1 Assignment 2 (Choice Under Uncertainty)

1. Verify the examples E1-E7 in the notes of preferences over lotteries if they satisfy the continuity and the independence axioms of expected utility theory.

   Answer. We give rough sketch of the answer.

   - E7: Expected Utility. Expected utility is a linear function of probabilities. Hence, it is continuous in probabilities. So, it will satisfy continuity. To be precise, pick $p \succeq q \succeq r$. This means that

     $$\sum_{z\in Z} u(z)p(z) \geq \sum_{z\in Z} u(z)q(z) \geq \sum_{z\in Z} u(z)r(z).$$

     Let

     $$U(p) = \sum_{z\in Z} u(z)p(z), U(q) = \sum_{z\in Z} u(z)q(z), U(r) = \sum_{z\in Z} u(z)r(z).$$

     Since $U(p) \geq U(q) \geq U(r)$, there is $\alpha \in [0,1]$ such that $U(q) = \alpha U(p) + (1-\alpha)U(r)$. Now define, the lottery q' as follows:

     $$q'(z) = \alpha p(z) + (1-\alpha)r(z)$$

     for all $z \in Z$. Notice that

     $$\begin{aligned}\sum_{z\in Z} u(z)q'(z) &= \sum_{z\in Z} \big[u(z)\alpha p(z) + u(z)(1-\alpha)r(z)\big] \\ &= \alpha U(p) + (1-\alpha)U(r) = U(q).\end{aligned}$$

     Hence, $q' \sim q$. This establishes Continuity.

     To show independence, again use linearity. Take two lotteries p, q with $p \succeq q$ and let $\alpha \in [0,1]$. As before, define $U(p) = \sum_{z\in Z} u(z)p(z)$ and $U(q) = \sum_{z\in Z} u(z)q(z)$. Now, for every lottery r,

     $$U(\alpha p \oplus (1-\alpha)r) = \alpha U(p) + (1-\alpha)U(r)$$

     and

     $$U(\alpha q \oplus (1-\alpha)r) = \alpha U(q) + (1-\alpha)U(r).$$

     Since $p \succeq q$, $U(p) \geq U(q)$. Hence,

     $$U(\alpha p \oplus (1-\alpha)r) \geq U(\alpha q \oplus (1-\alpha)r),$$

     which implies that $\alpha p \oplus (1-\alpha)r \succeq \alpha q \oplus (1-\alpha)r$.

- E3: Good Outcomes. This can be represented as an expected utility maximization. In particular, if G is the set of good outcomes, then define $u(z) = 1$ for all $z \in G$ and $u(z) = 0$ for all $z \notin G$. Now, notice that for all $p \in L(Z)$, we have $\sum_{z \in Z} u(z)p(z) = \sum_{z \in G} p(z)$. Hence, for any $p, q \in L(Z)$, we have $\sum_{z \in Z} u(z)p(z) \geq \sum_{z \in Z} u(z)q(z)$ if and only if $\sum_{z \in G} p(z) \geq \sum_{z \in G} q(z)$.

  Since this particular preference over lotteries is an expected utility maximizer, it satisfies both continuity and independence.

- E6: Lexicographic Preferences. Suppose the DM orders the outcomes using the strict ordering P as $z_1 P z_2 P \dots P z_n$. As we have seen, this fails continuity. To see this suppose, $[z_1] \succ [z_2] \succ [z_3]$. Now, any mixture of z_1 and z_3 will either have positive probability on z_1 or probability one on z_3. In the first case, such a mixture will be better than the degenerate lottery $[z_2]$. In the latter case, it is the degenerate lottery $[z_3]$, which is worse than $[z_2]$. Hence, no mixture of z_1 and z_3 can be indifferent to $[z_2]$.

  However, this preference satisfies independence. To verify this, pick $p \succeq q$, $r \in L(Z)$, and $\alpha \in [0, 1]$. Since $p \succeq q$, either $p = q$, in which case we are done or $p \succ q$. If $p \succ q$, then there is some k such that $p(z_j) = q(z_j)$ for all $j < k$ and $p(z_k) > q(z_k)$. But when we consider the mixture $p' := \alpha p \oplus (1 - \alpha)r$ and $q' := \alpha q \oplus (1 - \alpha)r$, we see that $p'(z_j) = q'(z_j)$ for all $j < k$ and $p'(z_k) > q'(z_k)$. Hence, $p' \succ q'$.

- E1: Most Likelihood. This satisfies continuity. To see this, suppose $p \succeq q \succeq r$. Let $m_p = \max_{z \in Z} p(z)$, $m_q = \max_{z \in Z} q(z)$, and $m_r = \max_{z \in Z} r(z)$. By definition, $m_p \geq m_q \geq m_r$. By definition, there exists $\alpha \in [0, 1]$ such that $m_q = \alpha m_p + (1 - \alpha)m_r$. Hence, the lottery $\alpha p \oplus (1 - \alpha)r \sim q$.

  However, this preference violates independence. To see this, let $Z = \{z_1, z_2, z_3\}$, $p(z_1) = 0.3, p(z_2) = 0.5, p(z_3) = 0.2$, and $q(z_1) = 0.4, q(z_2) = q(z_3) = 0.3$. By definition, $p \succ q$. Now, choose the degenerate lottery $[z_1]$ and consider the mixtures $0.5[z_1] \oplus 0.5p$ and $0.5[z_1] \oplus 0.5q$. Denote the first mixture as p' and the second mixture as q'. It is easy to check that $p'(z_1) = 0.65, p'(z_2) = 0.25, p'(z_3) = 0.1$ and $q'(z_1) = 0.7, q'(z_2) = q'(z_2) = 0.15$. Hence, $q' \succ p'$. This violates independence.

- E2: Size of Positive Support. This violates both continuity and independence. Let $Z = \{z_1, z_2, z_3\}$. Let p be the degenerate lottery $[z_1]$, q be the lottery $q(z_1) = q(z_2) = 0.5, q(z_3) = 0$ and r be the lottery $r(z_1) = r(z_2) = r(z_3) = \frac{1}{3}$. Now, $r \succ q \succ p$. Any mixture of r and p will either produce a lottery whose size of positive support is either 3 or 1. Hence, it cannot be indifferent to q. So, continuity is violated.

For independence, $r \succ q$. But any mixture with p will produce lotteries which will have size of positive support 3 and hence, will be indifferent. This violates independence.

- E4: Worst Case. Consider $Z = \{z_1, z_2, z_3\}$. Let $u(z_1) = 1, u(z_2) = 0.5, u(z_3) = 0$. Then, using the worst case criteria, $[z_1] \succ [z_2] \succ [z_3]$. Now, choose $\alpha \in [0,1]$ and consider $\alpha[z_1] \oplus (1-\alpha)[z_3]$. For $\alpha = 0$ and $\alpha = 1$, this mixture is same as $[z_1]$ and $[z_3]$ respectively, and hence, cannot be indifferent to $[z_2]$. For any $\alpha \in (0,1)$, we see that the mixture gives positive probability to both z_1 and z_3. Hence, the worst case criteria will evaluate its utility at $u(z_3) = 0$. This will be less that the utility for $[z_2]$. Hence, no mixture can be indifferent to $[z_2]$. This shows that the worst case criteria does not satisfy continuity.

  The worst case criteria also violates independence. To see this, we see that $0.5[z_1] \oplus 0.5[z_3] \sim 0.5[z_2] \oplus 0.5[z_3]$ - this is because both the lotteries will put some positive probability on z_3, and this immediately means the utility of both the lotteries according to the worst case is $u(z_3) = 0$. But note that $[z_1] \succ [z_2]$. Hence, independence is violated.

- E5: Most Likely Comparisons. Consider $Z = \{z_1, z_2, z_3\}$ and assume that $[z_1] \succ [z_2] \succ [z_3]$. If we take a mixture $p \equiv 0.2[z_1] \oplus 0.8[z_3]$ and $q \equiv 0.2[z_2] \oplus 0.8[z_3]$, then independence requires that $p \succ q$. But note that the most likely outcome in both p and q are z_3. Hence, it must be $p \sim q$. So, independence is violated.

  This relation also violates continuity. To see this, note that $[z_1] \succ [z_2] \succ [z_3]$ implies that by continuity there is some α such that $\alpha[z_1] \oplus (1-\alpha)[z_3] \sim [z_2]$. But $\alpha[z_1] \oplus (1-\alpha)[z_2]$ will have either z_1 or z_2 as the most likely outcome, and hence will either be better or worse than $[z_3]$. Hence, it can never be indifferent to $[z_3]$.