Ranking distributions of an ordinal attribute[1]

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Abstract

We establish an equivalence between three criteria for comparing distributions of an ordinally measurable attribute taking finitely many values. The first criterion is the possibility of going from one distribution to the other by a finite sequence of increments and Hammond transfers. The latter transfers are like the Pigou-Dalton ones, but without the requirement that the amount transferred be fixed. The second criterion is the unanimity of all comparisons of the distributions performed by a large class of additively separable social evaluation functions. The third criterion is a new statistical test based on a weighted recursion of the cumulative distribution.

1 Introduction

When can we say that one distribution of a *cardinally meaningful* attribute among a group of agents is more equal than another? One of the greatest achievements of the modern theory of inequality measurement is the demonstration, made by Hardy, Littlewood, and Polya (1952) and popularized among economists by Kolm (1969), Atkinson (1970), Dasgupta, Sen, and Starrett (1973), Sen (1973) and Fields and Fei (1978), that the following *three* answers to this question are *equivalent*:

1) When one distribution has been obtained from the other by a finite sequence of Pigou-Dalton transfers.

2) When one distribution would be considered better than the other by all utilitarian planners who assume that agents convert income into utility by the same increasing and concave function.

3) When the Lorenz curve associated with one distribution lies nowhere below, and at least somewhere above, that of the other.

The equivalence of these three answers is important because it ties together three *a priori* distinct aspects of inequality measurement. The first - Pigou-Dalton transfer - is an *elementary transformation* of the distribution that captures, in a crisp fashion, the nature of the equalization at stake. The second aspect is the *ethical principle* underlying utilitarianism or, more generally, *additively separable* social evaluation. The third aspect is the *empirically implementable criterion* underlying Lorenz dominance.

The current paper addresses the very same question in the case where the distributed attribute is *ordinal* in nature. Over the last twenty years, there has been extensive use of data involving distributions of attributes such as access to basic services, educational achievements, health outcomes, and self-declared happiness, to mention just a few. When comparing distributions of such attributes, it is not uncommon for researchers to disregard the ordinal nature of the attribute and to treat it, just like income, as a variable that can be "summed", or "transferred" across agents. Examples include Castelló-Clement and Doménech (2002) and Castelló-Clement and Doménech (2008) (discussing inequality indices on human capital) and Pradhan, Sahn, and

Younger (2003) (decomposing Theil indices applied to the heights children under 36 month interpreted as a measure of health). Yet, following the contribution by Allison and Foster (2004), there has been a growing awareness of the need to duly account for the ordinal nature of the numerical information conveyed by the indicators. Examples of studies taking the ordinal nature of the attribute into account when normatively appraising its distribution include Abul-Naga and Yalcin (2008), Apouey (2007), Zheng (2008), Zheng (2011) and Cowell and Flachaire (2017).

A difficulty raised by the normative evaluation of distributions of an ordinal attribute is that of defining an adequate notion of inequality reduction. What does it mean for a distribution of an ordinal attribute to be "more equal" than another? It is no use invoking the notion of Pigou-Dalton transfer for answering that question. A Pigou-Dalton transfer is, in effect, the operation by which an agent transfers a *given quantity* of the attribute to another agent. This notion of "given quantity" is meaningless when applied to an attribute measured in an ordinal fashion.

Some forty years ago, Peter J. Hammond (1976) proposed, in the context of social choice theory, a "minimal equity principle" that is explicitly concerned with distributions involving an ordinally measurable variable. According to Hammond's principle, a change in the distribution that reduces the gap between two agents endowed with different quantities of the ordinal attribute is a good thing, whether or not the gain from the poor recipient is equal to the loss from the rich giver. The purely ordinal nature of Hammond transfers qualifies them, in our view, as highly plausible instances of clear inequality reduction.

The main contribution of this paper is to identify a normative dominance criterion and a statistically implementable criterion that are each equivalent to the notion of equalization underlying Hammond transfer. It does so in the specific but empirically important case where the ordinal attribute can take only a *finite* number of different values. Our choice of this finite case has specific implications for the Hammond equity principle because, as is well known in social choice theory (see e.g. D'Aspremont and Gevers (1977), D'Aspremont (1985), Hammond (1979) and Sen (1977)), when attribute quantities can vary continuously, the principle is closely related to the lexicographic extension of the Maximin - or Leximin - ordering. Hammond (1979) has even shown that any anonymous, Pareto-inclusive and transitive ranking of all vectors in $\mathbb{R}^n$ that is strictly sensitive to Hammond transfers must be the Leximin ordering. As shown in this paper, this tight connection between the Leximin criterion and Hammond transfers becomes significantly looser when attention is restricted to distributions of an attribute that can take finitely many different values.

Concerning the class of normative principles in the spirit of answer 2) above, we stick to the tradition of comparing distributions by means of an additively separable social evaluation function. Each attribute quantity is thus assigned a numerical value by some function, and distributions are compared on the basis of the sum - taken over all agents - of these values. While this normative approach *can* be considered utilitarian (if the value assigned to the attribute is interpreted as "utility"), it *does not need to* be. One could also interpret the function more generally as an *advantage function* reflecting the value assigned to each agent's attribute quantity by some "ethical observer". If the attribute is considered to be good for the agent, the advantage function can be assumed to be increasing with respect to the attribute. We show in this paper that, in order for a ranking of distributions based on an additively separable social evaluation function to be sensitive to Hammond transfers, it is necessary and sufficient for the advantage function to satisfy a somewhat *strong concavity* property. Specifically, any increase in the quantity of the attribute obtained from some initial level must increase the advantage more than any increase obtained at some higher level of the attribute, no matter what the latter increase is. Because of this result, we consider the ranking of distributions provided by the unanimity of all additively separable rankings based on an advantage function that is strongly concave in this sense.

The empirical implementable criterion that we consider is, to the best of our knowledge, a new one. Its construction is based on a curve that we call the H-curve, by reference to the Hammond principle of transfer to which it is, as it turns out, closely related. The H-curve is defined recursively as follows. It starts by assigning to the smallest possible quantity of the ordinal attribute the fraction of the population that are endowed with this quantity. It then proceeds recursively, for any quantity of the attribute strictly larger than this smallest quantity, by adding together the relative frequency of the population endowed with this quantity of the attribute *and* twice the value assigned by the curve to the immediately preceding category. The criterion that we propose, and that we call H-dominance, is for the dominating distribution to have an H-curve nowhere above and somewhere below that of the dominated one. As illustrated in the paper, the construction of these curves, and the resulting implementation of the criterion, is extremely easy.

This paper provides some justification for the use of such an H-curve. It does so by proving that having a distribution that H-dominates another is equivalent to the possibility of going from the latter to the former by a finite sequence of Hammond transfers and/or increments of the attribute. The paper also shows that H-dominance coincides with the unanimity of all additively separable aggregations of advantage functions that are strongly concave in the manner described above.

While these results justify comparing distributions of an ordinal attribute on the basis of H-dominance, they do not readily lead to a definition of what it means for a distribution to be more equal than another. In effect, H-dominance combines *both* Hammond transfers *and* increments. While the former transformation is a plausible candidate for a definition of an inequality reduction in an ordinal setting, the latter is not. Is it possible to identify an operational criterion that coincides with Hammond transfers *only* and that could thus serve as an operational definition of pure inequality reduction in an ordinal setting?

This point can be illustrated through a parallel with the classical cardinal case. In the classical setting, it is well known that the Lorenz domination of one distribution over another is equivalent, when the two distributions have the same mean, to the possibility of going from the dominated to the dominating distribution by a finite sequence of Pigou-Dalton transfers. If there was a meaningful analogue - in the ordinal setting - to the requirement that two distributions have the "same mean", one could consider paralleling - for that analogue - the standard approach and identifying H-dominance with Hammond transfers only. Unfortunately there is no such meaningful analogue to the mean in the ordinal setting. However, there is another (less well-known) approach in the theory of majorization (see e.g. Marshall, Olkin, and Arnold (2011)) that can be followed to handle that issue. This theory (see Marshall, Olkin, and Arnold (2011) (chapter 1, p. 13, equation (14)) makes it clear that Lorenz dominance between two distributions of a cardinal attribute with the same mean is no more than the *intersection* of *two* independent majorization quasi-orderings that do not assume anything about the mean of the attribute. The first one, called *submajorization* by Marshall, Olkin, and Arnold (2011), defines a distribution of income d as better than a distribution d' if, for any rank k, the sum of the k *highest incomes* is weakly *smaller* in d than in d'. The second criterion, called *supermajorization* by Marshall, Olkin, and Arnold (2011), corresponds to the usual (generalized) Lorenz domination criterion (see e.g. Shorrocks (1983)) according to which d is better than d' if the sum of the k *lowest incomes* is weakly *larger* in d than in d' no matter what k is.

In this paper, we parallel the route taken by Marshall, Olkin, and Arnold (2011) by exploring a *dual* dominance criterion based on what we call the $\overline{H}$-curve of a distribution. This curve is constructed just like the H-curve, except that it starts from "above" (that is from the highest ordinal category) rather than from "below", and iteratively cumulates the (discrete) *survival* function rather than the (discrete) *cumulative* function. We then establish that this notion of $\overline{H}$-dominance coincides with the possibility of going from the dominated to the dominating distribution by a finite sequence of either Hammond transfers and/or decrements. Just as with the submajorization and supermajorization criteria, the ranking of distributions generated by the intersection of the dominance criteria H and $\overline{H}$ could plausibly constitute a clear inequality reduction in an ordinal setting. Indeed, as established in this paper, any finite sequence of Hammond transfers would be recorded as an improvement by this intersection ranking. While we cannot - unfortunately - supply a proof of the converse implication, we do have results that point towards it.

One such result is the equivalence between, on the one hand, the intersection of the H and the $\overline{H}$-dominance criteria and, on the other, the intersection of all rankings produced by additively separable social evaluation functions that use a strongly concave - in the above sense - advantage function. Another such result comes from our examination of the H- and $\overline{H}$-dominance criteria when the finite grid used to define the categories is *refined*. We show in effect that there exists a level of grid refinement above which the H-dominance criterion coincides with the Leximin ranking of the two distributions. Similarly, we show that there exists a level of grid refinement above which the $\overline{H}$-domination criterion coincides with the (anti) Leximax ranking of those two distributions. Note that these results are consistent with those obtained in classical social choice theory. The Leximin criterion can be viewed as the limit of the H-criterion when the number of different categories of the attribute becomes large. Similarly the anti-Leximax criterion can be viewed as the limit of the $\overline{H}$-criterion when the number of different categories becomes large. It follows that the intersection of the anti-Leximax and the Leximin criteria is the limit of the intersection of the H and the $\overline{H}$-criteria when the number of categories becomes large. In Gravel, Magdalou, and Moyes (2017), we prove that a vector in $\mathbb{R}^n$ being ranked above another by both the anti-Leximax and the Leximin criteria is equivalent to the possibility of going from the dominated to the dominating vector by a finite sequence of Hammond transfers. Taken together, these results strongly suggest that the possibility of going from a distribution B to a distribution A by a finite sequence of Hammond transfers is closely related to the dominance of B by A according to both the H- and the $\overline{H}$-criterion.

The plan of the rest of the paper is as follows. The next section introduces the notation, and presents the elementary transformations, the normative criteria and the implementation criteria when the attribute can take finitely many different values. The main results identifying the elementary transformations underlying the H- and the $\overline{H}$-curves are stated and proved in the third section. The fourth section compares the discrete setting and the classical social choice setting originally used for the Hammond equity principle and examines the behavior of the H- (and the $\overline{H}$-) criterion when the number of categories is enlarged. The fifth section illustrates the usefulness of the criteria for comparing the distributions of self-reported health indicators in regions of Switzerland examined in Abul-Naga and Yalcin (2008). The sixth section concludes.

2 Three perspectives for comparing distributions of an ordinal attribute

2.1 Notation

We consider distributions of an ordinal attribute among a *given* number - n say - of agents.[1] We assume that there are k (with $k \geq 3$) different values that the attribute can take. These values, which can be interpreted as "categories" (e.g. "being gravely ill", "being mildly ill", "being in perfect health"), are indexed by h. These categories are assumed to be ordered from the worst (e.g. being gravely ill) to the best (e.g. being in perfect health). We let $\mathcal{C} = \{1, ...k\}$ denote the set of **c**ategories. The fact that the attribute is ordinal means that the integers $1, ..., k$ assigned to the different categories have no significance other than reflecting the ordering of the categories from worst to best. Hence, any comparative statement made on two distributions in which the attribute quantity is measured by the list of numbers $1, ..., k$ would be unaffected if this list was replaced by the list $f(1), ..., f(k)$ generated by any strictly increasing real valued function f. We adopt throughout an *anonymous* perspective according to which "the names of the agents do not matter". This enables us to describe any distribution or ***society*** s as a particular list $(n_1^s, ..., n_k^s)$ of k non-negative integers satisfying $n_1^s + n_2^s + ... + n_k^s = n$, where n_j^s denotes the number of agents in society s who are in category j (for $j = 1, ..., k$).

[1] As is standard in distributional analysis since at least Dalton (1920), distributions of the attribute among a *varying* number of agents can be compared by means of the principle of *population replication* (replicating a distribution any number of times is a matter of social indifference).

2.2 Elementary transformations

The definition of these transformations lies at the very heart of the problem of comparing alternative distributions of an ordinally measured attribute. These transformations are intended to capture in a crisp and concise fashion intuitions about the meaning of "equalizing" or "gaining in efficiency" (amongst others). In defining the transformations, it is important to ensure that they use only ordinal properties of the attribute. In this paper, we discuss *three* such transformations.

The *first* - increment - is hardly new. It captures the idea - somewhat related to efficiency - that moving an individual from one category to a better category is a good thing *ceteris paribus*. We actually formulate this principle in the following minimalist fashion.

Definition 1 *(Increment) We say that society s has been obtained from society s' by means of an increment, if there exist $j \in \{1, \ldots, k-1\}$ such that:*

$$n_h^s = n_h^{s'}, \; \forall \; h \neq j, j+1\,; \tag{1}$$

$$n_j^s = n_j^{s'} - 1\;;\; n_{j+1}^s = n_{j+1}^{s'} + 1\,. \tag{2}$$

In words, society s has been obtained from society s' by an increment if the move from s' to s is the sole result of the move of one agent from a category j to the immediately *superior* category $j+1$.

For the second transformation, in a somewhat reverse fashion, we can introduce the notion of a *decrement* as follows.

Definition 2 *(Decrement) We say that society s has been obtained from society s' by means of a decrement if and only if society s' has been obtained from s by an increment in the sense of Definition 1.*

The *third* elementary transformation considered is the one underlying the *equity principle* put forward by Peter J. Hammond (1976) some forty years ago. This principle, captured in the following definition, considers that a reduction in someone's endowment of the attribute compensated by an increase in the endowment of another person is a good thing if the loser remains, after the reduction, better off than the winner.

Definition 3 *(Hammond Transfer) We say that society s is obtained from society s' by means of a Hammond' transfer, if there exist four categories $1 \leq g < i \leq j < l \leq k$ such that:*

$$n_h^s = n_h^{s'}, \ \forall \ h \neq g, i, j, l\, ; \tag{3}$$

$$n_g^s = n_g^{s'} - 1 \ ; \ n_i^s = n_i^{s'} + 1\, ; \tag{4}$$

$$n_j^s = n_j^{s'} + 1; \ n_l^s = n_l^{s'} - 1\, . \tag{5}$$

While a reduction in someone's endowment that is compensated by an increase in that of someone else may be viewed as the result of a "transfer" of the attribute between the two, it should be noted that unlike standard Pigou-Dalton transfers, this *does not* require the "quantity" given by to be equal to that received by the recipient. Since comparing the gains and losses of an ordinal attribute is meaningless, the Hammond transfer can be viewed as the natural analogue in the ordinal setting of the Pigou-Dalton transfer.[2]

2.3 Normative evaluation

We assume that alternative societies are compared by some ethical observer who uses an additively separable criterion. Such an ethical observer would consider that society s is normatively better than society s' if:

$$\sum_{j=1}^{k} n_j^s \alpha_j \geq \sum_{j=1}^{k} n_j^{s'} \alpha_j \tag{6}$$

holds for some list of k real numbers α_j (for $j = 1, ..., k$). The numbers $(\alpha_1, ..., \alpha_k)$ can, of course, be seen as numerical evaluations of the correspond-

[2] Note that a Pigou-Dalton transfer is nothing else than a Hammond transfer for which the indices g, i, j and l of Definition 3 satisfy the additional condition that $i - g = l - j$ (a given - by $i - g$ - quantity of the attribute is transferred). See Fishburn and Lavalle (1995) or Chakravarty and Zoli (2012) for analysis of Pigou-Dalton transfers in a discrete setting.

ing categories, in which case they would be required to satisfy $\alpha_1 \leq ... \leq \alpha_k$. These valuations may reflect subjective utility (if a utilitarian perspective is adopted) or a non-welfarist interpretation of an agent's falling into the different categories to which these numbers are assigned. If such a non-welfarist perspective is adopted, the specific additive form of the numerical representation (6) of the social ordering can be axiomatically justified (see e.g. Gravel, Marchant, and Sen (2011)).

The ordinal interpretation of the categories means that care should be taken to avoid the normative evaluation exercise being unduly sensitive to particular choices of numbers α_j (for $j = 1, ..., k$). A standard means of preventing this is to require *unanimity* of ranking of two societies as per (6) over a wide class of such lists of k numbers.

2.4 Implementation criteria

Three implementation criteria are considered in this paper. The *first* criterion - first order (stochastic) dominance - is standard. As is well known, it is based on comparing the values taken by the *cumulative distribution* function $F(i; s)$ associated with every society s and every category $i \in \{1, ..., k\}$ and defined by:

$$F(i; s) = \sum_{h=1}^{i} n_h^s / n \,. \tag{7}$$

A society s would then be considered to dominate society s' at the first order if the inequality $F(i; s) \leq F(i; s')$ is observed for every category i.

The *second* implementation criterion examined herein is based on the following H-curve, defined for any society s and any $i \in \{1, ..., k\}$, by:

$$H(i; s) = \sum_{h=1}^{i} \left(2^{i-h}\right) n_h^s / n \ . \tag{8}$$

A few remarks can be made about this curve. First, it verifies:

$$H(1; s) = F(1; s) \tag{9}$$

and:

$$H(i;s) = \sum_{h=1}^{i-1} \left(2^{i-h-1}\right) F(h;s) + F(i;s)\,,\ \forall i = 2,3,\ldots,k. \tag{10}$$

The different values of $H(\cdot;s)$ are therefore nested. Moreover, for any $i = 2,3,\ldots,k$, we have:

$$H(i;s) = 2\,H(i-1;s) + F(i;s) - F(i-1;s) = 2\,H(i-1;s) + n_i^s\,/n. \tag{11}$$

Hence, by successive decomposition, we obtain, for all $i = 2,3,\ldots,k$:

$$H(i;s) = \left(2^j\right) H(i-j;s) + \sum_{h=0}^{j-1} \left(2^h\right) \frac{n_{i-h}^s}{n}\,,\ \forall j = 1,2,\ldots,i-1\,. \tag{12}$$

In plain English, $H(i;s)$ is a (specifically) weighted sum of the fractions of the population in s that are in weakly worse categories than i. The weight assigned to the fraction of the population in category h (for $h < i$) in that sum is 2^{i-h}. Hence the weights are (somewhat strongly) decreasing with respect to the categories. A nice feature of the H-curve - striking in formula (11) - is its recursive construction, which is quite similar to that underlying the cumulative distribution curve. The cumulative distribution F can indeed be defined recursively by:

$$F(1;s) = n_1^s/n \tag{13}$$

and, for $i = 2,...,k$, by:

$$F(i;s) = F(i-1;s) + n_i^s/n \tag{14}$$

The recursion that defines H starts just in the same way as in (13) with:

$$H(1;s) = n_1^s/n$$

but with the iteration formula (14) replaced by:

$$H(i;s) = 2H(i-1;s) + n_i^s/n$$

The H- curve gives rise to a natural notion of dominance: Society s H-dominates society s' if and only if the inequality

$$H(i;s) \leq H(i;s') \tag{15}$$

holds for every category i. Observe that the definition of the H-curve provided by (10) makes it clear that first-order dominance implies H-dominance.

The *last* implementation criterion examined in this paper is somewhat dual to H-dominance. Its formal definition makes use of the *complementary cumulative distribution* function - also known as the survival function - associated with a society s denoted, for every category $i \in \{1,...,k\}$, by $\overline{F}(i;s)$ and defined by:

$$\begin{aligned} \overline{F}(i;s) &= 1 - F(i,s) && (16)\\ &= \sum_{h=i+1}^{k} n_h^s/n \text{ for } i = 1,...,k-1 \text{ and,} && (17)\\ &= 0 \text{ for } i = k && (18) \end{aligned}$$

Hence $\overline{F}(i;s)$ is the fraction of the population in s that is in a *strictly better* category than i. For technical reasons, we find it useful to extend the domain of the definition of $\overline{F}(.;s)$ from $\{1,...,k\}$ to $\{0,...,k\}$ and to set $\overline{F}(0;s) = 1$. With this notation, the following $\overline{H}$-curve can be defined for any society s and any $i \in \{1,...,k\}$ by:

$$\overline{H}(i;s) = \sum_{h=i+1}^{k-1} \left(2^{h-i-1}\right) n_h^s/n \text{ for } i = 1,...k-2 \tag{19}$$

and:

$$\overline{H}(k;s) = 0 \tag{20}$$

This curve is thus constructed under exactly the same recursive principle as the H-curve, but starting with the highest category, and iterating with the complementary cumulative distribution function rather than with the standard cumulative one. The $\overline{H}$-curve therefore starts at category $k-1$:

$$\overline{H}(k-1;s) = \overline{F}(k-1;s) = n_k^s/n \tag{21}$$

and satisfies:

$$\overline{H}(i;s) = \sum_{h=i+1}^{k-1} \left(2^{h-i-1}\right) \overline{F}(h;s) + \overline{F}(i;s)\,,\ \forall i = 1, 2, \ldots, k-2. \tag{22}$$

so that the different values of $\overline{H}(\cdot;s)$ are nested starting from above and going below. Moreover, for any $i = 1, 2, \ldots, k-2$ we have:

$$\overline{H}(i;s) = 2\,\overline{H}(i+1;s) + \overline{F}(i;s) - \overline{F}(i+1;s) = 2\,\overline{H}(i+1;s) + n_{i+1}^s/n. \tag{23}$$

Hence, just as in expression (12) above, we obtain, by successive decomposition, for any $j = 1, ..., k-2$:

$$\overline{H}(j;s) = \left(2^i\right) \overline{H}(j+i;s) + \sum_{h=1}^{i} \left(2^{h-1}\right) \frac{n_{j+h}^s}{n}\,,\ \forall i = 1, \ldots, k-j-1\,. \tag{24}$$

This curve gives rise to the obvious corresponding notion of $\overline{H}$-dominance. A society s $\overline{H}$-dominates society s' if and only if the inequality

$$\overline{H}(i;s) \leq \overline{H}(i;s') \tag{25}$$

holds for every category j.

As illustrated in Section 5, the H- and $\overline{H}$-curves are easy to use and draw. As will also be seen in the next section, the two dominance criteria that they generate serve as perfect diagnostic test of the possibility of moving from the dominated to the dominating distribution by Hammond transfers and increments (for H-dominance) or decrements (for $\overline{H}$-dominance). Moreover, the additional criterion provided by the intersection of $\overline{H}$- and H-dominance turns out to be tightly related to the notion of equalization contained in the Hammond principle of transfers.

We end this section by pointing out the links between some of these notions of dominance. Specifically, we show that first-order dominance of a society s' by a society s entails the H-dominance of society s' by s *and* the $\overline{H}$-dominance of society s by s'. The proof of this result is, like all proofs for this paper, in the Appendix.

Proposition 1 *Suppose s and s' are two societies such that $F(i;s) \leq F(i;s')$ for all categories $i \in \mathcal{C}$. Then, $H(i;s) \leq H(i;s')$ and $\overline{H}(i;s') \leq \overline{H}(i;s)$ for all $i \in \mathcal{C}$.*

3 Equivalence results

This section establishes a few theorems connecting, on the one hand, normative comparison of two societies as per Condition (6) over specific classes of collections of numbers $\alpha_1, ..., \alpha_k$ and, on the other hand, specific implementable criteria as well as the possibility of going from the dominated to the dominating distribution by appropriate elementary transformations.

We start with the notions of increment and decrement. Suppose that we are comparing two societies on the basis of Inequality (6) for some list $\alpha_1, ..., \alpha_k$ of real numbers. What properties must these numbers satisfy for such a comparison to always consider an increment (decrement) as a definite social improvement? It should come as no surprise that the answer to this question is that the k numbers must belong to the following sets:

$$\mathcal{A}_F = \{(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k : \alpha_1 \leq ... \leq \alpha_k\} \text{ (for increments)}$$

and

$$\mathcal{A}_{\overline{F}} = \{(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k : \alpha_1 \geq ... \geq \alpha_k\} \text{ (for decrements)}$$

Set $\mathcal{A}_F$ (resp. $\mathcal{A}_{\overline{F}}$) is the largest set of valuations of the k categories for which the ranking of two societies as per Inequality (6) will consider an increment (resp. a decrement) as a normative improvement. The following two propositions establish this formally.

Proposition 2 *For any two societies s and s', s being obtained from s' by an increment as per Definition 1 implies Inequality (6) for all lists of real* *numbers* $(\alpha_1, ..., \alpha_k) \in \mathcal{A}$ *if and only if* $\mathcal{A} = \mathcal{A}_F$.

Proposition 3 *For any two societies* s *and* s', s *being obtained from* s' *by a decrement as per Definition 2 implies Inequality (6) for all lists of real numbers* $(\alpha_1, ..., \alpha_k) \in \mathcal{A}$ *if and only if* $\mathcal{A} = \mathcal{A}_{\overline{F}}$.

We now use these propositions to establish the following two theorems, the proof of which makes use of the following technical decomposition result.

Lemma 1 *For any society* s *and any conceivable collection of* k *numbers* $(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k$, *we have:*

$$\frac{1}{n}\sum_{h=1}^{k} n_h^s \, \alpha_h = \alpha_k - \sum_{h=1}^{k-1} F(h;s)\,[\alpha_{h+1} - \alpha_h] \ , \tag{26}$$

or equivalently:

$$\frac{1}{n}\sum_{h=1}^{k} n_h^s \, \alpha_h = \alpha_1 + \sum_{h=1}^{k-1} \bar{F}(h;s)\,[\alpha_{h+1} - \alpha_h] \ . \tag{27}$$

Moreover, for all $t = 2, 3, \ldots, k-1$, *we have:*

$$\frac{1}{n}\sum_{h=1}^{k} n_h^s \, \alpha_h = \alpha_t - \sum_{h=1}^{t-1} F(h;s)\,[\alpha_{h+1} - \alpha_h] + \sum_{h=t}^{k-1} \bar{F}(h;s)\,[\alpha_{h+1} - \alpha_h] \ . \tag{28}$$

The first theorem, which links increment to dominance as per Inequality (6) for all lists of real numbers in $\mathcal{A}_F$, and to first-order dominance, has been known for quite a long time (see e.g. Lehmann (1955) or Quirk and Saposnik (1962)). We nonetheless provide a proof of part of it for completeness and for later use in the proof of the important Theorem 3 below.

Theorem 1 *For any two societies* s *and* s', *the following three statements are equivalent:*

(a) s *is obtained from* s' *by means of a finite sequence of increments,*

(b) *Inequality (6) holds for all* $(\alpha_1, ..., \alpha_k)$ *in* $\mathcal{A}_F$,

(c) *$F(h;s) \leq F(h;s')$ for every category h in $\mathcal{C}$.*

The second theorem is dual to the previous one. It links decrements to both normative dominance - as per Inequality (6) - for set $\mathcal{A}_{\overline{F}}$ of valuations of the k categories and (anti) first-order dominance. The formal statement of this theorem - whose proof, similar to that of Theorem 1, is left to the reader - is as follows.

Theorem 2 *For any two societies s and s', the following three statements are equivalent:*
(a) *s is obtained from s' by means of a finite sequence of decrements,*
(b) *Inequality (6) holds for all $(\alpha_1, ..., \alpha_k)$ in $\mathcal{A}_{\overline{F}}$,*
(c) *$F(h;s') \leq F(h;s)$ for every category h in $\mathcal{C}$.*

We now turn to Hammond transfers. Paralleling what was established before Propositions 2 and 3, we first seek the conditions on the numerical valuations of the categories under which a normative comparison of two societies based on Inequality (6) would be sensitive to Hammond transfers (as per Definition 3). It turns out that the conditions involve the following subset $\mathcal{H}$ of $\mathbb{R}^k$:

$$\mathcal{H} = \left\{(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k \,|\, (\alpha_i - \alpha_g) \geq (\alpha_l - \alpha_j) \text{ for } 1 \leq g < i \leq j < l \leq k\right\}. \tag{29}$$

In words, $\mathcal{H}$ contains all lists of categories' valuations that are "strongly concave" with respect to these categories in the sense that the utility gain from moving from one category to a better one is always larger when moving from categories in the bottom part of the scale than when moving within the upper part of it. The following proposition establishes that set $\mathcal{H}$ of categories' valuations is really the largest one for which the ranking of two societies based on Inequality (6) would consider favorably the notion of equalization underlying Hammond transfers.

Proposition 4 *For any two societies s and s', s being obtained from s' by a Hammond transfer as per Definition 3 implies Inequality (6) for all lists of real numbers $(\alpha_1, ..., \alpha_k) \in \mathcal{A}$ if and only if $\mathcal{A} = \mathcal{H}$.*

The intuition that set $\mathcal{H}$ captures a "strong concavity" property is, perhaps, better seen through the following proposition, which establishes that $\mathcal{H}$ contains all lists of categories' valuations that are "single-peaked" in the sense of admitting a largest value before which they are increasing (at a strongly decreasing rate) and after which they are decreasing (at a strongly increasing rate).

Proposition 5 *A list $(\alpha_1, ..., \alpha_k)$ of real numbers belongs to $\mathcal{H}$ if and only if there exists a $t \in \{1, ..., k\}$ such that $(\alpha_{i+1} - \alpha_i) \geq (\alpha_t - \alpha_{i+1})$ for all $i = 1, 2, \ldots, t-1$ (if any) and $(\alpha_{i'+1} - \alpha_{i'}) \leq (\alpha_{i'} - \alpha_t)$, for all $i' = t, t+1, \ldots, k-1$ (if any).*

Two "peaks" among those identified in Proposition 5 are of particular importance. One is when $t = k$ so that numbers $\alpha_1, ..., \alpha_k$ are increasing (at a strongly decreasing rate) with respect to the categories. In this case, the elements of $\mathcal{H}$ are also in $\mathcal{A}_F$, and we denote by $\mathcal{A}_H = \mathcal{H} \cap \mathcal{A}_F$ this set of increasing *and* strongly concave valuations of the categories. We then have the following immediate (and therefore unproved) corollary of Proposition 5 (applied to $t = k$).

Proposition 6 *A list of k real numbers $(\alpha_1, ..., \alpha_k)$ belongs to $\mathcal{A}_H$ if and only if it satisfies $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1}$ for all $i \in \{1, ..., k-1\}$.*

The other extreme of the possible peaks identified in Proposition 5 corresponds to the case where $t = 1$ so that numbers $\alpha_1, ..., \alpha_k$ are decreasing (at a strongly increasing rate) with respect to the categories. In this case, the elements of $\mathcal{H}$ are also in $\mathcal{A}_{\overline{F}}$, and we denote by $\mathcal{A}_{\overline{H}} = \mathcal{H} \cap \mathcal{A}_{\overline{F}}$ this subset of the set of all strongly concave valuations of the categories that are also decreasing with respect to these categories. We then have the following also immediate (and unproved) corollary of Proposition 5 (applied to $t = 1$).

Proposition 7 *A list of k real numbers $(\alpha_1, ..., \alpha_k)$ belongs to $\mathcal{A}_{\overline{H}}$ if and only if it satisfies $\alpha_{i+1} - \alpha_i \leq \alpha_i - \alpha_1$ for all $i \in \{1, ..., k-1\}$.*

We now establish what we view as the most important result of this paper: H-dominance is **the** implementable test to determine whether one distribution is obtained from another by a finite sequence of either Hammond transfers or increments. The formal statement of this result is as follows.

Theorem 3 *For any two societies s and s', the following three statements are equivalent:*
(a) *s is obtained from s' by means of a finite sequence of Hammond transfers and/or increments,*
(b) *Inequality (6) holds for all $(\alpha_1, ..., \alpha_k)$ in $\mathcal{A}_H$,*
(c) *$H(h; s) \leq H(h; s')$ for every category h in $\mathcal{C}$.*

Although a detailed proof of the equivalence of the three statements of Theorem 3 is provided in the Appendix, the main arguments are worth presenting here. The fact that Statement (a) implies Statement (b) is an immediate consequence of Propositions 2 and 4. These propositions actually imply that the ranking of two societies based on Inequality (6) is sensitive to Hammond transfers (if the list of valuations $(\alpha_1, ..., \alpha_k)$ belongs to $\mathcal{H}$) and to increments (if $(\alpha_1, ..., \alpha_k)$ belongs to $\mathcal{A}_F$). The proof that Statement (b) implies Statement (c) amounts to verifying that any list of k real numbers $(\alpha_1^i, \ldots, \alpha_k^i)$ defined, for any $i \in \{1, ..., k\}$, by:

$$\alpha_h^i = -(2^{i-h}) \text{ for } h = 1, .., i \quad (30)$$

$$\alpha_h^i = 0 \text{ for } h = i+1, ..., k \quad (31)$$

belongs to set $\mathcal{A}_H$. Indeed, it is apparent from Expression (8) that verifying the inequality:

$$\sum_{j=1}^{k} n_j^s \alpha_j^i \geq \sum_{j=1}^{k} n_j^{s'} \alpha_j^i$$

for any list $(\alpha_1^i, \ldots, \alpha_k^i)$ of real numbers defined as per (30) and (31) for any i is equivalent to the H-dominance of s' by s. Since Inequality (6) holds for all

$(\alpha_1, .., \alpha_k)$ in set $\mathcal{A}_H$, it must hold in particular for those $(\alpha_1^i, \ldots, \alpha_k^i)$ defined as per (30) and (31) for any i. The most difficult proof, that Statement (c) implies Statement (a), is obtained by first noting that if s first-order dominates s', then the possibility of going from s' to s by a finite sequence of increments is an immediate consequence of Theorem 1. The proof is then constructed under the assumption that s H-dominates s', but that no first-order dominance exists between the two societies. Hence there must be categories where the two cumulative distribution functions associated with s and s' "cross". In that case, we show that a Hammond transfer "above" the first category for which this crossing occurs can be made in such a way that the new society thereby obtained remains H-dominated by s. We also show that this Hammond transfer "brings to naught" at least one of the strict inequalities that distinguish $F(.; s)$ from $F(.; s')$. Hence if the final distribution s is not reached after this first transfer, then the same procedure can be applied again and again until s is reached. As the number of inequalities that distinguish $F(., s)$ from $F(., s')$ is finite, this proves the implication.

We now state a theorem that is the mirror image of Theorem 3, but with increments replaced by decrements, set $\mathcal{A}_H$ by $\mathcal{A}_{\overline{H}}$ and H-dominance by the $\overline{H}$-dominance. We omit the proof of this theorem because its logic and construction follow those of Theorem 3.[3]

Theorem 4 *For any two societies s and s', the following three statements are equivalent:*
(a) *s is obtained from s' by means of a finite sequence of Hammond transfers and/or decrements,*
(b) *Inequality (6) holds for all $(\alpha_1, ..., \alpha_k)$ in $\mathcal{A}_{\overline{H}}$,*
(c) *$\overline{H}(h; s) \leq \overline{H}(h; s')$ for every category h in $\mathcal{C}$.*

Theorem 3 (resp. 4) shows that H- (resp. $\overline{H}$-) dominance provides a perfect diagnostic tool to determine the possibility of going from one society to another by a finite sequence of Hammond transfers and/or increments (resp. decrements). But what about the possibility of going from one society to another by Hammond transfers *only*?

[3] The proof is however available upon request.

It follows clearly from Theorems 3 and 4 that where this possibility exists, the society from which these transfers originate is dominated by the society to which these transfers lead by both H- *and* $\overline{H}$- criteria. Unfortunately, we do not have proof of the converse implication that the dominance of one society by another by both the H- and the $\overline{H}$- criteria entails the possibility of going from the dominated to the dominating society by a finite sequence of Hammond transfers only. However, we can prove that the domination of a society s' by a society s by both the H- and the $\overline{H}$- criteria is equivalent to requiring Inequality (6) to hold for all list k real numbers $(\alpha_1, \ldots, \alpha_k)$ in $\mathcal{H}$. The next theorem summarizes all the relations between Hammond transfers and the relevant normative and implementable criteria that we are aware of.

The proof of this theorem makes some use of the following technical result that extends one step further the decomposition (28) of Lemma 1.

Lemma 2 *For any society s, any conceivable collection of k numbers $(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k$ and any $t = 2, 3, \ldots, k-1$, we have:*

$$\begin{aligned}\frac{1}{n}\sum_{h=1}^{k} n_h^s \alpha_h \quad = \quad & \sum_{i=1}^{t-2} -H(i;s)] \left[\theta_i - \sum_{h=i+1}^{t-1} \theta_h\right] \\ & -H(t-1;s)\,\theta_{t-1} \\ & +\overline{H}(t;s)\,\theta_t \\ & +\sum_{i=t+1}^{k-1} \overline{H}(i;s) \left[\theta_i - \sum_{h=t}^{i-1} \theta_h\right] \qquad (32)\end{aligned}$$

where $\theta_h = \alpha_{h+1} - \alpha_h$ for every $h = 1, ..., k-1$.

The proved theorem is the following.

Theorem 5 *Consider any two societies s and s' and the following three statements:*
(a) *s is obtained from s' by means of a finite sequence of Hammond transfers,*
(b) *Inequality (6) holds for all $(\alpha_1, ..., \alpha_k)$ in $\mathcal{H}$,*
(c) *$H(h, s) \leq H(h, s')$ and $\overline{H}(h, s) \leq \overline{H}(h, s')$ for all categories $h \in \mathcal{C}$.*
Then, statement (a) implies statement (b) and statement (b) and (c) are equivalent.

4 Sensitivity of the criteria to the grid of categories

As recognized in classical social choice theory (see e.g. Hammond (1976), Hammond (1976), Deschamps and Gevers (1978), D'Aspremont and Gevers (1977) and Sen (1977)), Hammond transfers, when combined with the "Pareto principle", are related to the *lexicographic extension* of the *Maximin* (or *Leximin*) criterion for ranking various ordered lists of n numbers. For example, Theorem 4.17 in Blackorby, Bossert, and Donaldson (2005) (ch. 4; p. 123) states that the Leximin criterion is the only monotonically increasing and anonymous ordering of $\mathbb{R}^n$ that is strictly sensitive to Hammond transfers. In an analogous vein, Bosmans and Ooghe (2013) and Miyagishima (2010) have shown that the Maximin criterion is the only continuous and Pareto-consistent reflexive and transitive ranking of $\mathbb{R}^n$ that is weakly sensitive to Hammond transfers. Since the H-dominance criterion coincides, by Theorem 3, with the possibility of going from the dominated to the dominating society by a finite sequence of Hammond transfers and/or increments - which are nothing more than anonymous Pareto improvements - it is of interest to understand the connection between the H-dominance criterion and the Leximin one.

We start by defining the latter criterion in the current setting as follows.

Definition 4 *Given two societies s and s', we say that s dominates s' according to the Leximin criterion, which we write $s \succsim_L s'$, if and only if there exists $i \in \{1, ..., k\}$ such that $n_i < n'_i$ and $n_h = n'_h$ for all integers h such that $1 \leq h < i$ (if any).*

It is clear (and well known) that the Leximin criterion provides a complete (and transitive) ranking of all lists of n real numbers. The following proposition establishes that the criterion of H-dominance is a strict subrelation of $\succsim_L$.

Proposition 8 *Assume that $n > 2$. Then, for any two societies s and s' for which $H(h, s) \leq H(h, s')$ holds for all categories $h \in \mathcal{C}$, we have $s \succsim_L s'$. However, the converse implication is false.*

A key difference between our framework and that of classical social choice theory is, of course, the *discrete* nature of the former.

In order to connect the two frameworks, it is useful to examine the sensitivity of the H-dominance criterion to the *level of refinement* of the finite grid over which it is defined. As it turns out, at a suitably high level of grid refinement, the H-dominance criterion becomes indistinguishable from the Leximin ordering. There are obviously many ways to refine a given finite grid. In this section, we consider the following notion of t-refinement of grid $\mathcal{C} = \{1, 2, \ldots, k\}$.

Definition 5 *The t-refinement of grid $\mathcal{C} = \{1, 2, \ldots, k\}$ for $t = 0, 1, ...$ is the set $\mathcal{C}(t)$ defined by:*

$$\mathcal{C}(t) = \left\{ i/2^t \, : \, i = 1, 2, \ldots, (2^t)k \right\} \tag{33}$$

or, equivalently,

$$\mathcal{C}(t) = \left\{ \frac{1}{2^t}, \frac{2}{2^t}, \frac{3}{2^t}, \ldots, \frac{(2^t)\, k}{2^t} \right\} . \tag{34}$$

Notice that $\mathcal{C}(0) = \mathcal{C}$ so that the initial grid corresponds to "zero" refinement. The grid obviously becomes finer as t increases, and it is clear that $\mathcal{C}(t) \subset \mathcal{C}(t+1)$ for all $t = 0, ...$ We also see that the finite set $\mathcal{C}(t)$ tends to the interval $]0, k]$ as t tends to infinity.

For any society s, and any real number x in the interval $]0, k]$, let us denote by $n^s(x)$ the (possibly null) number of agents in s who belong to category x. Clearly $n^s(x) = 0$ for all $x \notin \{1, ..., k\}$ and $n^s(j) = n^s_j$ for any $j \in \mathcal{C}$. Using these numbers $n^s(.)$ and applying the definition of the H-curve provided by Equation (8) to grid $\mathcal{C}(t)$ enables the t-refinement of the H-curve, denoted H^t, to be defined as follows (for any society s):

$$H^t\,(0; s) = 0$$

and:

$$H^t \left(\frac{i}{2^t}; s \right) = \frac{1}{n} \sum_{h=1}^{i} \left(2^{i-h} \right) n^s \left(\frac{h}{2^t} \right) , \quad \forall i = 1, 2, \ldots, (2^t)k \, . \tag{35}$$

We obviously define the notion of $H-$dominance of a society s over a society s' on a $t-$refined grid -referred to as H^t-dominance - as the fact that a society s has a H^t-curve (as defined by (35)) nowhere above and somewhere below that of a society s'. This definition produces a sequence of dominance quasi-orderings indexed by t which, as it turns out, converges to the complete ordering $\succsim_L$ when t becomes large enough.

The first notable effect of such a refinement of the grid is that it reduces the incompleteness of the quasi-ordering of societies induced by the H-dominance criterion. Specifically, the following proposition is proved in the Appendix.

Proposition 9 *For any two societies s and s' and any $t = 0, 1, ...,$ if society s H^t-dominates society s', then society s H^{t+1}-dominates society s'.*

Hence, refining the grid increases the discriminating power of the H^t-dominance criterion. The next theorem establishes that this increase eventually reaches a point where the H^t-dominance criterion becomes complete and equivalent to the Leximin ordering.

Theorem 6 *For any two societies s and s', the following two statements are equivalent:*

(a) *There exists an integer $t \in \mathbb{N}_+$ such that society s H^t-dominates society s'.*

(b) $s \succsim_L s'$.

We conclude this section by pointing out a similar relationship between the $\overline{H}$-dominance criterion and the Lexicographic extension of the *Minimax* criterion. The Minimax criterion compares alternative lists of n real numbers on the basis of their maximal elements: the lower the maximal element, the better the list. The lexicographic extension of the Minimax criterion - *anti-Leximax* for short - extends the principle to the second maximal element, and to the third and so on when the maximal, the second maximal and so on of two lists are identical. While the Leximin and the Maximin criteria can be seen as ethically favoring the "worst off", the Minimax or the anti-Leximax criteria disfavor the "best off".

The fact that $\overline{H}$-dominance converges to the anti-Leximax criterion and H-dominance converges to the Leximin one when the grid becomes sufficiently fine has obvious, but important, implication for the criterion defined in the preceding subsection as the intersection of the H- and the $\overline{H}$- dominance criteria. This intersection of H- and $\overline{H}$- dominance must converge to the intersection of the Leximin and the anti-Leximax criteria when the grid becomes sufficiently fine. Now Gravel, Magdalou, and Moyes (2017) show that the intersection of the Leximin and the anti-Leximax criteria is the smallest transitive relation that is strictly sensitive to a Hammond transfer. Taken together, these two results strike us as a compelling argument for the use of the intersection of H- and $\overline{H}$- dominance as a perfect test of the possibility of going from one society to another by a finite sequence of Hammond transfers.

5 Empirical Illustration

In this section, we use our criteria to compare the distributions of self-reported health status in Switzerland that are evaluated in Abul-Naga and Yalcin (2008) by means of inequality indices compatible with the incomplete ranking proposed by Allison and Foster (2004). Recall that this latter ranking applies to societies that have the same unique median category. For any two such societies s and s', Allison and Foster (2004) consider that the attribute is (weakly) more equally distributed in society s than in society s' if the inequality $F(h; s) \leq F(h; s')$ holds for every category h strictly below the median and the converse inequality $F(h; s) \geq F(h; s')$ holds for all categories h weakly above the median. While our criteria are not restricted to distributions with a unique common median, they can certainly be applied to such distributions, as we now illustrate. The health status considered by Abul-Naga and Yalcin (2008) lies in one of the five following categories: "very bad" (1), "bad" (2),"so-so" (3), "good" (4) and "very good" (5). The fractions n_i^s/n and the cumulated distribution $F(i; s)$ (for $i = 1, ..., 5$) are shown in the two following tables.

	n_1^s/n	n_2^s/n	n_3^s/n	n_4^s/n	n_5^s/n
$s =$ Leman	0.01	0.04	0.11	0.56	0.28
$s =$ North-West	0.01	0.04	0.13	0.63	0.19
$s =$ Central	0	0.02	0.11	0.63	0.24
$s =$ Middle-Land	0.01	0.03	0.13	0.60	0.23
$s =$ East	0	0.03	0.11	0.64	0.22
$s =$ Ticino	0.01	0.05	0.11	0.70	0.13
$s =$ Zurich	0	0.03	0.10	0.65	0.22

Table 1

	$F(1;s)$	$F(2;s)$	$F(3;s)$	$F(4;s)$	$F(5;s)$
Leman	0.01	0.05	0.16	0.72	1
North-West	0.01	0.05	0.18	0.81	1
Central	0	0.02	0.13	0.76	1
Middle-Land	0.01	0.04	0.17	0.77	1
East	0	0.03	0.14	0.78	1
Ticino	0.01	0.06	0.17	0.87	1
Zurich	0	0.03	0.13	0.78	1

Table 2

Using Formula (11) and (23), the values of $H(i;s)$ and $\overline{H}(i;s)$ (for $i = 1, ..., 5$) can then be calculated as follows.

	$H(1;s)$	$H(2;s)$	$H(3;s)$	$H(4;s)$	$H(5;s)$
Leman	0.01	0.06	0.23	1.02	2.32
North-West	0.01	0.06	0.25	1.13	2.45
Central	0	0.02	0.15	0.93	2.10
Middle-Land	0.01	0.05	0.23	1.06	2.35
East	0	0.03	0.17	0.98	2.18
Ticino	0.01	0.07	0.25	1.20	2.53
Zurich	0	0.03	0.16	0.97	2.16

Table 3

	$\overline{H}(0;s)$	$\overline{H}(1;s)$	$\overline{H}(2;s)$	$\overline{H}(3;s)$	$\overline{H}(4;s)$	$\overline{H}(5;s)$
Leman	9.49	4.74	2.35	1.12	0.28	0
North-West	8.69	4.34	2.15	1.01	0.19	0
Central	9.36	4.68	2.33	1.11	0.24	0
Middle-Land	9.07	4.53	2.25.	1.06	0.23	0
East	9.14	4.57	2.27	1.08	0.22	0
Ticino	8.23	4.11	2.03	0.96	0.13	0
Zurich	9.18	4.59	2.28	1.09	0.22	0

Table 4

As is clear, all the seven regions have "good" as their common median category. The H- and $\overline{H}$- curves associated to the Léman and the Central regions are depicted on Figure 1.

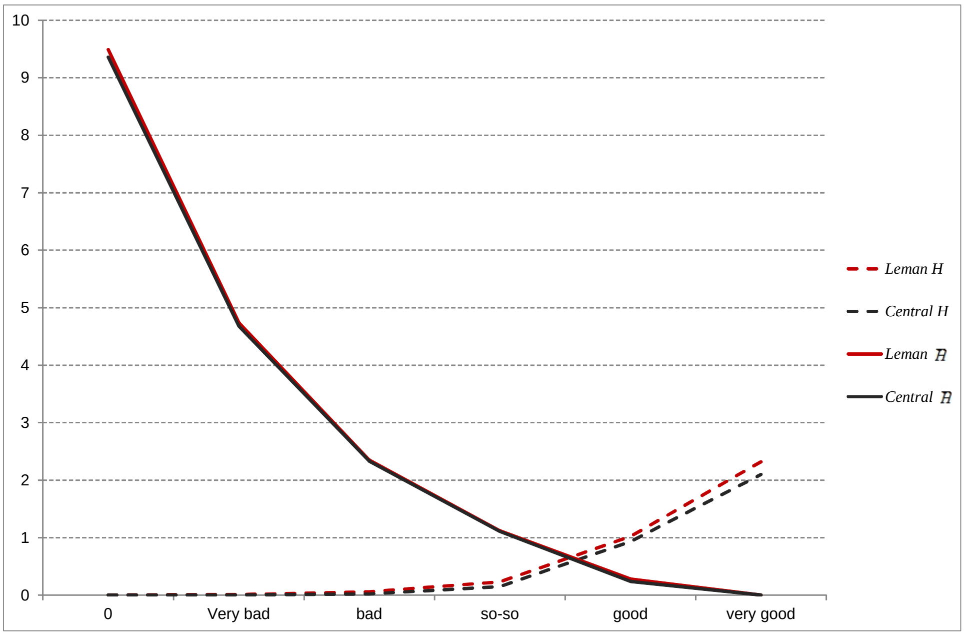

Figure 1: H- and $\overline{H}$- curves for the Leman and the Central region.

As can be seen, the Central region both H- and $\overline{H}$-dominates the Leman region. Hence, self-reported health status appears to be more equally distributed in the Central than in the Leman region. Observe that the same conclusion would obtain if one were using instead Allison and Foster (2004) criterion. The Hasse diagram showing the ranking of the seven regions by first-order dominance, H-dominance, and the intersection of H- and $\overline{H}$-dominance is provided in Figure 2. This figure shows that almost all regions of Switzerland can be compared by H-dominance. The only two regions that can not be compared are the Leman and the Middle land regions. As Figure 2 makes clear, a significant fraction of the H-rankings of the regions result from first-order dominance. Yet there are five additional comparisons that are obtained by adding Hammond transfers to increments. Among these five additional rankings, three result from "pure equality" comparisons in the sense of being obtained by the intersection of H- and $\overline{H}$-dominance while the two others (in red) necessitate the combination of increments and Hammond transfers. The diagram reveals that self-reported health is more equally distributed in either the Central, the Zurich or the East region than in the Leman region. For illustrative purpose, Figure 3 shows the dominance diagram associated with the Allison-Foster criterion. Observe that this criterion agrees with our conclusion that self-reported health is more equally distributed in the Central, the Zurich and the East region than in the Leman region. However, Allison-Foster criterion considers in addition - and somewhat surprisingly - that self-reported health is more equally distributed in the Zurich than in the East region, despite the fact that the dominance of Zurich over East (recognized by the H-criterion) results from first-order dominance. Allison-Foster criterion also agrees with H-dominance that East is better than Middle-Land. However, the Allison-Foster criterion can not compare the North-West and the Ticino region while H-dominance (but not first-order dominance) can. The reader can similarly compare the almost complete ranking of the seven regions provided by H-dominance with the complete rankings of those same regions produced by the ordinal inequality indices of Abul-Naga and Yalcin (2008) (see e.g. their Table 4).

6 Conclusion

The paper has laid the groundwork for comparing distributions of an ordinal attribute that takes finitely many values. The crux of our analysis is

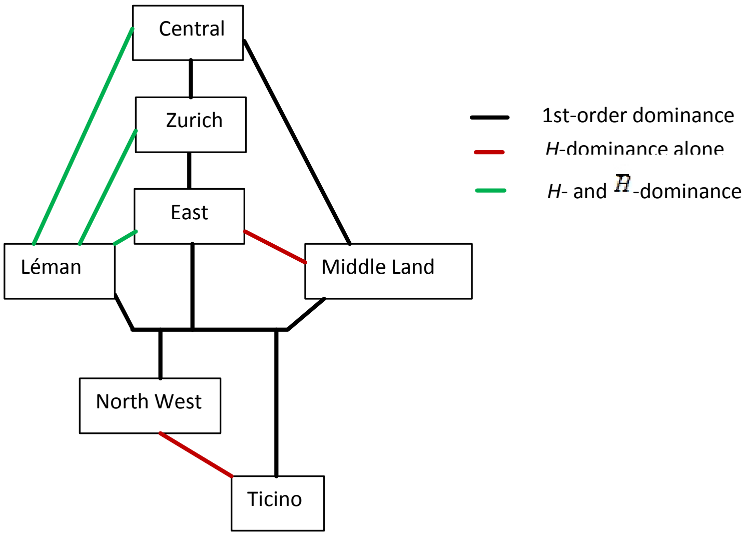

Figure 2: Hasse diagram of 1st order dominance, H-dominance, and the intersection of H- and $\overline{H}$-dominance of the seven regions.

an easy-to-use criterion, called H-dominance. This criterion can be viewed as the analogue, for comparing distributions of an ordinally measurable attribute, of the generalized Lorenz curve used for comparing distributions of a cardinally measurable one. It is well known (see e.g. Shorrocks (1983)) that one distribution of a cardinally measurable attribute dominates another for the generalized Lorenz criterion if and only if it is possible to go from the dominated distribution to the dominating one by a finite sequence of increments of the attribute and/or Pigou-Dalton transfers. The main result of this paper - Theorem 3 - establishes an analogous result for the H-dominance criterion. We show that the latter criterion ranks two distributions of an attribute in the same way as would going from the dominated to the dominating distribution by a finite sequence of increments and/or Hammond transfers. We also identified a dual $\overline{H}$-dominance criterion that ranks two distributions in the same way as would going from the dominated to the dominating distribution by a finite sequence of decrements and/or Hammond transfers of the attribute. We strongly suspect that the intersection of the H- and the $\overline{H}$- dominance criteria coincides with the possibility of going from the dominated to the dominating distribution by a finite sequence of Hammond

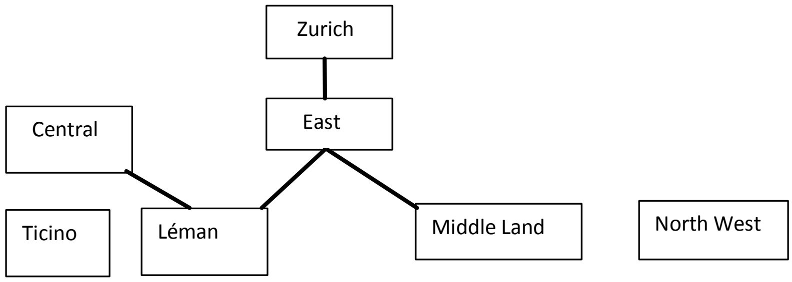

Figure 3: Hasse diagram of the ranking of the seven regions by the Allison-Foster criterion.

transfers only. We were, however, unable to prove this equivalence in the discrete framework considered, even though we succeeded in proving it indirectly for a sufficiently fine level of refinement of the grid of values taken by the attribute.

As illustrated with the data analyzed in Abul-Naga and Yalcin (2008), we believe that the H-dominance criterion, and the Hammond principle of transfers that justifies it along with increments, is a useful tool for comparing distributions of an attribute that cannot be meaningfully transferred *à la* Pigou-Dalton. Not only is the H-dominance criterion justified by clear and meaningful elementary transformations, it also has the advantage of being applicable to a much wider class of situations than the widely discussed criterion proposed by Allison and Foster (2004). The latter is limited to distributions that have the same median, and is not associated with clear and meaningful elementary transformations. Moreover, while we sought the criteria and transfers principles discussed here mainly so as to apply them to distributions of an ordinally measurable attribute, they can also be applied to a cardinally measurable attribute if the strong egalitarian flavour of Hammond transfers is deemed appropriate for that purpose.

Among the many possible extensions of the approach developed in this paper, three strike us as particularly important. First, it would be nice to obtain a proof that the intersection of H- and $\overline{H}$- dominance coincides with the possibility of going from the dominated to the dominating distribution by a finite sequence of Hammond transfers only in an arbitrary discrete setting.

For the moment, this equivalence is only a conjecture that is shown to be true only when the grid of values of the attribute is infinitely refined. Second, since the H-criterion is incomplete in the discrete setting, it would be interesting to obtain simple inequality indices that are compatible with Hammond transfers and, therefore, with the intersection of H- and $\overline{H}$- dominance. We believe that obtaining an axiomatic characterization of a family of such indices would not be too difficult. A good starting point would be to consider indices that can be written as per Expression (6) for some suitable choice of lists $(\alpha_1, ..., \alpha_k)$ of real numbers. A third extension, obviously more difficult, would be to consider multi-dimensional attributes.

A Appendix: Proofs

A.1 Proposition 1

Let s and s' be two societies such that $F(i; s) \leq F(i; s')$ holds for all $i \in \{1, ..., k\}$. It follows that:

$$F(1; s) \leq F(1; s')$$

and:

$$F(1; s) + F(2; s) \leq F(1; s') + F(2; s'),$$

$$2F(1; s) + F(2; s) + F(3; s) \leq 2F(1; s) + F(2; s) + F(3; s),$$

$$...$$

$$\sum_{h=1}^{i-1} \left(2^{i-h-1}\right) F(h; s) + F(i; s) \leq \sum_{h=1}^{i-1} \left(2^{i-h-1}\right) F(h; s) + F(i; s) \; \forall i = 2, 3, \ldots, k$$

so that, thanks to Expressions (9) and (10), inequality (15 that defines H-dominance of s' by s holds. To establish the $\overline{H}$-dominance of s by s', it suffices to notice that the requirement $F(i; s) \leq F(i; s')$ for all $i \in \{1, ..., k\}$ can alternatively be written (thanks to Expressions (16)-(18)) as :

$$\overline{F}(i; s) \geq \overline{F}(i, s')$$

for all $i \in \{1, ..., k-1\}$. This implies that:

$$\overline{F}(k-1; s) \geq \overline{F}(k-1; s')$$

and:

$$\begin{array}{rcl} \overline{F}(k-2; s) + \overline{F}(k-1; s) & \geq & \overline{F}(k-2; s') + \overline{F}(k-1; s'), \\ \overline{F}(k-3; s) + \overline{F}(k-2; s) + 2\overline{F}(k-1; s) & \geq & \overline{F}(k-3; s') + \overline{F}(k-2; s') + 2\overline{F}(k-1; s') \\ & & .... \end{array}$$

$$\sum_{h=i+1}^{k} \left(2^{h-i-1}\right) \overline{F}(h; s) + \overline{F}(i; s) \geq \sum_{h=i+1}^{k} \left(2^{h-i-1}\right) \overline{F}(h; s') + \overline{F}(i; s')\,, \ \forall i \in \mathcal{C}.$$

so that, thanks to Expressions (21 and (22), $\overline{H}(i; s') \leq H(i; s)$ for all $i \in \mathcal{C}$.

A.2 Propositions 2 and 3

For Proposition 2, let s be a society obtained from s' by an increment. By Definition 1, there exists some $j \in \{1, ..., k-1\}$ such that:

$$n_h^s = n_h^{s'}$$

for all $h \in \{1, ...k\}$ such that $h \neq j, j+1$,

$$n_j^s = n_j^{s'} - 1$$

and,

$$n_{j+1}^s = n_j^{s'} + 1.$$

Then Inequality (6) holds if and only if:

$$\sum_{h=1}^{k} n_h^s \alpha_h \geq \sum_{h=1}^{k} n_h^{s\prime} \alpha_h$$

$$\Longleftrightarrow$$
$$\alpha_{j+1}-\alpha_j \quad \geq \quad 0$$

by definition of an increment. As this inequality must hold for any $j \in \{1, ..., k-1\}$, this completes the proof of Proposition 2.The argument for Proposition 3 is similar (with Definition 1 replaced by Definition 2).

A.3 Lemma 1

Observe first that:

$$\sum_{h=1}^{k} n_h\,\alpha_h = \begin{cases} & n_1\,\alpha_1 \\ + & n_2\,\alpha_2 \\ + & \cdots \\ + & n_k\,\alpha_k\,, \end{cases} \tag{36}$$

or, equivalently:

$$\sum_{h=1}^{k} n_h\,\alpha_h = \begin{cases} & n_1\,\alpha_1 \\ + & n_2\,\alpha_1 + n_2\,[\alpha_2-\alpha_1] \\ + & n_3\,\alpha_1 + n_3\,[\alpha_2-\alpha_1] + n_3\,[\alpha_3-\alpha_2] \\ + & \cdots \\ + & n_k\,\alpha_1 + n_k\,[\alpha_2-\alpha_1] + n_k\,[\alpha_3-\alpha_2] + \ldots n_k\,[\alpha_k-\alpha_{k-1}]\,, \end{cases} \tag{37}$$

hence:

$$\sum_{h=1}^{k} n_h\,\alpha_h = \begin{cases} & n\,\alpha_1 \\ + & (n-n_1)\,[\alpha_2-\alpha_1] \\ + & [n-(n_1+n_2)]\,[\alpha_3-\alpha_2] \\ + & \cdots \\ + & \left[n-\sum_{h=1}^{k-1} n_h\right]\,[\alpha_k-\alpha_{k-1}]\,, \end{cases} \tag{38}$$

from which one obtains:

$$\frac{1}{n}\sum_{h=1}^{k} n_h\,\alpha_h = [\alpha_1+(\alpha_k-\alpha_1)] - \sum_{h=1}^{k-1} F(h;s)(\alpha_{h+1}-\alpha_h) \tag{39}$$

$$= \alpha_k - \sum_{h=1}^{k-1} F(h;s)(\alpha_{h+1}-\alpha_h) \tag{40}$$

as required by Equation (26). Now, by reconsidering Equation (38) and recalling that $\bar{F}(i;s) = 1 - F(i;s) = \left(n - \sum_{h=1}^{i} n_h\right)/n$ for every $i = 1, ..., k$, one immediately obtains Equation (27). We must now establish Equation (28). For this sake, one can notice that, for any $i \in \{2, ..., k-1\}$, one has:

$$\sum_{h=1}^{k} n_h\,\alpha_h = \sum_{h=1}^{i} n_h\,\alpha_h + \sum_{h=t+1}^{k} n_h\,\alpha_h\,. \tag{41}$$

If one successively decompose the two terms on the right hand of (41), one obtains for the first one:

$$\sum_{h=1}^{i} n_h\,\alpha_h = \begin{cases} & n_1\,\alpha_1 \\ + & n_2\,\alpha_1 + n_2\,\left[\alpha_2 - \alpha_1\right] \\ + & n_3\,\alpha_1 + n_3\,\left[\alpha_2 - \alpha_1\right] + n_3\,\left[\alpha_3 - \alpha_2\right] \\ + & \cdots \\ + & n_t\,\alpha_1 + n_k\,\left[\alpha_2 - \alpha_1\right] + n_i\left[\alpha_3 - \alpha_2\right] + \ldots n_i\left[\alpha_i - \alpha_{i-1}\right]\,, \end{cases}$$

One has therefore:

$$\sum_{h=1}^{i} n_h\,\alpha_h = \begin{cases} & \left(\sum_{h=1}^{i} n_h\right)\alpha_1 \\ + & \left[\sum_{h=1}^{i} n_h - n_1\right]\left[\alpha_2 - \alpha_1\right] \\ + & \left[\sum_{h=1}^{i} n_h - (n_1 + n_2)\right]\left[\alpha_3 - \alpha_2\right] \\ + & \cdots \\ + & \left[\sum_{h=1}^{i} n_h - \sum_{h=1}^{i-1} n_h\right]\left[\alpha_i - \alpha_{i-1}\right]\,, \end{cases}$$

or equivalently:

$$\frac{1}{n}\sum_{h=1}^{i} n_h\,\alpha_h = \left(\frac{1}{n}\sum_{h=1}^{i} n_h\right)\alpha_t - \sum_{h=1}^{i-1} F(h;s)\left[\alpha_{h+1} - \alpha_h\right]\,. \tag{42}$$

For the second term of (41), the successive decomposition yields:

$$\sum_{h=i+1}^{k} n_h\,\alpha_h = \begin{cases} & n_{i+1}\,\alpha_{i+1} \\ + & n_{i+2}\,\alpha_{i+1} + n_{i+2}\,[\alpha_{i+2} - \alpha_{i+1}] \\ + & n_{i+3}\,\alpha_{i+1} + n_{i+3}\,[\alpha_{i+2} - \alpha_{i+1}] + n_{i+3}\,[\alpha_{i+3} - \alpha_{i+2}] \\ + & \cdots \\ + & n_k\,\alpha_{i+1} + n_k\,[\alpha_{i+2} - \alpha_{i+1}] + n_k\,[\alpha_{i+3} - \alpha_{i+2}] + \ldots n_i\,[\alpha_k - \alpha_{k-1}]\ , \end{cases}$$

This can be written as:

$$\sum_{h=i+1}^{k} n_h\,\alpha_h = \begin{cases} & \left(\sum_{h=i+1}^{k} n_h\right)\alpha_{i+1} \\ + & \left(\sum_{h=i+2}^{k} n_h\right)[\alpha_{i+2} - \alpha_{i+1}] \\ + & \left(\sum_{h=i+3}^{k} n_h\right)[\alpha_{i+3} - \alpha_{i+2}] \\ + & \cdots \\ + & n_k\,[\alpha_k - \alpha_{k-1}]\ , \end{cases}$$

or equivalently:

$$\frac{1}{n}\sum_{h=i+1}^{k} n_h\,\alpha_h = \left(\frac{1}{n}\sum_{h=i+1}^{k} n_h\right)\alpha_{i+1} + \sum_{h=i+1}^{k-1} \bar{F}(h;s)\,[\alpha_{h+1} - \alpha_h]\ . \tag{43}$$

By summing Equations (42) and (43), one concludes that:

$$\begin{aligned} \frac{1}{n}\sum_{h=1}^{k} n_h\,\alpha_h &= \left(\frac{1}{n}\sum_{h=1}^{i} n_h\right)\alpha_i + \left(\frac{1}{n}\sum_{h=i+1}^{k} n_h\right)\alpha_{i+1} \\ &\quad - \sum_{h=1}^{i-1} F(h;s)\,[\alpha_{h+1} - \alpha_h] + \sum_{h=i+1}^{k-1} \bar{F}(h;s)\,[\alpha_{h+1} - \alpha_h] \end{aligned} \tag{44}$$

This equality can be further simplified, by observing that:

$$\begin{aligned} \left(\frac{1}{n}\sum_{h=1}^{i} n_h\right)\alpha_i + \left(\frac{1}{n}\sum_{h=i+1}^{k} n_h\right)\alpha_{i+1} &= \frac{1}{n}\left(n - \sum_{h=i+1}^{k} n_h\right)\alpha_i + \left(\frac{1}{n}\sum_{h=i+1}^{k} n_h\right)\alpha_{i+1} \\ &= \alpha_i + (\alpha_{i+1} - \alpha_i)\bar{F}(i;s) \end{aligned} \tag{45}$$

Equation (28) is then obtained from the reintroduction of (45) into (44).

A.4 Theorem 1

The equivalence between Statements (a) and (c) of this theorem is well-known in the literature. We therefore only prove the equivalence between Statements (b) and (c). Using equation (26) of Lemma 1, one has:

$$\frac{1}{n}\left[\sum_{h=1}^{k} n_h\,\alpha_h - \sum_{h=1}^{k} n'_h\,\alpha_h\right] = \sum_{h=1}^{k-1}\left[F(h;s') - F(h;s)\right]\left[\alpha_{h+1} - \alpha_h\right]\,. \tag{46}$$

Hence, if $F(h;s') - F(h;s) \geq 0$ for every $h \in \mathcal{C}$ and $(\alpha_1,...,\alpha_k) \in \mathcal{A}_F$, then $\sum_{h=1}^{k} n_h\,\alpha_h \geq \sum_{h=1}^{k} n'_h\,\alpha_h$. To establish the converse implication, define, for every $i \in \{1,...,k-1\}$, the list of k numbers $\alpha^i = (\alpha^i_1,\ldots,\alpha^i_k)$ to be such that $\alpha^k_h = 0$ for $h = 1,\ldots,i$ and $\alpha^i_h = 1$ for $h = i+1,\ldots,k$. We note that $\alpha^i \in \mathcal{A}_F$ for any $i \in \{1,...,k-1\}$. Since Inequality (6) holds for all lists of numbers $(\alpha_1,...,\alpha_k) \in \mathcal{A}_F$, one must therefore have, for any $i = 1,...,k-1$:

$$\begin{gathered}
\sum_{h=1}^{k} n_h\,\alpha^i_h \quad\geq\quad \sum_{h=1}^{k} n'_h\,\alpha^i_h \\
\Longleftrightarrow \\
\sum_{h=i+1}^{k} n_h \quad\geq\quad \sum_{h=i+1}^{k} n'_h \\
\Longleftrightarrow \\
n - \sum_{h=1}^{i} n_h \geq n - \sum_{h=1}^{i} n'_h \\
\Longleftrightarrow \\
\sum_{h=1}^{i} n_h \leq \sum_{h=1}^{i} n'_h
\end{gathered}$$

as required.

A.5 Proposition 4

Suppose that society s has been obtained from society s' by means of a Hammond transfer as per Definition 3. This means that there are categories $1 \leq g < i \leq j < l \leq k$ for which one has:

$$\sum_{h=1}^{k} n_h \, \alpha_h = \sum_{h=1}^{k} \alpha_h \, n'_h - \alpha_g + \alpha_i + \alpha_j - \alpha_l \, . \tag{47}$$

Hence if inequality (6) holds for s and s', one must have $\sum_{h=1}^{k} n_h^s \, \alpha_h - \sum_{h=1}^{k} n_h^{s\prime} \, \alpha_h = (\alpha_i - \alpha_g) - (\alpha_l - \alpha_j) \geq 0$ for all categories $1 \leq g < i \leq j < l \leq k$, which is precisely the definition of the set $\mathcal{A}_{\mathcal{H}}$.

A.6 Proposition 5

Assume that the list of numbers $(\alpha_1, ..., \alpha_k)$ belongs to $\mathcal{A}_{\mathcal{H}}$ and, therefore, satisfies $\alpha_i - \alpha_g \geq \alpha_l - \alpha_j$ for all $1 \leq g < i \leq j < l \leq k$. This implies in particular that $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1}$ for any $i \in \{1, 2, \dots, k-2\}$. Let $t = \min\{i = 1, ..., k : \alpha_{i+1} - \alpha_i \leq 0\}$ (using the convention that $\alpha_{k+1} = \alpha_k$). Such a t clearly exists under this convention, because $k \in \{i = 1, ..., k : \alpha_{i+1} - \alpha_i \leq 0\}$. If $t = k$, then the fact that $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1}$ holds for any $i \in \{1, 2, \dots, k-2\}$ implies that $\alpha_{i+1} - \alpha_i \geq \alpha_t - \alpha_{i+1}$ for all $i = 1, 2, \dots, t-1$ and (trivially) that $\alpha_{i'+1} - \alpha_{i'} \leq \alpha_{i'} - \alpha_t$ holds for all $i' \in \{t, ..., k-1\} = \varnothing$. Notice that if $t = k$, then, one has $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1} > 0$ for any $i \in \{1, 2, \dots, k-2\}$ (the alphas are increasing with respect to the categories). If $t = 1$, then the set $\{i = 1, 2, \dots, t-1\}$ is empty so that one must simply verify that $\alpha_{i'+1} - \alpha_{i'} \leq \alpha_{i'} - \alpha_1$, for $i' = 1, ..., k-1$. But this results immediately from the definition of t (if $i' = 1$) or from applying the requirement that $\alpha_i - \alpha_g \geq \alpha_l - \alpha_j$ for all $1 \leq g < i \leq j < l \leq k$ to the particular case where $g = 1$, $i = j = i' > 1$ and $l = i' + 1$ (otherwise). Notice that if $t = 1$, then one has by definition that $0 \geq \alpha_2 - \alpha_1 \geq \alpha_j - \alpha_{j-1}$ for every $j = 3, ..., k$ so that the alphas are decreasing with the categories. Assume now that $t \in \{2, ..., k-1\}$. We must check first that $\alpha_{i+1} - \alpha_i \geq \alpha_t - \alpha_{i+1}$ for all $i = 1, 2, \dots, t-1$. The case where $i = t-1$ is proved by observing that, by definition of t, one has $\alpha_t - \alpha_{t-1} > 0 = \alpha_t - \alpha_t$. The case

where $i < t-1$ (if any) is proved by applying the statement $\alpha_i - \alpha_g \geq \alpha_l - \alpha_j$ for all $1 \leq g < i \leq j < l \leq k$ to the particular case where $g = i \in \{1, ..., t-2\}$ $i = j = i+1$ and $l = t$. To check that the inequality $\alpha_{i'+1} - \alpha_{i'} \leq \alpha_{i'} - \alpha_t$ holds for all $i' \in \{t', ..., k-1\}$, simply observe that, for $i' = t$, the inequality is obtained from the very definition of t and, for $i' > t$, it results from applying the fact that $\alpha_i - \alpha_g \geq \alpha_l - \alpha_j$ for all $1 \leq g < i \leq j < l \leq k$ to the particular case where $g = t$, $i = j = i'$ and $l = i' + 1$.
Conversely, consider any list of numbers $(\alpha_1, ..., \alpha_k)$ for which there exists a $t \in \{1, ..., k\}$ such that:

$$\alpha_{i+1} - \alpha_i \geq \alpha_t - \alpha_{i+1} \tag{48}$$

holds for all $i \in \{1, 2, \ldots, t-1\}$ (if any) and:

$$\alpha_{i'+1} - \alpha_{i'} \leq \alpha_{i'} - \alpha_t \tag{49}$$

holds for all $i' \in \{t, ..., k-1\}$ (if any). Notice that applying inequality (48) to $i = t-1$ implies that $\alpha_t - \alpha_{t-1} \geq \alpha_t - \alpha_t = 0$. Combining this recursively with inequality (48) implies in turns that $\alpha_2 - \alpha_1 \geq \alpha_3 - \alpha_2 \geq ... \geq \alpha_t - \alpha_{t-1} \geq 0$ so that the list of numbers $(\alpha_1, ..., \alpha_k)$ is increasing from 1 up to t. Similarly, applying inequality (49) to $i' = t$ implies that $\alpha_{t+1} - \alpha_t \leq \alpha_t - \alpha_t = 0$. Combining this recursively with inequality (49) satisfied for all $i' \in \{t, ..., k-1\}$ (if any) leads to the conclusion that $\alpha_k - \alpha_{k-1} \leq \alpha_{k-1} - \alpha_{k-2} \leq ... \leq \alpha_{t+1} - \alpha_t \leq 0$ so that the list of numbers $(\alpha_1, ..., \alpha_k)$ is decreasing from t up to k. Consider then any four integers g, i, j and l satisfying $1 \leq g < i \leq j < l \leq k$. Five cases need to be distinguished:
(i) $g \geq t \geq 1$, then one has:

$$\begin{aligned}
\alpha_l - \alpha_j &= (\alpha_l - a_{l-1}) + (\alpha_{l-1} - \alpha_{l-2}) + ... + (\alpha_{j+1} - \alpha_j) \\
&\leq \alpha_{j+1} - \alpha_j \text{ (because the } \alpha_h \text{ are decreasing above } t) \\
&\leq \alpha_j - \alpha_t \text{ (by Inequality (49))} \\
&= \alpha_j - \alpha_i + \alpha_i - \alpha_g + \alpha_g - \alpha_t \text{ (for any integer } g,\ i,\ j) \\
&\leq \alpha_i - \alpha_g \text{ (because the } \alpha_h \text{ are decreasing above } t)
\end{aligned}$$

(ii) $g < t \leq i \leq j < l \leq k$. Then one has:

$$\begin{aligned}
\alpha_l - \alpha_j &= (\alpha_l - a_{l-1}) + (\alpha_{l-1} - \alpha_{l-2}) + ... + (\alpha_{j+1} - \alpha_j) \\
&\leq \alpha_{j+1} - \alpha_j \text{ (because the } \alpha_h \text{ are decreasing above } t) \\
&\leq \alpha_j - \alpha_t \text{ (by Inequality (49))} \\
&= \alpha_j - \alpha_i + \alpha_i - \alpha_g + \alpha_g - \alpha_t \text{ (for any integer } g,\ i,\ j) \\
&\leq \alpha_i - \alpha_g \text{ (because } \alpha_j - \alpha_i \leq 0 \text{ and } \alpha_g - \alpha_t \leq 0)
\end{aligned}$$

(iii) $g < i < t \leq j < l \leq k$. Then one has:

$$\begin{aligned}
\alpha_l - \alpha_j &= (\alpha_l - a_{l-1}) + (\alpha_{l-1} - \alpha_{l-2}) + ... + (\alpha_{j+1} - \alpha_j) \\
&\leq \alpha_{j+1} - \alpha_j \text{ (because the } \alpha_h \text{ are decreasing above } t) \\
&\leq \alpha_j - \alpha_t \text{ (by Inequality (49))} \\
&\leq 0 \text{ (because the } \alpha_h \text{ are decreasing above } t) \\
&\leq \alpha_i - \alpha_g \text{ (because the } \alpha_h \text{ are increasing below } t)
\end{aligned}$$

(iv) $g < i \leq j < t \leq l \leq k$. Then one has:

$$\begin{aligned}
\alpha_i - \alpha_g &= (\alpha_i - a_{i-1}) + (\alpha_{i-1} - \alpha_{i-2}) + ... + (\alpha_{g+2} - \alpha_{g+1}) + (\alpha_{g+1} - \alpha_g) \\
&\geq \alpha_{g+1} - \alpha_g \text{ (because the } \alpha_h \text{ are increasing below } t) \\
&\geq \alpha_t - \alpha_{g+1} \text{ (by Inequality (48))} \\
&= \alpha_t - \alpha_l + \alpha_l - \alpha_j + \alpha_j - \alpha_{g+1} \text{ (for any } g+1 \leq j < l \leq k) \\
&\geq \alpha_l - \alpha_j \text{ (because } \alpha_t - \alpha_l \geq 0 \text{ and } \alpha_j - \alpha_{g+1} \geq 0)
\end{aligned}$$

(v) $l < t \leq k$. In this case, one has:

$$\begin{aligned}
\alpha_i - \alpha_g &= (\alpha_i - a_{i-1}) + (\alpha_{i-1} - \alpha_{i-2}) + ... + (\alpha_{g+2} - \alpha_{g+1}) + (\alpha_{g+1} - \alpha_g) \\
&\geq \alpha_{g+1} - \alpha_g \text{ (because the } \alpha_h \text{ are increasing below } t) \\
&\geq \alpha_t - \alpha_{g+1} \text{ (by Inequality (48))} \\
&= \alpha_t - \alpha_l + \alpha_l - \alpha_j + \alpha_j - \alpha_{g+1} \text{ (for any } g+1 \leq j < l \leq k) \\
&\geq \alpha_l - \alpha_j \text{ (because because the } \alpha_h \text{ are increasing below } t)
\end{aligned}$$

Hence any list of k numbers $(\alpha_1, ..., \alpha_k)$ for which there exists a $t \in \{1, ..., k\}$ such that Inequalities (49) and (48) belongs to $\mathcal{A}_{\mathcal{H}}$.

A.7 Theorem 3.

A.7.1 Statement (a) implies statement (b)

Suppose s has been obtained from s' by means of an increment. It then follows from Proposition 2 that Inequality (6) holds for all ordered lists of k real numbers $(\alpha_1, ..., \alpha_k)$ in the set $\mathcal{A}_F$. This inequality holds therefore in particular for all such lists that belong to $\mathcal{A}_H \subset \mathcal{A}_F$. If, on the other hand, s has been obtained from s' by means of a Hammond transfer, we know from Proposition 4 that Inequality (6) holds for all ordered lists of k real numbers $(\alpha_1, ..., \alpha_k)$ in the set $\mathcal{H}$ and, therefore, for all ordered list of k real numbers in the set $\mathcal{A}_H \subset \mathcal{H}$. The implication then follows from any finite repetition of these two elementary implications.

A.7.2 Statement (b) implies statement (c)

Assume that the inequality

$$\sum_{i=1}^{k} n_i^s \alpha_i \geq \sum_{i=1}^{k} n_i^{s'} \alpha_i \tag{50}$$

holds for all $(\alpha_1, ..., \alpha_k) \in \mathcal{A}_{\mathcal{H}}$. For any $i \in \{1, ..., k\}$, define the ordered list of k numbers $(\alpha_1^i, \ldots, \alpha_k^i)$ by:

$$\begin{aligned} \alpha_h^i &= -(2^{i-h}) \text{ for } h = 1, .., i \\ \alpha_h^i &= 0 \text{ for } h = i+1, .., k \end{aligned}$$

Let us first show that the ordered list $(\alpha_1^i, \ldots, \alpha_k^i)$ of real numbers thus defined belongs to $\mathcal{A}_H$ for every $i \in \{1, ..., k\}$. Thanks to Proposition 6, this amounts to show that these real numbers satisfy

$$\alpha_{h+1}^i - \alpha_h^i \geq \alpha_k^i - \alpha_{h+1}^i \tag{51}$$

for every $h \in \{1, ..., k-1\}$. If $h \geq i+1$, then one has:

$$\alpha^i_{h+1} - \alpha^i_h = 0 - 0 \geq 0 - 0 = \alpha^i_k - \alpha^i_{h+1}$$

so that that Inequality (51) holds for that case. If $h = i$, then

$$\alpha^i_{i+1} - \alpha^i_i = 0 + 2^0 \geq 0 - 0 = \alpha^i_k - \alpha^i_{i+1}$$

so that (51) holds also for that case. If finally $h < i$, then one has:

$$\begin{aligned} \alpha^i_{h+1} - \alpha^i_h &= -2^{i-h-1} + 2^{i-h} \\ &= 2^{i-h-1} \\ &= 0 - (-2^{i-h-1}) \\ &= \alpha^i_k - \alpha^i_{h+1} \end{aligned}$$

so that (51) holds for this case as well. Since the ordered list $(\alpha^i_1, \ldots, \alpha^i_k)$ of real numbers belongs to $\mathcal{A}_H$ for every $i \in \{1, ..., k\}$, Inequality (50) must hold for any such ordered list of numbers. Hence, one has, for every $i \in \{1, ..., k\}$:

$$\begin{array}{ccc} \displaystyle\sum_{h=1}^{k} n^s_h \, \alpha^i_h & \geq & \displaystyle\sum_{h=1}^{k} n^{s'}_h \alpha^i_h \\ & \Longleftrightarrow & \\ \displaystyle\sum_{h=1}^{i} 2^{i-h} n^s_h & \leq & \displaystyle\sum_{h=1}^{i} 2^{i-h} n^{s'}_h \end{array}$$

which is nothing else than the condition for H-dominance, as expressed by Equation (8).

A.7.3 Statement (c) implies statement (a)

Assume that $H(i;s) \leq H(i;s')$ for all $i = 1, 2, \ldots, k-1$. We know from Proposition 1 that $F(i;s) \leq F(i;s')$ for all $i = 1, 2, \ldots, k-1$ implies that $H(i;s) \leq H(i;s')$ for all $i = 1, 2, \ldots, k-1$. If it is the case that, for all $i = 1, 2, \ldots, k-1$, one has both $H(i;s) \leq H(i;s')$ and $F(i;s) \leq F(i;s')$, we conclude from Theorem 1 that s can be obtained from s' by means of a finite sequence of increments and the proof is done. In the following, we therefore assume that $H(i;s) \leq H(i;s')$ holds for all $i = 1, 2, \dots, k-1$ but that there exists some $g \in \{1, 2, \dots, k-1\}$ for which one has $F(g;s) - F(g;s') > 0$. Define then the index h by:

$$h = \min \{g \mid F(g;s) - F(g;s') > 0\} \tag{52}$$

Given that index h, one can also define the index l by:

$$l = \min \{g > h \mid F(j;s) - F(j;s') \leq 0\,, \forall j \in [g,k]\}\,. \tag{53}$$

Such a l exists because $F(k;s) - F(k;s') = 0$. Notice that, by definition of l, one has:

$$F(l-1;s) - F(l-1;s') > 0 \text{ and } F(l;s) - F(l;s') \leq 0\,, \tag{54}$$

Hence, one has (using the definition of F provided by (7)), that $n_l^s < n_l^{s'}$. We now establish that there exists some $i \in \{1, 2, \dots, h-1\}$ such that:

$$F(i;s) - F(i;s') < 0 \text{ and } F(g;s) - F(g,s') = 0\,, \forall g < i\,. \tag{55}$$

Indeed, since $H(g;s) \leq H(g;s')$ for all $g = 1, 2, \dots, k-1$, one has either:

$$\begin{aligned} & H(1;s) < H(1;s') \\ \Longleftrightarrow \quad & \text{(thanks to expression (9))} \\ & F(1;s) < F(1;s') \end{aligned} \tag{56}$$

or:

$$F(1;s) = F(1;s') \tag{57}$$

If Case (56) holds, then the existence of some $i \in \{1, 2, \dots, h-1\}$ for which Expression (55) holds is established (with $i = 1$). If, on the other hand, Case (57) holds, then, since $H(2;s) \leq H(2;s')$ holds, we must have either:

$$H(2;s) < H(2;s')$$

$$\Longleftrightarrow \text{ (thanks to expression (10))}$$

$$2F(1;s) + F(2;s) < 2F(1;s') + F(2;s') \tag{58}$$

or:

$$2F(1;s) + F(2;s) = 2F(1;s') + F(2;s') \tag{59}$$

Again, if we are in case (58), we can conclude (since $F(1;s) = F(1;s')$) that $F(2;s) < F(2;s')$, which establishes the existence of some $i \in \{1, 2, \dots, h-1\}$ for which Expression (55) holds (with $i = 2$ in that case). If we are in case (59), we iterate in the same fashion using the definition of H provided by (10). We notice that the index i for which (55) holds must be strictly smaller than h because assuming otherwise will contradict, given the definition of h and the (iterated as above) definition of H, the fact that $H(g;s) \leq H(g;s')$ holds for all $g = 1, 2, \dots, k-1$. We finally note that, because of the definition of F provided by (7), the definition of the index i just provided entails that:

$$n_i^s < n_i^{s'} \tag{60}$$

and:

$$n_g^s = n_g^{s'}$$

for all $g = 1, ..., i-1$. We now proceed by defining a new society - s^1 say - obtained from s' by means of a Hammond transfer and such that $H(h;s) \leq H(h;s^1)$ for every $h = 1, ..., k-1$. For this sake, we define the numbers δ_1 and δ_2 and δ by:

$$\delta_1 = n_i^{s'} - n_i^s \ ; \ \delta_2 = n[F(l-1;s) - F(l-1;s')] \text{ and } \quad \delta = \min(\delta_1, \delta_2) \tag{61}$$

We note that, by the very definition of the index i, one has $\delta_1 > 0$. We notice also that, using (54) and the definition of the index l, one has $0 < \delta_2 \leq n_l^{s'} - n_l^s$. Define then the society s^1 by:

$$n_g^{s^1} = n_g^{s'}, \ \forall \ g \neq i, i+1, l \, ;$$

$$n_i^{s^1} = n_i^{s'} - \delta \ ; \ n_{i+1}^{s^1} = n_{i+1}^{s'} + 2\delta \ ; \ n_l^{s^1} = n_l^{s'} - \delta \, ;$$

It is clear that s^1 has been obtained from s' by δ Hammond transfers as per

Definition 3 where the indices g, i, j and l of this definition are, here, i, $i+1$, $i+1$ and l (respectively)). We observe that:

$$F(g, s^1) = F(g, s') \tag{63}$$

for all $g = 1, ..., i-1$. We also have that:

$$\begin{aligned} F(i, s^1) &= F(i-1, s') + n_i^{s^1}/n \\ &= F(i-1, s') + (n_i^{s'} - \delta)/n \\ &= F(i, s') - \delta/n \end{aligned} \tag{64}$$

$$\begin{aligned} F(i+1, s^1) &= F(i, s^1) + n_{i+1}^{s^1}/n \\ &= F(i, s') - \delta/n + n_{i+1}^{s^1}/n \\ &= F(i, s') - \delta/n + n_{i+1}^{s'}/n + 2\delta/n \\ &= F(i+1, s') + \delta/n \end{aligned} \tag{65}$$

Furthermore, for $g = i+2, ..., l-1$, one has:

$$\begin{aligned} F(g, s^1) &= F(i+1, s^1) + \sum_{e=i+2}^{g} n_e^{s^1}/n \\ &= F(i+1, s') + \delta/n + \sum_{e=i+2}^{g} n_e^{s^1}/n \\ &= F(i+1, s') + \delta/n + \sum_{e=i+2}^{g} n_e^{s'}/n \\ &= F(g, s') + \delta/n \end{aligned} \tag{66}$$

While finally, for $g = l, ..., k$:

$$\begin{aligned}
F(g, s^1) &= F(l-1, s^1) + \sum_{e=l}^{g} n_e^{s^1}/n \\
&= F(l-1, s') + \delta/n + \sum_{e=l}^{g} n_e^{s^1}/n \\
&= F(l-1, s') + \delta/n + n_l^{s'}/n - \delta/n + \sum_{e=l+1}^{g} n_e^{s'}/n \\
&= F(g, s')
\end{aligned} \tag{67}$$

Let us verify that $H(g; s) - H(g; s^1) \leq 0$ for all $g = 1, 2, \ldots, k-1$. We know already that $H(g; s) - H(g, s') \leq 0$ for all $h = 1, 2, \ldots, k-1$. We first observe that, by the definition just given of s^1, one has:

$$H(g, s^1) = \begin{cases} H(g, s') \text{ for } g = 1, .., i-1, \\ H(g, s') - \delta/n \text{ for } g = i, \\ H(g, s') \text{ for } g = i+1, ..., l-1, \\ H(g, s') - 2^{g-l}\delta/n \text{ for } g = l, ..., k. \end{cases} \tag{68}$$

The first line of (68) is indeed clear given Expression (63) and the definition of H provided by (8).The second line of (68) results from (64) and the definition of H

provided by (10). Consider now $g = i + 1$. One has (using (10) again):

$$\begin{aligned}
H(i+1,s^1) &= \sum_{g=1}^{i} \left(2^{i-g}\right) F(g;s^1) + F(i+1;s^1) \\
&= \sum_{g=1}^{i-1} \left(2^{i-g}\right) F(g;s') + F(i;s') - \delta/n + F(i+1;s^1) \text{ (by (64))} \\
&= \sum_{g=1}^{i-1} \left(2^{i-g}\right) F(g;s') + F(i;s') - \delta/n + F(i+1;s') + \delta/n \text{ (by (65))} \\
&= \sum_{g=1}^{i-1} \left(2^{i-g}\right) F(g;s') + F(i;s') + F(i+1;s') \\
&= \sum_{g=1}^{i} \left(2^{i-g}\right) F(g;s') + F(i+1;s') = H(i+1,s') \qquad (69)
\end{aligned}$$

Combined with (11) and the fact that $n_g^{s^1} = n_g^{s'}$ for all $g = i+2, ..., l-1$, Equality (69) establishes the third line of Expression (68). As for the last line of (68), we start with $g = l$ and we use (11) to write:

$$\begin{aligned}
H(l,s^1) &= 2H(l-1;s^1) + n_l^{s^1}/n \\
&= 2H(l-1;s') + (n_l^{s'} - \delta)/n \\
&= H(l,s') - \delta/n \qquad (70)
\end{aligned}$$

Iterating on this expression using (11) yields:

$$\begin{aligned}
H(l+1,s^1) &= 2H(l;s^1) + n_{l+1}^{s^1} \\
&= 2(H(l;s') - \delta/n) + n_{l+1}^{s'} \\
&= H(l+1,s') - 2\delta/n \qquad (71)
\end{aligned}$$

and therefore, for any $g \in \{l, ..., k\}$:

$$H(g,s^1) = H(g,s') - 2^{g-l}\delta/n$$

as required by the last line of Expression (68). We now notice that Expression

(68) entails that:

$$H(g,s)-H(g,s^1)=\begin{cases} H(g,s)-H(g,s') \text{ for } g=1,..,i-1, \\ H(g,s)-H(g,s')+\delta/n \text{ for } g=i, \\ H(g,s)-H(g,s') \text{ for } g=i+1,...,l-1, \\ H(g,s)-H(g,s')+2^{g-l}\delta/n \text{ for } g=l,...,k. \end{cases} \tag{72}$$

The fact that $H(g;s)-H(g,s') \leq 0$ holds for all $h=1,2,\ldots,k-1$ then immediately entails that $H(g;s)-H(g;s^1) \leq 0$ for all $g \in \{1,2,\ldots,i-1\} \cup \{i+1,...,l-1\}$. Consider now the case $g=i$. Using (11), we know that:

$$H(i;s)-H(i,s')=2\left(H(i-1;s)-H(i-1;s')\right)+(n_i^s-n_i^{s'})/n \tag{73}$$

By definition of i, one has $F(h;s)-F(h;s')=0$ for all $h<i$, so that the first term in the right hand side of Equation (73) is 0. Recalling then from (61) that $\delta_1=n_i^{s'}-n_i^s>0$ and that $\delta=\min(\delta_1,\delta_2)$, it follows that:

$$n_i^s-n_i^{s'}+\delta \leq 0$$

By combining Equations (72) and (73), we conclude that:

$$H(i,s)-H(i,s^1)=H(i,s)-H(i,s')+\delta/n=\frac{n_i^s-n_i^{s'}+\delta}{n} \leq 0 \tag{74}$$

Consider finally the case where $g=l,\ldots,k-1$. By using Equation (11) (and recalling that $\delta_2=n[F(l-1;s)-F(l-1;s')]$), one has:

$$H(l;s)-H(l;s')=2(H(l-1;s)-H(l-1;s'))+F(l;s)-F(l;s')-\delta_2/n. \tag{75}$$

Combining (75) with the last line of (72), and remembering that $\delta \leq \delta_2$, one obtains:

$$H(l,s)-H(l,s^1)=2[(H(l-1;s)-H(l-1;s')]+F(l;s)-F(l;s')+(\delta-\delta_2)/n \leq 0. \tag{76}$$

Finally, using successive applications of Equation (11), one obtains, for any $g=$

$l+1,\ldots,k-1$:

$$\begin{aligned} H(g;s)-H(g;s') &= 2^{g-l+1}[H(l-1;s)-H(l-1;s')] \\ &\quad + \sum_{e=l}^{g-1} 2^{g-1-e}[F(e;s)-F(e;s')] \\ &\quad + F(g;s)-F(g;s')-2^{g-l}\delta_2/n \\ &\leq 0 \end{aligned}$$

by assumption. Combined with the last line of (72) and the fact that $\delta \leq \delta_2$, this completes the proof that $H(g;s)-H(g;s^1) \leq 0$ for all $g=1,2,\ldots,k-1$. Hence, we have found a society s^1 obtained from society s' by means of a non-trivial Hammond transfers that is H-dominated by s. We now show that, in moving from s' to s, one has "brought to naught" at least one of the differences $|F(h;s)-F(h;s')|$ that distinguishes s from s'. That is to say, we establish the existence of some $h \in \{1,...,k-1\}$ for which one has:

$$\left|F(h;s)-F(h;s^1)\right| = 0$$

and:

$$|F(h;s)-F(h;s')| > 0$$

This is easily seen from the fact that, in the construction of s^1, one has either:

$$\delta = \delta_1 = n_i^{s'} - n_i^{s} \tag{77}$$

or:

$$\delta = \delta_2 = n[F(l-1;s)-F(l-1;s')] \tag{78}$$

If we are in the case (77), one has by definition of the index i and the function F:

$$F(i;s)-F(i;s^1) = 0$$

and:

$$F(i;s)-F(i;s') < 0$$

If on the other hand we are in case (78), then, we have (using (65)):

$$\begin{aligned} F(l-1;s) - F(l-1;s^1) &= F(l-1,s) - [F(l-1,s') + \delta_2/n] \\ &= 0 \end{aligned}$$

while, by definition of the index l, one has:

$$F(l-1,s) - F(l-1,s') > 0$$

Now, if $s = s^1$, then the proof is complete. If s is distinct from s^1 but s first order dominates s^1, then we conclude that society s can be obtained from society s' by means of a finite sequence of one Hammond transfer and a collection of increments (using Theorem 1). If s is distinct from s^1 and s does not first order dominates s^1, then we can find three categories i, h and l just as in the preceding step and construct a new distribution - say s^2 - that can be obtained from distribution s^1 by means of an (integer number of) Hammond transfers and that is H-dominated by s and so on. More generally, after a finite number - t say - of iterations, we will find a distribution s^t obtained from s' by means of t Hammond transfers such that s H-dominates s^t. In that case, we will have either $s = s^t$ or s first-order dominates s^t. Since there are finitely many differences of the kind $|F(h;s) - F(h;s')|$ to bring to naught, the number t must be finite. This completes the proof.

A.8 Lemma 2

Using (26) in Lemma 1, we have, for any society s:

$$\frac{1}{n}\sum_{h=1}^{k} n_i^s \alpha_i = \alpha_k - \sum_{h=1}^{k-1} F(h;s)\left[\alpha_{h+1} - \alpha_h\right] . \tag{79}$$

or:

$$\frac{1}{n}\sum_{i=1}^{k} n_i^s \alpha_i = \alpha_k - \sum_{h=1}^{k-1} F(h,s)\,\theta_h . \tag{80}$$

with $\theta_h = \alpha_{h+1} - \alpha_h$ for every $h = 1, ..., k-1$. Letting $\vartheta_i = \theta_i - \sum_{j=i+1}^{k-1} \theta_j$ for all $i = 1, 2, \ldots, k-2$ and $\vartheta_{k-1} = \theta_{k-1}$, we rewrite each term of the sum

$\sum_{i=1}^{k-1} F(i,s)\,\theta_i$ in (80) as follows:

For $i = 1$:

$$\begin{aligned}
F(1;s)\theta_1 &= F(1;s)[\theta_1 - \sum_{h=2}^{k-1}\theta_h] + F(1;s)\theta_2 + ... + F(1;s)\theta_{k-1} \\
&= F(1;s)[\theta_1 - \sum_{h=2}^{k-1}\theta_h] + F(1;s)[\theta_2 - \sum_{h=3}^{k-1}\theta_h] \\
&\quad +2F(1;s)\theta_3 + ... + 2F(1;s)\theta_{k-1} \\
&= F(1;s)[\theta_1 - \sum_{h=2}^{k-1}\theta_h] + F(1;s)[\theta_2 - \sum_{h=3}^{k-1}\theta_h] \\
&\quad +2F(1;s)[\theta_3 - \sum_{h=4}^{k-1}\theta_h] + 4\, F(1;s)\theta_4 ... + 4F(1;s)\theta_{k-1} \\
&= .... \\
&= F(1;s)\vartheta_1 + F(1;s)\vartheta_2 + 2F(1;s)\vartheta_3 \\
&\quad + 2^2 F(1;s)\vartheta_4 + ... + 2^{k-3}F(1;s)\vartheta_{k-1}
\end{aligned} \tag{81}$$

For $i = 2$

$$\begin{aligned}
F(2;s)\theta_2 &= F(2;s)[\theta_2 - \sum_{h=3}^{k-1}\theta_h] + F(2;s)\theta_3 + ... + F(2;s)\theta_{k-1} \\
&= F(2;s)[\theta_2 - \sum_{h=3}^{k-1}\theta_h] + F(2;s)[\theta_3 - \sum_{h=4}^{k-1}\theta_h] \\
&\quad +2F(2;s)\theta_4 + ... + 2F(2;s)\theta_{k-1} \\
&= F(2;s)[\theta_2 - \sum_{h=3}^{k-1}\theta_h] + F(2;s)[\theta_3 - \sum_{h=4}^{k-1}\theta_h] \\
&\quad +2F(2;s)[\theta_4 - \sum_{h=5}^{k-1}\theta_h] + 4F(2;s)\theta_5 ... + 4F(2;s)\theta_{k-1} \\
&= .... \\
&= F(2;s)\vartheta_2 + F(2;s)\vartheta_3 + 2F(2;s)\vartheta_4 \\
&\quad +2^2 F(2;s)\vartheta_5 + ... + 2^{k-4}F(2;s)\vartheta_{k-1}
\end{aligned} \tag{82}$$

More generally, one has $F(k-1,s)\theta_{k-1} = F(k-1;s)\vartheta_{k-1}$ and, for all $i = 1, 2, \ldots, k-2$:

$$F(i;s)\theta_i = F(i;s)\,\vartheta_i + F(i;s) \sum_{h=i+1}^{k-1} \left(2^{h-i-1}\right) \vartheta_h \, . \tag{83}$$

Hence, one can write:

$$\begin{array}{rclllllll}
F(1;s)\theta_1 & = & F(1;s)\vartheta_1 + & F(1;s)\vartheta_2 + & 2F(1;s)\vartheta_3 + & 2^2F(1;s)\vartheta_4 + & ... + & 2^{k-4}F(1;s)\vartheta_{k-2} & + 2^{k-3}F(1;s)\vartheta_{k-1} \\
F(2;s)\theta_2 & = & & F(2;s)\vartheta_2 + & F(2;s)\vartheta_3 + & 2F(2;s)\,\vartheta_4 & ... + & 2^{k-5}F(2;s)\vartheta_{k-2} & + 2^{k-4}F(2;s)\vartheta_{k-1} \\
F(3;s)\theta_3 & = & & & F(3;s)\vartheta_3 + & F(3;s)\vartheta_4 & ... + & 2^{k-6}F(3;s)\vartheta_{k-2} & + 2^{k-5}F(3;s)\vartheta_{k-1} \\
F(4;s)\theta_4 & = & & & & F(4;s)\vartheta_4 & ... + & 2^{k-7}F(4;s)\vartheta_{k-2} & + 2^{k-6}F(4;s)\vartheta_{k-1} \\
... & ... & ... & ... & ... & ... & ... & ... & ... \\
F(k-2;s)\theta_{k-2} & = & & & & & & + F(k-2;s)\vartheta_{k-2} & + F(k-2;s)\vartheta_{k-1} \\
F(k-1;s)\theta_{k-1} & = & & & & & & & + F(k-1;s)\vartheta_{k-1}
\end{array} \tag{84}$$

Remembering that $\vartheta_{k-1} = \theta_{k-1}$ and $\vartheta_i = \theta_i - \sum_{h=i+1}^{k-1} \theta_h$ for all $i = 1, 2, \ldots, k-2$, one can use Equation (10) and sum vertically the decomposition (84) to obtain:

$$\sum_{i=1}^{k-1} F(i;s)\,\theta_i = \sum_{i=1}^{k-2} H(i;s)[\theta_i - \sum_{h=i+1}^{k-2} \theta_h] + H(k-1;s)\theta_{k-1} \tag{85}$$

Since $\frac{1}{n}\sum_{i=1}^{k} n_i^s \alpha_i = \alpha_k - \sum_{i=1}^{k-1} F(i;s)\,\theta_i$, one obtains finally:

$$\frac{1}{n}\sum_{i=1}^{k} n_i^s \alpha_i = -\sum_{i=1}^{k-2} H(i;s)[\theta_i - \sum_{h=i+1}^{k-1} \theta_h] - H(k-1;s)\theta_{k-1} + \alpha_k. \tag{86}$$

In a symmetric fashion, one obtains from Equation (27) in Lemma 1:

$$\frac{1}{n}\sum_{i=1}^{k} n_i^s\,\alpha_i = \alpha_1 + \sum_{i=1}^{k-1} \bar{F}(i;s)\,[\alpha_{i+1} - \alpha_i] \ . \tag{87}$$

Hence, letting $\beta_1 = \theta_1$ and $\beta_i = (\theta_i - \sum_{j=1}^{i-1} \theta_j)$ for all $i = 2, 3, \ldots, k-1$, we rewrite $\sum_{i=1}^{k-1} \bar{F}(i;s)\,[\alpha_{i+1} - \alpha_i]$ in (87) as follows:

For $i = k-1$:

$$
\begin{aligned}
\overline{F}(k-1;s)\theta_{k-1} &= \overline{F}(k-1;s)[\theta_{k-1}-\sum_{h=1}^{k-2}\theta_h]+\overline{F}(k-1;s)\theta_{k-2}+...+\overline{F}(k-1;s)\theta_1 \\
&= \overline{F}(k-1;s)[\theta_{k-1}-\sum_{h=1}^{k-2}\theta_h]+\overline{F}(k-1;s)[\theta_{k-2}-\sum_{h=1}^{k-3}\theta_h] \\
&\quad +2\overline{F}(k-1;s)\theta_{k-3}+...+2\overline{F}(k-1;s)\theta_1 \\
&= \overline{F}(k-1;s)[\theta_{k-1}-\sum_{h=1}^{k-2}\theta_h]+\overline{F}(k-1;s)[\theta_{k-2}-\sum_{h=1}^{k-3}\theta_h] \\
&\quad +2\overline{F}(k-1;s)[\theta_{k-3}-\sum_{h=1}^{k-4}\theta_h]+4\overline{F}(k-1;s)\theta_{k-4}...+4\overline{F}(k-1;s)\theta_1 \\
&= .... \\
&= \overline{F}(k-1;s)\beta_{k-1}+\overline{F}(k-1;s)\beta_{k-2}+2\overline{F}(k-1;s)\beta_{k-3} \qquad (88) \\
&\quad +2^2\overline{F}(k-1;s)\beta_{k-4}+...+2^{k-3}2\overline{F}(k-1;s)\beta_1
\end{aligned}
$$

<u>For $i=k-2$</u>:

$$
\begin{aligned}
\overline{F}(k-2;s)\theta_{k-2} &= \overline{F}(k-2;s)[\theta_{k-2} - \sum_{h=1}^{k-3}\theta_h] + \overline{F}(k-2;s)\theta_{k-3} \\
&\quad +\overline{F}(k-2;s)\theta_{k-4} + ... + \overline{F}(k-2;s)\theta_1 \\
&= \overline{F}(k-2;s)[\theta_{k-2} - \sum_{h=1}^{k-3}\theta_h] + \overline{F}(k-2;s)[\theta_{k-3} - \sum_{h=1}^{k-4}\theta_h] \\
&\quad +2\overline{F}(k-2;s)\theta_{k-4} + ... + 2\overline{F}(k-2;s)\theta_1 \\
&= \overline{F}(k-2;s)[\theta_{k-2} - \sum_{h=1}^{k-3}\theta_h] + \overline{F}(k-2;s)[\theta_{k-3} - \sum_{h=1}^{k-4}\theta_h] \\
&\quad +2\overline{F}(k-2;s)[\theta_{k-4} - \sum_{h=1}^{k-5}\theta_h] \\
&\quad +4\overline{F}(k-2;s)\theta_{k-5} + ... + 4\overline{F}(k-2;s)\theta_1 \\
&= .... \\
&= \overline{F}(k-2;s)\beta_{k-2} + \overline{F}(k-2;s)\beta_{k-3} + 2\overline{F}(k-2;s)\beta_{k-4} \\
&\quad +2^2\overline{F}(k-2;s)\beta_{k-5} + ... + 2^{k-4}2\overline{F}(k-2;s)\beta_1
\end{aligned} \tag{89}
$$

More generally, one has $\bar{F}(1;s)\theta_1 = \bar{F}(1;s)\beta_1$ and:

$$
\bar{F}(i;s)\,\theta_i = \bar{F}(i;s)\,\beta_i + \bar{F}(i;s)\sum_{h=1}^{i-1}\left(2^{i-h-1}\right)\beta_h\,, \quad \forall i = 2,3,\ldots,k-1\,. \tag{90}
$$

Hence, one can conclude that:

$$
\begin{aligned}
\overline{F}(k-1;s)\theta_{k-1} &= \overline{F}(k-1;s)\beta_{k-1} + \overline{F}(k-1;s)\beta_{k-2} + ... + 2^{k-4}\overline{F}(k-1;s)\,\beta_2 + 2^{k-3}\overline{F}(k-1;s \\
\overline{F}(k-2;s)\theta_{k-2} &= \overline{F}(k-2;s)\beta_{k-2} + \cdots + 2^{k-5}\overline{F}(k-2;s)\beta_2 + 2^{k-4}\overline{F}(k-2; \\
..&=... \\
\overline{F}(2;s)\theta_2 &= \overline{F}(2;s)\beta_2 + \overline{F}(2;s)\beta_1 \\
\overline{F}(1;s)\theta_1 &= + \overline{F}(1;s)\theta_1
\end{aligned} \tag{91}
$$

Using (22), and summing vertically the previous equation, one obtains:

$$\sum_{i=1}^{k-1} \overline{F}(i;s)\theta_i = \overline{H}(1;s)\theta_1 + \sum_{i=2}^{k-1} \overline{H}(1;s)[\theta_i - \sum_{h=1}^{i-1} \theta_h] \tag{92}$$

These decompositions being obtained, consider now an integer $t \in \{2, 3, \ldots, k-1\}$ such that:

$$\theta_{t-1} \geq 0 \text{ and } (\theta_i - \sum_{h=i+1}^{t-1} \theta_h) \geq 0 \text{ for } i = 1, ..., t-2 \tag{93}$$

and:

$$\theta_t \leq 0 \text{ and } (\theta_i - \sum_{h=t}^{i-1} \theta_h) \leq 0 \text{ for } i = t+1, ..., k-1 \tag{94}$$

From Equation (28) in Lemma 1, one has, for any such a $t \in \{2, 3, \ldots, k-1\}$:

$$\frac{1}{n} \sum_{i=1}^{k} n_i^s \, \alpha_i = \alpha_t - \sum_{h=1}^{t-1} F(i;s)\theta_i + \sum_{i=t}^{k-1} \bar{F}(i;s)\theta_i \,. \tag{95}$$

By using Equation (85) and replacing category k by category t in this equation, one obtains:

$$\sum_{i=1}^{t-1} F(i;s)\theta_i = \sum_{i=1}^{t-2} H(i;s) \left[\theta_i - \sum_{h=i+1}^{t-1} \theta_j \right] + H(t-1;s)\,\theta_{t-1} \,. \tag{96}$$

Symmetrically, replacing category 1 by category t in Equation (92) enables one to write:

$$\sum_{i=t}^{k-1} \overline{F}(i;s)\theta_i = \overline{H}(t;s)\,\theta_t + \sum_{i=t+1}^{k-1} \overline{H}(i;s) \left[\theta_i - \sum_{h=t}^{i-1} \theta_j \right] . \tag{97}$$

Combining equations (95), (96) and (97), one gets finally:

$$\begin{aligned}\frac{1}{n}\sum_{h=1}^{k} n_h^s \alpha_h \quad = \quad & -\sum_{h=1}^{t-2} H(i;s)\left[\theta_i - \sum_{h=i+1}^{t-1}\theta_j\right] \\ & -H(t-1;s)]\,\theta_{t-1} \\ & +\overline{H}(t;s)\,\theta_t \\ & +\sum_{i=t+1}^{k-1}\overline{H}(i;s)\left[\theta_i - \sum_{h=t}^{i-1}\theta_j\right] \qquad (98)\end{aligned}$$

as required.

A.9 Theorem 5

A.9.1 Statement (a) implies statement (b)

This results immediately from the definition of the set $\mathcal{H}$ (using Proposition 4).

A.9.2 Statement (b) implies statement (c)

Assume that the inequality $\sum_{h=1}^{k} n_h^s \alpha_h \geq \sum_{h=1}^{k} n_h^{s'} \alpha_h$ holds for all lists of real numbers $(\alpha_1, ..., \alpha_l) \in \mathcal{H}$. This implies in particular that the inequality holds for all $(\alpha_1, ..., \alpha_l) \in \mathcal{A}_H$. It then follows from Theorem 3 that society s H-dominates society s'. Similarly, the fact that the inequality $\sum_{h=1}^{k} n_h^s \alpha_h \geq \sum_{h=1}^{k} n_h^{s'} \alpha_h$ holds for all lists of real numbers $(\alpha_1, ..., \alpha_l) \in \mathcal{H}$ implies in particular that it holds for all $(\alpha_1, ..., \alpha_l) \in \mathcal{A}_{\overline{H}}$. Hence, thanks to Theorem 4, society s $\overline{H}$-dominates society s'. Hence society s both H-dominates and $\overline{H}$-dominates society s'.

A.9.3 Statement (c) implies statement (b)

Assume that society s both H-dominates and $\overline{H}$-dominates society s'. Thanks to Proposition 5, one needs to show that $\sum_{h=1}^{k} n_h^s \alpha_h \geq \sum_{h=1}^{k} n_h^{s'} \alpha_h$ holds for all $(\alpha_1, .., \alpha_k) \in \mathbb{R}_+^k$ for which there exists an integer $t \in \{1, ..., k\}$ such that $(\alpha_{i+1} - \alpha_i) \geq (\alpha_t - \alpha_{i+1})$, for all $i = 1, 2, \ldots, t-1$ (if any) and $(\alpha_{i'+1} - \alpha_{i'}) \leq (\alpha_{i'} - \alpha_t)$, for all $i' \in \{t, t+1, \ldots, k-1\}$ (again if this set is non-empty). Since s both H-dominates and $\overline{H}$-dominates s', we know at once from Theorems 3 and

4 that $\sum_{h=1}^{k} n_h^s \alpha_h \geq \sum_{h=1}^{k} n_h^{s\prime} \alpha_h$ holds for all list of real numbers $(\alpha_1, .., \alpha_k) \in \mathcal{A}_H \cap \mathcal{A}_{\overline{H}}$. These lists of real numbers are associated to an integer $t \in \{1, k\}$. The only thing that remains to be shown is therefore that $\sum_{h=1}^{k} n_h^s \alpha_h \geq \sum_{h=1}^{k} n_h^{s\prime} \alpha_h$ must hold as well for $(\alpha_1, .., \alpha_k) \in \mathbb{R}_+^k$ for which there exists an integer $t \in \{2, ..., k-1\}$ such that $(\alpha_{i+1} - \alpha_i) \geq (\alpha_t - \alpha_{i+1})$, for all $i = 1, 2, \ldots, t-1$ and $(\alpha_{i'+1} - \alpha_{i'}) \leq (\alpha_{i'} - \alpha_t)$ for all $i' = \{t, t+1, \ldots, k-1\}$. For this sake, we resort to the decomposition result of Lemma 2, and, setting $\theta_h = \alpha_{h+1} - \alpha_h$ for every $h = 1, ..., k-1$. we write:

$$\begin{aligned}
\frac{1}{n}\sum_{h=1}^{k}(n_h^s - n_h^{s'})\alpha_h \quad = \quad & \sum_{i=1}^{t-2}[H(i, s') - H(i; s)] \left[\theta_i - \sum_{g=i+1}^{t-1} \theta_g\right] \\
& [H(t-1, s') - H(t-1; s)]\, \theta_{t-1} \\
& + [\overline{H}(t; s) \ - \overline{H}(t; s')]\theta_t \\
& + \sum_{i'=t+1}^{k-1} [\overline{H}(i; s) - \overline{H}(i; s') \left[\theta_i - \sum_{g=t}^{i-1} \theta_g\right] \qquad (99)
\end{aligned}$$

for any integer $t \in \{2, 3, \ldots, k-1\}$ such that:

$$\theta_{t-1} \geq 0 \text{ and } (\theta_i - \sum_{h=i+1}^{t-1} \theta_h) \geq 0 \text{ for } i = 1, ..., t-2 \qquad (100)$$

and:

$$\theta_t \leq 0 \text{ and } (\theta_{i'} - \sum_{h=t}^{i-1} \theta_h) \leq 0 \text{ for } i = t+1, ..., k-1 \qquad (101)$$

Since society s both H-dominates and $\overline{H}$-dominates society s', one has $H(h, s') - H(h; s) \geq 0$ and $\overline{H}(h; s) - \overline{H}(h; s') \leq 0$ for all $h = 1, ..., k$. Combining this information with Equations (100) and (101) leads to the required conclusion that Expression (99) is positive.

A.10 Proposition 8

Assume that s and s' are two distinct societies for which $H(h, s) \leq H(h, s')$ holds for all categories $h \in \mathcal{C}$. It follows from the recursive definition of the H-curve provided by Equations (9) and (11) that the smallest $i \in \{1, 2, \ldots, k\}$ for which

$n_i^s \neq n_i^{s\prime}$ is such that $n_i^s < n_i^{s\prime}$. But this implies that $s \succsim_L s'$. To show that the converse implication is false, one just needs to consider the following example for $k = n = 3$, and societies s and s' such that:

$$n_1^{s\prime} = 1, n_2^{s\prime} = 0,\ s_3^{s\prime} = 2$$

and:

$$n_1^s = n_3^s = 0,\ n_2^s = 3.$$

It is clear that $s \succsim_L s'$. The conclusion that $H(2, s) > H(2, s')$ and, therefore, that s does not H-dominate s' follows then from the following table which gives the values of $H(j; s)$ and $H(j; s')$ as per expression (11) for $j = 1, 2, 3$.

	category 1	category 2	category 3
$F(;s)$	0	1	1
$H(:s)$	0	1	2
$F(;s\prime)$	1/3	1/3	1
$H(;s\prime)$	1/3	2/3	2

A.11 Proposition 9

As a preliminary of the proof, we first notice that, for any society s, and any $t \in \{0, 1, ...\}$ one has:

$$n^s(\frac{2\,i+1}{2^{t+1}}) = 0 \tag{102}$$

and:

$$H^{t+1}(\frac{2\,i+1}{2^{t+1}}; s) = 2\,H^{t+1}(\frac{i}{2^t}; s) \tag{103}$$

Indeed, for any $i = 1, 2, \ldots, (2^t)k$, one has:

$$\frac{i}{2^t} = \frac{2\,i}{2^{t+1}}\,, \tag{104}$$

Equation (102) then follows from the fact that $n^s(\frac{h}{2^{t+1}}) = 0$ for all $\frac{h}{2^{t+1}} \notin \mathcal{C}(t)$, while Equation (103) is an immediate consequence of Equations (102) and (104) and the fact that, thanks to Expression (11), one has:

$$H^{t+1}(\frac{2\,i+1}{2^{t+1}};s) = 2\,H^{t+1}(\frac{2\,i}{2^{t+1}};s) + n^s(\frac{2\,i+1}{2^{t+1}})/n$$

for every $i = 0, 1, \ldots, (2^t)k - 1$. We also observe that:

$$H^{t+1}(\frac{i}{2^t};s) = \sum_{h=1}^{i-1} \left(2^{2(i-h)-1}\right) H^t(\frac{h}{2^t};s) + H^t(\frac{i}{2^t};s), \tag{105}$$

for any society s. Indeed, from Equation (35) applied to the grid $\mathcal{C}(t+1)$, we know that:

$$H^{t+1}\,(x;s) = \frac{1}{n}\sum_{h=1}^{j} \left(2^{j-h}\right) n^s(h/2^{t+1}),$$

for any $x \in \mathcal{C}(t+1)$, and $j = x2^{t+1}$. Applying this to $x = \frac{i}{2^t}$ for any $i = 1, ..., (2^t)k$ yields:

$$H^{t+1}(\frac{i}{2^t};s) = \frac{1}{n}\sum_{h=1}^{2i} \left(2^{2i-h}\right) n^s(h/2^{t+1}), \quad . \tag{106}$$

for any such i. Expression (105) can then be obtained from (106) and the following observations (made only for $i = 1, 2, 3$, but easily extendable to any other i). For $i = 1, 2, 3$ (106) writes indeed as:

$$\begin{aligned} H^{t+1}(\frac{1}{2^t};s) &= \frac{1}{n}[2n^s(\frac{1}{2^{t+1}}) + n^s(\frac{2}{2^{t+1}})] \\ &= \frac{1}{n}n^s(\frac{2}{2^{t+1}}) \text{ (since } n^s(\frac{1}{2^{t+1}}) = 0) \\ &= \frac{1}{n}n^s(\frac{1}{2^t}) \end{aligned} \tag{107}$$

$$\begin{aligned} H^{t+1}(\frac{2}{2^t};s) &= \frac{1}{n}[8n^s(\frac{1}{2^{t+1}}) + 4n^s(\frac{2}{2^{t+1}}) + 2n^s(\frac{3}{2^{t+1}}) + n^s(\frac{4}{2^{t+1}})] \\ &= \frac{1}{n}[4n^s(\frac{2}{2^{t+1}}) + n^s(\frac{4}{2^{t+1}})] \\ &= \frac{1}{n}[4n^s(\frac{1}{2^t}) + n^s(\frac{2}{2^t})] \end{aligned} \tag{108}$$

(since again $n^s(\frac{h}{2^{t+1}}) = 0$ for all $\frac{h}{2^{t+1}} \notin \mathcal{C}(t)$)

$$
\begin{aligned}
H^{t+1}(\frac{3}{2^t};s) &= \frac{1}{n}[32n^s(\frac{1}{2^{t+1}}) + 16n^s(\frac{2}{2^{t+1}}) + 8n^s(\frac{3}{2^{t+1}}) \\
&\quad +4n^s(\frac{4}{2^{t+1}}) + 2n^s(\frac{5}{2^{t+1}}) + n^s(\frac{6}{2^{t+1}})] \\
&= \frac{1}{n}[16n^s(\frac{2}{2^{t+1}}) + 4n^s(\frac{4}{2^{t+1}}) + n^s(\frac{6}{2^{t+1}})] \\
&= \frac{1}{n}[16n^s(\frac{2}{2^t}) + 4n^s(\frac{4}{2^t}) + n^s(\frac{6}{2^t})] \qquad (109)
\end{aligned}
$$

(because again $n^s(\frac{h}{2^{t+1}}) = 0$ for all $\frac{h}{2^{t+1}} \notin \mathcal{C}(t)$). Now, applying Equation (35) to the grid $\mathcal{C}(t)$, one has:

$$
\begin{aligned}
H^t(\frac{1}{2^t};s) &= \frac{1}{n}n^s(\frac{1}{2^t}) \qquad (110) \\
H^t(\frac{2}{2^t};s) &= \frac{1}{n}[2n^s(\frac{1}{2^t}) + n^s(\frac{2}{2^t})] \qquad (111) \\
H^t(\frac{3}{2^t};s) &= \frac{1}{n}[4n^s(\frac{1}{2^t}) + 2n^s(\frac{2}{2^t}) + \frac{1}{n}n^s(\frac{3}{2^t})] \qquad (112)
\end{aligned}
$$

so that Expression (105) for $i = 1, 2, 3$ results from combining (107)-(109) with (110)-(112).

In order to prove the result, consider two societies s and s' and assume that society s H^t dominates society s' so that:

$$H^t\left(\frac{i}{2^t};s\right) \leq H^t\left(\frac{i}{2^t};s'\right)$$

holds for all $i \in \{1, ..., (2^t)k\}$. Taking any such i, one has in particular:

$$H^t\left(\frac{h}{2^t};s\right) \leq H^t\left(\frac{h}{2^t};s'\right)$$

for any $h \in \{1, ..., i\}$. Hence, using (105):

$$H^{t+1}(\frac{i}{2^t};s) \leq H^{t+1}(\frac{i}{2^t};s')$$

for all $i \in \{1, ..., (2^t)k\}$ which implies, thanks to (103), that society s H^{t+1}-

dominates society s'.

A.12 Theorem 6

The proof that statement (a) of the theorem implies statement (b) has been established in Proposition 8 (by using $t = 0$). In order to prove the converse implication, consider two arbitrary societies s and s' such that $s \succsim_L s'$. Because of Proposition 9, we only have to show that there exists a non-negative integer t for which s H^t-dominates s' holds or, equivalently thanks to Theorem 3, that s can be obtained from s' by means of a finite sequence of increments and/or Hammond transfers on the grid $\mathcal{C}(t)$. Since $s \succsim_L s'$, there is by Definition 4 an index $i \in \{1, 2, \dots, k\}$ such that $n_h^s = n^s(h) = n^{s'}(h) = n_h^{s'}$ for all $h = 1, 2, \dots, i-1$ and $n_i^s = n^s(i) < n^{s'}(i) = n_i^{s'}$. Given this index i, consider a society s'' such that:

$$n_h^{s''} = n_h^s, \ \forall h = 1, \dots, i\,;$$

$$n_{i+1}^{s''} = \sum_{h=i+1}^{k} n_h^s\,;$$

$$n_h^{s''} = 0, \ \forall h = i+2, \dots, k\,.$$

Notice that $\sum_{h=1}^{k} n_h^{s''} = n$ and that $F(i; s) \leq F(i; s'')$ for all $i = 1, ..., k$ so that, by Theorem (1), s can be obtained from s'' by means of a finite sequence of increments. We also observe that:

$$n_h^{s'}(h) = n_h^{s''}, \ \forall h = 1, \dots, i-1\,; \tag{114a}$$

$$n_i^{s'} - n_i^{s''} > 0\,; \quad n_{i+1}^{s'} - n_{i+1}^{s''} < 0\,; \tag{114b}$$

$$n_h^{s'} \geq n_h^{s''} = 0, \ \forall h = i+2, \dots, k\,. \tag{114c}$$

Define, for any $h \in \mathcal{C}$, the number δ_h by:

$$\delta_h = n_h^{s'} - n_h^{s''}$$

It is clear that δ_h so defined is an integer (which may be positive or negative). Since $\sum_{h=i}^{k} n_h^{s\prime} = \sum_{h=i}^{k} n_h^{s''}$, one can write:

$$\delta_i + \delta_{i+1} = - \sum_{h=i+2}^{k} \delta_h \, . \tag{115}$$

Since, by (114c), $\delta_h \geq 0$ for all $h = i+2, \ldots, k$, one observes that $\delta_i + \delta_{i+1} \leq 0$. We consider two cases.

Case 1: $\delta_i + \delta_{i+1} = 0$. In that case, we conclude from (115) that $\sum_{h=i+2}^{k} \delta_h = 0$ and, thanks to (114c), that $n_h^{s\prime} = n_h^{s\prime\prime}$ for all $h = i+2, \ldots, k$. Hence, one has $n_h^{s\prime} = n_h^{s\prime\prime}$ for all $h = \{1, \ldots, i-1\} \cap \{i+2, \ldots, k\}$, and $\delta_i = n_i^{s\prime} - n_i^{s\prime\prime} = n_{i+1}^{s''} - n_{i+1}^{s'} > 0$. Hence, s'' can be obtained from s' by means of δ_i increments from i to $i+1$ and we conclude that $s \succsim_1 s'' \succsim_1 s'$ which implies that s H^t-dominates s' for all $t = 0, ...$

Case 2: $\delta_i + \delta_{i+1} < 0$. In that case, we deduce from (115) that there is an $h \in \{i+2, \ldots, k\}$ such that $\delta_h > 0$ or, equivalently, that $n_h^{s\prime} > n_h^{s\prime\prime} = 0$. From (114a)-(114c), one immediately observes that s'' can be obtained from s' by means of δ_i increments from category i to category $i+1$, and $(-\delta_{i+1})$ decrements (δ_{i+1} is a negative integer), from each category $h > i+1$ for which $n_h^{s\prime} > 0$ to category $i+1$. However more decrements than increments are required ($(-\delta_{i+1}) > \delta_i$), so that increments and decrements can not be matched one by one to produce Hammond transfers – and only Hammond transfers – in order to obtain, on the initial grid $\mathcal{C}$, s'' from s'. Yet we can match the increments with the decrements if an appropriate refinement of the grid between i and $i+1$ can be performed. First, staying on the initial grid $\mathcal{C}$, and starting from s', we can combine $(\delta_i - 1)$ increments (from i to $i+1$) to the same number of decrements starting from one or several categories h above $i+1$ and bringing the agents from these categories to $i+1$. This generates immediately $(\delta_i - 1)$ Hammond transfers. In order complete the move from s' to s by means of Hammond transfers, we need to match the last $[\delta_i - (\delta_i - 1)] = 1$ increment from i to $i+1$ with the remaining $[(-\delta_{i+1}) - (\delta_i - 1)] > 1$ decrements that are required from each category $h > i+1$ where the number of agents remains strictly positive to the category $i+1$. Whatever is the number $[(-\delta_{i+1}) - (\delta_i - 1)] > 1$, it is clearly possible to refine the grid $\mathcal{C}$ in such a way as to obtain at least $[(-\delta_{i+1}) - (\delta_i - 1)]$ adjacent categories between i and $i+1$. Once this refinement is obtained, one can then proceed in decomposing the last increment from i to $i+1$ into $[(-\delta_{i+1}) - (\delta_i - 1)]$ "small" increments between adjacent intermediate categories, each of which being matched with a decrement from each category $h > i+1$ for which there is a strictly positive number of agents. Hence, it is possible to achieve s'' from s by using Hammond transfers only (provided that a suitable refinement of the grid be performed). Hence, there exists a non-negative integer t such that s'' can be obtained from s' by means of exactly $(-\delta_{i+1})$ Hammond transfers on the grid $\mathcal{C}(t)$ (recalling that a transformation on the grid $\mathcal{C}$ is also a transformation on the grid $\mathcal{C}(t)$). We then conclude that society s first order dominates society s'' which in turn H^t-dominates society s' and this completes the proof.

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