Final examination

Mathematical programming with applications to economics

November 02, 2018; Time: **10 AM to 1 PM**.

Be precise and clear in your arguments. Arguments which skip some steps will be treated as incomplete.

1. A dictionary of the simplex method for a linear program is the following:

$$\begin{aligned} x_2 &= 14 - 2x_1 - 4x_3 - 5x_5 - 3x_7 \\ x_4 &= 5 - x_1 - x_3 - 2x_5 - x_7 \\ x_6 &= 1 + 5x_1 + 9x_3 + 21x_5 + 11x_7 \\ z &= 29 - x_1 - 2x_3 - 11x_5 - 6x_7. \end{aligned}$$

Suppose x_1, x_2, x_3, x_4 are the original variables in the linear program.

(a) Write down the value of the variables and the objective function in the optimal solution of this linear program. **(2 marks)**

Answer. This dictionary is optimal because non-basic variables have negative coefficient in the objective function row. The optimal solution: $x_1 = 0, x_2 = 14, x_3 = 0, x_4 = 5$ and $z = 29$.

(b) Write down the original linear program in standard form. **(3 marks)**

Answer. Multiplying the second equation by 3 and subtracting from the first equation, we get

$$x_2 - 3x_4 = (-1) + x_1 - x_3 + x_5.$$

So, we get $x_5 = 1 - x_1 + x_2 + x_3 - 3x_4$. Since x_5 is a slack variable, we get our first inequality,

$$x_1 - x_2 - x_3 + 3x_4 \leq 1.$$

Substituting the equation for x_5 in the first row of the dictionary gives us an expression for x_7. $x_7 = 3 + x_1 - 2x_2 - 3x_3 + 5x_4$. Since x_7 is a slack variable, we get our next inequality:

$$-x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3.$$

Finally, substituting x_5 and x_7 is the last row of the dictionary gives us: $x_6 = 55 - 5x_1 - x_2 - 3x_3 - 8x_4$. Since x_6 is a slack variable, we get our last inequality:

$$5x_1 + x_2 + 3x_3 + 8x_4 \leq 55.$$

So, the linear program (primal) is the following:

$$\begin{aligned} \max 4x_1 + x_2 + 5x_3 + 3x_4 & \\ \text{subject to} & \\ x_1 - x_2 - x_3 + 3x_4 &\leq 1 \\ 5x_1 + x_2 + 3x_3 + 8x_4 &\leq 55 \\ -x_1 + 2x_2 + 3x_3 - 5x_4 &\leq 3 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

(c) Write down the dual of this linear program. **(3 marks)**

Answer. The dual of this linear program is:

$$\begin{aligned} \min y_1 + 55y_2 + 3y_3 & \\ \text{subject to} & \\ y_1 + 5y_2 - y_3 &\geq 4 \\ -y_1 + y_2 + 2y_3 &\geq 1 \\ -y_1 + 3y_2 + 3y_3 &\geq 5 \\ 3y_1 + 8y_2 - 5y_3 &\geq 3 \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

(d) Without explicitly solving, write down the value of the variables and the objective function in the optimal solution of the dual of this linear program. **(2 marks)**

Answer. This can be seen from the coefficients of slack variables in the objective function row of the final dictionary: $y_1 = 11, y_2 = 0, y_3 = 6$. By strong duality, $z = 29$ for the dual also.

2. **Birkhoff-von-Neumann Theorem**. A $n \times n$ matrix A is **doubly stochastic** if each entry is non-negative, each row sum is 1 and each column sum is 1. A doubly stochastic matrix is a **permutation** matrix if each entry is $\{0, 1\}$. Show that every doubly stochastic matrix is a convex combination of a finite number of permutation matrices. **(10 marks)**

Answer. Denote the entry in the i-th row and j-th column of a doubly stochastic matrix as x_{ij}. By definition,

$$\begin{aligned}
\sum_{i=1}^{n} x_{ij} &= 1 \qquad && \forall\, j \in \{1, \ldots, n\} \\
\sum_{j=1}^{n} x_{ij} &= 1 \qquad && \forall\, i \in \{1, \ldots, n\} \\
x_{ij} &\geq 0 \qquad && \forall\, i, j \in \{1, \ldots, n\}.
\end{aligned}$$

We know that the constraint matrix of this polytope is TU, and hence, it has integral extreme points. Since $x_{ij} \leq 1$ for all i, j, it must be that $x_{ij} \in \{0, 1\}$ for any extreme point. Thus the extreme points (which are finite) of this polytope correspond to the permutation matrices. Hence, any doubly stochastic matrix can be expressed as convex combination of a finite number of permutation matrices.

3. Let $G = (N, E)$ be a directed graph with no cycles. Let s and t be two nodes in G. Two $s - t$ paths from s to t are disjoint if they do not share an edge. The maximum disjoint path (MDP) problem is to find the maximum number of $s - t$ paths in G.

   (a) Formulate the MDP as an integer program whose LP relaxation gives integral optimal solution (argue your answer). **(5 marks)**

   Answer. For every edge $(i, j) \in E$, let $x_{ij} \in \{0, 1\}$ be the binary variable denoting if edge (i, j) belongs to an $s - t$ path. The constraints for choosing a path is

$$\sum_{i \in N:(i,j) \in E} x_{ij} - \sum_{i \in N:(j,i) \in E} x_{ji} = 0 \qquad \forall\, j \in N \setminus \{s, t\}.$$

   A feasible solution to this will be an integral flow on in the associated network. Since there are no cycles in the graph, this gives feasible flows along $s - t$ paths. Since each edge has capacity one, these are edge-disjoint flows. So, the corresponding integer program is as follows.

$$\begin{aligned}
\max \sum_{j \in N:(s,j) \in E} & x_{sj} \\
\sum_{i \in N:(i,j) \in E} x_{ij} - \sum_{i \in N:(j,i) \in E} & x_{ji} = 0 \qquad && \forall\, j \in N \setminus \{s, t\} \\
& x_{ij} \in \{0, 1\} \qquad && \forall\, (i, j) \in E.
\end{aligned}$$

The LP relaxation is as follows.

$$\max \sum_{j \in N:(s,j)\in E} x_{sj}$$

$$\sum_{i\in N:(i,j)\in E} x_{ij} - \sum_{i\in N:(j,i)\in E} x_{ji} = 0 \qquad \forall\ j \in N \setminus \{s,t\}$$

$$x_{ij} \le 1 \qquad \forall\ (i,j) \in E$$

$$x_{ij} \ge 0 \qquad \forall\ (i,j) \in E.$$

Notice that the LP relaxation has integral extreme points since the constraint matrix is TU (network flow constraint).

There are other formulations but their LP relaxation may not give integral solution. The current formulation has the property that its LP relaxation satisfies integrality property.

(b) Write the dual of this LP relaxation. **(3 marks)**

Answer. The dual has two types of variables z_j for each vertex $j \in N \setminus \{s,t\}$ and y_{ij} for every edge (i,j).

$$\min \sum_{(i,j)\in E} y_{ij}$$

$$y_{ij} + z_j - z_i \ge 0 \qquad \forall\ (i,j) \in E, i \ne s, j \ne t$$

$$y_{it} - z_i \ge 0 \qquad \forall\ (i,t) \in E$$

$$y_{sj} + z_j \ge 1 \qquad \forall\ (s,j) \in E$$

$$y_{ij} \ge 0 \qquad \forall\ (i,j) \in E.$$

(c) Give a precise statement of the complementary slackness theorem for this problem. **(2 marks)**

Answer. It is important to write down the complentary slackness **theorem** (not just equations).

CS Theorem. If x^* is a feasible solution of primal and (y^*, z^*) is a feasible solution of dual then, they are optimal if and only if

$$(y^*_{ij} + z^*_j - z^*_i)x^*_{ij} = 0 \qquad \forall\ (i,j) \in E, i \ne s, j \ne t$$

$$(y^*_{it} - z^*_i)x^*_{it} = 0 \qquad \forall\ (i,t) \in E$$

$$(y^*_{sj} + z^*_j - 1)x^*_{sj} = 0 \qquad \forall\ (s,j) \in E$$

$$(1 - x^*_{ij})y^*_{ij} = 0 \qquad \forall\ (i,j) \in E.$$

4. Let E be a finite set and $\mathcal{P}(E)$ be all non-empty subsets of E. Let $r : \mathcal{P}(E) \to \mathbb{Z}_+$ be a non-negative integer-valued function ($\mathbb{Z}_+$ is the set of all non-negative integers) satisfying $r(S) \leq |S|$ for all $S \in \mathcal{P}(E)$. Let $\mathbb{I} = \{S \in \mathcal{P}(E) : r(S) = |S|\}$.

Typo. There is a typo in the question - $\mathcal{P}(E)$ should be the set of all subsets (including the $\emptyset$). This automatically defines $r(\emptyset) = 0$ since $r(S) \leq |S|$ for all S and r is non-negative valued. Every matroid must have $\emptyset \in \mathbb{I}$ - else, hereditary property will fail.

Though I announced r to be positive integer valued, this does not help in any way. I have ignored this typo in grading your answerscripts.

I give solutions below with this typo corrected.

(a) Show that if r is non-decreasing and submodular, then $(E, \mathbb{I})$ is a matroid. **(5 marks)**

Answer. $\mathbb{I}$ is non-empty because $\emptyset \in \mathbb{I}$. For hereditary, pick $T \in \mathbb{I}$. Since $r(T) = |T|$, for any $S \subseteq T$, we see

$$r(S) \leq |S| = |T| - |T \setminus S| \leq |T| - r(T \setminus S).$$

So, we have $r(S) + r(T \setminus S) \leq |T| = r(T)$. But submodularity gives $r(S) + r(T \setminus S) \geq r(T)$. Hence, we get $r(S) + r(T \setminus S) = r(T)$. Since $r(T \setminus S) \geq 0$, we get $r(S) \leq r(T)$.

For augmentation, pick $S, T \in \mathbb{I}$ with $|S| + 1 = |T|$. We know that if r is submodular (see Lemma 28 in notes),

$$r(T) - r(S) \leq \sum_{x \in T \setminus S} \big[r(S \cup \{x\}) - r(S) \big]$$

Since S and T are in $\mathbb{I}$ and $|T| = |S| + 1$, we get $r(T) - r(S) = 1$. So, for some $x \in T \setminus S$ we have $r(S \cup \{x\}) - r(S) \geq 1$. So, $r(S \cup \{x\}) \geq |S| + 1$. But $r(S \cup \{x\}) \leq |S| + 1$ by definition. Hence, $r(S \cup \{x\}) = |S| + 1$, which implies that $(S \cup \{x\}) \in \mathbb{I}$.

(b) Define a new function $r^* : \mathcal{P}(E) \to \mathbb{Z}_+$ as follows:

$$r^*(S) = |S| - r(E) + r(E \setminus S) \qquad \forall \ S \in \mathcal{P}(E).$$

Show that r^* is the rank function of a matroid $(E, \mathbb{I}^*)$. **(5 marks)**

Answer.

Set $\mathbb{I}^* := \{S : r^*(S) = |S|\}$. Again $r^*(\emptyset) = 0$ implies $\mathbb{I}^*$ is non-empty.

Now, note that r^* is non-negative: for any S, $r^*(S) = |S| - r(E) + r(E \setminus S) \geq r(S) + r(E \setminus S) - r(E) \geq 0$, where the last inequality comes from submodularity of r. Also, $r^*(S) = |S| - r(E) + r(E \setminus S) \leq |S|$, where the last inequality follows because $r(E) \geq r(E \setminus S)$. Hence, r^* is non-negative integer and $r^*(S) \leq |S|$. If we can prove that r^* is non-decreasing and submodular, we will be done by the previous question.

Non-decreasing. Take $S \subseteq T$. $r^*(T) - r^*(S) = |T| - |S| + r(E \setminus T) - r(E \setminus S) \geq |T \setminus S| - r(T \setminus S)$, where the last inequality follows from submodularity of r: $r(E \setminus T) + r(T \setminus S) \geq r(E \setminus S)$. This gives $r^*(T) - r^*(S) \geq |T \setminus S| - r(T \setminus S) \geq 0$, by definition of r.

Submodularity. Fix $S \subseteq T$ and $x \notin T$. Note the following:

$$\begin{aligned} r^*(S \cup \{x\}) - r^*(S) &= 1 - \big(r(E \setminus S) - r(E \setminus (S \cup \{x\}))\big) \\ &\geq 1 - \big(r(E \setminus T) - r(E \setminus (T \cup \{x\}))\big) && \text{(Using submodularity of } r\text{)} \\ &= r^*(T \cup \{x\}) - r^*(T). \end{aligned}$$

(c) What is the relation between the independent sets of the two matroids $(E, \mathbb{I})$ and $(E, \mathbb{I}^*)$? **(5 marks)**

Answer. If S is an independent set of r^*, then $r^*(S) = |S|$ and this means $r(E) = r(E \setminus S)$. Since $r(E)$ is the size of maximal independent set (basis) of the matroid, it follows that $E \setminus S$ must contain a basis of the matroid. So the set of independent sets $\mathbb{I}^*$ is derived from $\mathbb{I}$ as follows:

$$\mathbb{I}^* := \{S : (E \setminus S) \text{ contains a maximal independent set of } \mathbb{I}\}.$$