \succ on the set of alternatives and a strict linear order \triangleleft on the set of preferences \mathcal{D} such that for all $a, b \in A$ and for all $P, P' \in \mathcal{D}$,

- $a \succ b, P \triangleleft P'$, and aPb implies $aP'b$
- $a \succ b, P \triangleleft P'$, and $bP'a$ implies bPa .

Single crossing domains are a well studied domain in voting and political economy since they ensure existence of a Condorcet winner ([Saporiti, 2009](#)).

For any ordering P over A and any ordering \succ over A , let $X(P, \succ) := \{(a, b) : a \succ b, aPb\}$. Clearly, a set of preferences \mathcal{D} is a **single-crossing** domain if and only if there exists a strict linear order \succ on the set of alternatives and a strict linear order \triangleleft on the set of preferences \mathcal{D} such that for any $P, P' \in \mathcal{D}$ with $P \triangleleft P'$, we have $X(P, \succ) \subsetneq X(P', \succ)$ (notice the strict inclusion). We will denote a single-crossing domain as $\mathcal{D}^{\succ, \triangleleft}$.

A single-crossing domain $\mathcal{D}^{\succ, \triangleleft} := \{P^1, \dots, P^l\}$ with $P^1 \triangleleft \dots \triangleleft P^l$ is a **successive single-crossing domain** if for every $j \in \{1, \dots, l-1\}$, $|X(P^j, \succ)| + 1 = |X(P^{j+1}, \succ)|$. Successive single-crossing domains were introduced in [Carroll \(2012\)](#).

Lemma 2. *A successive single-crossing domain is connected without restoration.*

Proof. Let $\mathcal{D}^{\succ, \triangleleft} := \{P^1, \dots, P^l\}$ be a successive single-crossing domain with $P^1 \triangleleft P^2 \triangleleft \dots \triangleleft P^l$. Pick $P^j, P^k \in \mathcal{D}^{\succ, \triangleleft}$ with $j < k$. The sequence of preferences $(P^j, P^{j+1}, \dots, P^k)$ satisfies the fact that for any $j' \in \{j, j+1, \dots, k-1\}$, $P^{j'}$ and $P^{j'+1}$ are adjacent – this follows from the definition of a successive single-crossing domain. Now, assume for contradiction, there is some pair of alternatives $x, y \in A$ such that they are swapped more than once in this sequence. But the single-crossing property requires that if $x \succ y$, once $xP^{k'}y$ for some $P^{k'}$ in the sequence, it must remain $xP^{l'}y$ for all $l' > k'$. Hence, getting swapped more than once will violate the single-crossing property. This means that every successive single-crossing domain is connected without restoration. \square

In [Theorem 2](#), we require the domain to be rich and connected without restoration. A connected domain without restoration need not be rich. For instance, the single-dipped domain is not a rich domain. However, rich successive single-crossing domains exist – see [Mishra et al. \(2016\)](#).

6. Discussions on elementary monotonicity

The elementary monotonicity condition plays an important role in our analysis. The scf f^* discussed in [Proposition 1](#) is an example of an scf that satisfies G-LOBIC but fails elementary monotonicity. Hence, we cannot hope to drop elementary monotonicity in [Theorem 2](#).

As discussed in the literature, elementary monotonicity is satisfied by a variety of scfs ([Moulin, 1983; Majumdar and Sen, 2004](#)). However, it is admittedly a strong enough condition along with

G-LOBIC to imply DSIC in local domains. The positive responsiveness property is equivalent to elementary monotonicity in a large class of local domains under G-LOBIC. Both these conditions are weaker versions of Maskin monotonicity, which is known to be necessary and sufficient for dominant strategy incentive compatibility in the unrestricted domain (Kalai and Muller, 1977). To see why elementary monotonicity or positive responsiveness is much weaker than Maskin monotonicity, note that these axioms are satisfied by a large class of scfs but Maskin monotonicity and unanimity imply dictatorship in the unrestricted domain.

Since the additional axioms that we have used are weak, they are useful to rule out scfs that are not G-LOBIC in many domains. We give two examples. Consider the *status-quo* scf, which is specified by a status-quo alternative. It chooses the status-quo alternative at all the preference profiles except when everyone's top ranked alternative is the same. When everyone's top ranked alternative is the same, then that alternative is chosen. This scf clearly satisfies positive responsiveness, elementary monotonicity, and unanimity. However, it is not DSIC in many domains (including the single-peaked domain). We can then conclude from our results that the status-quo scf cannot be G-LOBIC in those domains.

Similarly, some scoring rules cannot be G-LOBIC. It is easy to see that all scoring rules satisfy elementary monotonicity. But scoring rules may not be DSIC in many interesting domains. To see this, consider the single-peaked domain in Table 2. Consider a scoring rule in this domain where the top ranked alternative gets a score of 2, the second ranked alternative gets a score of 1, and the last ranked alternative gets a score of 0. If there is a tie in the score, we break the tie using a linear order \triangleright , where $a \triangleright b \triangleright c$. Now, suppose agent 1 has the preference where a is top ranked, b is second ranked, and c is third ranked. Further, agents 2 and 3 have identical preference, where c is top, b is second, and a is third. The outcome of the scoring rule at this profile is c . But agent 1 can get the outcome b if he reports a preference ordering where b is top. Hence, this scoring rule is not DSIC. Using our results, we can conclude that such a scoring rule is not G-LOBIC in the single-peaked domain.

We also point out that elementary monotonicity plays an important role in the analysis of OBIC scfs with uniform priors and correlated priors. Majumdar and Sen (2004) show that when agents have *uniform* priors, every *neutral* scf satisfying elementary monotonicity is OBIC in the unrestricted domain. Thus, they show that a large class of scfs that are not DSIC are OBIC with uniform priors in the unrestricted domain. A similar result is shown with correlated priors in Bhargava et al. (2015). In contrast, our results show a negative implication of elementary monotonicity under independent generic priors.

Appendix

Proof of Proposition 1. Clearly, f^* is unanimous and anonymous. However, f^* is not DSIC. To see this, consider a profile \mathbf{P} such that for all agents $i \neq 3$, we have $P_i(1) = a$ – note that since a is the leftmost alternative, there is a unique preference ordering where a is top ranked. For agent 3, pick any preference ordering where b is preferred to a . As a result, $f^*(\mathbf{P}) = b$. A possible profile is shown in Table 4 – note that $a P_1 b$. Now, fixing the preference profile of all agents except agent 1, if agent 1 reports a preference ordering P'_1 such that $P'_1(1) = b$, $P'_1(2) = a$, then f^* will choose a – see Table 4. Hence, agent 1 can manipulate.

However, we show that f^* is G-OBIC. Let the prior of each agent i be given by the map $\mu_i : \mathcal{S} \rightarrow (0, 1)$. To show that f^* is OBIC with respect to $\{\mu_i\}_{i \in N}$ such that these are generic priors, we will compute the interim outcome probabilities of every agent in $\{1, 2, 3\}$.

Table 4
Failure of DSIC of f^* .

P_1	P_2	P_3	P'_1	P_2	P_3
a	a	b	b	a	b
b	b	a	a	b	a
c	c	c	c	c	c

If an agent $i \in \{1, 2, 3\}$ has preference P_i then define

$$O_i^{f^*}(P_i) := \{x \in A : f^*(P_i, P_{-i}) = x \text{ for some } P_{-i}\}.$$

By definition of f^* , for every $i \in \{1, 2, 3\}$ and for every $P_i \in \mathcal{S}$, $O_i^{f^*}(P_i) \subseteq \{a, b, P_i(1)\}$, and if $P_i(1) \neq c$, then $O_i^{f^*}(P_i) = \{a, b\}$. Let \bar{P} be the unique preference ordering where the leftmost alternative a is top ranked. Denote the probability that agent i has preference \bar{P} as $q_i \equiv \mu_i(\bar{P})$. Similarly, denote by $\hat{q}_i \equiv \mu_i(\hat{P})$, where \hat{P} is the unique ordering where alternative c is top ranked. We fix an agent $i \in \{1, 2, 3\}$ and denote the other two agents in $\{1, 2, 3\}$ as j and k . We consider three possible cases.

CASE 1. Suppose $P_i = \bar{P}$. Note that $O_i^{f^*}(P_i) = \{a, b\}$. Then his interim outcome probability for alternative a can be computed as follows. Note that $f^*(P_i, P_j, P_k) \neq a$ if P_j and P_k are such that either $(aP_jb$ and $bP_ka)$ or $(aP_kb$ and $bP_ja)$. The probability of this event is

$$q_j(1 - q_k) + q_k(1 - q_j).$$

Here, we used the fact that the probability that agent j has preference P_j such that bP_ja is just $(1 - q_j)$ and, similarly, the probability that agent k has preference P_k such that bP_ka is $(1 - q_k)$. Since $f^*(P_i, P_j, P_k) \in \{a, b\}$, the interim outcome probability of choosing a at P_i for agent i is

$$1 - q_j - q_k + 2q_jq_k,$$

and the interim outcome probability of choosing b at P_i for agent i is

$$q_j + q_k - 2q_jq_k.$$

CASE 2. Suppose P_i is such that $P_i(1) = b$ – this is possible for two preference orderings. Note that $O_i^{f^*}(P_i) = \{a, b\}$. Then, his interim outcome probability for alternative b can be computed as follows. Note that $f^*(P_i, P_j, P_k) \neq b$ if P_j and P_k are such that either $(aP_jb$ and $bP_ka)$ or $(aP_kb$ and $bP_ja)$. The probability of this event is

$$q_j(1 - q_k) + q_k(1 - q_j).$$

Hence, the interim outcome probability of choosing b at P_i for agent i is

$$1 - q_j - q_k + 2q_jq_k.$$

Since $O_i^{f^*}(P_i) = \{a, b\}$, the interim outcome probability of choosing a at P_i for agent i is

$$q_j + q_k - 2q_jq_k.$$

CASE 3. Suppose P_i is such that $P_i(1) = c$, i.e., $P_i = \hat{P}$. Then, $O_i^{f^*}(P_i) = \{a, b, c\}$. His interim outcome probability for alternative c can be computed straightforwardly – c is chosen if and only if $P_j(1) = P_k(1) = P_i(1) = c$. This is possible if and only if both agents j and k have the

Table 5

Interim outcome probabilities.

	Case 1: $P_i = \bar{P}$	Case 2: $P_i(1) = b$	Case 3: $P_i = \hat{P}$
$\pi_i^{f^*}(a, P_i)$	$1 - q_j - q_k + 2q_j q_k$	$q_j + q_k - 2q_j q_k$	$q_j + q_k - 2q_j q_k$
$\pi_i^{f^*}(b, P_i)$	$q_j + q_k - 2q_j q_k$	$1 - q_j - q_k + 2q_j q_k$	$1 - q_j - q_k + 2q_j q_k - \hat{q}_j \hat{q}_k$
$\pi_i^{f^*}(c, P_i)$	0	0	$\hat{q}_j \hat{q}_k$

preference \hat{P} . The probability of this event is $\hat{q}_j \hat{q}_k$. Hence, the interim outcome probability for alternative $P_i(1)$ at P_i for agent i is $\hat{q}_j \hat{q}_k$. Next, the interim outcome probability for agent i for alternative a can be computed as follows. For this, note that $c P_i b P_i a$. Hence, $f^*(P_i, P_j, P_k) = a$ if P_j and P_k are such that either $(a P_j b$ and $b P_k a)$ or $(a P_k b$ and $b P_j a)$ – notice that these events ensure that tops of all agents are not the same and a is not dominated by b . The probability of this event is

$$q_j(1 - q_k) + q_k(1 - q_j).$$

Hence, the interim outcome probability of choosing a at P_i for agent i is

$$q_j + q_k - 2q_j q_k.$$

Since $O_i^{f^*}(P_i) = \{a, b, P_i(1)\}$, the interim outcome probability of choosing b at P_i for agent i is

$$1 - q_j - q_k + 2q_j q_k - \hat{q}_j \hat{q}_k.$$

We enumerate all the interim outcome probabilities in Table 5 by considering the three cases for an agent $i \in \{1, 2, 3\}$ by denoting the other two agents as j and k .

Now, notice from Table 5 that if agent i has preference $P_i \equiv \bar{P}$, the only OBIC constraint to satisfy is

$$1 - q_j - q_k + 2q_j q_k \geq q_j + q_k - 2q_j q_k.$$

Alternatively, we must have

$$(1 - 2q_j)(1 - 2q_k) \geq 0. \quad (1)$$

This prevents any manipulation of agent i to a preference in Case 2 or Case 3. Also, Inequality (1) ensures OBIC constraints when agent i has a preference P_i such that $P_i(1) = b$. Finally, Inequality (1) also ensures OBIC constraints when agent i has a preference P_i such that $P_i(1) = c$ – this can be verified by checking from Table 5 that the truthtelling lottery first-order stochastic dominates other lotteries as long as Inequality (1) is satisfied.

Hence, f^* is OBIC if and only if the priors of agents 1, 2, and 3 satisfy

$$(1 - 2q_2)(1 - 2q_3) \geq 0,$$

$$(1 - 2q_1)(1 - 2q_3) \geq 0,$$

$$(1 - 2q_1)(1 - 2q_2) \geq 0.$$

This is satisfied if and only if $q_1, q_2, q_3 \in (0, 0.5)$ or $q_1, q_2, q_3 \in (0.5, 1)$. It also puts no restriction on the probabilities of orderings in $\mathcal{S} \setminus \{\bar{P}\}$. Hence, the set of priors of all agents satisfying $q_1, q_2, q_3 \in (0, 0.5)$ or $q_1, q_2, q_3 \in (0.5, 1)$ is a full dimensional subset of the set of all independent priors. As a result, it must have a non-empty intersection with the set of independent generic priors. \square

Proof of Theorem 2. The proof is done by establishing two important lemmas. We start by identifying a property that is implied by G-LOBIC.

Definition 18. An scf f satisfies **swap monotonicity** if for every $i \in N$, for every $P_i, P'_i \in \mathcal{D}$, where P'_i is an (a, b) -swap of P_i , we have for every $P_{-i} \in \mathcal{D}^{n-1}$,

- $f(P'_i, P_{-i}) = f(P_i, P_{-i})$ if $f(P_i, P_{-i}) \notin \{a, b\}$,
- $f(P'_i, P_{-i}) \in \{a, b\}$ if $f(P_i, P_{-i}) \in \{a, b\}$.

Our first claim shows the necessity of swap monotonicity.

Lemma 3. *If an scf is G-LOBIC, then it satisfies swap monotonicity.*

Proof. Let f be an LOBIC scf with respect to independent generic priors $\{\mu_i\}_{i \in N}$. For this, consider agent $i \in N$, and pick two preference orderings P_i and P'_i such that P'_i is an (a, b) swap of P_i . By definition $P_i(k) = a$, $P_i(k+1) = b$ and $P'_i(k+1) = a$, $P'_i(k) = b$ for some k and $P_i(j) = P'_i(j)$ for all $j \notin \{k, k+1\}$. We will do the proof in three steps.

STEP 1. Consider an alternative $x \in A \setminus \{a, b\}$ such that $P_i(k') = P'_i(k') = x$, where $k' < k$. We will show that $\{P_{-i} \in \mathcal{D}^{n-1} : f(P_i, P_{-i}) = x\} = \{P_{-i} \in \mathcal{D}^{n-1} : f(P'_i, P_{-i}) = x\}$. We do this using induction on k' . If $k' = 1$, by observing that $P_i(k'') = P'_i(k'')$ for all $k'' < k$, LOBIC implies that

$$\begin{aligned} \sum_{P_{-i} : f(P_i, P_{-i}) = P_i(1)} \mu(P_{-i}) &\geq \sum_{P_{-i} : f(P'_i, P_{-i}) = P_i(1)} \mu(P_{-i}) \\ \sum_{P_{-i} : f(P'_i, P_{-i}) = P'_i(1)} \mu(P_{-i}) &\geq \sum_{P_{-i} : f(P_i, P_{-i}) = P'_i(1)} \mu(P_{-i}). \end{aligned}$$

Combining these inequalities, we get

$$\sum_{P_{-i} : f(P_i, P_{-i}) = P_i(1)} \mu(P_{-i}) = \sum_{P_{-i} : f(P'_i, P_{-i}) = P_i(1)} \mu(P_{-i}).$$

Since priors are generic, we get that

$$\{P_{-i} : f(P_i, P_{-i}) = P_i(1)\} = \{P_{-i} : f(P'_i, P_{-i}) = P_i(1)\}.$$

Now, suppose the claim is true for all $k'' < k'$. Notice that the top k' alternatives in P_i and P'_i are the same – denote this set as B . Now, we apply LOBIC to top k' alternatives in P_i and P'_i to get

$$\begin{aligned} \sum_{P_{-i} : f(P_i, P_{-i}) \in B} \mu_i(P_{-i}) &\geq \sum_{P_{-i} : f(P'_i, P_{-i}) \in B} \mu_i(P_{-i}) \\ \sum_{P_{-i} : f(P'_i, P_{-i}) \in B} \mu_i(P_{-i}) &\geq \sum_{P_{-i} : f(P_i, P_{-i}) \in B} \mu_i(P_{-i}). \end{aligned}$$

Using genericity of μ_i gives us

$$\{P_{-i} : f(P_i, P_{-i}) \in B\} = \{P_{-i} : f(P'_i, P_{-i}) \in B\}.$$

Using the induction hypothesis, we have for all $k'' < k'$,

$$\{P_{-i} : f(P_i, P_{-i}) = P_i(k'')\} = \{P_{-i} : f(P'_i, P_{-i}) = P_i(k'')\}.$$

Hence, we get

$$\{P_{-i} : f(P_i, P_{-i}) = P_i(k')\} = \{P_{-i} : f(P'_i, P_{-i}) = P_i(k')\}.$$

STEP 2. In this step, we show that $\{P_{-i} : f(P_i, P_{-i}) \in \{a, b\}\} = \{P_{-i} : f(P'_i, P_{-i}) \in \{a, b\}\}$. Applying LOBIC, we get

$$\begin{aligned} \sum_{P_{-i} : f(P_i, P_{-i}) \in B(b, P_i)} \mu_i(P_{-i}) &\geq \sum_{P_{-i} : f(P'_i, P_{-i}) \in B(b, P_i)} \mu_i(P_{-i}) \\ \sum_{P_{-i} : f(P'_i, P_{-i}) \in B(a, P'_i)} \mu_i(P_{-i}) &\geq \sum_{P_{-i} : f(P_i, P_{-i}) \in B(a, P'_i)} \mu_i(P_{-i}). \end{aligned}$$

Since $B(b, P_i) = B(a, P'_i)$, by genericity we get

$$\{P_{-i} : f(P_i, P_{-i}) \in B(b, P_i)\} = \{P_{-i} : f(P'_i, P_{-i}) \in B(b, P_i)\}.$$

By Step 1, this implies that $\{P_{-i} : f(P_i, P_{-i}) \in \{a, b\}\} = \{P_{-i} : f(P'_i, P_{-i}) \in \{a, b\}\}$.

STEP 3. Consider an alternative $x \in A \setminus \{a, b\}$ such that $P_i(k') = P'_i(k') = x$, where $k' > k + 1$. Using the facts in Steps 1 and 2, we can mimic the method in Step 1 to show that $\{P_{-i} \in \mathcal{D}^{n-1} : f(P_i, P_{-i}) = x\} = \{P_{-i} \in \mathcal{D}^{n-1} : f(P'_i, P_{-i}) = x\}$.

Steps 1, 2, and 3 show that f satisfies swap monotonicity. \square

Note that [Lemma 3](#) holds in any arbitrary domain. We now use this to prove the following result.

Lemma 4. *Suppose \mathcal{D} is any domain and $f : \mathcal{D}^n \rightarrow A$ is an scf on this domain. Then, the following statements are equivalent.*

1. *f is G-LOBIC and satisfies elementary monotonicity.*
2. *f is LDSIC.*

Proof. First, we show that a LDSIC scf f satisfies elementary monotonicity. To see this, consider $i \in N$ and P_{-i} . Let P_i and P'_i be two preferences in \mathcal{D} such that P'_i is an (a, b) -swap of P_i and $f(P_i, P_{-i}) = b$. Assume for contradiction that $f(P'_i, P_{-i}) = c \neq b$. If $c P_i b$, then i can manipulate from P_i to P'_i . If $b P_i c$, then, by construction, $b P'_i c$, and again, agent i can manipulate from P'_i to P_i . This is a contradiction.

Further, an LDSIC scf is LOBIC with respect to all priors, and hence, it is G-LOBIC. Now, we show that if $f : \mathcal{D}^n \rightarrow A$ is G-LOBIC and satisfies elementary monotonicity, then it is LDSIC. Fix an agent i and P_{-i} . Now, pick two preference orderings P_i and P'_i such that P'_i is an (a, b) -swap of P_i , where $P_i(k) = a$, $P_i(k+1) = b$ and $P'_i(k+1) = a$, $P'_i(k) = b$ for some k . Suppose $f(P_i, P_{-i}) = x$ and $f(P'_i, P_{-i}) = y$. Suppose $P_i(k') = x$. If $k' < k$ or $k' > k+1$, then by [Lemma 3](#), we have $y = x$ (swap monotonicity). So, agent i cannot manipulate from P_i to P'_i . If $k' = k$, then $x = a$, and by swap monotonicity $f(P'_i, P_{-i}) \in \{a, b\}$. Since $a P_i b$, agent i cannot manipulate from P_i to P'_i . The other possibility is $k' = k+1$. In that case, $x = b$, and elementary monotonicity ensures that $f(P'_i, P_{-i}) = b$. Hence, agent i cannot manipulate from P_i to P'_i . This shows that f is LDSIC. \square

Proof of Theorem 2. EQUIVALENCE OF (1) AND (2). This equivalence follows from Lemma 4 because in local domains an LDSIC scf is DSIC.

EQUIVALENCE OF (2) AND (3). If \mathcal{D} is a rich connected domain without restoration, we first show that a G-LOBIC f satisfying elementary monotonicity also satisfies positive responsiveness. To see this, fix agent $i \in N$, $P_{-i} \in \mathcal{D}^{n-1}$ and $P_i, P'_i \in \mathcal{D}$ such that $f(P_i, P_{-i}) = a$ and $P'_i(1) = a$. Since \mathcal{D} is connected without restoration there is a connected sequence $(P^0 \equiv P_i, P^1, \dots, P^k \equiv P'_i)$. Consider P^j and P^{j+1} in this sequence. By construction, P^{j+1} is a (x^j, y^j) -swap of P^j for some pair of alternatives x^j, y^j . If $f(P^j, P_{-i}) \notin \{x^j, y^j\}$, then $f(P^j, P_{-i}) = f(P^{j+1}, P_{-i})$ by Lemma 3. By elementary monotonicity, if $f(P^j, P_{-i}) = y^j$, then $f(P^{j+1}, P_{-i}) = y^j = f(P^j, P_{-i})$. We argue that $f(P^j, P_{-i}) \neq x^j$, and we will be done.

To see this, note that since \mathcal{D} is a connected domain without restoration, no swap along the sequence $(P^0 \equiv P_i, P^1, \dots, P^k \equiv P'_i)$ is repeated. Since $P'_i(1) = a$, connectedness without restoration implies that the rank of a along this sequence can never go down. Hence, $a \neq x^j$ for any j . Hence, $f(P^0, P_{-i}) \neq x^0$. This implies that $f(P^1, P_{-i}) = a$. Using induction, if we assume $f(P^j, P_{-i}) = a$ and since $a \neq x^j$, we get $f(P^{j+1}, P_{-i}) = a$, and we are done.

For the converse, suppose f is G-LOBIC and satisfies positive responsiveness. We will show that f satisfies elementary monotonicity. Fix agent $i \in N$, $P_{-i} \in \mathcal{D}^{n-1}$ and $P_i, P'_i \in \mathcal{D}$ such that P'_i is an (a, b) -swap of P_i with $f(P_i, P_{-i}) = b$. Note that $bP'_i a$. Assume for contradiction that $f(P'_i, P_{-i}) \neq b$. By Lemma 3, $f(P'_i, P_{-i}) = a$. Since \mathcal{D} is rich, there is a preference ordering P''_i with $P''_i(1) = b$. Further, since the domain is connected without restoration, there is a connected sequence $(P^0 \equiv P'_i, P^1, \dots, P^K \equiv P''_i)$.

For any P^j in this sequence, define as before $B(b, P^j) = \{x \in A : x P^j b \text{ or } x = b\}$. We show that for any P^j in this sequence, $f(P^j, P_{-i}) \notin B(b, P^j)$. Since $f(P^0, P_{-i}) = a$, the claim is true for $j = 0$. Suppose the claim is true for all the preferences in the sequence till P^k . Suppose P^{k+1} is a (x, y) -swap of P^k .

Consider the case when $\{x, y\} \subseteq B(b, P^k)$. Since $f(P^k, P_{-i}) \notin B(b, P^k)$, Lemma 3 implies that $f(P^k, P_{-i}) = f(P^{k+1}, P_{-i})$, which further implies that $f(P^{k+1}, P_{-i}) \notin B(b, P^{k+1})$.

Now, consider the case when $\{x, y\} \subseteq (A \setminus B(b, P^k))$. By Lemma 3, $f(P^{k+1}, P_{-i}) \notin B(b, P^{k+1})$.

The only case that remains is $x = b$ and $y \notin B(b, P^k)$, i.e., y is just below b in P^k . Since \mathcal{D} is connected without restoration and the connected sequence $(P^0 \equiv P'_i, P^1, \dots, P^k \equiv P''_i)$ satisfies $P''_i(1) = b$, the rank of b can never go down along this sequence. Hence, $b \neq x$. Hence, this case is not possible.

Hence, by induction $f(P''_i, P_{-i}) \notin B(b, P''_i) = \{b\}$. But the fact that $f(P_i, P_{-i}) = b$ and $P''_i(1) = b$ implies that f does not satisfy positive responsiveness, a contradiction. \square

Proof of Theorem 3. Let f be a unanimous and G-LOBIC scf satisfying weak elementary monotonicity. The proof goes in many steps.

STEP 1. We start by stating a fact from Sato (2013).

Fact 1. (Sato, 2013) *The single-peaked domain is connected without restoration.*

We now prove a claim using this fact.

Table 6

Profiles (P_i, P_{-i}) and (P'_i, P_{-i}) .

P_i	P_{-i}				P'_i	P_{-i}			
.
c	$c\dots$	$c\dots$	$b\dots$	$a\dots$	c	$c\dots$	$c\dots$	$b\dots$	$a\dots$
.
a	$a\dots$	$b\dots$	$c\dots$	$c\dots$	b	$a\dots$	$b\dots$	$c\dots$	$c\dots$
b	a
.	$b\dots$	$a\dots$	$a\dots$	$b\dots$.	$b\dots$	$a\dots$	$a\dots$	$b\dots$
.

Claim 1. Suppose $P_i \in \mathcal{S}$ is a preference ordering such that $a P_i b$. Then, there exists a preference ordering $P'_i \in \mathcal{S}$ such that $P'_i(1) = a$ and $B(b, P_i) = B(b, P'_i)$. Moreover, for all $P_{-i} \in \mathcal{S}^{n-1}$ with $f(P_i, P_{-i}) = b$, we have $f(P'_i, P_{-i}) = b$.

Proof. The first part of the claim follows from the single-peaked domain – if $a P_i b$, then we can always lift a to the top and keep all the alternatives that are above b in P_i between a and b and all others below b in the new preference ordering. Let P'_i be such an ordering. By Fact 1, we know that there is a distinct sequence of preferences $(P_i = P^0, P^1, P^2, \dots, P^k, P^{k+1} = P'_i)$ without restoration such that consecutive preferences in the sequence are swaps of each other. Since $B(b, P_i) = B(b, P'_i)$ and the sequence is without restoration, none of these swaps involve b . By repeatedly applying swap monotonicity along the sequence, we get $f(P'_i, P_{-i}) = f(P_i, P_{-i}) = b$. \square

STEP 2. In this step, we show that if an scf is G-LOBIC and unanimous, then it must be Pareto efficient. For this consider a profile \mathbf{P} with $f(\mathbf{P}) = b$. Assume for contradiction that there exists $a \neq b$ such that $a P_i b$ for all $i \in N$. By repeated application of Claim 1, there exists a preference profile \mathbf{P}' such that $f(\mathbf{P}') = b$ and $P'_i(1) = a$ for all $i \in N$. This is a contradiction since unanimity implies that $f(\mathbf{P}') = a$.

STEP 3. Now, consider an agent $i \in N$ and $P_{-i} \in \mathcal{S}^{n-1}$. Let $P_i, P'_i \in \mathcal{S}$ be such that P'_i is an (a, b) -swap of P_i and $f(P_i, P_{-i}) = b$. We will show that $f(P'_i, P_{-i}) = b$. This will show that f satisfies elementary monotonicity and we will be done by Theorem 2. Note that if agents in P_{-i} have the same ranking of a and b then the claim is obvious. Hence, we assume that agents in P_{-i} do not have the same ranking of a and b . We consider two cases.

CASE 3-1. Suppose $P_i(k) = a$ and $k > 1$, i.e., the swap from P_i to P'_i is not happening at the top of the preference ordering. Since a and b are consecutively ranked in P_i and P'_i and neither of them are top ranked in P_i and P'_i , it must be that a and b are not neighbors (in \succ). This is because if a and b are neighbors then they can only be swapped if they are at the top.

Hence, consider a neighbor c of a such that c is between a and b in \succ (i.e., if $a \succ b$, then $a \succ c, c \succ b$ and if $b \succ a$ then $b \succ c, c \succ a$). By single-peakedness, $c P_i a$ and $c P'_i b$. Further, for any other agent $j \neq i$, there are four possible rankings between a, b, c in P_j : (1) $c P_j a P_j b$, (2) $c P_j b P_j a$, (3) $b P_j c P_j a$, and (4) $a P_j c P_j b$. The two profiles (P_i, P_{-i}) and (P'_i, P_{-i}) are shown in Table 6. Table 6 shows that there are four possible groups of agents in P_{-i} with different rankings between a, b, c . We now modify the profile (P_i, P_{-i}) in a sequence of steps to reach the profile (P'_i, P_{-i}) .

Table 7
Profile in Step 3-1-a.

P_i	Other agents			
.	$a\dots$
c	$c\dots$	$c\dots$	$b\dots$	$c\dots$
.
a	$a\dots$	$b\dots$	$c\dots$...
b
.	$b\dots$	$a\dots$	$a\dots$	$b\dots$
.

Table 8
Profile in Step 3-1-b.

P_i	Other agents			
.	$c\dots$
c	$c\dots$	$c\dots$	$b\dots$	$a\dots$
.
a	$a\dots$	$b\dots$	$c\dots$...
b
.	$b\dots$	$a\dots$	$a\dots$	$b\dots$
.

Table 9
Profile in Step 3-1-c.

P'_i	Other agents			
.	$c\dots$
c	$c\dots$	$c\dots$	$b\dots$	$a\dots$
.
b	$a\dots$	$b\dots$	$c\dots$...
a
.	$b\dots$	$a\dots$	$a\dots$	$b\dots$
.

STEP 3-1-a. In this step, we modify the preferences of agents in P_{-i} who rank a better than c better than b . For each such agent j , we construct P'_j such that $P'_j(1) = a$, $P'_j(2) = c$ and $B(b, P'_j) = B(b, P_j)$. Notice that since a and c are neighbors, single-peakedness implies that such a P'_j can be constructed such that it is single-peaked. Further, using a reasoning similar to [Claim 1](#), we can argue that we can go from P_j to P'_j using a without restoration sequence and since $B(b, P'_j) = B(b, P_j)$, b will not be involved in any swaps. As a result, the outcome at the new profile will be b . The new profile is shown in [Table 7](#).

STEP 3-1-b. In this step, we modify the profile in [Table 7](#) as follows. For every agent, who ranks a at the top, c second, we perform the (a, c) -swap. Notice that this leads to a feasible single-peaked preference ordering since a and c are neighbors. The new profile is shown in [Table 8](#). By swap monotonicity, the outcome at the new profile is b .

STEP 3-1-c. In this step, we perform (a, b) -swap of P_i to reach P'_i . The new profile is shown in [Table 9](#). By swap monotonicity, the outcome at the new profile is in $\{a, b\}$. Note that at the new profile $c P_j a$ for all $j \in N$. Hence, by Pareto efficiency the outcome at the new profile is b .

Table 10
Profile in Step 3-1-d.

P'_i	Other agents			
.
c	$c\dots$	$c\dots$	$b\dots$	$a\dots$
.
b	$a\dots$	$b\dots$	$c\dots$	$c\dots$
a
.	$b\dots$	$a\dots$	$a\dots$	$b\dots$
.

Table 11
Profile in Case 3-2.

P_i	P_{-i}
a	...
b	$a\dots$
.	...
.	$b\dots$
.	...

Table 12
Profile in Step 3-2-a.

P_i	Other agents	
a	$a\dots$...
b	...	$b\dots$
.
.	$b\dots$	$a\dots$
.

STEP 3-1-d. We can now consider the profile in [Table 9](#) and alter the preferences of last (fifth) column of agents by performing a (c, a) -swap and then doing a sequence of without restoration swaps to go to their preference at the start of Step 3-1-a (see preferences in [Table 6](#)). The new profile is shown in [Table 10](#).

Since none of these swaps involve alternative b , the outcome at this new profile is b due to swap monotonicity. But this profile is exactly (P'_i, P_{-i}) . Hence, $f(P'_i, P_{-i}) = b$.

CASE 3-2. The other case is $P_i(k) = a$ and $k = 1$, i.e., the swap from P_i to P'_i is occurring at the top. The profile (P_i, P_{-i}) is shown below in [Table 11](#). We now do the proof in many steps.

STEP 3-2-a. Now, consider every agent $j \neq i$ such that aP_jb (agents in second column of [Table 11](#)). By [Claim 1](#), we can construct a preference ordering from P_j such that a is top ranked and the outcome remains b . We change the preferences of all the agents in the profile of [Table 11](#) who prefer a to b in this manner to arrive at the new profile. The new profile is shown in [Table 12](#) and the outcome at the new profile is b .

STEP 3-2-b. Now, for every agent $j \neq i$ such that a is top ranked in the profile of [Table 12](#), we consider a preference ordering where a is top ranked and b is second ranked. The new profile is shown in [Table 13](#).

Table 13
Profile in Step 3-2-b.

P_i	Other agents	
a	$a\dots$	\dots
b	$b\dots$	$b\dots$
\cdot	\dots	\dots
\cdot	\dots	$a\dots$
\cdot	\dots	\dots

Table 14
Profile in Step 3-2-c.

P_i	Other agents	
a	$a\dots$	$b\dots$
b	$b\dots$	\dots
\cdot	\dots	\dots
\cdot	\dots	$a\dots$
\cdot	\dots	\dots

Table 15
Profile in Step 3-2-d.

P_i	Other agents	
a	$a\dots$	$b\dots$
b	$b\dots$	$a\dots$
\cdot	\dots	\dots
\cdot	\dots	\dots
\cdot	\dots	\dots

By [Sato \(2013\)](#), such a preference ordering can be reached by swaps without restoration. Hence, a will not be involved in such swaps. Swap monotonicity implies that the outcome at the new profile is not a .

STEP 3-2-c. Now, consider every agent $j \neq i$ such that b is preferred to a in the preference profile in Step 3-2-b (agents in third column of [Table 13](#)). By [Claim 1](#), we can construct a preference ordering from this preference ordering such that b is top ranked and the alternatives that were below a do not change. The new profile is shown in [Table 14](#). By definition, we can go to this new profile by doing a sequence of without restoration swaps that do not involve a . Hence, by swap monotonicity, the outcome at the new profile is not a . Then, by Pareto efficiency, the outcome at this new profile must be b .

STEP 3-2-d. In this step, we consider all the agents who have b top-ranked in the profile in Step 3-2-c (agents in the third column in [Table 14](#)). For every such agent, we consider another preference ordering where b is top-ranked and a is second ranked. The new profile is shown in [Table 15](#). By [Sato \(2013\)](#), we can go to this new preference ordering by doing a sequence of without restoration swaps. Hence, by swap monotonicity, the outcome at the new profile is b .

Now, consider any agent who ranks a at the top in the profile in Step 3-2-d. If his preference ordering is not the same as P_i , then we can transform it to P_i by a sequence of without restoration swaps. Since this will not involve any swaps of b , by swap monotonicity the outcome will remain

Table 16
Profile (P_i, P_{-i}) .

P_i	P_{-i}
a	...
b	...
.	$a \dots b \dots$
.	...
.	...
.	$b \dots a \dots$
.	...
.	...

b at the new profile. A similar argument can be made to transform the preference ordering of every agent who ranks b at the top to the preference ordering P'_i . As a consequence, the profile in Table 15 can be transformed to a profile where every agent $j \neq i$ has preference ordering P_i or P'_i and the outcome at this profile is b . Denote this profile as (P_i, \bar{P}_{-i}) . This, we conclude that $f(P_i, \bar{P}_{-i}) = b$.

Now, assume for contradiction $f(P'_i, P_{-i}) = a$. We now repeat the above procedure at the profile (P'_i, P_{-i}) to arrive at the profile (P'_i, \bar{P}_{-i}) and the outcome at this profile is $f(P'_i, \bar{P}_{-i}) = a$. But $(P'_i, \bar{P}_{-i}), (P_i, \bar{P}_{-i}) \in \mathcal{D}^n(2)$. Hence, $f(P_i, \bar{P}_{-i}) = b$ and $f(P'_i, \bar{P}_{-i}) = a$ is a contradiction. \square

Proof of Theorem 4. Let f be a tops-only, unanimous, G-LOBIC scf satisfying weak elementary monotonicity. We will show that f satisfies elementary monotonicity and we will be done by Theorem 2. Consider an agent $i \in N$ and a preference profile (P_i, P_{-i}) . Let P'_i be an (a, b) -swap of P_i and $f(P_i, P_{-i}) = b$. By swap monotonicity, $f(P'_i, P_{-i}) \in \{a, b\}$. Assume for contradiction that $f(P'_i, P_{-i}) = a$. By tops-only property $P_i(2) = P'_i(1) = b$ and $P_i(1) = P'_i(2) = a$.

Notation: For any pair of adjacent preferences, P and P' , we say P' is a **top-swap** of P if P' is a $(P(1), P(2))$ -swap of P . We now do the proof in several steps, where each step achieves some technical milestone towards the eventual claim.

STEP 1. The profile (P_i, P_{-i}) consists of agents who rank a above b and agents who rank b above a – see Table 16.

Pick an agent $j \neq i$ such that $a P_j b$. Since the domain is connected without restoration, there exists a sequence of preferences without restoration (P^1, \dots, P^k) such that $P^1 \equiv P_j$ and $P^k \equiv P_i$. By the without restoration property, for every P^ℓ in the sequence $a P^\ell b$, and hence, $P^\ell(1) \neq b$. This also implies that b is never involved in a top-swap along the sequence. By tops-only property and swap monotonicity, we conclude that $f(P^k, P_{-j}) = b$. Repeating this argument for all agents $j \neq i$ such that $a P_j b$, we reach a profile shown in Table 17, where the outcome of f is b .

STEP 2. Now, we consider an agent j such that $b P_j a$. Since the domain is connected without restoration, there exists a sequence of preferences without restoration (P^1, \dots, P^k) such that $P^1 \equiv P_j$ and $P^k \equiv P'_i$. Note that $P'_i(1) = b$ and $P'_i(2) = a$. Also, if $P_j(1) = b$, then $P_j = P'_i$, and this sequence has exactly one preference. Else, this is a sequence without restoration, and b will be involved in a top-swap at most once along the sequence. Let $P^\ell(1) = x, P^\ell(2) = b$ and $P^{\ell+1}(1) = b, P^{\ell+1}(2) = x$ for some P^ℓ along the sequence. Since this is a sequence without

Table 17
Profile reached at the end of Step 1.

P_i	Agents with preference P_i	Other agents
a	$a \dots$	\dots
b	$b \dots$	\dots
\cdot	\dots	$b \dots$
\cdot	\dots	\dots
\cdot	\dots	\dots
\cdot	\dots	$a \dots$
\cdot	\dots	\dots
\cdot	\dots	\dots

Table 18
Profile at the end of Step 2.

P_i	Agents with preference P_i	Other agents
a	$a \dots$	$x \dots$
b	$b \dots$	$b \dots$
\cdot	\dots	\dots
\cdot	\dots	$a \dots$
\cdot	\dots	\dots
\cdot	\dots	\dots

Table 19
Profile in Step 3.

P_i	\hat{P}_{-i}
a	$a \dots \quad b \dots$
b	$b \dots \quad x \dots$
\cdot	$\dots \quad \dots$
\cdot	$\dots \quad a \dots$
\cdot	$\dots \quad \dots$
\cdot	$\dots \quad \dots$

restoration, we conclude that b is always ranked higher than a along the sequence, and hence, $x \neq a$.

Now, for every j such that $b P_j a$, we change the preference along the sequence to P^ℓ . Since these changes do not involve top-swap of b , the outcome at the new profile is b . The new profile is shown in Table 18.

STEP 3. Now, we consider all the agents in the third column of the preference profile in Table 18 and change their preference to $P^{\ell+1}$. The new profile is shown in Table 19. By swap monotonicity the outcome of f at this profile is either x or b . We argue that it is b . Assume for contradiction that it is x . Then, we can change the preferences of agent i and all the agents in the second column of the preference profile in Table 19 to P'_i (note that all these agents have preference P_i at the profile in Table 19). By swap monotonicity, the outcome of f is at this new profile is still x , which will contradict unanimity.

STEP 4. Finally, we can change the preference of all the agents in the third column of the profile in Table 19 along the sequence $(P^{\ell+1}, \dots, P^k)$, where $P^k = P'_i$. The new profile is shown in

Table 20
Profile in Step 4 – a top-2 profile.

P_i	Agents with preference P_i	Agents with preference P'_i
a	$a\dots$	$b\dots$
b	$b\dots$	$a\dots$
\cdot	\dots	\dots
\cdot	\dots	\dots

Table 21
Profiles (P_i, P_{-i}) and (P'_i, P_{-i}) .

P_i	P_{-i}	P'_i	P_{-i}
\cdot	\dots	\cdot	\dots
\cdot	\dots	\cdot	\dots
a	$b\dots$	$a\dots$	$b\dots$
b	\dots	a	\dots
\cdot	$a\dots$	$b\dots$	$a\dots$
\cdot	\dots	\cdot	\dots

Table 20. Since no swaps along this sequence involves b , swap monotonicity ensures that the outcome at this top-2 profile is b . Denote this preference profile as (P_i, \hat{P}_{-i}) . An analogous argument starting from the profile (P'_i, P_{-i}) and ending at (P'_i, \hat{P}_{-i}) can be made to show that $f(P'_i, \hat{P}_{-i}) = a$. But $f(P_i, \hat{P}_{-i}) = b$ contradicts weak elementary monotonicity. \square

Proof of Theorem 5. We first show that if $f : \mathcal{P}^n \rightarrow A$ is G-LOBIC and unanimous, then it is Pareto efficient. Suppose f is G-LOBIC and unanimous but assume for contradiction that it is not Pareto efficient. For this, we consider a profile \mathbf{P} such that $f(\mathbf{P}) = b$ but there exists $a \in A$ such that $a P_i b$ for all $i \in N$. Consider an agent $i \in N$ such that $P_i(k) = a$ and $k \neq 1$. Suppose $P_i(k-1) = x$. Consider P'_i which is a (x, a) -swap of P_i . By swap monotonicity, $f(P'_i, P_{-i}) = b$. We can repeat such swaps to reach a preference ordering P''_i for agent i such that $P''_i(1) = a$ and $f(P''_i, P_{-i}) = b$. Now, we can repeat this procedure for every agent j such that $P_j(k) = a$ and $k \neq 1$ to arrive at a profile \mathbf{P}'' such that $f(\mathbf{P}'') = b$. But this will contradict unanimity since $P''_j(1) = a$ for all $j \in N$.

Hence, we show that any $f : \mathcal{P}^n \rightarrow A$ that is G-LOBIC and Pareto efficient must satisfy elementary monotonicity. By [Theorem 2](#), we will be done.

To do so, we consider an agent $i \in N$, a preference profile P_{-i} of other agents, and P_i, P'_i such that P'_i is an (a, b) -swap of P_i and $f(P_i, P_{-i}) = b$. The two profiles are shown in [Table 21](#). Notice that there are some agents in P_{-i} who prefer a to b and some prefer b to a .

We will now show that $f(P'_i, P_{-i}) = b$. By swap monotonicity, $f(P'_i, P_{-i}) \in \{a, b\}$. Assume for contradiction that $f(P'_i, P_{-i}) = a$. Now, we do the proof in several steps.

STEP 1. We modify the profile (P_i, P_{-i}) to bring one of the alternatives not in $\{a, b\}$ (such an alternative exists since $|A| \geq 3$) just below $\{a, b\}$ for all the agents. Let $x \notin \{a, b\}$ be some alternative. If $a P_j x$ and $b P_j x$ for some $j \in N$, then we can do a series of swaps to lift x up such that it is just below b if $a P_j b$ or just below a if $b P_j a$ (note that none of these swaps will involve b). By swap monotonicity, the outcome at the new profile continues to be b . Using a similar argument, if $b P_j x$ and $x P_j a$ for some $j \in N$, then we can come to a preference ordering where x is just below a maintaining the outcome to be b .

Table 22
Profile $\bar{\mathbf{P}}$.

$\bar{\mathbf{P}}$			
...
...
a	b	a	x
...	b
...
b	a	x	...
x	x	b	a
...
...

Table 23
Profiles $(\bar{P}'_i, \bar{P}'_{-i})$ and $(\bar{P}''_i, \bar{P}'_{-i})$.

\bar{P}'_i	\bar{P}'_{-i}	\bar{P}''_i	\bar{P}'_{-i}
...
...
a	$b \dots$	$a \dots$	b
b	a
x	$a \dots$	$b \dots$	x
...	$x \dots$	$x \dots$	$a \dots$
...	$b \dots$
...	$x \dots$
...

Now, consider $j \in N$, such that $x P_j b$. If x and b are not consecutive in P_j , then again we can do a series of swaps to come to a preference ordering such that x is just above b (note again that none of these swaps will involve b). By swap monotonicity, the outcome at the new profile continues to be b . Let us denote this new profile by $\bar{\mathbf{P}}$.

So, we have reached a profile $\bar{\mathbf{P}}$, where for every $j \in N$, either x is just above b in \bar{P}_j or [x is just below b if $a \bar{P}_j b$ and x is just below a if $b \bar{P}_j a$]. [Table 22](#) shows the profile $\bar{\mathbf{P}}$.

Now, for every $j \in N$ such that x is just above b in \bar{P}_j (Columns 3 and 4 in [Table 22](#)). We do a (x, b) -swap. By swap monotonicity the outcome at the new profile is either x or b . But b is preferred to x by all the agents, and hence, Pareto efficiency implies that the outcome at this profile is b . For every j belonging to Column 4 in [Table 22](#), we then do a sequence of swaps to get x just below a . Denote this new preference ordering by \bar{P}'_j . By swap monotonicity the outcome at the new profile is b . Denote the new profile as $\bar{\mathbf{P}'}$ and note that $f(\bar{\mathbf{P}'}) = b$.

Now, consider the (a, b) -swap of \bar{P}'_i and denote this preference ordering as \bar{P}''_i . Since $f(P'_i, P_{-i}) = a$, an analogous argument will show that $f(\bar{P}''_i, \bar{P}'_{-i}) = a$. The two profiles $(\bar{P}'_i, \bar{P}'_{-i})$ and $(\bar{P}''_i, \bar{P}'_{-i})$ are shown in [Table 23](#).

STEP 2. In this step, we modify the profile $(\bar{P}'_i, \bar{P}'_{-i})$ in a particular way. First, we look at an agent $j \in N$, such that $a \bar{P}'_j b$ and $b \bar{P}'_j x$. We perform a (b, x) swap for each of these agents. The new profile is shown in [Table 24](#). By swap monotonicity, the outcome at the new profile must be in $\{b, x\}$. But since a is ranked higher than x for all the agents, Pareto efficiency implies the outcome at the new profile must be b .

STEP 3. In this step, we modify the profile in [Table 24](#) further. In particular, we lift x just above a . For agent i and for all $j \neq i$ such that x is just below a , this can be done by a (a, x) -swap. For

Table 24
New profile in Step 2.

Agent i	Other agents	
.
.
a	$b\dots$	$a\dots$
x
b	$a\dots$	$x\dots$
.	$x\dots$	$b\dots$
.

Table 25
New profile in Step 3.

Agent i	Other agents	
.
.
x	$b\dots$	$x\dots$
a	...	$a\dots$
b	$x\dots$...
.	$a\dots$	$b\dots$
.

Table 26
New profile in Step 4.

Agent i	Other agents	
.
.
x	$b\dots$	$x\dots$
b	...	$a\dots$
a	$x\dots$...
.	$a\dots$	$b\dots$
.

all other agents, this requires a series of swaps – note that these swaps can be done by without involving b . The new profile is shown in Table 25. Since none of the swaps involve b , swap monotonicity implies that the outcome at the new profile remains b .

STEP 4. In this step, we modify the profile in Step 3 by changing only agent i 's preference ordering. We do this by doing an (a, b) -swap of the preference ordering of agent i in the profile shown in Table 25. The new profile is shown in Table 26. By swap monotonicity, the outcome at the new profile is in $\{a, b\}$. But x is better than a for all agents, and hence, Pareto efficiency implies the outcome at the new profile is b .

STEP 5. In this step, we modify the profile in Step 4 by changing the preferences of those agents who prefer x to a and a to b (the third column of agents in Table 26). We perform a series of swaps to bring x just one position above b . The new profile is shown in Table 27. By swap monotonicity, the outcome at this profile remains b .

STEP 6. Now, we perform an (x, b) -swap of preferences of those agents who rank x just above b in the profile in Step 5 – this will be agent i and agents in the third column in Table 27. The new profile is shown in Table 28. By swap monotonicity, the outcome at the new profile is in $\{x, b\}$.

Table 27
New profile in Step 5.

Agent i	Other agents	
.
.
x	$b\dots$	$a\dots$
b
a	$x\dots$	$x\dots$
.	$a\dots$	$b\dots$
.

Table 28
New profile in Step 6.

Agent i	Other agents	
.
.
b	$b\dots$	$a\dots$
x
a	$x\dots$	$b\dots$
.	$a\dots$	$x\dots$
.

Table 29
New profile in Step 7.

Agent i	Other agents	
.
.
b	$b\dots$	$a\dots$
a
x	$a\dots$	$b\dots$
.	$x\dots$	$x\dots$
.

But b is preferred to x for all the agents. Hence, Pareto efficiency implies the outcome at the new profile remains b .

STEP 7. Finally, we perform a (x, a) -swap for the preferences of all agents in the profile in Step 6 who rank x just above a – this will include agent i and agents in the second column of Table 28. The new profile is shown Table 29. By swap monotonicity, the outcome at this profile remains b .

But the profile shown in Table 29 is exactly the profile $(\bar{P}_i'', \bar{P}_{-i}')$ (see Table 23) and we had assumed that $f(\bar{P}_i'', \bar{P}_{-i}') = a$. This is a contradiction. \square

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