Optimal Mechanism for Selling Two Goods with Uniformly Distributed Valuations

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The setup

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- \( z_i \sim f_i, \ z \sim f = f_1 f_2 \), where \( f \) is common knowledge. The buyer bids \( \hat{z} \).
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- $z_i \sim f_i$, $z \sim f = f_1 f_2$, where $f$ is common knowledge. The buyer bids $\hat{z}$.
- Based on the value of $\hat{z}$, the auctioneer decides the allocation probability $q : D \to [0, 1]^2$, and the payment from the buyer $t : D \to \mathbb{R}$.
- Consider a quasi-linear mechanism, where the utility function of the buyer is given by $u(z, \hat{z}) = z \cdot q(\hat{z}) - t(\hat{z})$. 
Optimal Auctions

- **Auctioneer’s objective**: Maximize $\mathbb{E}_z t(z)$, subject to IC and IR constraints.
- **IC**: $u(z) \geq u(z, \hat{z}) \forall z, \hat{z} \in D$.
- **IR**: $u(z) \geq 0 \forall z \in D$.

\[ \phi(z) = z - 1 - F(z) f(z). \]

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- Define $\phi(z) = z - \frac{1-F(z)}{f(z)}$. Assume that $\phi$ is increasing. Then the optimal allocation is given by

\[ q(z) = \begin{cases} 
1 & \text{if } \phi(z) \geq 0 \\
0 & \text{if } \phi(z) \leq 0.
\end{cases} \]

- Myerson has also provided the solution for the single-item $n$-buyer setting.

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- The solution is known for cases when the distributions $f_1$ and $f_2$ give rise to a so-called well-formed partition of the support set $D$.
- Cases such as $f = \text{Unif}[0, b_1] \times [0, b_2]$, $f = \exp(\lambda_1) \times \exp(\lambda_2)$, and $f = \text{Beta} \times \text{Beta}$ belong to this category.

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  - The optimal solution for all these cases was given by Daskalakis et al.\(^2\)

- In this talk, we shall derive the formulation of the optimization problem as done by Daskalakis et al., discuss their solutions when $f = \text{Unif}[0, b_1] \times [0, b_2]$, and then the work that we have done to find the optimal solution for the case when $f = \text{Unif}[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$.

Primal problem

- Recall that the auctioneer’s objective was to maximize $\mathbb{E}_z t(z)$ w.r.t. IC and IR constraints.
- Rochet’s theorem provides a necessary and sufficient condition for a mechanism to be IC and IR.

**Theorem**

A quasi-linear mechanism satisfies IC and IR, iff $u(\cdot)$ is convex, $\nabla u(z) = q(z)$ a.e. $z \in D$, and $u(z) \geq 0 \forall z \in D.$

- This theorem helps us formulate the optimization problem completely in terms of $u$.

$$\max_u \int_D (z \cdot \nabla u(z) - u(z))f(z) \, dz$$

s.t. $\{ u \text{ convex}, \nabla u(z) \in [0, 1]^2 \text{ a.e. } z, \text{ and } u(z) \geq 0 \forall z \}$

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The problem undergoes a series of simplifications as follows:

- \( \nabla u(z) \in [0,1]^2 \iff u(x) - u(y) \leq c(x,y) \forall x, y \in D \), where,

  \( c(x,y) = (x_1 - y_1) + (x_2 - y_2) \).

- \( u(z) \geq 0 \iff u(0,0) \geq 0 \), since \( u(0,0) \geq 0 \) combined with \( \nabla u \geq 0 \) implies \( u(z) \geq 0 \).

- We further consider \( u(0,0) = 0 \), since fixing \( u(0,0) > 0 \) only reduces the objective function.

- The optimization problem now becomes

\[
\begin{align*}
\max & \quad u \int_D (z \cdot \nabla u(z) - u(z)) f(z) \, dz \\
\text{s.t.} & \quad u \text{ convex}, \\
& \quad u(x) - u(y) \leq c(x,y) \forall (x,y), \text{ and} \\
& \quad u(0,0) = 0.
\end{align*}
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Primal problem (contd...)

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s.t. \{ $u$ convex, $u(x) - u(y) \leq c(x, y) \forall (x, y)$, and $u(0, 0) = 0$.\}
Primal problem (contd...)

- Using integration by parts, the objective function can be written as,

\[ \int_D (z \cdot \nabla u(z) - u(z)) f(z) \, dz = \int_D u \, d(\mu + \mu_s) \]

- \( \mu(z) := -z \cdot \nabla f(z) - 3f(z), \mu_s(z) := f(z)(z \cdot n)1(z \in \partial D) \).
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\[ \mu \] is the density of a measure absolutely continuous w.r.t. \( \mathcal{L}_2 \).
\[ \mu_s, \text{ w.r.t. } \mathcal{L}_1, \] \( n \) is the normal to the surface \( \partial D \).
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- Using integration by parts, the objective function can be written as,

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- \( \mu_s \), w.r.t. \( L_1 \), \( n \) is the normal to the surface \( \partial D \).

- We have \( \int_{D} d(\mu + \mu_s) = -1 \).
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We have \( \int_D d(\mu + \mu_s) = -1 \).

To make this 0, we add a point measure \( \mu_p \) of 1 at \((0,0)\).

Defining \( \bar{\mu} = \mu + \mu_s + \mu_p \), we have the objective function to be \( \int_D u \, d\bar{\mu} \). Observe that defining \( \mu_p \) causes no harm to the objective function since \( u(0,0) = 0 \).
The Primal problem:

$$\max_u \int_D u \, d\bar{\mu}$$

s.t. \( \{ u \text{ convex}, \ u(x) - u(y) \leq c(x, y) \ \forall (x, y), \ u(0, 0) = 0. \} \)

The Dual problem:

$$\inf_\gamma \int_{D \times D} c(x, y) \, d\gamma(x, y)$$

s.t. \( \{ \gamma(\cdot, D) = \gamma_1(\cdot), \ \gamma(D, \cdot) = \gamma_2(\cdot), \ \text{and} \ \gamma_1 - \gamma_2 \succeq_{\text{cvx}} \bar{\mu}. \} \)

where we say the measure \( \alpha \) convex dominates measure \( \beta \) \( (\alpha \succeq_{\text{cvx}} \beta) \) if for every convex and increasing function, we have \( \int_D f \, d\alpha \geq \int_D f \, d\beta. \)
The Optimal Transport problem

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- \( c(x, y) \rightarrow \) Cost of transporting unit mass from \( x \) to \( y \).
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- The dual problem is a version of *optimal transport* problem.
- \( c(x, y) \) → Cost of transporting unit mass from \( x \) to \( y \).
- \( \gamma(x, y) \) → The differential mass transported from \( x \) to \( y \).
- We need to find a way to minimize the cost of transportation subject to the constraint that \( \gamma_1 - \gamma_2 \preceq_{cvx} \bar{\mu} \).
The solution by Daskalakis et al.

- The problem of optimal transport was solved by Daskalakis et al. for $f_1$ and $f_2$ that give rise to a well-formed canonical partition of the support set $D$. 

$$z_1 + z_2 = 2b_1 + 2b_2 - \sqrt{2b_1b_2}.$$ 

The optimal $\gamma$ that they provide is such that $\gamma_1 - \gamma_2 = \bar{\mu}$. 
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- The solution when $f = Unif[0, b_1] \times [0, b_2]$ is:

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\text{where the line joining } P \text{ and } Q \text{ is } z_1 + z_2 = \frac{2b_1 + 2b_2 - \sqrt{2b_1 b_2}}{3}.
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- The problem of optimal transport was solved by Daskalakis et al. for $f_1$ and $f_2$ that give rise to a well-formed canonical partition of the support set $D$.

- The solution when $f = \text{Unif} [0, b_1] \times [0, b_2]$ is:

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\begin{align*}
(0, b_2) & \quad (0,1) & \quad s_1(z_1) & \quad (1,1) & \quad s_2(z_2) & \quad (1,0) \\
(2/3* b_1,0) & \quad (0,0) & \quad P & \quad (1,1) & \quad (2/3* b_1,0) & \quad (b_1,0) \\
\end{align*}
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- We aim to find a measure \(\bar{\alpha}\) such that \(\bar{\alpha} \succeq_{cvx} 0\), and then to construct a \(\gamma\) such that \(\gamma_1 - \gamma_2 = \bar{\mu} + \bar{\alpha} \succeq_{cvx} \bar{\mu}\).
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One can prove that \(\bar{\alpha} \succeq_{cvx} 0\) for any \(m_2 \geq m_1 \geq 0, \ m_3 \geq 0\).
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One can prove that $\bar{\alpha} \succeq_{cvx} 0$ for any $m_2 \geq m_1 \geq 0$, $m_3 \geq 0$.

We thus find a solution which does not relax the convexity constraint.
The general structure
Conjecture

Consider $z \sim \text{Unif}[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$. The structure of the optimal solution for any $(c_1, c_2, b_1, b_2) \in \mathbb{R}_+^4$ is one among the eight structures (a)-(h).

In other words, defining $E_x$ to be the set of all $(c_1, c_2, b_1, b_2)$ such that the optimal solution has the structure “$x$”, “$x$” taking any alphabet from (a) to (h), we conjecture that $\bigcup_{x=a}^{h} E_x = \mathbb{R}_+^4$. 
Structures when $c_1 = 0$

**Theorem**

Consider $z \sim \text{Unif}[0, b_1] \times [c_2, c_2 + b_2]$. Then the optimal solution has one of the following structures when $\frac{b_1}{b_2} \geq 2$:

![Diagram showing the structures of the optimal solution](image)

1. $(0,0) \rightarrow (1,1)$
2. $(c_1, c_2 + b_2) \rightarrow (1,1)$
3. $(c_1, c_2 + b_2) \rightarrow (0,1)$
4. $(c_1, c_2) \rightarrow (c_1 + b_1)/2, (c_1 + b_1, c_2)$
Structures when $c_1 = 0$

The optimal solution has one of the following structures when $\frac{b_1}{b_2} \in [0.6541, 2]$:
Structures when $c_1 = 0$

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Summary

- Formulated the two-item single-buyer auction as an optimization problem.

- Found its dual to be the problem of optimal transport.

- Provided the optimal solution when the underlying distribution of the buyer's valuations are uniform in $[0, b_1] \times [0, b_2]$.

- Identifying that the solution was found by relaxing the convexity constraint, we found a "shuffle measure" $\bar{\alpha}$, and tried to construct a $\gamma$ that has its convexity constraint tight.

- We conjecture that the optimal solution satisfies one of the eight structures given by the shuffle measure.

- The optimal solution was proved to have one of those structures when $c_1 = 0$. 
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Future Work

- Can the conjecture be proved? Or can we find a counter-example of a $(c_1, c_2, b_1, b_2)$ whose structure of optimal solution is different from the eight?
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Future Work

- Can the conjecture be proved? Or can we find a counter-example of a \((c_1, c_2, b_1, b_2)\) whose structure of optimal solution is different from the eight?
- How do we find the optimal solution when the distribution of valuations is not uniform? Can we derive a general solution for finding the solution for any distribution?
- What happens when the number of buyers is more than 1?