Measuring Vulnerability Using the Counting Approach*

Indranil Dutta
University of Manchester, UK

Ajit Mishra
University of Bath, UK

ABSTRACT

Vulnerability has become an integral part of any deprivation assessment. In this paper we take a fresh look at measuring vulnerability, where we separate out the identification part of whether an individual is vulnerable from the aggregation part as has been done in the multi-dimensional poverty context. In doing so, we have also been able to deal with one of the crucial problems that we see in the multi-dimensional context, which is that of weights used on the different dimensions under aggregation. In this case the probabilities are used as the natural weights. We axiomatically characterize this new measure of vulnerability and thus also provide a theoretical underpinning to many of the empirical applications in this field.

Key Words: Poverty, Vulnerability, Uncertainty.

September, 2014.

*We would like to thank James Foster, Prasanta Pattanaik and Tony Shorrocks for comments and encouragement.

1 Introduction

Vulnerability has become an integral part of any deprivation assessment. In recent years there has been several measures of vulnerability (see Calvo and Dercon 2013; Dutta Foster and Mishra, 2011; Pritchett et al. 2000 among others), which study vulnerability as an ex-ante forward looking measure. In most of these studies vulnerability is assumed as the expected level of poverty in the future. In this paper we take a fresh look at measuring vulnerability, where we separate out the identification part of whether an individual is vulnerable from the aggregation part which involves aggregating the overall future deprivation.

Most of the previous measures of the vulnerability implicitly considers an individual as vulnerable if in future there is any possibility of falling below the poverty line. This is broadly in line with what is known as the union approach in the literature. This approach suffers from the undesirable possibility of considering individuals as vulnerable who may be really rich in general but in certain situations, which may not have a high probability of occurring in the future, may actually fall in to poverty. On the other hand, as is done in the 'intersection' approach, one could also consider the possibility of counting those individuals as vulnerable who would be deprived in the future no matter what. In such case we would miss out on a substantial number of individuals who may not fall in to poverty for sure in the future but may still have a high probability of doing so.

One way out of this situation is to consider a more general approach where one can choose the cut off level at which we can consider whether the individual is vulnerable or not. This approach, therefore, is very much similar to that of Chakravarty and D'Ambrosio (2006), Foster (2007) and Alkire and Foster (2011) which have been applied in different contexts. While Chakravarty and D'Ambrosio (2006) and Alkire and Foster (2011) focussed on multi-dimensional deprivation, Foster (2007) used a similar approach for chronic poverty. To the best of our knowledge, however, this framework has so far not been applied to the context of vulnerability.

The cut-off that we suggest is based on the probability weighted sum of the number of states that the individual is going to be deprived in the future. If the sum of the number of deprived states exceeds the cut-off we would consider the individual as vulnerable. This way, we first identify the individuals who may be vulnerable in the future and then apply some sort of aggregation. As suggested in Sen (1976) and Alkire and Foster (2010), the distinction between identification and aggregation is important in the context of poverty and multi-dimensional deprivation respectively. In a similar vein we would first want to identify the vulnerable individuals in a society. We can then apply some aggregation rule to find the individuals vulnerability and from their to the society's vulnerability.

This paper also provides a theoretical underpinning to many of the applied work in the context of vulnerability. For instance, Chaudhuri et al. (2003, p17) have used a threshold level of expected poverty to classify individuals as vulnerable. Similar threshold criteria has also been applied by Imai et al. (2011). Here we provide an axiomatic framework which justifies the empirical approach.

In the next section we illustrate our concept of a vulnerability measure. Once we identify the individuals who are vulnerable, we engage in two levels of aggregation. First we aggregate the vulnerable individuals deprivation across the different states to find the individuals overall level of vulnerability. Next, given the individuals vulnerability, we use another aggregation rule to find the society's vulnerability. In Section 3 we discuss some axioms and characterize the individual aggregation rule and in Section 4 we characterize the societal aggregation rule. In the last section we conclude the paper with some brief remarks.

2 The concept of vulnerability measure

2.1 Notation

Consider a society of n individuals. Suppose there be m states of the world. For any one individual there is a finite level of future deprivation associated with each state which is denoted as $\mathbf{d} = \{d^1, d^2, ..., d^m\}$. The deprivation associated with each state can come from a multi-dimensional exercise such as Alkire and Foster (2010) or from a unidimensional measure. Although in this context we consider the deprivations as primitive, these can be derived from deeper variables which track the achievements in different dimensions. In fact derivation of these deprivations involve the use of a cut-off, which we assume to be given. If we were to focus on income poverty then one consider the poverty line to this cut-off income. Dutta, Foster and Mishra (2011) use a different cut-off where a person is considered deprived in a particular state if his income falls below a certain reference income level which depends on, but which is not the same as, the poverty line. In the current paper we focus on the second cut-off problem: the number of states in which the individual has to be deprived so as to be counted as vulnerable. We ignore the first cut-off exercise and focus on the deprivation levels, $d^s \in [0, 1]$.

In this paper, our focus is on the identification part too. Therefore, associated with the deprivation profile we create a $(1 \times m)$ vulnerability identification vector $\mathbf{r} = \{r^1, r^2, ..., r^m\}$ based on the following rule

$$\forall s, r^s = \begin{cases} 1 & \text{if } d^s > 0 \\ 0 & \text{otherwise} \end{cases}.$$

It is clear that it basically partitions the states in to deprived and non-deprived states. We would use $\mathbf{r}$ to identify whether the individual will be considered vulnerable in the future.

An individual faces a lottery $L = (p_1^L, d^1; p_2^L, d^2; ...; p_m^L, d^m)$. The probability vector associated with lottery L is represented as $\mathbf{p}^L = \{p_1^L, p_2^L, ..., p_m^L\} \in P$ where P is the set of probability distributions such that for all $s = 1, .., m$, $p_s \geq 0$, $m \geq 2$ and $\sum_{s=1}^{m} p_s = 1$.

Let $\mathcal{L}$ be the set of all lotteries, L, L', faced by the individual. A 'counting' function is defined as $\rho : \mathcal{L} \longrightarrow [0, 1]$, where $\rho(L) = 1$ would mean that individual facing the lottery L, is deprived in all states whereas $\rho(L) = 0$, that the individual is not deprived in any state. We use a specific counting function, ρ^E, which is defined as

Definition 1 *ρ^E is a counting function such that $\rho^E(L) \geq \rho^E(L')$ iff $\mathbf{p}^L \cdot \mathbf{r} \geq \mathbf{p}^{L'} \cdot \mathbf{r}$.*

We consider any individual as vulnerable if the scalar product $\mathbf{p}^L \bullet \mathbf{r} \geq \theta$, where $\theta \in (0, 1]$. This means when $\theta \rightarrow 0$, we consider the individual vulnerable if they are poor in any one future state with positive probability. On the other hand when $\theta = 1$ the individual is deemed vulnerable if he is poor with certainty in the future. Once we identify the individual as vulnerable, we then use the lottery the individual faces to come up with the level of vulnerability an individual faces. Thus for each individual i, vulnerability is measured by $V^i : (0, 1] \times \mathcal{L} \longrightarrow \mathbb{R}_+$. In addition if $\rho(L) = 0$, then $V^i(L) = 0$. Throughout the paper when we discuss the vulnerability of one individual, we shall denote the vulnerability measure is, $V^i(L)$ as $V(L)$.

For the society with n individuals and m future states of the world, we have a $n \times m$ deprivation matrix which we denote as $M^{n,m}$. Each row of the deprivation matrix lists the deprivation of one individual over all the states. Each column on the other hand lists the deprivation of all the individuals in one state. Let Φ denote the set of all such matrices. The societal vulnerability is a function $V^S : \Phi \longrightarrow \mathbb{R}_+$.

2.2 The measure

In this section we present our vulnerability measure in three steps and then illustrate it with an example. Suppose an individual faces a lottery L. The first step focuses on identifying whether an individual is vulnerable or not and hence included in our overall vulnerability index. If the individual is identified as vulnerable from the first step i.e. $\mathbf{p}^L \bullet \mathbf{r} \geq \theta$, then in the second step, we aggregate his deprivation across all the states and his vulnerability is given as

$$V^i(L) = \sum_{s=1}^{m} p_s^L (d^s)^\alpha \quad \text{if } \mathbf{p}^L \bullet \mathbf{r} \geq \theta \ . \tag{1}$$

where $\alpha \geq 1$.

Once each individual's vulnerability has been computed, in the third step, we aggregate over all individuals to find out the societal level of vulnerability. Let $M^{n,m}$ represent the societal deprivation matrix of n individuals over m future states. Then the societal vulnerability measure can be represented as

$$V^S(M^{n,m}) = \frac{1}{n}\sum_{i=1}^{n} V^i(L) \tag{2}$$

Before we proceed to the axiomatic characterization of our measure, we can highlight an important feature of the identification strategy. When we talk about vulnerability, we often refer to the *probability that the individual will be deprived* in future. The common intutiton uses the crudest (coarsest) partition of states; deprived or not deprived. Our identification formalizes this intuition. Even when there are several states, while indentifying whether an individual is vulnerable or not, we supress the extent of deprivations in different states and look the total probability that an individual is likely to be in any of the deprived state.

2.3 Illustrated Examples

Example 1 *Consider a society of* $n = 3$ *with deprivation vector* $d = (0.8, 1, 0)$ *over three states. The probability matrix is given below*

$$M^{3,3} = \begin{pmatrix} 0.1 & 0 & 0.9 \\ 0.6 & 0.3 & 0.1 \\ 0.5 & 0.5 & 0 \end{pmatrix}.$$

Each row of the matrix represent the probabilities faced by an individual over the three states. Thus the lotteries faced by each of the individuals are, $L^1 = (0.1, 0.8; 0, 1; 0.9, 0)$, $L^2 = (0.6, 0.8; 0.3, 1; 0.1, 0)$ *and* $L^3 = (0.5, 0.8; 0.5, 1; 0, 0)$. *Let* $\theta = 0.5$, *and* $\alpha = 1$. *Note* $\mathbf{r} = \{1, 1, 0\}$.
First let us begin with Stage 1. $\mathbf{p}^{L^1} = \{0.1, 0, 0.9\}$ *and the scalar product* $\mathbf{p}^{L^1} \bullet \mathbf{r} = 0.1 < 0.5$. *Hence individual* 1 *is not considered vulnerable. On the other hand* $\mathbf{p}^{L^2} = \{0.6, 0.3, 0.1\}$ *and* $\mathbf{p}^{L^2} \bullet \mathbf{r} = 0.9 > 0.5$ *which implies that individual* 2 *is vulnerable. Similarly for the third individual,* $\mathbf{p}^{L^3} = \{0.5, 0.5, 0\}$ *and* $\mathbf{p}^{L^3} \bullet \mathbf{r} = 1 > 0.5$ *which indicates that individual* 3 *is also vulnerable.*
In stage 2 using (1) $V^1(L^1) = 0$, $V^2(L^2) = 0.78$ *and* $V^3(L^3) = 0.90$.
In stage 3, the overall societal vulnerability will be $V(M^{3,3}) = 1/3(V^1(L^1) + V^2(L^2) + V^3(L^3)) = 0.56$.

In the next example we show why the proposed measure will be different from expected poverty. The measure proposed here is not the same as expected poverty.

Example 2 *Consider two lotteries* $L^1 = (5/6, 0.3; 1/6, 0; 0, 0.95)$, $L^2 = (0, 0.3; 2/3, 0; 1/3, 0.95)$. *Let* $\theta = 0.5$ *as in the previous example.*
The Expected Poverty can be calibrated as $E_p(L^1) = 0.25 < 0.316 = E_p(L^1)$.
Given $\theta = 0.5$, $p^{L^1} \bullet r = 5/6 > 0.5$ *and* $p^{L^2} \bullet r = 0.33 < 0.5$. *Thus* $V(L^1) > 0$, *where as* $V(L^2) = 0$.

Therefore, the ranking of lotteries by Expected Poverty indices would be very different from the ranking provided by the vulnerability measure proposed in this paper.

3 Individual's measure of vulnerability

Properties of the vulnerability measures will differ based on whether they are aimed at the individual level or the societal level. We first postulate the axioms for the vulnerability at the individual level and then the axioms related to the overall society measure is considered. The individual vulnerability measure will have an identification rule and also the aggregation rule. Here we characterize these rules separately.

3.1 Axioms for the Identification function

The first axiom just captures the notion that if two lotteries that have only one state of positive deprivation with positive probability, then the lottery which has a higher probability of the deprived state should be ranked higher in terms of identification compared to the other lottery. Thus if the lottery with the lower probability in the deprived state is considered to be vulnerable, so should the other.

Axiom 1 *Axiom of Single State Deprivation (A1): Consider a lottery L be such that $p_k^L.r^k > 0$, and $\forall s \neq k, p_s^L.r^s = 0$. Let L' be such that $p_{k'}^{L'}.r^{k'} > 0$, and $\forall s \neq k', p_s^{L'}.r^s = 0$. If $p_{k'}^{L'} \geq p_k^L > 0$ then $\rho(L') \geq \rho(L)$.*

The intuition for the next axiom is quite straight forward. Consider a lottery with deprivation in k states. If we have another lottery with deprivation in an additional state besides the k states, where probability of the additional state of deprivation has been transferred from one of the previous k deprived states, then identification of this lottery should be the same as the previous lottery.

Before we state the axiom, let us provide a further definition. Let any given lottery L, with k deprived states with positive probability as L_k.

Definition 2 *Suppose* $L_k = (p_1^L, d^1; p_2^L, d^2; ...; p_k^L, d^k; 0, d^{k+1}; ...p_m^L, d^m)$, *where* $\forall s = 1, ..., k$, $p_s^L > 0, d_{t+1}^s > 0$. *We say* L'_{k+1} *is derived through a probability transfer from* L_k *if* $L'_{k+1} = (p_1^L, d^1; p_2^L, d^2; ...; p_k^L - \varepsilon, d^k; \varepsilon, d^{k+1}; ...p_m^L, d^m)$, *where* $p_k^L > \varepsilon > 0$.

Given this definition we can now state the axiom formally.

Axiom 2 *Axiom of Invariance to Probability Transfers (A2): Suppose* L'_{k+1} *is derived through a probability transfer from* L_k, *where* $k < m$. *Then* $\rho(L') = \rho(L)$.

Given these two axioms we can show the following result.

Theorem 1 *A counting function* ρ *satisfies Axioms of Single State Deprivation (A1) and Axiom of Invariance to Probability Transfers (A2) iff* $\rho = \rho^E$.

Proof: Only if.

Suppose ρ satisfies A1 and A2. Then given any two lotteries L and L' we show the following:

Case I:

$$\mathbf{p}^{L'} \cdot \mathbf{r} > \mathbf{p}^L \cdot \mathbf{r} \quad \Longrightarrow \rho(L) > \rho(L') \ ,$$

We apply the method of induction. Let n stand for the number of deprived states with positive probability.

Suppose $n = 1$. Consider two lotteries L_1 and L'_1. Suppose for L_1 the deprived state with positive probability is s and for L'_1 it is s'. In this case

$$\mathbf{p}^{L'} \cdot \mathbf{r} > \mathbf{p}^L \cdot \mathbf{r} \quad \Longrightarrow p_{s'}^{L'} > p_s^L \ .$$

Then from axiom A1 we can show that

$$p_s^L > p_{s'}^{L'} \quad \Longrightarrow \rho(L) > \rho(L') \ \ .$$

Suppose it is true for $n = k < m$ states, which implies that for any lottery L and L'

$$\sum_{n=1}^{k} p_n^L r^n > \sum_{n=1}^{k} p_n^{L'} r^n \Longrightarrow \rho(L) > \rho(L'). \tag{3}$$

Now consider $n = k + 1 \leq m$, we have to show that

$$\sum_{n=1}^{k+1} p_n^L r^n > \sum_{n=1}^{k+1} p_n^{L'} r^n \Longrightarrow \rho(L) > \rho(L').$$

Consider any lottery with $k+1$ deprived states with positive probability, $L_{k+1} : (p_1^L, d^1; p_2^L, d^2; ...; p_k^L, d^k; p_{k+1}^L, d^{k+1}; ...; p_m^L, d^m)$ such that for any state $s > k+1$, either $p_s^L = 0$ or $d^s = 0$. Now construct a lottery $\widehat{L}_k$ from L_{k+1} such that $\widehat{L}_k : (p_1^L, d^1; p_2^L, d^2; ...; \widehat{p}_k^L, d^k; 0, d^{k+1}; ...; p_m^L, d^m)$, where $\widehat{p}_k^L = (p_k^L + p_{k+1}^L)$. Then from axiom A2, $\rho(L_{k+1}) = \rho(\widehat{L}_k)$. Suppose by repeated transfer of probability we arrive at $\widetilde{L}_1 : (p_1^{\widetilde{L}}, d^1; 0, d^2; ...; 0, d^m)$, where $p_1^{\widetilde{L}} = \sum\limits_{n=1}^{k+1} p_n^L$. From axiom A2 we can derive,

$$\rho(L_{k+1}) = \rho(\widetilde{L}_1). \tag{4}$$

Similarly for a lottery $L'_{k+1} : (p_1^{L'}, d^1; p_2^{L'}, d^2; ...; p_k^{L'}, d^k; p_{k+1}^{L'}, d^{k+1}; ...; p_m^{L'}, d^m)$ through repeated transfer of probability we can arrive at $\widetilde{L}'_1 : (\widetilde{p}_1^{L'}, d^1; 0, d^2; ...; 0, d^m)$, where $p_1^{L'} = \sum\limits_{n=1}^{k+1} p_n^{L'}$. From axiom A2 we know

$$\rho(L'_{k+1}) = \rho(\widetilde{L}'_1). \tag{5}$$

Suppose $p_1^{\widetilde{L}} \geq p_1^{\widetilde{L}'}$, then using the axiom A1, (4) and (5)we can claim

$$\begin{aligned} p_1^{\widetilde{L}} &> p_1^{L'} \Longrightarrow \rho(\widetilde{L}_1) > \rho(\widetilde{L}'_1), \\ \sum_{n=1}^{k+1} p_n^L &> \sum_{n=1}^{k+1} p_n^{L'} \Longrightarrow \rho(L_{k+1}) > \rho(L'_{k+1}). \end{aligned}$$

Case II:

$$\mathbf{p}^L \cdot \mathbf{r} = \mathbf{p}^{L'} \cdot \mathbf{r} \quad \Longrightarrow \rho(L) = \rho(L') \ \ .$$

The proof is similar to Case I and is omitted.

It can be easily checked that the sufficient conditions are satisfied. ■

3.2 Axioms for the individual vulnerability measure

Our next set of axioms captures the notion of counting the number of deprived states in a lottery to identify an individual as vulnerable or not. We bring the identification part along with the notion of vulnerability in the following two axioms. The intuition is that if we identify one lottery L as to have a higher deprivation count than L' through ρ^E, then $V(L)$ should not be less than $V(L')$. Thus, the axiom emphasizes a monotonic relation between the $\rho^E(L)$ and $V(L)$.

Axiom 3 *Consistency Axiom (A3) Consider two lotteries L and L', such that $\rho^E(L) > \rho^E(L')$ then $V(L) \geq V(L')$*

The next axiom identifies whether individuals should considered as vulnerability or not for the extreme values of $\rho^E(L)$. When there is no possibility of deprivation in the future, then the individual should not be identified as vulnerable. On the other hand, if the individual is definitely going to deprived in the future then we should identify him as vulnerable.

Axiom 4 *Axiom (A4) Consider two lotteries L and L' such that $\rho^E(L) = 1$, and $\rho^E(L') = 0$, then $V(L) > 0$, and $V(L') = 0$ respectively.*

Our next axiom follows the equivalent of the focus axiom that has been put forth in Alkire and Foster (2010) and Foster (2007), in the context of multi-dimensional deprivation and chronic poverty . What we are trying to capture through this axiom is the equivalent of the focus axiom in the context of vulnerability. In the Alkire and Foster (2010) approach there is a cut-off based on the number of dimensions one must be poor to be considered as deprived in that context. In the current context, we have a cut-off based on the probability that the person is poor in the future. So for instance, consider a lottery $L = (0.15, 1; 0.2, 1; 0.1, 1; 0.25, 0; 0.2, 0)$ faced by an individual. The individuals is deprived in the first three states only. Now if we consider the cut-off level of $\rho^E(L)$ to be deemed vulnerable as 0.4 , then obviously the person is vulnerable. This is because $\rho^E(L)$ that individual will be deprived in the future is 0.45.

Axiom 5 *Focus Axiom (A5): Let* $L = (p_1, d^1; p_2, d^2; ...; p_m, d^m)$ *such that* $\rho^E(L) \leq \theta$. *Then* $V(L) = 0$.

The axiom states that vulnerability emanating from any lottery not making the cutoff is zero.

The next two axioms that we present is similar to those of Dutta Foster and Mishra (2011). First, we define the convex combination of two lotteries as follows:

Definition 3 *Suppose* $L^i = (p^i_1, d^1; p^i_2, d^2; ...; p^i_m, d^m)$ *and* $L^j = (p^j_1, d^1; p^j_2, d^2; ...; p^j_m, d^m)$. *Then* $\lambda L^i + (1-\lambda)L^j = (\lambda p^i_1 + (1-\lambda)p^j_1, d^1; \lambda p^i_2 + (1-\lambda)p^j_2, d^2; ...; \lambda p^i_m + (1-\lambda)p^j_m, d^m)$, *where* $0 < \lambda < 1$.

The following axiom states that the vulnerability of a convex combination of lotteries should be the same as the convex combination of the vulnerability of each of the lotteries.

Axiom 6 *Axiom of Decomposability (A6): Consider any two deprivation lotteries L and L' such that $V(L) > 0$ and $V(L') > 0$. Then $V(\lambda L + (1-\lambda)L') = \lambda V(L) + (1-\lambda)V(L')$.*

The implication of this axiom would be to make the vulnerability measure linear in probabilities. It will thus generate the von Nueman-Morgenstern expected utility structure for the vulnerability measure.

The intuition for our next axiom comes from the well known monotonicity axiom which states that vulnerability should increase when the probability of a bad state occurring increases relative to a better state.

Axiom 7 *Axiom of Monotonicity (A7): Consider two lotteries $L = (p_1, d^1; ..., p_i, d^i; p_j, d^j; ...; p_m, d^m)$ and $L' = (p_1, d^1; ...; p_i + t, d^i; p_j - t, d^j; ...; p_m, d^m)$, such that $p_i > 0$; $p_j > 0$, $d^i \geq d^j > 0$. Then $V(L) < V(L')$.*

Our next axiom is closer to the standard transfer axiom which captures the notion that as more probability is transferred from less deprived to more deprived states vulnerability increases at an increasing rate. A formal definition of the transfer axiom is as follows:

Axiom 8 *Axiom of Transfer (A8): Suppose $L = (p_1, d^1; ..., p_i, d^i; p_j, d^j; ...; p_m, d^m)$, $\widetilde{L} = (p_1, d^1; ...; p_i+t, d^i; p_j-t, d^j; ...; p_m, d^m)$ and $\widehat{L} = (p_1, d^1; ...; p_h+t, d^h; p_i-t, d^i; p_j, d^j ...; p_m, d^m)$ such that $p_h > 0$, $p_i > 0$, $p_j > 0$, $d^h \geq d^i \geq d^j > 0$. Then $V(\widetilde{L}) - V(L) < V(\widehat{L}) - V(\widetilde{L})$.*

The intuition of the next axiom comes from the notion of homotheticity of the vulnerability function. The intuition here is that if we transfer probability between two states with different deprivation, then any changes in the vulnerability due to this can be mitigated compensated by an appropriate adjustment to the probabilities transferred based on the differences in the deprivations.

Axiom 9 *Axiom of Homotheticity (A9): Consider* $L = (p_1, d^1; ...; p_i + t, d^i; p_j, d^j; p_k - t, d^k; ...; p_m, d^m)$ *and* $L' = (p_1, d^1; ...; p_i, d^i; p_j + g(\lambda)t, d^j; p_k, d^k; p_l - g(\lambda)t, d^l; ...; p_m, d^m)$ *such that* $d^j = \lambda d^i$, $d^l = \lambda d^k$ *and* $0 < \lambda \leq 1$. *Then* $V(L) = V(L')$.

The final axiom is the normalization axiom, which reflects the intuition that when for all states the deprivation is the highest, then vulnerability should be maximum. Similarly for all states if the deprivation is at the lowest then vulnerability should be also at minimum.

Axiom 10 *Axiom of Normalization (A10): Let* $L = (p_1, d^1; ..; p_m, d^m)$, *with,* $d^i = 1$ *and* $d^j = 0$. *Then if* $p_i = 1$, $V(L) = 1$. *If* $p_j = 1$ *then* $V(L) = 0$.

3.3 Characterization of the individual measure

In this section we characterize the two broad class of vulnerability measures presented in Section 2. Before we characterize the measure we demonstrate that the focus axiom, which is one of the key to the counting based approaches, can be derived from other more basic properties.

Lemma 1 *Axiom of Consistency (A3) and Axiom of Identification (A4)*$\Longrightarrow$ *Focus (A5).*

Proof: Suppose lottery L' is such that $\rho^E(L') = 0$, then from A4 we know that $V(L') = 0$. Consider $\rho^E(\widetilde{L}) > \rho^E(L) = 0$, axiom A3 implies that $V(\widetilde{L}) \geq V(L')$. In particular, when $\rho^E(\overline{L}) = 1$, $V(\overline{L}) > 0$. Thus, there exist $0 \leq \theta \leq 1$, such that $\rho^E(L) < \theta$, $V(L) = 0$ and for $\rho^E(L) \geq \theta$, such that $V(L) > 0$. ■

We first characterize a general measure based on (1) in the following Theorem.

Theorem 2 *A vulnerability index V of a given lottery L satisfies Axiom of Consistency (A3), Axiom of Identification (A4), Axiom of Decomposability (A6), Axiom of Monotonicity (A7) and Axiom of Transfer (A8) iff*

$$V(L) = \begin{cases} \sum_{s=1}^{m} p_s f(d^s) & \text{if } \mathbf{p}^L \cdot \mathbf{r} \geq \theta \\ 0 & \text{otherwise} \end{cases}.$$

where $f(d^s)$ is monotonic and convex with respect to d^s.

Proof: We start by proving the necessary condition. First consider any lottery $L = (p_1, d^1; ..; p_m, d^m)$ faced by individual i, such that $\mathbf{p}^L \cdot \mathbf{r} \geq \theta$. From Lemma 1 of Dutta et al. (2011) we know that A6 implies that

$$V(L^i) = \sum_{s=1}^{m} p_s f(d^s). \tag{6}$$

Consider two lotteries $L = (p_1, d^1; ..., p_i, d^i; p_j, d^j; ...; p_m, d^m)$ and $L' = (p_1, d^1; ...; p_i + t, d^i; p_j - t, d^j; ...; p_m, d^m)$, such that $p_i > 0$; $p_j > 0$, $d^i \geq d^j > 0$. Then using A7 and (6) and cancelling terms, we can show

$$p_i f(d^i) + p_j f(d^j) < (p_i + t) f(d^i) + (p_j - t) f(d^j)$$

$$\implies f(d^j) < f(d^i). \tag{7}$$

Given that $d^i \geq d^j$, and (7) holds for any arbitrary i and j, one can infer that $f(d^s)$ is monotonic.

Suppose $L = (p_1, d^1; ..., p_i, d^i; p_j, d^j; ...; p_m, d^m)$, $\widetilde{L} = (p_1, d^1; ...; p_i + t, d^i; p_j - t, d^j; ...; p_m, d^m)$ and $\widehat{L} = (p_1, d^1; ...; p_h + t, d^h; p_i - t, d^i; p_j, d^j ...; p_m, d^m)$ such that $p_h > 0$, $p_i > 0$, $p_j > 0$, $d^h \geq d^i \geq d^j > 0$. Applying A8 and (6) and cancelling terms we will get

$$tf(d^i) - t(d^j) < ((p_h + t) f(d^h) + (p_i - t) f(d^i) - (p_h f(d^h) + p_j f(d^j))$$

$$f(d^i) - f(d^j) < f(d^h) - f(d^i)$$

Since $d^h \geq d^i \geq d^j > 0$, we can claim that $f(d^s)$ is convex.

We know from Lemma 1, A3 and A4 implies A5. For any lottery L' such that $\mathbf{p}^{L'} \cdot \mathbf{r} < \theta$, thus, $V(L') = 0$. It can be easily checked that the sufficient conditions are satisfied. ■

Next we characterize a more specific functional form of the deprivation function $f(d^s)$ which is represented in (1).

Theorem 3 *A measure of vulnerability, V, of an individual satisfies Axiom of Focus (A4), Axiom of Monotonicity (A7), Axiom of Transfer (A8), Axiom of Homotheticity (A9) and Axiom of Normalization (A10) iff:*

$$V(L) = \begin{cases} \sum_{s=1}^{m} p_s (d^s)^{\alpha} & \textit{if } \mathbf{p}^L \cdot \mathbf{r} \geq \theta \\ 0 & \textit{otherwise} \end{cases}$$

Proof: First lets start with the necessary conditions. From axioms A6, A7 and A8, as in Theorem 1, we can show for any lottery, $L = (p_1, d^1; ..; p_m, d^m)$ such that $\mathbf{p}^L \cdot \mathbf{r} \geq \theta$,

$$V(L) = \sum_{s=1}^{m} p_s f(d^s). \tag{8}$$

where $f(d^s)$ is monotonic and convex . Using A10 we can demonstrate $f(0) = 0$ and $f(1) = 1$.

Now consider $L = (p_1, d^1; ...; p_m, d^m)$, $\widehat{L} = (p_1, d^1; ...; p_i + g(\lambda)t, d^i; p_j, d^j; p_k - g(\lambda)t, d^k; ...; p_m, d^m)$ and $\widetilde{L} = (p_1, d^1; ...; p_i, d^i; p_j + t, d^j; p_k, d^k; p_l - t, d^l ...; p_m, d^m)$ such that $p_i > 0$, $p_j > 0$, $p_k > 0$, $p_l > 0$; $d^j = \lambda d^i$ and $d^l = \lambda d^k$, $0 < \lambda \leq 1$. Using A9 and (8), we can show

$$t(f(d^i) - f(d^k)) = tg(\lambda)(f(d^j) - f(d^l)).$$

This can be written as

$$h(\lambda)(f(d^i) - f(d^k)) = (f(d^j) - f(d^l))$$

where $h(\lambda) = 1/g(\lambda)$. Given $d^j = \lambda d^i$ and $d^l = \lambda d^k$, from the above equation we can obtain

$$f(\lambda d^k) - h(\lambda)f(d^k) = f(\lambda d^i) - h(\lambda)f(d^i) \tag{9}$$

Since $f(d)$ is convex and monotonic and (9) should hold for all d^i and d^k, therefore it must be the case that for all d

$$f(\lambda d) = h(\lambda)f(d) \tag{10}$$

Suppose $d = 1$, given $f(1) = 1$, it implies that $f(\lambda) = h(\lambda)$. Replacing this in (10) we get

$$f(\lambda d) = f(\lambda)f(d). \tag{11}$$

(11) is a Pexider equation, whose general solution is given by (Aczel 1966)

$$f(d) = d^\alpha \tag{12}$$

where $\alpha \geq 1$, given that $f(d)$ is convex. Thus from (8) and (12) we can show $V(L) = \sum_{s=1}^{m} p_s(d^s)^\alpha$, $\alpha \geq 1$. As earlier for any lottery L' such that $\mathbf{p}^{L'} \bullet \mathbf{r} < \theta$, we know from A4, $V(L') = 0$. And again it can be easily checked that the sufficient conditions are satisfied.■

4 Societal Measure of Vulnerability

Once the individual vulnerability measure is computed, we can then measure the societal level of vulnerability. Although the domain of the societal measure of vulnerability is the set of deprivation matrices denoted by Φ, in calculating the overall vulnerability we take the approach recommended in Dutta, Pattanaik and Xu (2003). The societal vulnerability measure would be an aggregation over the individual vulnerability measures. For any individual i, $0 < V^i(L) < 1$, then the societal vulnerability measure $V : [0,1]^N \longrightarrow \mathbb{R}_+$.

We next consider the axioms on the societal vulnerability measure and then we characterize the societal measure.

4.1 Axioms on the societal measure

Since each row of the deprivation matrix represents the probabilities faced by an individual over the different states, an interchanging of the rows should not affect the overall vulnerability. We say that matrix $M'^{n,m}$ is obtained from $M^{n,m}$ by the permutation of rows if only the rows are interchanged with everything else remaining same.

Axiom 11 *Axiom of Symmetry (A11): Consider two matrices $M^{n,m}$ and $M'^{n,m}$ where $M'^{n,m}$ is obtained from $M^{n.m}$ through a permutation of rows. Then $V(M^{n,m}) = V(M'^{n,m})$.*

This axiom will imply that the individual vulnerability functions are the same for all the people. In other words if two individuals face the same probability lottery then their vulnerability should be same.

The next axiom captures the notion that we divide the individuals in to different groups then the overall vulnerability should be the sum of the vulnerability of the different groups.

Axiom 12 *Axiom of Societal Decomposability (A12): Consider three matrices $M^{n,m}$, $M^{n_1,m}$ and $M^{n_2,m}$, where $n = n_1 + n_2$. Then $V(M^{n.m}) = \frac{n_1}{n}V(M^{n_1,m}) + \frac{n_2}{n}V(M^{n_2,m})$.*

4.2 Characterization of the societal measure

We now characterize the overall societal measure which is similar to (2).

Theorem 4 *A societal measure of vulnerability, V, satisfies Axiom of Symmetry (A11) and Axiom of Societal Decomposability (A12) iff :*

$$V = \frac{1}{n}\sum_{i=1}^{n} V(L^i),$$

where L^i is the lottery faced by individual i.

Proof: Repeated application of axiom A12, will yield

$$V = \frac{1}{n}\sum_{i=1}^{n} V^i(L^i). \tag{13}$$

Due to axiom A11, we can show that $V^i(L^i) = V(L^i)$. Applying this in (13) would yield the result. ■

5 Conclusion

In this paper we have attempted to conceptualize and characterize a new class of vulnerability measures. The main innovation of the paper is in bringing a clear identification part to the measurement of vulnerability as has been done in the literature on multi-dimensional deprivation. In doing so, we have also been able to deal with one of the crucial problems that we see in the multi-dimensional context, which is that of weights used on the different dimensions under aggregation. In this case the probabilities are used as natural weights.

References.

Aczel, J (2003) Lectures on Functional Equations and their Applications, Academic Press, New York

Alkire, S. and J.E. Foster (2011),Counting and Multidimensional Poverty Measurement, Journal of Public Economics, 95, 476–487.

Bourguignon, F. and S. Chakravarty (2003), The Measurement of Multidimensional Poverty, Journal of Economic Inequality, 1, 25-19.

Calvo, C. and S. Dercon (2013), Vulnerabilityto Individual and Aggregate Poverty, Social Choice and Welfare, 41, 721-740.

Chakravarty, S.R., D'Ambrosio, A., (2006), The Measurement of Social Exclusion, Review of Income and Wealth, 523, 377-398.

Chaudhuri, S., J. Jalan and A. Suryahadi (2002), Assessing Household Vulnerability to Poverty from Cross-Sectional Data: A Methodology and Estimates from Indonesia, Working Paper, Columbia University.

Chaudhuri, S. (2003), Assessing Vulnerability to Poverty: Concepts, Empirical Methods and Illustrative Examples,Working Paper, Columbia University.

Dutta, I., J. Foster and A. Mishra (2011), On Measuring Vulnerability to Poverty, Social Choice and Welfare, 37, 729-741.

Ersado, L. (2008), Rural Vulnerability in Serbia, Working Paper 4010, World Bank.

Foster, J. E. (2007), A Class of Chronic Poverty Measures, Working Paper No. 07-W01, Department of Economics, Vanderbilt University.

Imai, K., R. Gaiha and W. Kang (2011) Vulnerability and Poverty Dynamics in Vietnam, Applied Economics, 43, 3603-3618.

Ligon, E. and L. Schecter (2003), Measuring Vulnerability, Economic Journal, 113, C95-C102.

Morduch, J. (1994), Poverty and Vulnerability, American Economic Review Papers and Proceedings, 84, 221-225.

Pritchett, L., A. Suryahadi and S. Sumarto (2000), Quantifying Vulnerability to Poverty, World Bank, Policy Research Working Paper 2437.

Sen, A. (1976), Poverty: An Ordinal Approach to Measurement, Econometrica, 44, 219-231.

Sen, A (1981), Poverty and Famines, Oxford University Press, Oxford.