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An Axiomatic Approach to "Preference for Freedom of Choice"[1]

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The paper axiomatically characterizes a notion of "freedom of choice." The central condition considered in this paper is the following. In every non-empty opportunity set there exists at least one alternative such that its exclusion would reduce an agent's freedom. It turns out that this condition induces a formal structure which is familiar from the theory of revealed preference. Under a somewhat different interpretation, some well-known rationality conditions are used to characterize models of freedom. Furthermore, a link is established between the notion of "preference for freedom" and Kreps' concept of "preference for flexibility." *Journal of Economic Literature* Classification Number: D71. © 1996 Academic Press, Inc.

1. INTRODUCTION

In a number of papers, Sen has argued that the concept of freedom of choice plays a fundamental role for our understanding of rational behaviour and consistent choice (see Sen [7–10]). While the *instrumental* importance of freedom—as a means towards other ends—is widely recognized, Sen has in particular drawn attention to its *intrinsic* value, i.e., to its value as an end on its own right (see, e.g., Sen [8, p. 270]). Inspired by the work of Sen and others, there have been some recent attempts to formally define a notion of freedom of choice, including Bossert *et al.* [1], Klemisch-Ahlert [2], Pattanaik and Xu [4, 5], and Suppes [11]. The purpose of this paper is to provide an axiomatic foundation for a class of models of "preference for freedom of choice."

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Since the concept of freedom is a very complex phenomenon every attempt to formalize it must neglect important aspects. As most authors have done, this paper concentrates on the *opportunity aspect* of freedom. Under this aspect, the extent of freedom is determined by the opportunities which a decision maker may or may not have in a certain decision situation. The basic model in this context is a two-stage decision model, where in the first stage a non-empty opportunity set (or menu) is chosen from which, in a second stage, exactly one alternative is chosen as the final outcome. A simple example of this framework, suggested in Kreps [3], is that of making reservations at a restaurant. Consider an individual who plans to go out for dinner. What the individual will eventually choose is a meal, but the initial choice is of a restaurant, or a menu, from which the individual will later choose a meal. Obviously, in such a model a desire for freedom of choice will manifest itself in the first stage when the individual is to choose the set which determines his or her opportunities in the second stage. Consequently, this paper focuses on the question of how the first stage choices among menus are to be modelled.

Some of the axioms suggested in the literature imply that the assessment of the freedom offered by a particular menu is *independent* of an underlying preference relation among its elements. In particular, the axiomatic characterization given in Pattanaik and Xu [4] implies that a menu A offers at least as much freedom as another menu B if and only if A contains at least as many elements as B. This *cardinality-based* comparison of opportunity sets is in sharp contrast to the standard procedure of comparing opportunity sets on the basis of the indirect utilities derived from an underlying preference relation among the basic alternatives. According to traditional consumer theory, an opportunity set A is to be (weakly) "preferred" to B if and only if the maximal attainable utility level in A is at least as high as the maximal attainable utility level in B. For example, if $x \in A$ is a best alternative in A, an individual is assumed to be "indifferent" between the singleton-menu $\{x\}$ and the menu A, no matter what the size of A is. Thus, these different ways of ranking opportunity sets are in a sense opposite extremes. The first completely neglects the role of the prevailing preferences among the basic alternatives, whereas the second leaves no room for a desire for freedom of choice.

The models suggested in this paper are in between these two extreme cases. On the one hand, they are consistent with an underlying preference relation among the basic alternatives in the sense that the menu $\{x\}$ is preferred to the menu $\{y\}$ if and only if the alternative x is preferred to the alternative y. Thus, the models considered here respect the fundamental interrelation of freedom and preference referred to in Sen [9]. On the other hand, the models of this paper are distinguished from the standard approach by the following condition. For every non-empty menu A there exists an alternative $x \in A$ such that A is ranked strictly above $A \backslash \{x\}$. In the following, this condition is referred to as Axiom F. Note that the standard approach of comparing opportunity sets does not in general satisfy this axiom. For example, if a decision maker is indifferent between the alternatives x and y he or she is assumed to be "indifferent" between the three menus $\{x, y\}$, $\{x\}$, and $\{y\}$. Clearly, this violates Axiom F. It is argued that Axiom F is a *necessary* condition for a ranking of menus to display a "preference for freedom of choice." Note that the cardinality-based approach satisfies a much stronger property than Axiom F. Indeed, the characterization by Pattanaik and Xu [4] implies that A is ranked strictly above $A \backslash \{x\}$ for *every* $x \in A$. However, we believe that the latter is too strong a condition for "preference for freedom of choice." Again, this follows from the interrelation of freedom and preference. It could be argued that an available alternative can only contribute to the freedom of a decision maker if it is in some (weak) sense *valuable* to him or her. For instance, does it expand your freedom when you are given the additional option of suffering from a serious desease? The answer seems to be, no. Therefore, it would be desirable to have models available which do not imply that *every* available alternative constitutes an essential contribution to the freedom in a certain decision situation. Such models are provided in this paper.

The outline of the paper is as follows. Section 2 introduces some notation and our basic conditions. It is shown that Axiom F induces in a natural way a correspondence on the set of opportunity sets which associates to every non-empty menu A a non-empty subset $E(A)$ of its *essential* alternatives. Section 3 provides a general representation theorem in terms of the correspondence E. The formal similarities between the correspondence E and choice correspondences as considered in the theory of revealed preference are exploited in Section 4. Under a somewhat different interpretation, some of the well-known rationality conditions introduced in Sen [6] are used to characterize a class of models of "preference for freedom of choice." These models have the following common representation. For all non-empty menus A and B, *A dominates B in terms of freedom of choice* if and only if for some reflexive and complete binary relation R_E on the set of basic alternatives one has

$$\max_{R_E} (A \cup B) \subseteq A.$$

Here, $\max_{R_E} A$ denotes the set of maximal alternatives in A with respect to R_E, i.e., the set of all $x \in A$ such that $x R_E y$ for all $y \in A$. The interpretation of the relation R_E is as follows. For all alternatives x and y, $x R_E y$ if and only if x is essential in a situation where y is the only additional alternative which is available. With other words, xR_Ey if and only if ($x=y$ or $\{x, y\} \succ \{y\}$). In the context of the present paper, some of the conditions used in Sen [6] have remarkably strong implications. For example, Property (α) implies that the relation R_E corresponding to a ranking of menus in terms of freedom of choice is not only acyclic, but quasi-transitive. Thus, under the particular interpretation intended here, Properties (α) and (γ) together imply Property (δ). Note that, in general, Property (δ) is an independent condition (see Sen [6]). As in the general case, full transitivity of R_E corresponds to the conjunction of Properties (α) and (β) (see also Sen [6]).

Section 5 investigates the relation between the models considered in Sections 3 and 4 and a given preference relation among the basic alternatives. In Section 6, a link is established between the models of freedom of choice presented here and the concept of flexibility under uncertainty about future tastes introduced in Kreps [3]. Our main result in this context is the following. Let R be a quasi-transitive preference relation on the set of basic alternatives. Assume that the decision maker—while being sure about his or her *strict* preference judgements—is uncertain about his or her *indifference* judgements. Also, assume that for all alternatives x and y, the decision maker regards x as essential in $\{x, y\}$ if and only if xRy, i.e., $R_E = R$. Then a menu A dominates another menu B in terms of freedom of choice if and only if A dominates B in terms of flexibility.

Section 7 provides a brief summary and some concluding remarks. All proofs are found in the Appendix.

2. Notations, Definitions, and Basic Conditions

Let $\Omega = \{x, y, z, \ldots\}$ be a finite non-empty set of basic alternatives with $\#\Omega \geqslant 3$, and let $\mathscr{X} := 2^{\Omega} \setminus \{\varnothing\}$ denote the set of all non-empty subsets of Ω. Generic elements of $\mathscr{X}$ will be denoted by $A, B, C, \ldots$, and are referred to as opportunity sets or menus. Let $R \subseteq \Omega \times \Omega$ be a binary relation on Ω. As usual, P and I denote the asymmetric and symmetric part of R, respectively. Furthermore, for $A \in \mathscr{X}$ let $\max_R A$ denote the set of maximal elements in A with respect to R, i.e.,

$$\max_R A := \{x \in A : xRy \text{ for all } y \in A\}.$$

Throughout this paper, the symbol R denotes a reflexive and complete binary relation on Ω, i.e., for all $x, y \in \Omega$, xRx and (xRy or yRx). It is well known that $\max_R A$ is always non-empty if and only if R is *acyclic*, i.e., if and only if there are no P-cycles. The relation R is called *quasi-transitive* if its asymmetric part P is transitive. Furthermore, R is a *weak order* if R

itself is transitive, i.e., if for all $x, y, z \in \Omega$ (xRy and yRz) implies xRz. Obviously, transitivity implies quasi-transitivity, which in turn implies acyclicity.

According to the traditional theory of consumer's behaviour a weak order R on Ω induces a binary relation $\succcurlyeq_S$ on $\mathscr{X}$ by

$$A \succcurlyeq_S B :\Leftrightarrow [\max_R (A \cup B)] \cap A \neq \varnothing. \tag{1}$$

Obviously, $\succcurlyeq_S$ is a weak order on $\mathscr{X}$.

Let $\succcurlyeq$ be a *preorder* on $\mathscr{X}$, i.e., a reflexive and transitive (but not necessarily complete) binary relation. The asymmetric and symmetric part of $\succcurlyeq$ are denoted by $\succ$ and $\sim$, respectively. Note that $\succ$ and $\sim$ are transitive. The intended interpretation of $A \succcurlyeq B$ is that menu A offers at least as much freedom as menu B. It should be emphasized that in our formal framework we do not distinguish between indifference and incomparability on the level of basic alternatives. On the other hand, we want to allow for this distinction at the level of menus. This is the reason why rankings of menus are formally described by possibly incomplete preorderings. In this respect we thus follow Sen who writes: "Comparisons of freedom must frequently take the form of incomplete orderings. While some set comparisons would be obvious enough, others would remain undecidable [10, p. 19]." An illustration of the distinction between indifference and incomparability with respect to freedom of choice is given in the next section.

A rather uncontroversial condition for a ranking of menus in terms of freedom of choice is the following.

Axiom M (Monotonicity with Respect to Set Inclusion). For all $A, B \in \mathscr{X}$, $B \subseteq A \Rightarrow A \succcurlyeq B$.

Thus, if B is a subset of A then A offers at least as much freedom as B.[2] Note that even the weak order $\succcurlyeq_S$ defined in (1) satisfies Axiom M although $\succcurlyeq_S$ is not concerned with freedom of choice. Thus, Axiom M is certainly not sufficient for a ranking of opportunity sets to display a "preference for freedom of choice."

In order to come closer to a substantial description of "preference for freedom of choice" we introduce the following condition. It is the condition on which all the models considered in this paper are based. For notational convenience, the relation $\succcurlyeq$ is extended to $2^{\Omega} = \mathscr{X} \cup \{\varnothing\}$ by defining $A \succ \varnothing$ for all $A \in \mathscr{X}$.

[2] Of course, in assuming M we neglect "thinking costs" and similar considerations which could suggest that expanding the opportunities of a decision maker might not always be advantageous.

Axiom F (Preference for Freedom of Choice). For all $A \in \mathscr{X}$, there exists $x \in A$ such that $A \succ A \backslash \{x\}$.

The intuition behind Axiom F is as follows. Consider a decision maker who cares about his or her freedom to choose. Intuitively, to say that a decision maker cares about freedom is to say that he or she attaches importance to the *availability* of certain alternatives without necessarily planning to choose one particular of these alternatives. The set of alternatives the availability of which would contribute to an agent's freedom may contain various items on very different grounds. However, our intuition is that in any case this set will contain those alternatives which in the decision maker's opinion deserve serious consideration for later choice. Of course, this is not to deny that the presence of other alternatives might expand a decision maker's freedom as well. Consider a specific choice situation in which a decision maker's opportunities are given by the set A. By assumption, A contains *all* the alternatives which a decision maker can possibly choose. Consequently, there must be some alternatives in A which the decision maker seriously considers for later choice. Let x be one of these alternatives. Then by the foregoing argument, A will be ranked strictly above $A \backslash \{x\}$ in terms of freedom of choice. In particular, there always *exists* an alternative the availability of which contributes to the agent's freedom, which is exactly what is required by Axiom F.

Whether or not for a specific alternative $x \in A$ one has $A \succ A \backslash \{x\}$ will in general depend on the decision maker's evaluation of x vis-à-vis the other alternatives available in A. However, in asserting $A \succ A \backslash \{x\}$ it is not necessary that a decision maker has a definite *preference ordering* among the alternatives of A. Indeed, in order to determine whether the exclusion of a certain alternative would reduce one's freedom it may be sufficient to have a more or less *vague* notion of the "value" of that alternative relative to all other available alternatives. Nevertheless, the meaning of Axiom F becomes particularly clear by considering specific examples in which the decision maker is assumed to have preferences among the basic alternatives. Thus, consider a two-element menu $A = \{x, y\}$. If one of the alternatives is strictly preferred to the other then it seems safe to assume that the exclusion of the preferred alternative would make the agent worse off. Next, suppose that the decision maker is indifferent between x and y. In this case, it seems plausible to assume that the exclusion of *either* alternative would reduce the agent's freedom (even if it does *not* reduce the derived indirect utility). While this would certainly verify Axiom F, it is worth emphasizing that also in this case Axiom F only requires ($\{x, y\} \succ \{x\}$ *or* $\{x, y\} \succ \{y\}$). Hence, Axiom F by itself does not exclude the possibility that information which is not reflected by the agent's preference relation might allow the agent to discriminate between the exclusion of indifferent alternatives. Additional conditions on the ranking $\succcurlyeq$ will, however, imply $A \succ A \backslash \{x\} \Leftrightarrow A \succ A \backslash \{y\}$, whenever x and y are *equivalent* with respect to the decision maker's preferences in the sense that $xRz \Leftrightarrow yRz$ for all $z \in A$ (cf. Corollary 3 below). In general, it is argued that $A \succ A \backslash \{x\}$ holds at least for some x which is undominated in A. Note that given acyclicity of the underlying preference relation undominated alternatives always exist. Clearly, this argument does not rule out the possibility that the exclusion of other (dominated) alternatives might reduce an agent's freedom as well.[3]

Intuitively, the extent to which an alternative can contribute to the freedom offered by a menu will not only depend on the characteristics of that alternative but also on the size of the specific menu. Suppose for instance that a menu contains many alternatives between all of which the agent is indifferent. In such a case, the role of any single element might diminish when the menu becomes large. It is thus conceivable that "in the limit" any *single* alternative would be insignificant. However, since the set X of basic alternatives is assumed to be finite, the size of menus is bounded. Axiom F is therefore justified by the assumption that the decision maker has a sufficiently high (though not necessarily perfect) discrimination power with respect to the number of available alternatives.

There are two respects in which Axiom F is considered to be a rather weak condition of "preference for freedom." First, it requires for any menu A only the *existence* of some $x \in A$ such that $A \succ A \backslash \{x\}$. In contrast, the ranking characterized in Pattanaik and Xu [4], which is given by

$$A \succcurlyeq_{\#} B :\Leftrightarrow \# A \geqslant \# B, \tag{2}$$

has the much stronger property that $A \succ A \backslash \{x\}$ for *every* $x \in A$. Intuitively, this would correspond to the case in which *every* alternative is—at least in some weak sense—valuable to the decision maker. However, it is conceivable that in some situations there is in fact only one alternative $x \in A$ such that $A \succ A \backslash \{x\}$. Suppose for instance that in a certain opportunity set A there is only one acceptable alternative x among other terrible and dreadful alternatives. Then it could be argued that x is the only alternative which contributes to the freedom offered by menu A.

Second, Axiom F is weak since it compares every set only with some of its subsets, implying no restrictions on the comparison of menus which are incomparable with respect to set inclusion. Thus, Axiom F alone is certainly not sufficient to single out one specific ranking of freedom. However, the intended interpretation of Axiom F is as a property characterizing the

[3] For example, one could argue that the presence of an alternative x may contribute to the agent's freedom if there are *other* agents who consider x to be valuable. For an analysis of such an approach see Pattanaik and Xu [5].

class of all rankings in which one would have to search for appropriate specifications of "preference for freedom of choice" by imposing further restrictions. Thus, it is contended that any reasonable notion of "freedom of choice" would have to satisfy Axiom F. In this sense, we consider Axiom F to be a *necessary* condition for "freedom of choice."[4]

To conclude the discussion of our central condition, we note that although Axiom F is a very weak property it is not completely innocuous. Indeed, it implicitly assumes that every opportunity set (containing at least two elements) does offer *some* freedom. One could imagine an extreme case in which an opportunity set contains only terrible and dreadful alternatives between all of which a decision maker is indifferent. In such a desperate situation it might be disputable whether a decision maker would appreciate the presence of any of the alternatives. The applicability of Axiom F might therefore be limited by such extreme cases.

In the following sections it will be shown that, despite its generality, Axiom F induces a very tractable structure on rankings of opportunity sets. If further conditions are imposed on this structure the class of rankings characterized by Axiom F will considerably shrink and may eventually contain only one single element (see Corollary 3 below). The key to the description of this structure is, given any menu A, the distinction between those alternatives x in A for which $A \succ A \setminus \{x\}$ and those for which $A \sim A \setminus \{x\}$. Formally, for every $A \in \mathscr{Z}$, let

$$E(A) := \{x \in A : A \succ A \setminus \{x\}\}. \tag{3}$$

Obviously, $E(A) \subseteq A$ for all $A \in \mathscr{Z}$. By Axiom F, $E(A) \neq \varnothing$ for all $A \in \mathscr{Z}$. Thus, given Axiom F, (3) defines a mapping E: $\mathscr{Z} \to \mathscr{Z}$ with $E(A) \subseteq A$ for all $A \in \mathscr{Z}$. An alternative $x \in A$ will be called *essential* in A if $x \in E(A)$. If $x \notin E(A)$, x is called *non-essential* in A.

3. A General Model

Let $\succcurlyeq$ be a preorder on $\mathscr{Z}$. There are in principle two different ways in which a menu A can offer at least as much freedom as B. First, $A \succcurlyeq B$ can hold in the very strong sense that joining B to A is of no value, i.e., $A \succcurlyeq A \cup B$. In that case, we will say that A *dominates* B. On the other hand, $A \succcurlyeq B$ may hold without $A \succcurlyeq A \cup B$ being true, i.e., A may offer at least as much freedom as B and at the same time joining B to A may be valuable.

[4] This view is supported by the fact that all the rankings studied in Bossert *et al.* [1], Klemish-Ahlert [2], and Pattanaik and Xu [4, 5] satisfy Axiom F. Our intuition is that Axiom F is indeed the essential common property of the different rankings of freedom suggested by these authors.

According to this observation we define for every preorder $\succcurlyeq$ on $\mathscr{X}$ the induced *domination relation* $\succcurlyeq^*$ on $\mathscr{X}$ (see also Kreps [3]) by

$$A \succcurlyeq^* B :\Leftrightarrow A \succcurlyeq A \cup B.$$

Obviously, $\succcurlyeq^*$ is reflexive, and $B \subseteq A$ implies $A \succcurlyeq^* B$. The induced domination relation $\succcurlyeq^*$ is a restriction of $\succcurlyeq$, i.e., for all $A, B \in \mathscr{X}$, $A \succcurlyeq^* B \Rightarrow A \succcurlyeq B$, if and only if $\succcurlyeq$ satisfies Axiom M. Of course, in this case $A \succcurlyeq^* B$ is equivalent to $A \sim A \cup B$. We note that $\succcurlyeq^*$ is not necessarily transitive. Conditions which guarantee transitivity of $\succcurlyeq^*$ will be given later.

In this section, we characterize the class of rankings in terms of freedom of choice such that A dominates B if and only if every essential alternative in $A \cup B$ is available in A. It can be shown that a necessary condition for this equivalence is the following.

Axiom I (Independence of Non-essential Alternatives). For all $A \in \mathscr{X}$, $E(A) \sim A$.

Axiom I is a regularity condition. It states that the assessment of the freedom offered by menu A depends only on the subset $E(A)$ of essential alternatives. Thus, by Axiom I, $A \succcurlyeq B \Leftrightarrow E(A) \succcurlyeq B \Leftrightarrow A \succcurlyeq E(B)$ for all $A, B \in \mathscr{X}$.

PROPOSITION 1. *Let $\succcurlyeq$ be a preorder on $\mathscr{X}$ which satisfies Axiom* F, *and let E: $\mathscr{X} \rightarrow \mathscr{X}$ be the corresponding mapping defined by* (3). *Then $\succcurlyeq$ satisfies Axioms* M *and* I *if and only if the induced domination relation $\succcurlyeq^*$ is a restriction of $\succcurlyeq$ and for all $A, B \in \mathscr{X}$,*

$$A \succcurlyeq^* B \Leftrightarrow E(A \cup B) \subseteq A. \tag{4}$$

The main content of Proposition 1 is that it provides simple conditions under which the induced domination relation is completely determined by the correspondence E: $\mathscr{X} \rightarrow \mathscr{X}$ which associates to each opportunity set the subset of essential alternatives. The class of rankings characterized by Proposition 1 is, however, very large. Indeed, a ranking of menus satisfying Axioms F, M and I is unspecified in two respects. First, there is no restriction on the correspondence E: $\mathscr{X} \rightarrow \mathscr{X}$ other than $E(A) \subseteq A$ for all $A \in \mathscr{X}$. This issue will be dealt with in the next section where the structure of the mapping E is examined in more detail. Second, even when such a mapping E is given, Proposition 1 does not provide a rule for ranking two menus in a situation where neither menu dominates the other. Consider the following examples.

The cardinality-based ranking $\succcurlyeq_{\#}$ defined by (2) satisfies Axioms F, M, and I. The corresponding mapping E is the identity on $\mathscr{Z}$, i.e., $E(A)=A$ for all $A \in \mathscr{Z}$. Therefore, the induced domination relation $\succcurlyeq^*_{\#}$ reduces to set inclusion, i.e., $A \succcurlyeq^*_{\#} B \Leftrightarrow B \subseteq A$. But also the ranking of menus given by set inclusion itself satisfies F, M, and I, with $E(A)=A$ for all $A \in \mathscr{Z}$. Thus, either ranking, the cardinality-based weak order $\succcurlyeq_{\#}$ or the ranking given by the partial order of set inclusion, induces the same domination relation.

As we have argued, there might be situations in which not every alternative is considered to be essential. Thus, in general for some $A \in \mathscr{Z}$, $E(A) \neq A$. Suppose for instance that in the set $\Omega = \{x, y, z, v, w\}$ only the alternatives x, y, and z, are considered to be essential for the freedom offered by menu Ω. Let $\succcurlyeq$ be a ranking of the non-empty subsets of Ω which satisfies Axioms F, M, and I, such that $E(\Omega) = \{x, y, z\}$. By Proposition 1, it follows that $\{x, y, z, v\} \sim^* \{x, y, z, w\}$, hence also $\{x, y, z, v\} \sim \{x, y, z, w\}$. Indeed, it seems reasonable to assume that two sets which differ only in non-essential alternatives are treated as indifferent. On the other hand, it is not clear how, e.g., the sets $\{x, y\}$ and $\{y, z\}$ are to be compared in terms of freedom of choice. It is possible that in some special situations there do exist reasonable rules for ranking two menus when neither menu dominates the other. For instance, one could think of a rule based on the respective cardinalities *of essential alternatives*. Indeed, under some restrictive assumptions this modified cardinality-based approach yields a *weak order* on the set of non-empty menus (see Section 7). However, it is not obvious that in the general framework of this paper there exists a satisfactory *universal* rule for ranking undominated menus. The problem of ranking undominated menus is clearly an interesting and important issue, but it is not the aim of the present paper to advocate a particular solution to this problem. Some further remarks on the difficulty of systematically extending the domination relation are therefore deferred to Section 7.

There are two possible ways to avoid these difficulties. As was done in Proposition 1, one can leave the relation $\succcurlyeq$ *unspecified* on the domain of all pairs (A, B) such that neither A dominates B nor B dominates A. On the other hand, one could claim that two menus are *incomparable* with respect to $\succcurlyeq$ whenever neither dominates the other. Formally, this claim amounts to assuming the following condition (see also [3]).

Axiom D (Domination Principle). For all $A, B \in \mathscr{Z}$, $A \succcurlyeq B \Leftrightarrow A \sim A \cup B$.

Obviously, D implies M and $\succcurlyeq^* = \succcurlyeq$. Thus, the following corollary follows at once from Proposition 1.

Corollary 1. *Let* $\succcurlyeq$ *be a preorder on* $\mathscr{X}$ *which satisfies Axiom* F, *and let* $E\colon \mathscr{X} \to \mathscr{X}$ *be the corresponding mapping defined by* (3). *Then* $\succcurlyeq$ *satisfies Axioms* D *and* I *if and only if, for all* $A, B \in \mathscr{X}$,

$$A \succcurlyeq B \Leftrightarrow E(A \cup B) \subseteq A.$$

Note that Corollary 1 yields a very simple characterization of the class of preorders satisfying Axioms F, D, and I. It is the class of preorders $\succcurlyeq$ for which there *exists* a mapping $E\colon \mathscr{X} \to \mathscr{X}$ with $E(A) \subseteq A$ for all $A \in \mathscr{X}$, such that $A \succcurlyeq B$ if and only if $E(A \cup B) \subseteq A$. The price for the simplicity of this characterization is of course the fact that a ranking which satisfies D is, in general, rather incomplete.[5] Because of this ambivalence of Axiom D the following sections will provide two alternative types of results: One type which uses Axiom D, and another type which does not assume this axiom and which specifies the induced domination relation only.

4. Contraction and Expansion Consistency Conditions for Sets of Essential Alternatives

Let $\succcurlyeq$ be a preorder on $\mathscr{X}$ satisfying Axiom F. In this section we want to exploit the formal similarities between the induced correspondence $E\colon \mathscr{X} \to \mathscr{X}$ and choice correspondences as considered in the theory of revealed preference. Indeed, we will make use of some well-known rationality conditions which have been proposed in that context. Specifically, we will apply some of the results established in Sen [6]. It should be emphasized, however, that despite the formal similarities between the following treatment of the correspondence E and the treatment of a choice correspondence C in the theory of revealed preference, the set $E(A)$ and the choice set $C(A)$ have somewhat different interpretations. It may well be that even though $x \in E(A)$ a decision maker would never *choose* the alternative x from A in a later stage of choice. Indeed, $x \in E(A)$ just means that a decision maker considers the presence of x in A to be an essential contribution to the freedom offered by menu A.

Define a binary relation $R_E \subseteq \Omega \times \Omega$ as follows. For $x, y \in \Omega$,

$$xR_E y\colon \Leftrightarrow x \in E(\{x, y\}). \tag{5}$$

[5] Note, however, that the "degree of completeness" of $\succcurlyeq$ depends on the structure of the mapping $E\colon \mathscr{X} \to \mathscr{X}$. Roughly speaking, the smaller the sets $E(A)$ for $A \in \mathscr{X}$ the "more complete" is the relation $\succcurlyeq$. For instance, let $\succcurlyeq$ satisfy Axioms F, D, and I. Then $\succcurlyeq$ is complete, i.e., a weak order, if and only if for every $A \in \mathscr{X}$, the set $E(A)$ consists of exactly one element.

Thus, xR_Ey if and only if x is essential in a situation where y is the only additional alternative. Obviously, R_E is reflexive, and by Axiom F, it is complete. The following condition states that x is essential in A if and only if x is essential in $\{x, y\}$ for every $y \in A$.

Axiom B (Binariness). For all $A \in \mathscr{X}$, $E(A) = \max_{R_E} A$.

By Axiom B, the set $E(A)$ is determined by *binary* comparisons of the alternatives in A with respect to the relation R_E. It is well known from the theory of revealed preference that B is equivalent to the conjunction of the following two conditions (see Sen [6]). For all $A, B \in \mathscr{X}$, $x \in \Omega$,

(α) $[x \in B \subseteq A \text{ and } x \in E(A)] \Rightarrow x \in E(B)$,

(γ) $x \in E(A) \cap E(B) \Rightarrow x \in E(A \cup B)$.[6]

The first condition, Property (α), is a contraction consistency condition. In our framework, it states that if x is essential in A, then x is essential in every subset B of A which contains x. The second condition, Property (γ), is an expansion consistency condition. It states that an alternative which is essential in A and in B must also be essential in the union of these sets.

It is known that (α) implies acyclicity of the relation R_E defined by (5). However, under the interpretation intended here, i.e., under Axioms F and M, Property (α) has stronger implications.

LEMMA 1. *Let* $\succcurlyeq$ *be a preorder on* $\mathscr{X}$ *satisfying Axioms* F *and* M. *Then* $\succcurlyeq$ *satisfies Property* (α) *if and only if the induced domination relation* $\succcurlyeq^*$ *is transitive.*

The equivalence established by Lemma 1 implies the following result. Let $\succcurlyeq$ satisfy Axioms F, M, and (α). Then the induced relation R_E according to (5) is quasi-transitive. Indeed, for all $x, y \in \Omega$ with $x \neq y$, $xP_Ey \Leftrightarrow \{x\} \succcurlyeq^* \{y\}$. Therefore, transitivity of the domination relation implies quasi-transitivity of R_E. In particular, under Axioms F and M, Properties (α) and (γ) together imply Property (δ), which in the general case is an independent condition (see Sen [6]).[7]

The following lemma shows that a ranking satisfying Axioms F, M, and (γ) is independent of non-essential alternatives.

[6] Properties (α) and (γ) are formulated here as conditions on the correspondence E. However, since E is determined by the ranking $\succcurlyeq$, Properties (α) and (γ) readily translate into conditions on the ranking $\succcurlyeq$.

[7] In fact, in the Appendix it is shown that under Axioms F and M, Property (γ) *alone* is sufficient for a ranking to satisfy Property (δ) (see Corollary 5).

LEMMA 2. *Let* $\succcurlyeq$ *be a preorder on* $\mathscr{X}$. *If* $\succcurlyeq$ *satisfies Axioms* F, M, *and* (γ) *then it satisfies Axiom* I.

We are now ready to state the main result of this section.

PROPOSITION 2. *Let* $\succcurlyeq$ *be a preorder on* $\mathscr{X}$ *which satisfies Axiom* F, *and let* R_E *be the induced relation on* Ω *according to* (5). *Then* $\succcurlyeq$ *satisfies Axiom* M *and Properties* (α) *and* (γ)—*or, equivalently, Axioms* M *and* B—*if and only if* R_E *is quasi-transitive and the induced domination relation* $\succcurlyeq^*$ *is a restriction of* $\succcurlyeq$ *such that for all* $A, B \in \mathscr{X}$,

$$A \succcurlyeq^* B \Leftrightarrow \max_{R_E} (A \cup B) \subseteq A.$$

By Proposition 2, the induced domination relation of a preorder which satisfies Axioms F, M, and B is completely determined by the relation R_E defined by (5). Moreover, the domination relation is necessarily transitive, and the relation R_E is quasi-transitive.

We now want to state a result corresponding to Proposition 2 under the assumption that $\succcurlyeq$ satisfies the domination principle D. It has been already noted that D implies M and $\succcurlyeq^* = \succcurlyeq$. In particular, $\succcurlyeq^*$ is transitive. Therefore, given Axiom F, the domination principle implies Property (α). An example of a preorder satisfying Axioms F and D but not Property (γ) is the following.

EXAMPLE. Choose three pairwise different elements $x, y_1, y_2 \in \Omega$, and define a correspondence $E: \mathscr{X} \to \mathscr{X}$ by

$$E(A) = \begin{cases} A & \text{if } \{y_1, y_2\} \not\subseteq A \\ A \backslash \{x\} & \text{if } \{y_1, y_2\} \subseteq A. \end{cases}$$

Furthermore, define a binary relation $\succcurlyeq$ on $\mathscr{X}$ as follows. For all $A, B \in \mathscr{X}$,

$$A \succcurlyeq B \Leftrightarrow E(A \cup B) \subseteq A.$$

It is easily verified that $\succcurlyeq$ is a preorder which satisfies Axioms F and D. However, $\succcurlyeq$ does not satisfy (γ). For example, let $A = \{x, y_1\}$ and $B = \{x, y_2\}$. Then $E(A) = A$ and $E(B) = B$, thus $E(A) \cap E(B) = \{x\}$, but $E(A \cup B) = E(\{x, y_1, y_2\}) = \{y_1, y_2\}$.

Note that $\succcurlyeq$ also satisfies Axiom I. Hence, Property (γ) is not implied by any combination of Axioms F, M, D, I, and (α).

The example shows that in Proposition 2 and in the following corollary, Property (γ) cannot be omitted.

Corollary 2. *Let $\succcurlyeq$ be a preorder on $\mathscr{Z}$ which satisfies Axiom* F, *and let R_E be the induced relation on Ω according to* (5). *Then $\succcurlyeq$ satisfies Axiom* D *and Property* (γ) *if and only if R_E is quasi-transitive and for all $A, B \in \mathscr{Z}$,*

$$A \succcurlyeq B \Leftrightarrow \max_{R_E} (A \cup B) \subseteq A. \tag{6}$$

It should be emphasized that every given quasi-transitive relation R_E on Ω defines via (6) a preorder $\succcurlyeq$ on $\mathscr{Z}$ which satisfies Axioms F, D, and (γ). Therefore, by Corollary 2, these axioms are in fact equivalent to the *existence* of a quasi-transitive relation R_E such that $\succcurlyeq$ is defined by (6).

We conclude this section by investigating the consequences of replacing Property (γ) with the following condition, which is well known as Property (β). For all $A, B \in \mathscr{Z}$, $x, y \in \Omega$,

$$(\beta) \quad [B \subseteq A \text{ and } \{x, y\} \subseteq E(B)] \Rightarrow [y \in E(A) \Rightarrow x \in E(A)].$$

The correspondence $E: \mathscr{Z} \to \mathscr{Z}$ satisfies (α) and (β) if and only if R_E is a weak order and for all $A \in \mathscr{Z}$, $E(A) = \max_{R_E} A$ (see Sen [6]). Thus, in Proposition 2 and Corollary 2 one may replace Property (γ) by Property (β) and, at the same time, quasi-transitivity of R_E by transitivity of R_E. Formally, there are no difficulties in replacing Property (γ) by Property (β). Note, however, that Property (β) is a rather strong requirement, especially in the interpretation intended here (see also the example in the following section).

5. Freedom and Preference

In this section we examine the relation between rankings of menus satisfying the proposed axioms and a given preference relation among the basic alternatives. It is shown that one simple additional axiom allows the reinterpretation of some of the results of the previous section in terms of a given preference relation on Ω.

Thus, let R be a complete and reflexive binary relation on Ω representing the decision maker's preferences among the basic alternatives. As before, let $\succcurlyeq$ be a preorder on $\mathscr{Z}$ reflecting the decision maker's estimation of the freedom offered by the elements of $\mathscr{Z}$. According to the remarks in the introduction and following Sen [9, p. 25] we require the following condition.

Axiom P (Consistency with Preference). For all $x, y \in \Omega$, $xPy \Leftrightarrow \{x\} \succ \{y\}$.

Note that Axiom P implies that R is quasi-transitive. Also, it should be emphasized that Axiom P is weaker than the requirement that $\succcurlyeq$ be an extension of R in the sense that for all $x, y \in \Omega$, $xRy \Leftrightarrow \{x\} \succcurlyeq \{y\}$. However, in our framework Axiom P is more appropriate since it allows for the possibility that xIy and at the same time $\{x\}$ and $\{y\}$ are incomparable with respect to $\succcurlyeq$. Given the basic conditions of the previous sections, Axiom P implies that the relation R_E defined in (5) is an *extension* of the preference relation R. Formally, one has the following lemma.

LEMMA 3. *Let R and $\succcurlyeq$ be given as above. Assume that $\succcurlyeq$ satisfies Axioms* F *and* M. *Then Axiom* P *implies that* $R \subseteq R_E$.

EXAMPLE. Let R be a weak order on Ω with utility representation $u\colon \Omega \to \mathbf{R}$. Hence, $xRy \Leftrightarrow u(x) \geqslant u(y)$. For every $A \in \mathcal{X}$, let $U(A)$ denote the derived indirect utility of A, i.e., $U(A) := \max_{x \in A} u(x)$. Let $\succcurlyeq$ be a preorder representing a decision maker's estimation of the freedom offered by the non-empty subsets of Ω. Suppose that $\succcurlyeq$ satisfies Axioms F, M, and P. Furthermore, assume that the decision maker regards an alternative $x \in A$ as essential in $A \in \mathcal{X}$ if and only if the utility of x differs from $U(A)$ by no more than some positive constant $\varepsilon > 0$. Hence, for all $A \in \mathcal{X}$,

$$E(A) = \{x \in A : u(x) \geqslant U(A) - \varepsilon\}.$$

It is easily verified that the correspondence $E\colon \mathcal{X} \to \mathcal{X}$ satisfies Properties (α) and (γ), but in general not (β). Obviously, one has

$$xR_E y \Leftrightarrow u(x) \geqslant u(y) - \varepsilon.$$

Thus, in accordance with Lemma 3, R_E is a (proper) extension of R. Note that in this example, the preorder $\succcurlyeq$ does in general not satisfy Axiom D. Indeed, suppose that x and y are such that $u(x) > u(y) \geqslant u(x) - \varepsilon$. Then, $\{x\} \succ \{y\}$ and $\{x, y\} \succ \{x\}$.

The following lemma shows that under the domination principle, Axiom P is equivalent to $R_E = R$.

LEMMA 4. *Let R and $\succcurlyeq$ be given. Assume that $\succcurlyeq$ satisfies Axioms* F *and* D. *Then Axiom* P *is satisfied if and only if* $R = R_E$.

Thus, if $\succcurlyeq$ satisfies the domination principle and Axiom P an alternative x is essential in $\{x, y\}$ if and only if xRy. By the results of the previous section, one has the following corollary.

COROLLARY 3. *Let R be a given quasi-transitive preference relation on Ω. There exists one and only one preorder $\succcurlyeq$ on $\mathscr{X}$ which satisfies Axioms* F, D, P, *and* (γ). *It is given by*

$$A \succcurlyeq B \Leftrightarrow \max_R (A \cup B) \subseteq A,$$

for all $A, B \in \mathscr{X}$.

6. "PREFERENCE FOR FREEDOM" AS "PREFERENCE FOR FLEXIBILITY"

In this section we consider a particular interpretation of the models introduced in this paper. The interpretation is in terms of *uncertainty about future tastes* and provides a link between the notion of "preference for freedom" and the concept of "preference for flexibility" introduced in Kreps [3].

Suppose that in the first stage of the two-stage decision model the decision maker is *uncertain* about his or her preferences which will prevail in the second stage of choice. Formally, there is a finite state space S, and for every $s \in S$, there is a reflexive and complete binary relation $R(s) \subseteq \Omega \times \Omega$ describing the decision maker's preferences among the basic alternatives in state s. The decision maker regards every state in S as possible but is uncertain about which state will obtain in the second stage of choice. For simplicity, we will assume that for every $s \in S$, $R(s)$ is a weak order on Ω. Let $u: \Omega \times S \to \mathbf{R}$ be a state-dependent utility function representing the weak orders $R(s)$, i.e., for all $x, y \in \Omega$ and all $s \in S$, $xR(s)y \Leftrightarrow u(x, s) \geqslant u(y, s)$. Furthermore, for every $A \in \mathscr{X}$ let $U(A, s)$ denote the derived indirect utility of menu A in state s, i.e., $U(A, s) = \max_{x \in A} u(x, s)$.

A decision maker displays a "preference for flexibility" vis-à-vis the uncertainty if he or she tries to choose a menu in such a way that the chosen menu contains valuable alternatives no matter which state will obtain. More specifically, we will say that a menu A *dominates* another menu B with respect to its flexibility if and only if for all $s \in S$, $U(A, s) \geqslant U(B, s)$.[8] Note that if there is no uncertainty, i.e., if $S = \{s\}$, then this approach to comparing menus reduces to the standard comparison defined by (1). The following lemma can be derived from the proof of Theorem 1' in Kreps [3].

[8] Unlike Kreps [3] we do not assume menus to be completely ordered. Indeed, there is nothing to suggest that one could completely order menus based on the notion of flexibility *alone*. Thus, it seems that the difficulties to systematically rank undominated menus in terms of their flexibility correspond to the difficulties in ranking undominated menus in terms of the freedom they offer.

LEMMA 5. *Let $\succcurlyeq$ be a reflexive binary relation on $\mathscr{X}$ and let $\succcurlyeq^*$ be the induced domination relation. There exist a finite state space S and a state-dependent utility function u: $\Omega \times S \to \mathbf{R}$ such that for all $A, B \in \mathscr{X}$,*

$$A \succcurlyeq^* B \Leftrightarrow \text{for all } s \in S,\ U(A, s) \geqslant U(B, s),$$

if and only if $\succcurlyeq^*$ is transitive.

COROLLARY 4. *Let $\succcurlyeq$ be a preorder on $\mathscr{X}$. There exist a finite state space S and a state-dependent utility function u: $\Omega \times S \to \mathbf{R}$ such that for all $A, B \in \mathscr{X}$,*

$$A \succcurlyeq B \Leftrightarrow \text{for all } s \in S,\ U(A, s) \geqslant U(B, s),$$

if and only if $\succcurlyeq$ satisfies Axiom D.

The class of rankings which allow for a representation in terms of flexibility under uncertainty about future tastes as in Lemma 5 is very large. For example, every ranking satisfying the conditions in Proposition 2 induces a transitive domination relation. It turns out that the class of rankings characterized by Proposition 2 corresponds to a very natural assumption about the uncertainty which the decision maker faces in the first stage of choice. Suppose that in the first stage of choice the decision maker's preferences on Ω are described by a quasi-transitive preference relation $R \subseteq \Omega \times \Omega$. The assumption is that the decision maker—while being *sure* about his or her current *strict preference* judgements—is uncertain about his or her *indifference* or *incomparability* judgements. That is, if x is strictly preferred to y at the time when the choice of the menu is performed, then, according to the decision maker's beliefs, x will also be strictly preferred to y at the second stage. On the other hand, if x and y are indifferent, or incomparable, in the first stage, then the decision maker regards *any* ranking of x and y in the second stage as a possibility.

Formally, we will say that a preference relation $\tilde{R}$ is (*weakly*) *more differentiated* than a given preference relation R if and only if for all $x, y \in \Omega$, $xPy \Rightarrow x\tilde{P}y$. Thus, in terms of the underlying state space S and the corresponding weak orders $R(s)$, the above assumption reads as follows.

Assumption 1. Let R be a given quasi-transitive preference relation on Ω. The set of possible future preferences consists of all weak orders on Ω which are more differentiated than R; i.e., $\tilde{R} = R(s)$ for some $s \in S$ if and only if for all $x, y \in \Omega$, $xPy \Rightarrow x\tilde{P}y$.

It should be emphasized that the kind of uncertainty addressed in Assumption 1 is not meant to be caused by the fact that the decision maker might receive some new information about the alternatives or the environment in between the two stages of choice. Rather, we propose to think of this uncertainty as an *intrinsic* phenomenon inherent in the concept of "indifference" or "incomparability." Indeed, there is an important conceptual difference between strict preference judgements and indifference judgements, which is reflected by the fact that—unlike strict preference judgements—indifference judgements are not *stable* with respect to "small preference perturbations." In our view, the reason for this is the following. Any preference judgement—a strict preference judgement as well as an indifference judgement—is the result of a (more or less) systematic weighing of reasons for and against particular alternatives. In general, there is no natural termination point for this process of weighing reasons. There will always be reasons and aspects which the decision maker has not yet taken into account. Obviously, however, at some point the decision maker has to stop the process. The stopping point might in principle be chosen quite arbitrarily. It seems reasonable, however, to assume that a decision maker will terminate the process of weighing reasons at a point where he or she is convinced that any further reasoning will only marginally alter the causal basis of his or her judgement. While such a "small perturbation" of the causal basis may not affect a strict preference judgement, it certainly can affect an indifference judgement. Therefore, indifference judgements are in a sense *tentative* or *provisional* and can be subject to revision at any *arbitrarily close* future time. This is, in a nutshell, what is captured by Assumption 1.

Clearly, there might be *additional* uncertainty caused, e.g., by incomplete information about the alternatives or the environment. However, what Assumption 1 describes is in a sense a "minimal uncertainty" which seems to be present in any multiple-stage decision process.

PROPOSITION 3. *Suppose that a decision maker's preferences on* Ω *are given by* $R \subseteq \Omega \times \Omega$, *and suppose that Assumption* 1 *applies. Then for all* $A, B \in \mathscr{X}$,

$$\max_R (A \cup B) \subseteq A \Leftrightarrow \textit{for all } s \in S,\ U(A, s) \geqslant U(B, s).$$

The interpretation of Proposition 3 is that a "preference for flexibility" is equivalent to a "preference for freedom of choice" if in addition to Assumption 1 the following assumption applies.

Assumption 2. Let R be a given quasi-transitive preference relation. Then for every $x, y \in \Omega$, the decision maker regards x as essential for the freedom offered by $\{x, y\}$ if and only if xRy. Thus, in terms of the relation R_E defined in (5) one has $R_E = R$.

We note that Assumption 2 is somewhat restrictive. There is an alternative interpretation of Proposition 3 which dispenses with Assumption 2. Let $\succcurlyeq$ be a given preorder on $\mathscr{X}$ satisfying Axioms F and M, and let R_E

be the corresponding relation on Ω defined by (5). Suppose that R_E is quasi-transitive and that, according to Lemma 3, R_E is an extension of R, i.e., $R \subseteq R_E$. Assume that the decision maker not only is uncertain about his or her indifference judgements, but also is uncertain about any preference judgement xRy such that xI_Ey. In other words, suppose that Assumption 1 does not apply with respect to the preference relation R but with respect to its extension R_E. Then, by Proposition 3,

$$\max_{R_E} (A \cup B) \subseteq A \Leftrightarrow \text{for all } s \in S,\ U(A, s) \geqslant U(B, s),$$

i.e., A dominates B in terms of flexibility if and only if A dominates B in terms of freedom of choice.

There are two extreme cases covered by Proposition 3. If R is a *linear* order on Ω, i.e., for all $x \neq y$, either xPy or yPx, then by Assumption 1, there is no uncertainty at all. The set of possible future preferences consists of R only. Proposition 3 reduces in this case to the assertion that $\max_R (A \cup B) \subseteq A \Leftrightarrow A \succcurlyeq_S B$ where $\succcurlyeq_S$ is the standard way of comparing opportunity sets defined in (1).

The other extreme case arises when R equals $\Omega \times \Omega$, i.e., when the decision maker is indifferent between all alternatives. In that case, $\max_R(A \cup B) \subseteq A$ if and only if $B \subseteq A$. Assumption 1 implies that the decision maker regards *any* weak order on Ω as a future possibility. Thus, Proposition 3 reduces to the assertion that B is a subset of A if and only if for every function $u: \Omega \to \mathbf{R}$ one has $U(A) \geqslant U(B)$ for the indirect utility U derived from u.

7. Summary and Concluding Remarks

In this paper we have suggested a class of models of "preference for freedom of choice." The crucial condition in all those models is Axiom F. This condition induces in a natural way the correspondence $E: \mathscr{X} \to \mathscr{X}$, which associates to every menu the subset of its essential alternatives. It has been shown that under two additional conditions, Axioms M and I, a menu A dominates another menu B in terms of freedom of choice if and only if all essential alternatives of $A \cup B$ are available in A (Proposition 1). Thus, the induced domination relation is completely determined by the correspondence E. A ranking of menus coincides with its domination relation if and only if it satisfies Axiom D. It is admitted that Axiom D is a rather strong requirement which might not be satisfied in some applications. However, in some cases it is possibly satisfied. In these cases, one obtains a simple characterization of the class of all preorders $\succcurlyeq$ on $\mathscr{X}$ which satisfy Axioms F, D, and I (Corollary 1).

Properties (α) and (γ) together imply that the correspondence E is determined by the relation R_E, i.e.,by the comparison of all sets of the form $\{x, y\}$ with $\{x\}$ for $x, y \in \Omega$. Under Axioms F, M, and (α), the domination relation is transitive and R_E is quasi-transitive (Lemma 1). Axioms F, M, and (γ) together imply Axiom I (Lemma 2). Consequently, if a ranking of menus satisfies Axioms F, M, and B, its domination relation is transitive and completely determined by R_E (Proposition 2). Axioms F and D together imply Property (α), hence Axioms F, D, and (γ) are equivalent to the existence of a quasi-transitive relation R_E such that $A \succcurlyeq B \Leftrightarrow \max_{R_E}(A \cup B) \subseteq A$ (Corollary 2).

If a ranking of menus is consistent with a given quasi-transitive preference relation R among the basic alternatives, i.e., if it satisfies Axiom P, the relation R_E is necessarily an extension of R (Lemma 3). If, in addition, the ranking satisfies Axiom D then R_E coincides with R (Lemma 4). Thus, there is one and only one ranking of menus satisfying Axioms F, D, and (γ) which is consistent with a given preference relation R. It is the ranking given by $A \succcurlyeq B \Leftrightarrow \max_R (A \cup B) \subseteq A$ (Corollary 3).

It has been shown that there is a link between these models of "preference for freedom" and Kreps' concept of "preference for flexibility." Let R be a given quasi-transitive preference relation among the basic alternatives. Assume that

(i) the decision maker is uncertain about his or her indifference-judgements, and

(ii) the decision maker regards x as essential in $\{x, y\}$ if and only if xRy.

Then a menu A dominates another menu B in terms of freedom if and only if A dominates B in terms of flexibility (Proposition 3).

Finally, we want to briefly resume the discussion of Section 3 about the difficulties in ranking undominated menus. The following rule combines the ideas of this paper with the cardinality-based approach of Pattanaik and Xu [4] defined by (2). Let $E\colon \mathscr{X} \to \mathscr{X}$ be a correspondence with $E(A) \subseteq A$. For all $A, B \in \mathscr{X}$ set

$$A \succcurlyeq_{\#, E} B \Leftrightarrow \#[E(A \cup B) \cap A] \geqslant \#[E(A \cup B) \cap B]. \tag{7}$$

Obviously, $\succcurlyeq_{\#, E}$ is complete and reflexive. The induced domination relation is given by

$$A \succcurlyeq^*_{\#, E} B \Leftrightarrow E(A \cup B) \subseteq A.$$

Thus, $\succcurlyeq_{\#, E}$ belongs to the class of rankings characterized in Corollary 1. In general, however, the relation $\succcurlyeq_{\#, E}$ is not transitive. But there is also another problem with the extension (7). Suppose that (7) holds. If both alternatives, x and y, are essential in $\{x, y\}$, then it necessarily follows that $\{x\}$ and $\{y\}$ are indifferent with respect to $\succcurlyeq_{\#, E}$. However, it may well be that one of the two alternatives, x or y, is strictly preferred to the other although *both* are essential in $\{x, y\}$. Thus, in general the definition (7) is in conflict with Axiom P. To further illustrate this problem, suppose that the correspondence E in (7) is binary, i.e., $E(A) = \max_{R_E} A$ for all $A \in \mathscr{Z}$ and some binary relation R_E on Ω. Furthermore, assume that R is an independently given preference relation on Ω. The following equivalences are easily verified.

(i) The relation $\succcurlyeq_{\#, E}$ is transitive if and only if R_E is a weak order, i.e., if and only if E satisfies, in addition to Axiom B, also Property (β).[9]

(ii) The relation $\succcurlyeq_{\#, E}$ is consistent with R, i.e., it satisfies Axiom P with respect to R, if and only if $R_E = R$.

Thus, if E is binary, a weak order extension of a given preference relation R by means of (7) is only possible in the very special case where R is a weak order *and* Assumption 2 applies. Nevertheless, for that case it would be interesting to have an axiomatic characterization of the relation defined in (7). This is, however, beyond the scope of the present paper and needs a separate investigation.

APPENDIX: PROOFS

Proof of Proposition 1. First, we prove sufficiency of the axioms. Let $\succcurlyeq$ satisfy Axioms F, M, and I, and let $E: \mathscr{Z} \to \mathscr{Z}$ be the induced correspondence according to (3). Suppose that $A \succcurlyeq^* B$, i.e., $A \succcurlyeq A \cup B$. Let $x \in E(A \cup B)$, i.e., $A \cup B \succ (A \cup B) \backslash \{x\}$. We have to show that $x \in A$. Suppose to the contrary that $x \in B \backslash A$. Then $A \subseteq (A \cup B) \backslash \{x\}$; thus by Axiom M and transitivity, $(A \cup B) \backslash \{x\} \succcurlyeq A$ and $A \cup B \succ A$. But this contradicts the assumption that $A \succcurlyeq^* B$, hence $x \in A$. Conversely, let $E(A \cup B) \subseteq A$. By Axiom M, $A \succcurlyeq E(A \cup B)$. Axiom I implies $E(A \cup B) \sim A \cup B$, hence $A \succcurlyeq A \cup B$, i.e., $A \succcurlyeq^* B$.

To prove the necessity of Axioms M and I, suppose that (4) holds and that $\succcurlyeq^*$ is a restriction of $\succcurlyeq$. First, let $B \subseteq A$. The equivalence (4) implies $A \succcurlyeq^* B$, and since $\succcurlyeq^*$ is a restriction of $\succcurlyeq$ one has $A \succcurlyeq B$. Hence, $\succcurlyeq$ satisfies M. Finally, for all $A \in \mathscr{Z}$, $E(A) \subseteq A$, hence $E(E(A) \cup A) \subseteq E(A)$. This implies $E(A) \succcurlyeq^* A$ and $E(A) \succcurlyeq A$, hence $E(A) \sim A$. Thus, $\succcurlyeq$ satisfies Axiom I.

[9] Moreover, it can be shown that even if R_E is quasi-transitive the ranking $\succcurlyeq_{\#, E}$ defined by (7) will, in general, not be acyclic.

Proof of Lemma 1. Let $\succcurlyeq$ satisfy Axioms F and M, and let $E: \mathscr{X} \to \mathscr{X}$ be the induced correspondence according to (3). Suppose that $\succcurlyeq$ satisfies Property (α). First, we show that if for $A \in \mathscr{X}$ and $z_i \in \Omega$ one has $A \sim A \cup \{z_i\}$ for all $i \in \{1, ..., n\}$, then $A \sim A \cup \{z_1, ..., z_n\}$. Indeed, if for all $i \in \{1, ..., n\}$, $z_i \notin E(A \cup \{z_i\})$, then one has for all $i \in \{1, ..., n-1\}$,

$$A \cup \{z_1, ..., z_i\} \sim A \cup \{z_1, ..., z_{i+1}\}.$$

Otherwise one would obtain $z_{i+1} \in E(A \cup \{z_1, ..., z_{i+1}\})$ for some i, which by (α) would imply $z_{i+1} \in E(A \cup \{z_{i+1}\})$. Hence by induction,

$$A \sim A \cup \{z_1\} \sim A \cup \{z_1, z_2\}, \sim \cdots \sim A \cup \{z_1, ..., z_n\},$$

which by transitivity implies $A \sim A \cup \{z_1, ..., z_n\}$.

Now assume that $A \succcurlyeq^* B$ and $B \succcurlyeq^* C$, i.e., $A \sim A \cup B$ and $B \sim B \cup C$. In order to verify transitivity of the domination relation we must show that $A \succcurlyeq^* C$, i.e., $A \sim A \cup C$. Clearly, $A \sim A \cup C$ if $C \backslash A$ is empty. Thus, let $C \backslash A$ be non-empty, say $C \backslash A = \{z_1, ..., z_n\}$. We will show that $A \sim A \cup \{z_i\}$ for all $i \in \{1, ..., n\}$. Let z_i be given. Then one has $A \cup B \sim A \cup B \cup \{z_i\}$. This follows at once if $z_i \in B$. If, on the other hand, $z_i \notin B$, then $A \cup B \cup \{z_i\} \succ A \cup B$ would imply $z_i \in E(A \cup B \cup \{z_i\})$; hence by Property ($\alpha$), $z_i \in E(B \cup \{z_i\})$. But this is not possible, since $B \succcurlyeq^* C$. Thus, one can conclude $A \sim A \cup B \sim A \cup B \cup \{z_i\}$. Hence, by Axiom M and transitivity, $A \sim A \cup \{z_i\}$. Therefore, we have shown that for all $i \in \{1, ..., n\}$, $A \sim A \cup \{z_i\}$. By the first part of this proof this implies $A \sim A \cup \{z_i, ..., z_n\} = A \cup C$, i.e., $A \succcurlyeq^* C$. Thus, Property (α) implies transitivity of the domination relation.

Assume now that, conversely, the induced domination relation $\succcurlyeq^*$ is transitive. Let $x \in B \subseteq A$ and $x \in E(A)$. In order to verify Property (α) we have to show that $x \in E(B)$. Suppose to the contrary that $x \notin E(B)$, i.e., $B \backslash \{x\} \sim B$. Then $B \backslash \{x\} \succcurlyeq^* B$. Also, one has $A \backslash \{x\} \succcurlyeq^* B \backslash \{x\}$. Hence, by transitivity of $\succcurlyeq^*$, $A \backslash \{x\} \succcurlyeq^* B$. Since $x \in B$, this implies $A \backslash \{x\} \succcurlyeq A$. But this contradicts the assumption that $x \in E(A)$. Therefore, x must be in $E(B)$.

Proof of Lemma 2. Let $\succcurlyeq$ satisfy Axioms F and M and Property (γ). Furthermore, let $E: \mathscr{X} \to \mathscr{X}$ be the induced correspondence according to (3). By Axiom M, $A \succcurlyeq E(A)$ for all $A \in \mathscr{X}$. Obviously, $E(A) \sim A$ if $E(A) = A$. Thus, suppose that $A \backslash E(A)$ is non-empty, say $A \backslash E(A) = \{x_1, ..., x_n\}$. By Axiom F, the subset of essential alternatives of any non-empty set is non-empty. Therefore, we may assume without loss of generality that the x_i are ordered in such a way that for all $i \in \{1, ..., n\}$, $x_i \in E(\{x_i, ..., x_n\})$. This implies $E(A) \sim E(A) \cup \{x_1\}$. Otherwise, if $E(A) \cup \{x_1\} \succ E(A)$, one would have $x_1 \in E[E(A) \cup \{x_1\}]$ and $x_1 \in E(\{x_1, ..., x_n\})$.

By Property (γ), this would imply $x_1 \in E(A)$ which is false by assumption. By the same argument, one has for every $i \in \{1, ..., n-1\}$,

$$E(A) \cup \{x_1, ..., x_i\} \sim E(A) \cup \{x_1, ..., x_{i+1}\}.$$

This implies

$$E(A) \sim E(A) \cup \{x_1\} \sim E(A) \cup \{x_1, x_2\} \sim \cdots \sim E(A) \cup \{x_1, ..., x_n\},$$

hence by transitivity, $E(A) \sim E(A) \cup \{x_1, ..., x_n\} = A$. Thus, $\succcurlyeq$ satisfies Axiom I.

COROLLARY 5. *Let $\succcurlyeq$ be a preorder on $\mathscr{X}$. If $\succcurlyeq$ satisfies Axioms* F *and* M *and Property* (γ) *then it satisfies Property* (δ).

Proof. Property (δ) states that for all $A, B \in \mathscr{X}$, and for all $x, y \in \Omega$ with $x \neq y$,

$$[B \subseteq A \text{ and } \{x, y\} \subseteq E(B)] \Rightarrow \{y\} \neq E(A).$$

We verify this property by contraposition. Therefore, suppose that $B \subseteq A$ and $\{y\} = E(A)$. By Lemma 2, one has $A \sim \{y\}$. Since $y \in B \subseteq A$, it follows that $B \sim \{y\}$. If $x \in E(B)$, one would obtain $\{y\} \sim B \succ B \backslash \{x\}$ which is not possible by Axiom M. Hence, $x \notin E(B)$.

Proof of Proposition 2. Sufficiency of the axioms follows from Proposition 1 together with Lemmas 1 and 2. Necessity follows from the observation that for all $A \in \mathscr{X}$ and all $x \in \Omega$,

$$A \succ A \backslash \{x\} \Leftrightarrow A \succ^* A \backslash \{x\},$$

where $\succ^*$ is the asymmetric part of the domination relation. Thus, the representation for $\succcurlyeq^*$ given in Proposition 2 implies for all $A \in \mathscr{X}$, $E(A) = \max_{R_E} A$, i.e., Axiom B.

Proof of Lemma 3. Let $\succcurlyeq$ satisfy Axioms F and M. Furthermore, let R be a quasi-transitive preference relation on Ω such that Axiom P is satisfied. We show $R \subseteq R_E$ by contraposition. Thus, suppose that not $(xR_E y)$, i.e., $yP_E x$. Then, $\{y\} \sim \{x, y\} \succ \{x\}$, in particular, $\{y\} \succ \{x\}$. By Axiom P, yPx, i.e., not (xRy).

Proof of Lemma 4. Let $\succcurlyeq$ satisfy Axioms F and D, and let R be quasi-transitive. By Lemma 3, Axiom P implies $R \subseteq R_E$. Thus, it suffices to show $R_E \subseteq R$. Again, this is shown by contraposition. Suppose that not (xRy), i.e., yPx. By Axiom P, $\{y\} \succ \{x\}$, hence by Axiom D, $\{y\} \sim \{x, y\}$. This implies $yP_E x$, i.e., not $(xR_E y)$.

Now, conversely, let $R_E = R$. Then xPy implies xP_Ey, which in turn implies $\{x\} \sim \{x, y\}$. By Axiom F, $\{x, y\} \succ \{y\}$, hence $\{x\} \succ \{y\}$. On the other hand, $\{x\} \succ \{y\}$ implies by Axiom D, $\{x\} \sim \{x, y\}$; hence xP_Ey, and therefore xPy. Thus, $xPy \Leftrightarrow \{x\} \succ \{y\}$, i.e., $\succcurlyeq$ satisfies Axiom P.

Proof of Lemma 5. The following proof uses the technique introduced in the proof of Theorem 1' in Kreps [3]. Note that transitivity of the domination relation $\succcurlyeq^*$ is clearly necessary for the representation in Lemma 5. To prove sufficiency define, as in [3], a function $f\colon \mathscr{X} \to \mathscr{X}$, which associates to each menu the union of all menus which it dominates, i.e.,

$$f(A) := \bigcup_{A \succcurlyeq^* B} B.$$

First, we prove the following statements. For all $A, B, C \in \mathscr{X}$,

(i) $(A \succcurlyeq^* B$ and $A \succcurlyeq^* C) \Rightarrow A \succcurlyeq^* B \cup C$,

(ii) $A \succcurlyeq^* f(A)$,

(iii) $A \succcurlyeq^* B \Leftrightarrow B \subseteq f(A)$,

(iv) $f(f(A)) = f(A)$,

(v) $A \succcurlyeq^* B \Leftrightarrow f(B) \subseteq f(A)$.

To show (i), observe that, by reflexivity of $\succcurlyeq$, one has $A \succcurlyeq^* B$ whenever $B \subseteq A$. Thus $A \cup B \succcurlyeq^* A \succcurlyeq^* C$, which implies $A \cup B \succcurlyeq^* C$ by transitivity of $\succcurlyeq^*$. Hence, $A \cup B \succcurlyeq^* A \cup B \cup C$. Thus, $A \succcurlyeq^* A \cup B \succcurlyeq^* A \cup B \cup C$, which implies $A \succcurlyeq^* A \cup B \cup C$. This implies the desired conclusion $A \succcurlyeq^* B \cup C$. Part (ii) follows from (i) by induction.

If $A \succcurlyeq^* B$ then $B \subseteq f(A)$ by definition. To show the converse, note that $B \subseteq f(A)$ implies $f(A) \succcurlyeq^* B$ by reflexivity of $\succcurlyeq$. Therefore, by part (ii) and transitivity of $\succcurlyeq^*$, one has $A \succcurlyeq^* f(A) \succcurlyeq^* B$ and $A \succcurlyeq^* B$. This shows (iii).

To show (iv), note that $A \succcurlyeq^* f(A) \succcurlyeq^* f(f(A))$ implies $A \succcurlyeq^* f(f(A))$, hence by (iii), $f(f(A)) \subseteq f(A)$. Reflexivity of $\succcurlyeq^*$ implies that $f(A) \subseteq f(f(A))$, hence $f(f(A)) = f(A)$.

Finally, $A \succcurlyeq^* B$ if and only if $A \succcurlyeq^* f(B)$, and by part (iii), $A \succcurlyeq^* f(B)$ if and only if $f(B) \subseteq f(A)$. This shows (v).

Now define the state space S as the set of fixed points of the mapping f, i.e.,

$$S = \{A \in \mathscr{X} : A = f(A)\}.$$

Furthermore, set

$$u(x, s) := \begin{cases} 1 & \text{if} \quad x \notin s \\ 0 & \text{if} \quad x \in s. \end{cases}$$

Then,

$$U(A, s) = \max_{x \in A} u(x, s) = \begin{cases} 1 & \text{if } \ A \backslash s \neq \varnothing \\ 0 & \text{if } \ A \subseteq s, \end{cases}$$

and one has

$$A \succcurlyeq^* B \Leftrightarrow \text{for all } s \in S, \ U(A, s) \geqslant U(B, s).$$

Indeed, if not ($A \succcurlyeq^* B$) then by part (iii) above, $B \nsubseteq f(A)$. Thus, one has $U(A, f(A)) = 0$ and $U(B, f(A)) = 1$. On the other hand, if $A \succcurlyeq^* B$ then $U(A, s) = 0$ implies $A \subseteq s$, which in turn implies $f(A) \subseteq f(s) = s$ by (iv) and (v). Hence, $B \subseteq f(B) \subseteq f(A) \subseteq s$ and therefore $U(B, s) = 0$. This completes the proof of Lemma 5.

Proof of Proposition 3. For the proof of Proposition 3 we use the following result, due to Szpilrajn [12]. Let Ω be a (not necessarily finite) set, and let P be a strict partial order on Ω, i.e., an irreflexive and transitive binary relation on Ω. Then there exists a linear order $\tilde{R}$ on Ω such that $xPy \Rightarrow x\tilde{P}y$, where $\tilde{P}$ is the asymmetric part of $\tilde{R}$. Recall that a linear order is a weak order $\tilde{R}$ such that for all $x \neq y$, either $x\tilde{P}y$ or $y\tilde{P}x$.

Suppose that for some $s_0 \in S$, $U(B, s_0) > U(A, s_0)$. Then there exists a $\tilde{y} \in B \backslash A$ such that for the weak order $R(s_0)$ represented by $u(\cdot, s_0)$, one has $\tilde{y}R(s_0)x$ for all $x \in A \cup B$. By Assumption 1, this implies $\tilde{y}Rx$ for all $x \in A \cup B$. Since $\tilde{y} \notin A$, it follows that $\max_R (A \cup B) \nsubseteq A$. Thus, by contraposition, the left-hand side of the representation in Proposition 3 implies the right-hand side.

Conversely, suppose that there exists an alternative $\tilde{y} \in B \backslash A$ such that $\tilde{y}$ is maximal in $A \cup B$ with respect to R. Define a binary relation on Ω by

$$R^0 := R \backslash \{(z, \tilde{y}) \in \Omega \times \Omega : z \neq \tilde{y} \text{ and } zI\tilde{y}\},$$

where I denotes the symmetric part of R. Thus, R^0 differs from R only in that it ranks $\tilde{y}$ above all alternatives to which $\tilde{y}$ is R-indifferent. Obviously, $\tilde{y}P^0x$ for all $x \in A \cup B$, $x \neq \tilde{y}$, hence $\tilde{y}$ is the *unique* maximal element of $A \cup B$ with respect to R^0. It is easily verified that for all $x, y \in \Omega$, $xPy \Rightarrow xP^0y$. The relation R^0 is not necessarily quasi-transitive but it is acyclic. Therefore the transitive closure P^0_t of P^0 is a strict partial order, and one has $xP^0y \Rightarrow xP^0_ty$ for all $x, y \in \Omega$. By Szpilrajn's theorem, there exists a linear order $\tilde{R}$ such that $xP^0_ty \Rightarrow x\tilde{P}y$.[10] Thus, $xPy \Rightarrow x\tilde{P}y$ for all $x, y \in \Omega$. By Assumption 1, the order $\tilde{R}$ belongs to the set of possible future

[10] Note that we do not use the full strength of Szpilrajn's theorem here. Indeed, it would suffice to know that $\tilde{R}$ is a *weak order* such that $xP^0_ty \Rightarrow x\tilde{P}y$. Also, we need the result only for finite sets in which case it can be proved by elementary methods.

preferences, i.e., $\tilde{R} = R(s_0)$ for some $s_0 \in S$. Obviously, $\tilde{y} \in B \backslash A$ is the unique best element of $A \cup B$ with respect to $\tilde{R}$. Thus, $\tilde{y} \in B \backslash A$ implies $U(B, s_0) > U(A, s_0)$ for a utility representation $u(\cdot, s_0)$ of $\tilde{R}$. Hence, the right-hand side of the representation in Proposition 3 implies the left-hand side by contraposition.

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