Core and Coalitional Fairness: The Case of Information Sharing Rule

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13. June 2014

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Abstract. We investigate two of the most extensively studied cooperative notions in a pure exchange economy with asymmetric information. One of them is the core and the other is known as coalitional fairness. The set of agents is modelled by a mixed market consisting of some large agents and an ocean of small agents; and the commodity space is an ordered Banach space whose positive cone has an interior point. The information system in our framework is the one introduced by Allen in [1]. Thus, the same agent can have common, private or pooled information when she becomes member of different coalitions. It is shown that the main results in Grodal [23], Schmeidler [30] and Vind [35] can be established when the economy consists of a continuum of small agents. We also focus on the information mechanism based on size of coalitions introduced in [21] and obtain a result similar to the main result in [21]. Finally, we examine the concept of coalitional fairness proposed in [24]. We prove that the core is contained in the set of coalitionally fair allocations under some assumptions. This result provides extensions of Theorem 2 in [24] to an economy with asymmetric information as well as a deterministic economy with infinitely many commodities. Although we consider a general commodity space, all our results were so far unsolved to the case of information sharing rule with finitely many commodities.

1. Introduction

The classical deterministic Arrow-Debreu-McKenzie model on an economic system consists of finitely many agents and commodities, refer to [3, 26]. In this model, the set of Walrasian allocations is properly contained in the core. To see whether any core allocation can be supported by prices so as to become a Walrasian allocation, Debreu and Scarf [11] expanded the original economy by replicating each agent m many times. They showed that each allocation in the core of any replicated economy assigns the same consumption bundle to all agents of the same type and as m becomes larger, more and more core allocations are ruled out and eventually only the competitive allocations remain. Since no agent prefers her net trade to that of another agent of the same type, Schmeildler and Vind [31] introduced the concept of *fair net trade* in an exchange economy with finitely many agents, where an agent was able to compare her net trade to that of another agent with a different type. A net trade is fair if the net trade of each agent is at least as good for her as the net trade of any other agent would be. Thus, each agent evaluates the other agent's position on the same terms that she judges her own. To define it formally, let $x = (x_1, \cdots, x_n)$ be an allocation of commodities among agents in an exchange economy with n agents. The *net trade* of agent i is $x_i - a_i$, where x_i is

JEL classification: D51; D82.
Keywords. Asymmetric information economy; coalitional fairness; core; information sharing rule.

the commodity bundle received by i at x and a_i is the initial endowment of agent i. The net trade $y = (y_1, \cdots, y_n)$, defined by $y_i = x_i - a_i$, is said to be *fair* if for all agents i and j, $y_i \succeq_i y_j$, where $\succeq_i$ denotes the preference relation of agent i. In other words, if a net trade is fair then the market does not discriminate among agents. It was shown in [31] that a fair net trade exists. An analogous idea of discrimination was considered in Jaskold-Gabszewicz [24] in terms of coalitions and it was termed as the *coalitional fairness*. The allocation x is called *coalitionally unfair* if there exist two disjoint coalitions S_1 and S_2 such that $\sum_{i\in S_1} y_i < \sum_{i\in S_2} y_i$. In this case, agents in S_1 could have benefited by achieving the net trade of S_2. Formally, there exists another allocation $z = (z_1, \cdots, z_n)$ such that $z_i \succ x_i$ for all $i \in S_1$ and $\sum_{i\in S_1}(z_i - a_i) = \sum_{i\in S_2} y_i$. So, S_1 is treated under x in a discriminatory way by the market. The allocation x is called *coalitionally fair* [1] if there does not exist any two such disjoint coalitions. It is known that any Walrasian allocation is coalitionally fair and the set of coalitionally fair allocations is a subset of the core.

In [4], Aumann remarked that in an economy with finitely many agents the influence of an agent is not negligible, thus the competition is imperfect. To achieve the perfect competition, he introduced the concept of non-atomic agents. The consequence of an economy with an atomless measure space of agents is that the influence of a single agent on market prices is insignificant and so, it leads to characterization of Walrasian allocations in terms of the core, refer to [4]. Thus, the core and the set of coalitionally fair allocations are two indistinguishable co-operative notions in atomless economies under standard assumptions. Eight years later, three notes in the same issue of *Econometrica* gave a sharper interpretation to Aumann's core-Walras equivalence theorem as a characterization of perfect competition. Firstly, Schmeidler [30] proved that if an allocation f is blocked by a coalition S via an allocation g, then for any $\varepsilon > 0$, f can be blocked via the same allocation g by a coalition $S' \subseteq S$ with $\mu(S') \leq \varepsilon$. Schmeidler's result was further generalized in [23] by restricting the set of coalitions to those consisting of finitely many arbitrarily small sets of agents with similar characteristics, which are presumably easier to form and also interpret. Precisely, Grodal proved that an allocation belongs to the core if and only if it cannot be blocked by a coalition which is the union of at most $\ell + 1$ sub-coalitions, each of which has measure and diameter less than ε, where ℓ denotes the number of commodities. Finally, Vind [35] showed that if some coalition blocks an allocation then there is a blocking coalition with any measure less than the measure of the grand coalition. These results imply that, for a finite-dimensional commodity space, the set of Walrasian allocations of an atomless economy coincides with the set of allocations that are not blocked by coalitions of arbitrarily given measure less than that of the grand coalition.

It is recognized by several researchers that Aumann's atomless model corresponds to an extreme situation since the consumption in real economic exchange is far from being perfect. An example of this kind of model is that an economy where some agents concentrate in their hands initial ownerships of some commodities which are large with respect to the aggregate endowments of those commodities. It was Aumann [4] who first pointed out that such a market is probably best represented by a mixed model, in which some agents are insignificant and others are individually significant. Interestingly, the equivalence relationship between the core and the set of Walrasian allocations fails to hold in this framework. However, the core

[1] See Shitovitz [33] for a similar concept.

is equivalent to the set of Walrasian allocations if there are at least two large agents and all large agents have the same characteristics, that is, the same initial endowments and the same preferences, refer to [32]. Thus, if the above assumptions violate then one cannot claim that any allocation in the core is also coalitionally fair. An interesting and weaker result in this direction was proved by Jaskold-Gabszewicz in [24]. Indeed, in a pure exchange mixed economy with finitely many commodities, Jaskold-Gabszewicz [24] showed that the core is contained in the set of coalitionally fair allocations if coalitions are restricted to those measurable sets which are either atomless or containing all atoms. The result may fail if coalitions are any arbitrary measurable sets, refer to Proposition 2 in [24].

In the past few decades, an economy involving uncertainty and asymmetric information is one of the most important research areas in the theoretical economics. It is well known that information structure within coalitions have major influence on the set of allocations which can have attainable alternative, refer to [21]. Due to different information and communication opportunities among agents, several alternative core concepts had been proposed in [36, 37]. Precisely, Wilson [36] introduced the concepts of fine and coarse cores, the first one takes into account that agents within a coalition, pool their initial private information, whereas the latter involves the common information of all agents within a coalition. The fine core may be empty, since blocking is "easy", whereas the coarse core is large, since blocking is "difficult". In the private core introduced by Yannelis [37], agents have no access to the communication system. Thus, the information of each agent is not modified when a coalition is formed and each member of the coalition uses only her own private information whenever a coalition blocks an allocation. It is worth pointing out that under standard assumptions, the private core is non-empty (see [37]). Thus, the initial private information of each agent can be susceptible to alter when she becomes a member of a coalition. Using Yannelis's approach, Graziano and Pesce [22] proposed an extension of the notion of coalitionally fair allocations[2] in asymmetric information economies. In fact, according to their definition, a function $x = (x_1, \cdots, x_n)$ is called an *allocation* if x_i is $\mathscr{P}_i$-measurable for all $1 \leq i \leq n$, and it is termed as *coalitionally fair* there are no coalitions S_1, S_2 and $z = (z_1, \cdots, z_n) \in \mathbb{R}_+^{\ell n}$ satisfying z_i is $\mathscr{P}_i$-measurable and $z_i \succ_i x_i$ for all $i \in S_1$ and $\sum_{i \in S_1}(z_i - a_i) = \sum_{i \in S_2}(x_i - a_i)$, where $\mathscr{P}_i$ is agent i's initial private information. One of the key results in [22] claims that in an asymmetric information economy with a mixed measure space of agents and a finite dimensional commodity space, the private core is a subset of the set of coalitionally fair allocations if coalitions are restricted to those measurable sets which are either atomless or containing all atoms. In their result, the allocations were restricted to a certain class of functions (refer to the assumption (A.6) in [22]) and the feasibility was taken as free disposal. Since joining a coalition has no direct consequences on information, it is necessary to define similar concepts by adopting the mechanism that agents within a coalition use either the pooled information or the common information. In all these concepts, the rule that allocates the information to agents within a coalition is fixed a priori and does not depend on any specific property of the coalition.

In this paper, we consider the notion of information sharing rule introduced by Allen in [1]. This includes various possibilities of the information available to an agent within different coalitions, which means the same agent can have common,

[2]See Donnini et al. [13] for an existence of a coalitionally fair allocation in the interim stage.

pool or private information depending on the coalition in which she is a member. We also restrict our attention to the information sharing rule based on size of coalitions, as proposed in [21]. According to their rule, there is a family of exogenous thresholds representing different sizes of coalitions, and each threshold is associated with some information sharing rule. If an agent is a member of some coalition then she can only access the information that is given by the information sharing rule of the corresponding threshold. The feasibility condition in our paper is defined to be exact, since when feasibility is defined with free disposal, the core allocations may not be incentive compatible and contracts may not be enforceable, refer to [2] for the case of private core. The commodity space in our model is an ordered Banach space having an interior point in its positive cone. As stated in [20], infinite dimensional commodity spaces arise if one allows an infinite variation in any of the characteristics describing commodities. These characteristics could be physical properties, locations or the time of delivery; and an infinite variation in time occurs whenever infinitely many time periods are considered in each state of nature.

The purpose of this paper is to explore the main results in [23, 24, 30, 35] to an asymmetric information economy whose commodity space is an ordered Banach space admitting an interior point its positive cone and feasibility is defined as exact, where the information of each agent is given by any information sharing rule. It is clear from Examples 3 and 4 in [21] that such extensions are impossible unless we use some assumptions on information sharing rules. It can be also checked that the approaches in [21] for the proof of Schmeidler's theorem are not directly applicable for the case of information sharing rule with infinite dimensional commodity spaces and the exact feasibility condition (see Bhowmik and Cao [6] for a similar result in the case of the private core). It is crucial to remark that if the number of commodities is finite then also the techniques for Vind's theorem under the exact feasibility condition cannot be the same as in [21], since blocking is difficult by large coalitions under any arbitrary information structure and the exact feasibility condition. We establish these results under mild assumptions. The extended version of Vind's theorem in our framework allows us to obtain an extension of the main result of Hervés-Beloso et al. [21]. For particular interests, we also establish Grodal's result in our framework. In a mixed economy, we define the concept of a coalitionally fair allocation using the information sharing rule. Thus, given an information sharing rule, an allocation is called *coalitionally fair* if no coalition could redistribute among its members the net trade of any other coalition in a way which would assign a preferred bundle to each of its members, where preferred bundles are measurable with respect to the given information sharing rule. We show that Jaskold-Gabszewicz's result can be extended to an asymmetric information economy whose commodity space is an ordered Banach space admitting an interior point in its positive cone and information structure is general enough like [1, 21]. It is worth pointing out that Jaskold-Gabszewicz's approach is not directly applicable if (i) the commodity space is of infinite dimension or (ii) agents have asymmetric information and the feasibility condition is defined as exact. In fact, in the first case, Lyapunov's convexity theorem does not hold in its standard form and it is only true in a weaker form. The last case deals with the information structure and thus, the blocking will be difficult under the exact feasibility condition. The rest of the paper is organized as follows. In Section 2, a general description of the model and the concept of information sharing rule are provided. Section 3 deals with some technical lemmas which are useful in the proofs of the main results. An atomless economy is considered in Section 4, where extensions of Grodal, Schmeidler and Vind's theorems are obtained under information sharing rules and it is shown that a result similar to the main result in [21] is also valid in our framework. In section 5, we establish a relation between the core and the set of coalitionally fair allocations in a mixed economy under the information sharing rule formation. Finally, we conclude our paper with some remarks and open questions which basically give the limitation of our main results.

2. Economic model and information sharing rule

In this section, we describe the basic model of a pure exchange asymmetric information economy and discuss the concept of an information sharing rule, which means the information that an agent can dispose of when she becomes a member of a coalition.

2.1. **Description of the model.** We consider a standard mixed model of a pure exchange economy with asymmetric information. The space of *economic agents* is denoted by a measure space $(T, \mathscr{T}, \mu)$ with a complete, finite, and positive measure μ. Since $\mu(T) < \infty$, the set T can be decomposed into two parts: one is atomelss and the other contains countably many atoms. That is, $T = T_0 \cup T_1$, where T_0 is the atomless part and T_1 is the countable union of atoms. Let

$$\mathscr{T}_0 = \{S \in \mathscr{T} : S \subseteq T_0\} \text{ and } \mathscr{T}_1 = \{S \in \mathscr{T} : T_1 \subseteq S\}.$$

Thus, $\mathscr{T}_0$ (resp. $\mathscr{T}_1$) is the subfamily of $\mathscr{T}$ containing no atoms (resp. all atoms). Denote by

$$\mathscr{T}_2 = \mathscr{T}_0 \cup \mathscr{T}_1 = \{S \in \mathscr{T} : S \in \mathscr{T}_0 \text{ or } S \in \mathscr{T}_1\}$$

the subfamily of $\mathscr{T}$ containing either no atoms or all atoms. The exogenous uncertainty is described by a measurable space $(\Omega, \mathscr{G})$, where Ω is a finite set denoting all possible states of nature and the σ-algebra $\mathscr{G}$ denotes all events. Thus, $\mathscr{G}$ is generated by a partition $\mathscr{G}(\mathscr{P}^*)$ of Ω. The *commodity space* is B^Ω, where B is an ordered Banach space whose positive cone has an interior point. The order on B is denoted by $\leq$, and $B_+ = \{x \in B : x \geq 0\}$ denotes the positive cone of B. The symbol $x \gg 0$ is employed to denote that x is an interior point of B_+, and put $B_{++} = \{x \in B_+ : x \gg 0\}$. Suppose that B^Ω is endowed with the point-wise algebraic operations, the point-wise order and the product norm. An element $y \in B_+^\Omega$ can be identified with the function $y : \Omega \to B_+$ and vise-versa. The economy extends over two periods. In the first period, agents arrange contracts that may be contingent on the realized state of nature. Consumption takes place in the second period when agents receive their private information.

Each agent $t \in T$ is associated with the *consumption set* B_+^Ω. The *initial and private information* of agent t is described by a partition $\mathscr{P}_t$ of Ω. Recall that a signal on Ω with values in some set X is just a mapping $f : \Omega \to X$. Note that any partition $\mathscr{P}$ can be seen as a signal $f : \Omega \to 2^\Omega$ defined by $f(\omega) = \mathscr{P}(\omega)$, where $\mathscr{P}(\omega)$ denotes the unique member of the partition $\mathscr{P}$ containing ω. Reciprocally, a signal $f : \Omega \to X$ induces a partition on Ω given by $\mathscr{P}_f = \{f^{-1}(s) : s \in f(\Omega)\}$ and the unique member of this partition containing ω is $f^{-1}(s)$ if $f(\omega) = s$. Thus, the partition $\mathscr{P}_t$ gives a signal to agent t and if ω_* is the true state of nature in the second period then agent t observes $\mathscr{P}_t(\omega_*)$. An *assignment* is a function $f : T \times \Omega \to B_+$ such that $f(\cdot, \omega)$ is Bochner integrable for all $\omega \in \Omega$. There is a fixed assignment a; $a(t,\omega)$ represents the *initial endowment density* of agent t in the state of nature ω. It is assumed that $a(t,\omega) \in B_{++}$ for all $(t,\omega) \in T \times \Omega$ and $a(t,\cdot)$ is $\mathscr{G}$-measurable for all $t \in T$. The *preference* of agent t is described by a correspondence $P_t : B_+^\Omega \rightrightarrows B_+^\Omega$. For any assignment f, defined a correspondence $P_f : T \rightrightarrows B_+^\Omega$ such that $P_f(t) = P_t(f(t,\cdot))$ for all $t \in T$. The *graph* of P_f is defined by

$$\mathrm{Gr}_{P_f} = \left\{(t,x) \in T \times B_+^\Omega : x \in P_f(t)\right\}.$$

We assume that $\mathrm{Gr}_{P_f} \in \mathscr{T} \otimes \mathscr{B}(B^\Omega)$, where $\mathscr{B}(B^\Omega)$ is the Borel σ-algebra generated by B^Ω. In addition, suppose that (i) for all $(t,x) \in T \times B_+^\Omega$, $P_t(x)$ is open in B_+^Ω; (ii) for all $t \in T$, P_t is *monotone* in the sense that $x + y \in P_t(x)$ for all $x \in B_+^\Omega$ and $y \in B_{++}^\Omega$; and (iii) for all $(t,x) \in T_1 \times B_+^\Omega$, $P_t(x)$ is convex. Thus, the economy $\mathscr{E}$ can be described as

$$\mathscr{E} = \left\{(T,\mathscr{T},\mu); B_+^\Omega; (\mathscr{P}_t, P_t, a(t,\cdot))_{t\in T}\right\}.$$

Now, consider a special case when each agent t is associated with a *state dependent utility function* $U_t : \Omega \times B_+ \to \mathbb{R}$ and a *prior belief*, which is given by a probability measure $\mathbb{Q}_t$ on Ω. The *ex ante expected utility* and *ex ante preference relation* of agent t for a random bundle $x : \Omega \to B_+$ are defined by

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, x(\cdot))) = \sum_{\omega\in\Omega} U_t(\omega, x(\omega))\mathbb{Q}_t(\omega)$$

and

$$P_t(x) = \left\{y \in B_+^\Omega : \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, y(\cdot))) > \mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, x(\cdot)))\right\},$$

respectively. For any $k \geq 1$, the $(k-1)$-simplex of $\mathbb{R}^k$ is defined as

$$\Delta^k = \left\{x = (x_1, \cdots, x_k) \in \mathbb{R}_+^k : \sum_{i=1}^k x_i = 1\right\}.$$

Consider a function $\varphi : (T,\mathscr{T},\mu) \to \Delta^{|\Omega|}$ defined by $\varphi(t) = \mathbb{Q}_t$ for all $t \in T$. For each $\omega \in \Omega$, define a function $\psi_\omega : T \times B_+ \to \mathbb{R}$ by $\psi_\omega(t,x) = U_t(\omega,x)$. Now we impose some assumptions in the case of ex ante expected utility formulation. The first two of these are similar to those in [9, 6, 7, 16], and the last two are standard.

($\mathbf{A}_1$) The function φ is measurable, where $\Delta^{|\Omega|}$ is endowed with the Borel structure.

($\mathbf{A}_2$) For each $\omega \in \Omega$, the function ψ_ω is Carathéodory, that is, $\psi_\omega(\cdot, x)$ is measurable for all $x \in B_+$, and $\psi_\omega(t,\cdot)$ is norm-continuous for all $t \in T$.

($\mathbf{A}_3$) For each $(t,\omega) \in T \times \Omega$, $U_t(\omega, x+y) > U_t(\omega,x)$ if $x, y \in B_+$ with $y \gg 0$.

($\mathbf{A}_4$) For each $(t,\omega) \in T_1 \times \Omega$, $U_t(\omega,\cdot)$ is concave.

By ($\mathbf{A}_1$) and ($\mathbf{A}_2$), it can be easily verified that $\mathrm{Gr}_{P_f} \in \mathscr{T} \otimes \mathscr{B}(B)$. Note that the conditions (i)-(iii) are also satisfied under the above assumptions.

2.2. **Information sharing rule.** Any set in $\mathscr{T}$ is called a *coalition* of $\mathscr{E}$. A *null coalition* of $\mathscr{E}$ is a coalition whose measure is zero. If S and S' are two coalitions of $\mathscr{E}$ with $S' \subseteq S$ then S' is termed as a *sub-coalition* of S. In a framework of asymmetric information, one of the natural questions is that how the initial information of an agent is altered when she becomes a member of a coalition S. In addition, one may also think about the information available for an agent in a sub-coalition S' of S. Are the information of a common agent in S and S' identical? In this subsection, we model these situations using the concept of an information sharing rule, introduced in introduced in [1, 21].

The family of partitions of Ω is denoted by $\mathfrak{P}$. Since Ω is finite, $\mathfrak{P}$ also has finitely many elements: $\mathscr{P}_1, \cdots, \mathscr{P}_n$. Throughout, by $\mathscr{P}_i$-measurability of a function, we mean the function is measurable with respect to the σ-algebra generated by $\mathscr{P}_i$. It is assumed that the set $T_i = \{t \in T : \mathscr{P}_t = \mathscr{P}_i\}$ is $\mathscr{T}$-measurable for all $1 \leq i \leq n$. For any non-null coalition S, let $S_i = S \cap T_i$ and $\mathfrak{P}(S) = \{i : \mu(S_i) > 0\}$. Thus, $\{\mathscr{P}_i : i \in \mathfrak{P}(S)\}$ is the structures of information available in the non-null coalition S. There are three well known information sharing rules in the literature: the coarse information sharing rule, the fine information sharing rule and the private information sharing rule for S. To define these information sharing rules, recall first that a partition $\mathscr{P}$ of Ω is *finer* than a partition $\mathscr{Q}$ of Ω, denoted by $\mathscr{P} \succeq \mathscr{Q}$, if for every $A \in \mathscr{P}$ there is some $B \in \mathscr{Q}$ such that $A \subseteq B$. In such a case, $\mathscr{Q}$ is termed as *coarser* than $\mathscr{P}$. Let $\mathfrak{Q}$ be a subfamily of $\mathfrak{P}$. The *meet* of $\mathfrak{Q}$, denoted by $\bigwedge \mathfrak{Q}$, is the finest partition that is coarser than every $\mathscr{P} \in \mathfrak{Q}$. It was given in Ore [28] that two points ω and ω' belong to the same element of $\bigwedge \mathfrak{Q}$ if there is a set $\{\omega_1, \cdots, \omega_k\}$ of states of nature such that $\omega_1 = \omega$, $\omega_k = \omega'$ and for each $1 \leq i \leq k-1$, ω_i and ω_{i+1} belong to the same element of some partition $\mathfrak{P} \in \mathfrak{Q}$. Moreover, the *join* of $\mathfrak{Q}$, denoted by $\bigvee \mathfrak{Q}$, is the coarsest partition that is finer than every $\mathscr{P} \in \mathfrak{Q}$. It can be shown that

$$\bigvee \mathfrak{Q} = \left\{ \bigcap_{\mathscr{P} \in \mathfrak{Q}} A_{\mathscr{P}} : A_{\mathscr{P}} \in \mathscr{P} \text{ for all } \mathscr{P} \in \mathfrak{Q} \text{ and } \bigcap_{\mathscr{P} \in \mathfrak{Q}} A_{\mathscr{P}} \neq \emptyset \right\}.$$

The *coarse information sharing rule*, the *fine information sharing rule* and the *private information sharing rule* are rules that assign to each non-null coalition S and each agent in S the information partition $\bigwedge\{\mathscr{P}_i : i \in \mathfrak{P}(S)\}$, $\bigvee\{\mathscr{P}_i : i \in \mathfrak{P}(S)\}$ and $\mathscr{P}_t$, respectively. Next, we give the formal definition of an information sharing rule.

Definition 2.1. An *information sharing rule* is a rule Υ that assigns a collection $\Upsilon(S) = \{\Upsilon_t(S) : t \in S\}$ of information partitions of Ω to every non-null coalition S, where $\Upsilon_t(S)$ denotes the information partition that an agent $t \in S$ can have under Υ.

The partition $\Upsilon_t(S)$ is intended as the signal that agent t receives when she becomes a member of S. Thus, it is the information that agent t is able to use once the coalition S has been formed. Given two information sharing rules Υ^1 and Υ^2, the rule Υ^1 is said to be *finer* than Υ^2, denoted by $\Upsilon^1 \succeq \Upsilon^2$, if $\Upsilon^1_t(S) \succeq \Upsilon^2_t(S)$ for each non-null coalition S and each $t \in S$. In what follows, we give an example of an information sharing rule which differs from the coarse, fine and private information sharing rules.

Example 2.2. Let $T = [0,1] \cup \{2\}$. Suppose that $(T, \mathscr{T}, \mu)$ is a measure space of agents with $\mu(2) = 1$ and [0, 1] is endowed with the Borel σ-algebra and the Lebesgue measure. Assume $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and define the initial information of each agent by

$$\mathscr{P}_t = \begin{cases} \{\{\omega_1, \omega_2\}, \{\omega_3\}\}, & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \{\{\omega_1, \omega_3\}, \{\omega_2\}\}, & \text{if } t \in \left(\frac{1}{2}, 1\right]; \\ \{\{\omega_2, \omega_3\}, \{\omega_1\}\}, & \text{if } t = \{2\}. \end{cases}$$

Consider an information sharing rule Υ defined by

$$\Upsilon_t(S) = \begin{cases} \bigwedge\{\mathscr{P}_t : t \in S\}, & \text{if } \mu(S) < \frac{1}{2}; \\ \mathscr{P}_t, & \text{if } \frac{1}{2} \leq \mu(S) \leq \frac{3}{4}; \\ \bigvee\{\mathscr{P}_t : t \in S\}, & \text{if } \mu(S) > \frac{3}{4}. \end{cases}$$

For any coalition S, let $\mathscr{T}_S = \{R \in \mathscr{T} : R \subseteq S\}$ and μ_S the restriction of μ on S. Throughout the rest of the paper, we use the following assumptions on an information sharing rule Υ.

$(\mathbf{P}_1)$ If S' is a non-null sub-coalition of a non-null coalition S with $\mathfrak{P}(S') = \mathfrak{P}(S)$, then $\Upsilon_t(S') = \Upsilon_t(S)$ for all $t \in S'$.

$(\mathbf{P}_2)$ If S' is a non-null sub-coalition of a non-null coalition S, then $\Upsilon_t(S) \succeq \Upsilon_t(S')$ for all $t \in S'$.

$(\mathbf{P}_3)$ For any non-null coalition S, the function $\xi^S : (S, \mathscr{T}_S, \mu_S) \to \mathfrak{P}$, defined by $\xi^S(t) = \Upsilon_t(S)$, is measurable when $\mathfrak{P}$ is endowed with the power set as its σ-algebra.

The following example shows that the assumptions $(\mathbf{P}_1)$ and $(\mathbf{P}_2)$ are independent.

Example 2.3. Let $\Omega = \{\omega_1, \omega_2\}$ and suppose that $T = [0, 1]$ is endowed with the Borel σ-algebra and the Lebesgue measure. The initial information of each agent is given by

$$\mathscr{P}_t = \begin{cases} \{\{\omega_1\}, \{\omega_2\}\}, & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \{\omega_1, \omega_2\}, & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Consider two information sharing rules Υ^1 and Υ^2 defined by

$$\Upsilon^1_t(S) = \bigwedge\{\mathscr{P}_t : t \in S\}$$

and

$$\Upsilon^2_t(S) = \begin{cases} \mathscr{P}_t, & \text{if } \mu(S) \leq \frac{1}{2}; \\ \bigvee\{\mathscr{P}_t : t \in S\}, & \text{if } \mu(S) > \frac{1}{2}. \end{cases}$$

Note that Υ^1 satisfies $(\mathbf{P}_1)$. However, it fails to satisfy $(\mathbf{P}_2)$. Indeed, if $t \in \left[0, \frac{1}{2}\right]$ then $\Upsilon^1_t(\left[0, \frac{1}{2}\right]) = \{\{\omega_1\}, \{\omega_2\}\}$ and $\Upsilon^1_t([0, 1]) = \{\omega_1, \omega_2\}$. On the other hand, $(\mathbf{P}_2)$ holds for Υ^2. But, Υ^2 does not satisfy $(\mathbf{P}_1)$ since $\Upsilon^2_t([0, 1]) = \{\{\omega_1\}, \{\omega_2\}\}$ if $t \in [0, 1]$ and

$$\Upsilon^2_t\left(\left[\frac{1}{3}, \frac{2}{3}\right]\right) = \begin{cases} \{\{\omega_1\}, \{\omega_2\}\}, & \text{if } t \in \left[\frac{1}{3}, \frac{1}{2}\right]; \\ \{\omega_1, \omega_2\}, & \text{if } t \in \left(\frac{1}{2}, \frac{2}{3}\right]. \end{cases}$$

Remark 2.4. As mentioned in Hervés-Beloso et al. [21], $(\mathbf{P}_1)$ claims that if one non-null coalition is contained in the other coalition and they have the same informational structure then any agent in the smaller coalition can use the same information as the information she can use in the larger coalition. The assumption $(\mathbf{P}_2)$ says that if we consider an initial coalition and some additional agents join in the later stage then the members in the original coalition cannot become worse off from an informational point of view. An information sharing rule satisfying $(\mathbf{P}_2)$ is referred to as *nested* by Allen [1] and using this assumption, she established the non-emptiness of the core for NTU games with finitely many players in the asymmetric information framework. The assumption $(\mathbf{P}_3)$ is equivalence to the $\mathscr{T}_S$-measurability of

$$S_k^{\Upsilon} = \{t \in S : \Upsilon_t(S) = \mathscr{P}_k\}$$

for all $\mathscr{P}_k \in \mathfrak{P}$ and any non-null coalition S. Thus, the assumption $(\mathbf{P}_3)$ is satisfied if Υ is either the coarse, fine or private information sharing rule. Note that the assumptions $(\mathbf{P}_1)$-$(\mathbf{P}_3)$ are restricted on only non-null coalitions since informational structures for null coalitions do not have influence on the proofs of our main results.

3. Blocking Mechanism

In this section, we first introduce one of the main concepts of this paper. Then we present some technical lemmas, and these lemmas will be employed to prove the main results in the next two sections.

For any information sharing rule Υ and non-null coalition S, an assignment f is termed as an $\Upsilon(S)$-assignment if $f(t,\cdot) - a(t,\cdot)$ is $\Upsilon_t(S)$-measurable μ-a.e. on S. Let $\mathscr{F} \subseteq \mathfrak{P}$ denote the informational structure that associates with each agent t a signal $\mathscr{F}_t$. An assignment f is called an *allocation* if $f(t,\cdot)-a(t,\cdot)$ is $\mathscr{F}_t$-measurable μ-a.e. and

$$\int_T f(\cdot,\omega)d\mu = \int_T a(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. An allocation f is said to be *Υ-blocked by a non-null coalition S* in $\mathscr{E}$ if there is an $\Upsilon(S)$-assignment g such that $g(t,\cdot) \in P_t(f(t,\cdot))$ μ-a.e. on S and

$$\int_S g(\cdot,\omega)d\mu = \int_S a(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. The *core*[3] of $\mathscr{E}$ under the information sharing rule Υ, denoted by $\mathscr{C}^{\Upsilon}(\mathscr{E})$, is the set of allocations that are not Υ-blocked by any non-null coalition. In particular, (i) if $\mathscr{F}_t = \mathscr{P}_t$ for all $t \in T$ and Υ is the private information sharing rule, then the corresponding core is known as the *private core* of $\mathscr{E}$; (ii) if $\mathscr{F}_t = \mathscr{P}_t$ for all $t \in T$ and Υ is the fine information sharing rule, then the corresponding core is termed as the *fine core* of $\mathscr{E}$; (iii) if $\mathscr{F}_t = \bigvee\{\mathscr{P}_t : t \in T\}$ for all $t \in T$ and Υ is the fine information sharing rule, then the corresponding core is called the *weak fine core* of $\mathscr{E}$. It is clear that the fine core is a subset of the weak fine core. Let $\mathbf{1}_\Omega$ denote the characteristic function on Ω, that is, $\mathbf{1}_\Omega(\omega) = 1$ for all $\omega \in \Omega$. For any two non-null coalitions S, R with $S \subseteq R$ and information sharing rule Υ satisfying $(\mathbf{P}_3)$, define the set

$$I^{\Upsilon}_{(S,R)} = \left\{k : 1 \leq k \leq n \text{ and } \mu\left(S \cap R^{\Upsilon}_k\right) > 0\right\}.$$

Lemma 3.1. *Assume $(\mathbf{P}_3)$ is satisfied for an information sharing rule Υ. Suppose that f is an assignment and S is a non-null coalition. If g is an $\Upsilon(S)$-assignment and $g(t,\cdot) \in P_t(f(t,\cdot))$ μ-a.e. on S, then there exist an $\lambda \in (0,1)$, a $z^k \in B_{++}$,*

[3]Our core concept is similar to that in Allen [1]. The only difference is that in her definition, f is an allocation means $f(t,\cdot)$ is an $\Upsilon_t(T)$-measurable μ-a.e. Thus, under the assumption that $\Upsilon_t(T) \succeq \mathscr{F}_t$ μ-a.e., our definition is exactly the same as that in Allen [1]. We do not need this assumption to prove some of our results and thus, we take a slightly different definition of the core.

and an assignment h^k such that $h^k(t,\cdot) \in P_t(f(t,\cdot))$ and $h^k(t,\cdot) - a(t,\cdot)$ is $\mathscr{P}_k$-measurable μ-a.e. on S_k^Υ, and

$$\int_{S_k^\Upsilon} (h^k - a)d\mu + z\mathbf{1}_\Omega = (1-\lambda)\int_{S_k^\Upsilon} (g-a)d\mu$$

for all $k \in I^\Upsilon_{(S,S)}$.

Proof. Since f and g are Bochner integrable, there exist a sub-coalition R of S and a separable closed linear subspace Z of B^Ω such that $f(R,\cdot)\cup g(R,\cdot) \subseteq Z$, $\mu(S\backslash R) = 0$ and $g(t,\cdot) \in P_t(f(t,\cdot))$ for all $t \in R$. Let $\{c_m : m \geq 1\}$ be a monotonically decreasing sequence in $(0,1)$ converging to 0. Define a function $g_m : R \to Z_+$ by $g_m(t) = (1-c_m)g(t,\cdot)$ for all $t \in R$. Note that $g_{m+1}(t) \geq g_m(t)$ for all $t \in R$ and $m \geq 1$. For all $k \in I^\Upsilon_{(R,R)}$, define $Q^k : R_k^\Upsilon \rightrightarrows Z_+$ such that $Q^k(t) = Z_+ \cap P_f(t)$ for all $t \in R_k^\Upsilon$. Since Gr_{P_f} is $\mathscr{T} \otimes \mathscr{B}(B^\Omega)$-measurable, $\mathrm{Gr}_{Q^k} \in \mathscr{T}_R \otimes \mathscr{B}(Z)$. For all $m \geq 1$, let

$$A_m^k = \left\{t \in R_k^\Upsilon : g_m(t) \in Q^k(t)\right\}$$

and

$$B_m^k = \mathrm{Gr}_{Q^k} \cap \left\{(t, g_m(t)) : t \in R_k^\Upsilon\right\}.$$

Obviously, A_m^k is the projection of B_m^k on R_k^Υ. Note that

$$\left\{(t, g_m(t)) : t \in R_k^\Upsilon\right\} \in \mathscr{T}_R \otimes \mathscr{B}(Z)$$

for all $m \geq 1$. Thus, by the measurable projection theorem, one has $A_m^k \in \mathscr{T}_R$ for all $m \geq 1$. Define

$$R_m^k = \bigcap \left\{A_r^k : r \geq m\right\}.$$

Since $P_f(t)$ is open in B_+^Ω for all $t \in R$, one obtains

$$R_k^\Upsilon = \bigcup \left\{R_m^k : m \geq 1\right\}.$$

Since $\{R_m^k : m \geq 1\}$ is monotonically increasing, there must exist some $m_0 \geq 1$ such that $\mu(R_{m_0}^k) > 0$ for all $k \in I^\Upsilon_{(R,R)}$. It is easy to verify that[4] there exist an element $b \in B_{++}$ and a non-null sub-coalition E^k of $R_{m_0}^k$ such that $a(t,\omega) \gg 2b$ for all $(t,\omega) \in E^k \times \Omega$ and $k \in I^\Upsilon_{(R,R)}$. Without any loss of generality, we assume that $\mu(E^k) < \mu(R_{m_0}^k)$. Thus, one can find some $m_1 \geq m_0$ such that

$$b - \frac{1}{\mu(E^k)}\int_{R_k^\Upsilon \backslash R_{m_1}^k} (g(\cdot,\omega) - a(\cdot,\omega))d\mu \gg 0$$

for all $\omega \in \Omega$ and $k \in I^\Upsilon_{(R,R)}$. Pick an $k \in I^\Upsilon_{(R,R)}$ and define $y^k : E^k \times \Omega \to B_+$ such that

$$y^k(t,\omega) = a(t,\omega) - b - \frac{1}{\mu(E^k)}\int_{R_k^\Upsilon \backslash R_{m_1}^k} (g(\cdot,\omega) - a(\cdot,\omega))d\mu.$$

[4] To see this, let $c \in B_{++}$ and define

$$D_m^k = \left\{t \in R_k^\Upsilon : a(t,\omega) \gg \frac{c}{m} \text{ for all } \omega \in \Omega\right\}$$

for all $m \geq 1$ and $k \in I^\Upsilon_{(R,R)}$. Then $\{D_m^k : m \geq 1\}$ is monotonically increasing and $R_k^\Upsilon = \bigcup\{D_m^k : m \geq 1\}$.

So, $y^k(t,\omega) \in B_{++}$ for all $(t,\omega) \in E^k \times \Omega$. Consider an assignment $h^k : T \times \Omega \to B_+$ defined by

$$h^k(t,\omega) = \begin{cases} (1-c_{m_0})g(t,\omega) + c_{m_0} y^k(t,\omega), & \text{if } (t,\omega) \in E^k \times \Omega; \\ (1-c_{m_0})g(t,\omega) + c_{m_0} a(t,\omega), & \text{if } (t,\omega) \in (R^k_{m_1} \setminus E^k) \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

It follows from the definition of $R^k_{m_1}$ and the monotonicity of preferences that $h^k(t,\cdot) \in P_t(f(t,\cdot))$ μ-a.e. on $R^k_{m_1}$. Thus, $h^k(t,\cdot) \in P_t(f(t,\cdot))$ μ-a.e. on R^Υ_k. Using the $\mathscr{P}_k$-measurability of $g(t,\cdot) - a(t,\cdot)$, one can easily show that $h^k(t,\cdot) - a(t,\cdot)$ is $\mathscr{P}_k$-measurable μ-a.e. on R^Υ_k. Put, $\lambda = c_{m_0}$ and $z^k = c_{m_0} b\mu(E^k)$. It can be checked that

$$\int_{R^\Upsilon_k} (h^k - a)d\mu + z^k \mathbf{1}_\Omega = (1-\lambda)\int_{R^\Upsilon_k} (g-a)d\mu.$$

Since $\mu(R^\Upsilon_k) = \mu(S^\Upsilon_k)$ and $I^\Upsilon_{(R,R)} = I^\Upsilon_{(S,S)}$, the proof has been completed. □ □

Corollary 3.2. *Under the hypothesis of Lemma 3.1, there exist a $\lambda \in (0,1)$, a $z \in B_{++}$, and an $\Upsilon(S)$-assignment h satisfying $h(t,\cdot) \in P_t(f(t,\cdot))$ μ-a.e. on S and*

$$\int_S (h-a)d\mu + z\mathbf{1}_\Omega = (1-\lambda)\int_S (g-a)d\mu,$$

where

$$z = \sum_{k \in I^\Upsilon_{(S,S)}} z^k$$

and the assignment h is defined by

$$h(t,\omega) = \begin{cases} h^k(t,\omega), & \text{if } (t,\omega) \in S^\Upsilon_k \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

Let S be a coalition of $\mathscr{E}$. For any allocation f, non-null coalition $R \supseteq S$ and information sharing rule Υ, define a correspondence $Q^{\{\Upsilon,R\}}_f : (S, \mathscr{T}_S, \mu_S) \rightrightarrows B^\Omega_+$ such that

$$Q^{\{\Upsilon,R\}}_f(t) = \{x \in P_f(t) : x - a(t,\cdot) \text{ is } \Upsilon_t(R)\text{-measurable}\}.$$

An *integrable selection* of the correspondence $Q^{\{\Upsilon,R\}}_f$ is a Bochner integrable function $g : (S, \mathscr{T}_S, \mu_S) \to B^\Omega_+$ such that $g(t) \in Q^{\{\Upsilon,R\}}_f(t)$ μ_S-a.e. The *integration* of $Q^{\{\Upsilon,R\}}_f$ over a sub-coalition S_0 of S in the sense of Aumann [5] is a subset of B^Ω_+, defined as

$$\int_{S_0} Q^{\{\Upsilon,R\}}_f d\mu = \left\{ \int_{S_0} g d\mu : g \text{ is an integrable selection of } Q^{\{\Upsilon,R\}}_f \right\}.$$

Since $P_f(t)$ is convex for all $t \in T_1$, one obtains the convexity of $\int_{S_0} Q^{\{\Upsilon,R\}}_f d\mu$. In proofs of the next two lemmas, this result will be used.

Lemma 3.3. *Suppose that the assumption* $(\mathbf{P}_3)$ *is satisfied for an information sharing rule Υ, $0 < \lambda < 1$ and $z \in B_{++}$. Let S, R be two non-null coalitions such that $S \subseteq R$. Assume f, g, h are three assignments satisfying*

$$\int_{S \cap R^\Upsilon_k} g d\mu, \int_{S \cap R^\Upsilon_k} h d\mu \in \mathrm{cl} \int_{S \cap R^\Upsilon_k} Q^{\{\Upsilon,R\}}_f d\mu$$

for all $k \in I^{\Upsilon}_{(S,R)}$. *Then there exists an assignment* y *such that* $y(t,\cdot) \in P_f(t)$ *and* $y(t,\cdot) - a(t,\cdot)$ *is* $\Upsilon_t(R)$*-measurable* μ*-a.e. on* S*, and*

$$\int_S (y-a)d\mu = \lambda \int_S (g-a)d\mu + (1-\lambda)\int_S (h-a)d\mu + z\mathbf{1}_\Omega.$$

Proof. Fix an $k \in I^{\Upsilon}_{(S,R)}$. Since cl $\int_{S\cap R^{\Upsilon}_k} Q_f^{\{\Upsilon,R\}} d\mu$ is convex,

$$\lambda \int_{S\cap R^{\Upsilon}_k} g d\mu + (1-\lambda)\int_{S\cap R^{\Upsilon}_k} h d\mu \in \mathrm{cl} \int_{S\cap R^{\Upsilon}_k} Q_f^{\{\Upsilon,R\}} d\mu.$$

Choose an open neighbourhood W of 0 in B such that

$$\frac{z}{|I^{\Upsilon}_{(S,R)}|} - W \subseteq B_{++},$$

where $|I^{\Upsilon}_{(S,R)}|$ denotes the number of elements of $I^{\Upsilon}_{(S,R)}$. It follows that

$$\left(\lambda \int_{S\cap R^{\Upsilon}_k} g d\mu + (1-\lambda)\int_{S\cap R^{\Upsilon}_k} h d\mu + W^{\Omega}\right) \bigcap \int_{S\cap R^{\Upsilon}_k} Q_f^{\{\Upsilon,R\}} d\mu \neq \emptyset.$$

So, there exist a function $w : \Omega \to W$ and an integrable selection x of $Q_f^{\{\Upsilon,R\}}$ such that

$$\lambda \int_{S\cap R^{\Upsilon}_k} g d\mu + (1-\lambda)\int_{S\cap R^{\Upsilon}_k} h d\mu + w = \int_{S\cap R^{\Upsilon}_k} x d\mu,$$

which is equivalent to

$$\lambda \int_{S\cap R^{\Upsilon}_k} (g-a) d\mu + (1-\lambda)\int_{S\cap R^{\Upsilon}_k} (h-a) d\mu + w = \int_{S\cap R^{\Upsilon}_k} (x-a) d\mu.$$

It follows from the last equation that w is $\mathscr{P}_k$-measurable. Define an assignment $y^k : T \times \Omega \to B_+$ such that

$$y^k(t,\omega) = \begin{cases} x(t,\omega) + \frac{1}{\mu(S\cap R^{\Upsilon}_k)}\left(\frac{z}{|I^{\Upsilon}_{(S,R)}|} - w(\omega)\right), & \text{if } (t,\omega) \in (S\cap R^{\Upsilon}_k)\times\Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

So, one has $y^k(t,\cdot) \in Q_f^{\{\Upsilon,R\}}(t)$ μ-a.e. on $S \cap R^{\Upsilon}_k$ and

$$\int_{S\cap R^{\Upsilon}_k} y^k d\mu = \lambda \int_{S\cap R^{\Upsilon}_k} g d\mu + (1-\lambda)\int_{S\cap R^{\Upsilon}_k} h d\mu + \frac{z}{|I^{\Upsilon}_{(S,R)}|}\mathbf{1}_\Omega.$$

Thus, the assignment $y : T \times \Omega \to B_+$, defined by

$$y(t,\omega) = \begin{cases} y^k(t,\omega), & \text{if } (t,\omega) \in (S\cap R^{\Upsilon}_k)\times\Omega,\ k \in I^{\Upsilon}_{(S,R)}; \\ h(t,\omega), & \text{otherwise,} \end{cases}$$

is desired. □ □

Corollary 3.4. *Suppose that the assumption* $(\mathbf{P}_3)$ *is satisfied for an information sharing rule* Υ*,* $0 < \lambda < 1$ *and* $z \in B_{++}$*. Let* f *be an assignment and* S *a non-null coalition. If* g *and* h *are two* $\Upsilon(S)$*-assignments such that* $g(t,\cdot), h(t,\cdot) \in P_f(t)$ μ*-a.e. on* S*, then there is an* $\Upsilon(S)$*-assignment* y *such that* $y(t,\cdot) \in P_f(t)$ μ*-a.e. on* S *and*

$$\int_S (y-a)d\mu = \lambda \int_S (g-a)d\mu + (1-\lambda)\int_S (h-a)d\mu + z\mathbf{1}_\Omega.$$

Corollary 3.5. *Assume* $(\mathbf{P}_2)$ *and* $(\mathbf{P}_3)$ *are satisfied for an information sharing rule* Υ, $0<\lambda<1$ *and* $z\in B_{++}$. *Suppose that* f *is an* $\Upsilon(T)$*-assignment and* S *is a non-null coalition. If* g *is an* $\Upsilon(S)$*-assignment such that* $g(t,\cdot)\in P_f(t)$ μ*-a.e. on* S*, then there is an assignment* y *such that* $y(t,\cdot)\in P_f(t)$ *and* $y(t,\cdot)-a(t,\cdot)$ *is* $\Upsilon_t(T)$*-measurable* μ*-a.e. on* S*, and*

$$\int_S (y-a)d\mu = \lambda\int_S (g-a)d\mu + (1-\lambda)\int_S (f-a)d\mu + z\mathbf{1}_\Omega.$$

Lemma 3.6. *Assume* $(\mathbf{P}_3)$ *is satisfied for an information sharing rule* Υ, $0<\lambda<1$ *and* $z\in B_{++}$. *Let* f *be an assignment and* $S\in\mathscr{T}_0$ *a non-null coalition. Suppose also that* R *is a coalition such that* $S\subseteq R$ *and* g *is an assignment such that*

$$\int_{S_i\cap R_k^\Upsilon} g d\mu \in \operatorname{cl}\int_{S_i\cap R_k^\Upsilon} Q_f^{\{\Upsilon,R\}}d\mu$$

for all $k\in I^\Upsilon_{(S_i,R)}$ *and* $i\in\mathfrak{P}(S)$. *Then there exist a sub-coalition* S' *of* S *and an assignment* h *such that* (i) $\mu(S')=\lambda\mu(S)$ *and* $\mathfrak{P}(S')=\mathfrak{P}(S)$*;* (ii) $h(t,\cdot)\in P_f(t)$ *and* $h(t,\cdot)-a(t,\cdot)$ *is* $\Upsilon_t(R)$*-measurable* μ*-a.e. on* S'*, and*

$$\int_{S'}(h-a)d\mu = \lambda\int_S (g-a)d\mu + z\mathbf{1}_\Omega.$$

Proof. Pick an $i\in\mathfrak{P}(S)$ and an $k\in I^\Upsilon_{(S_i,R)}$. Let W be an open neighbourhood of 0 in B such that

$$\frac{z}{2\lambda\sum_{i\in\mathfrak{P}(S)}|I^\Upsilon_{(S_i,R)}|} - W\subseteq B_{++}.$$

Applying an argument similar to that in the proof of Lemma 3.3, one obtains an assignment y^{ik} such that $y^{ik}(t,\cdot)\in Q_f^{\{\Upsilon,R\}}(t)$ μ-a.e. on $S_i\cap R_k^\Upsilon$ and

$$\int_{S_i\cap R_k^\Upsilon} y^{ik}d\mu = \int_{S_i\cap R_k^\Upsilon} gd\mu + \frac{z}{2\lambda\sum_{i\in\mathfrak{P}(S)}|I^\Upsilon_{(S_i,R)}|}\mathbf{1}_\Omega.$$

By Lemma 3.3 in Bhowmik and Cao [6], one can find a sequence $\{S_n^{ik}: n\geq 1\}\subseteq \mathscr{T}_{S_i\cap R_k^\Upsilon}$ such that $\mu(S_n^{ik})=\lambda\mu(S_i\cap R_k^\Upsilon)$ for all $n\geq 1$ and

$$\lim_{n\to\infty}\int_{S_n^{ik}}(y^{ik}-a)d\mu = \lambda\int_{S_i\cap R_k^\Upsilon}(y^{ik}-a)d\mu.$$

The function $x_n^{ik}:\Omega\to B$, defined by

$$x_n^{ik}(\omega)=\lambda\int_{S_i\cap R_k^\Upsilon}(y^{ik}(\cdot,\omega)-a(\cdot,\omega))d\mu - \int_{S_n^{ik}}(y^{ik}(\cdot,\omega)-a(\cdot,\omega))d\mu,$$

is $\mathscr{P}_k$-measurable for all $n\geq 1$ and $\lim_{n\to\infty}\|x_n^{ik}(\omega)\|=0$ for all $\omega\in\Omega$. Choose an $n_{ik}\geq 1$ such that

$$\frac{z}{2\sum_{i\in\mathfrak{P}(S)}|I^\Upsilon_{(S_i,R)}|} + x_{n_{ik}}^{ik}(\omega)\gg 0$$

for each $\omega\in\Omega$ and then consider a function $h^{ik}: S_{n_{ik}}^{ik}\times\Omega\to B_+$ defined by

$$h^{ik}(t,\omega)=y^{ik}(t,\omega)+\frac{1}{\mu(S_{n_{ik}}^{ik})}\left(\frac{z}{2\sum_{i\in\mathfrak{P}(S)}|I^\Upsilon_{(S_i,R)}|}+x_{n_{ik}}^{ik}(\omega)\right).$$

Obviously, $h^{ik}(t,\cdot) \in Q_f^{\{\Upsilon,R\}}(t)$ μ-a.e. on $S^{ik}_{n_{ik}}$ and

$$\int_{S^{ik}_{n_{ik}}} (h^{ik} - a)d\mu = \lambda \int_{S_i \cap R^{\Upsilon}_k} (g-a)d\mu + \frac{z}{\sum_{i \in \mathfrak{P}(S)} |I^{\Upsilon}_{(S,R)}|} \mathbf{1}_{\Omega}.$$

Put,

$$S' = \bigcup \left\{ S^{ik}_{n_{ik}} : i \in \mathfrak{P}(S), k \in I^{\Upsilon}_{(S_i,R)} \right\}.$$

Note that $\mu(S') = \lambda\mu(S)$ and $\mathfrak{P}(S') = \mathfrak{P}(S)$. Thus, the sub-coalition S' of S and the assignment $h : T \times \Omega \to B_+$, defined by $h(t,\omega) = h^{ik}(t,\omega)$, if $(t,\omega) \in S^{ik}_{n_{ik}} \times \Omega$; and $h(t,\omega) = g(t,\omega)$, otherwise, are desired. □ □

Corollary 3.7. *Suppose that the assumption* $(\mathbf{P}_3)$ *is satisfied for an information sharing rule* Υ*,* $0 < \lambda < 1$ *and* $z \in B_{++}$*. Let* f *be an assignment and* $S \in \mathscr{T}_0$ *a non-null coalition. If* g *is an* $\Upsilon(S)$*-assignment such that*

$$\int_{S_i \cap S^{\Upsilon}_k} g d\mu \in \mathrm{cl} \int_{S_i \cap S^{\Upsilon}_k} P_f d\mu$$

for all $k \in I^{\Upsilon}_{(S_i,S)}$ *and* $i \in \mathfrak{P}(S)$*, then there are a sub-coalition* S' *of* S *and an assignment* h *such that* (i) $\mu(S') = \lambda\mu(S)$ *and* $\mathfrak{P}(S) = \mathfrak{P}(S')$*;* (ii) $h(t,\cdot) \in P_f(t)$ *and* $h(t,\cdot) - a(t,\cdot)$ *is* $\Upsilon_t(S)$*-measurable* μ*-a.e. on* S'*, and*

$$\int_{S'} (h-a)d\mu = \lambda \int_S (g-a)d\mu + z\mathbf{1}_{\Omega}.$$

Moreover, if $(\mathbf{P}_1)$ *is also satisfied for* Υ*, then* h *is an* $\Upsilon(S')$*-assignment.*

Corollary 3.8. *Assume* $(\mathbf{P}_3)$ *is satisfied for an information sharing rule* Υ*,* $0 < \lambda < 1$ *and* $z \in B_{++}$*. Let* f *be an assignment and* $S \in \mathscr{T}_0$ *a non-null coalition. Suppose that* g *is an assignment such that* $g(t,\cdot) - a(t,\cdot)$ *is* $\Upsilon_t(T)$*-measurable* μ*-a.e. on* S *and*

$$\int_{S_i \cap T^{\Upsilon}_k} g d\mu \in \mathrm{cl} \int_{S_i \cap T^{\Upsilon}_k} P_f d\mu$$

for all $k \in I^{\Upsilon}_{(S_i,T)}$ *and* $i \in \mathfrak{P}(S)$*. Then there exist a sub-coalition* S' *of* S *and an assignment* h *such that* (i) $\mu(S') = \lambda\mu(S)$ *and* $\mathfrak{P}(S') = \mathfrak{P}(S)$*;* (ii) $h(t,\cdot) \in P_f(t)$ *and* $h(t,\cdot) - a(t,\cdot)$ *is* $\Upsilon_t(T)$*-measurable* μ*-a.e. on* S'*, and*

$$\int_{S'} (h-a)d\mu = \lambda \int_S (g-a)d\mu + z\mathbf{1}_{\Omega}.$$

4. Core Solutions in Atomless Economies

In this section, we put our attention to only atomless economies. It is well known that the information transmission within coalitions is costly: the larger the coalition, the more difficult to communicate among its members. Thus, it is reasonable to consider small coalitions. As mentioned in Hervés-Beloso et al. [21], one can argue in a symmetric way whenever coalitions are large. In fact, if a coalition becomes a member of a large coalition then she believes that her private information is negligible and/ irrelevant as it is already available within the coalition. As a result, she makes her private information public within the coalition. Thus, it is also important to consider large coalitions. This section explores the idea of finding a coalition of any size as well as a characterization of the core in terms of the core for higher information structure.

4.1. **Blocking coalition for a given measure.** Recall that the result in [30] rely heavily on Lyapunov's convexity theorem, which is not true in its exact form in an infinite dimensional setting. Thus, the exact extension of Schmeidler's result is not possible in an economy with infinitely many commodities, as mentioned in [18]. Indeed, Núñez [27] gave an example of an atomless economy, with infinitely many commodities, where an assignment f is blocked by the grand coalition via an assignment g, but there is no other different coalition blocking f via the same allocation g. Despite the impossibility for obtaining the result in the exact strong form, Hervés-Beloso [18] first established a variation of Schmeidler's result in an infinite dimensional setting. In particular, they showed that in continuum economies whose commodity space is the space of bounded sequences if an assignment f is blocked by a coalition S via g then for every $\varepsilon \in (0, \mu(S))$ there is a sub-coalition S' and an assignment g' such that f is blocked by S' via g'. In the case of asymmetric information, Hervés-Beloso et al. [19, 20] obtained results similar to those in [30, 35] in an economy with either finite dimensional commodity space or the real bounded sequences as the commodity spaces. Later, these results were generalized to an atomless economy with an ordered Banach space whose positive cone has an interior point as the commodity space, refer to [16]. Since the results obtained so far in an asymmetric economy without exact feasibility condition, Bhowmik and Cao [6] proved these results in an asymmetric information economy with an atomless measure space of agents, an ordered Banach space whose positive cone has an interior point as the commodity space and the exact feasibility condition. Recently, Hervés-Beloso et al. [21] established similar results under information sharing rule in economies with finitely many commodities. We now give an extension of Proposition 5.1 in an economy with infinitely many commodities and the exact feasibility condition.

Theorem 4.1. *Suppose that the assumptions* $(\mathbf{P}_1)$ *and* $(\mathbf{P}_3)$ *are satisfied for an information sharing rule* Υ *and that* $T = T_0$*. If an allocation* f *is* Υ*-blocked by a non-null coalition* S*, then* f *is also* Υ*-blocked by a coalition* S_ε *with* $\mu(S_\varepsilon) = \varepsilon$ *for any* $\varepsilon \in (0, \mu(S))$*.*

Proof. Suppose that f is Υ-blocked by a non-null coalition S via g. Choose an $\varepsilon \in (0, \mu(S))$. Let $\alpha \in (0,1)$ be such that $\varepsilon = \alpha\mu(S)$. It follows from Corollary 3.2 that there are an $\Upsilon(S)$-assignment h, a $z \in B_{++}$ and a $\lambda \in (0,1)$ such that $h(t,\cdot) \in P_f(t)$ μ-a.e. on S and

$$\int_S (h-a)d\mu + \frac{z}{\alpha}\mathbf{1}_\Omega = (1-\lambda)\int_S (g-a)d\mu = 0.$$

By Corollary 3.7, there exist a sub-coalition S' of S with $\mu(S') = \alpha\mu(S)$ and an $\Upsilon(S')$-assignment y such that $y(t,\cdot) \in P_f(t)$ μ-a.e. on S' and

$$\int_{S'} (y-a)d\mu = \alpha\int_S (h-a)d\mu + z\mathbf{1}_\Omega.$$

Combining the last two equalities, one obtains $\int_{S'}(y-a)d\mu = 0$. Thus, f is Υ-blocked by the coalition S' via y. This completes the proof. □ □

Remark 4.2. The assumption $(\mathbf{P}_1)$ is essential to extend Schmeidler's theorem to an economy with finitely many commodities under any information sharing rule, as noted in Hervés-Beloso et al. [21]. Similar to Hervés-Beloso et al. [21], it can be verified that the assumption $(\mathbf{P}_1)$ is enough to prove Theorem 4.1 for an economy with finitely many commodities and the exact feasibility condition. However, to get a positive result in an infinite dimensional setting, the assumption ($\mathbf{P}_3$) plays a crucial role to overcome the difficulty with weak form of Lyapunov's convexity theorem. Thus, at this stage, it is unclear that whether the conclusion of Theorem 4.1 is positive in an infinite dimensional setting without this additional assumption.

Next, we derive an extension of the main result in Grodal [23] under the formulation of an information sharing rule.

Theorem 4.3. *Suppose that* ($\mathbf{P}_1$) *and* ($\mathbf{P}_3$) *are satisfied for an information sharing rule* Υ *and that* $T = T_0$. *Let* T *be endowed with a pseudo-metric which makes* T *a separable topological space such that* $\mathscr{B}(T) \subseteq \mathscr{T}$. *If an allocation* f *is* Υ*-blocked by a non-null coalition, then for every* $\varepsilon, \delta > 0$ *there is a coalition* R *such that* $\mu(R) \leq \varepsilon$ *and* f *is* Υ*-blocked by* R *via some assignment* y*; and* $R = \bigcup\{R_i : 1 \leq i \leq m\}$ *for a finite collection of coalitions* $\{R_1, \cdots, R_m\}$ *with the diameter of* R_i *is smaller than* δ *and* y *is* $\Upsilon(R_i)$*-assignment for all* $i = 1, \cdots, m$.

Proof. By Theorem 4.1, there are a non-null coalition S and an assignment g such that f is Υ-blocked by S via g and $\mu(S) \leq \varepsilon$. By Lemma 3.1, there exist a $\lambda \in (0,1)$, a $z^k \in B_{++}$ and an assignment h^k such that $h^k(t,\cdot) \in P_t(f(t,\cdot))$ and $h^k(t,\cdot) - a(t,\cdot)$ is $\mathscr{P}_k$-measurable μ-a.e. on S_k^Υ, and

$$\int_{S_k^\Upsilon} (h^k - a) d\mu + z^k \mathbf{1}_\Omega = (1-\lambda) \int_{S_k^\Upsilon} (g - a) d\mu$$

for all $k \in I^\Upsilon_{(S,S)}$. For every $k \in I^\Upsilon_{(S,S)}$ and non-null sub-coalition E of S_k^Υ, let

$$b_E^k = \frac{1}{\mu(E)} \left[\int_{S_k^\Upsilon \setminus E} (h^k - a) d\mu + z^k \mathbf{1}_\Omega \right].$$

Choose an $\alpha > 0$ such that for all $k \in I^\Upsilon_{(S,S)}$ and non-null coalition $E \subseteq S_k^\Upsilon$ with $\mu(S_k^\Upsilon \setminus E) < \alpha$, one has $b_E^k \in B_{++}$. Pick an $k \in I^\Upsilon_{(S,S)}$ and let E_k be a sub-coalition of S_k^Υ such that $\mu(S_k^\Upsilon \setminus E_k) < \alpha$. Define $y^{E_k} : E_k \times \Omega \to B_+$ by letting

$$y^{E_k}(t,\omega) = h^k(t,\omega) + b^k_{E_k}(\omega)$$

for all $(t,\omega) \in E_k \times \Omega$. Clearly, $y^{E_k}(t,\cdot) \in P_f(t)$ and $y^{E_k}(t,\cdot) - a(t,\cdot)$ is $\mathscr{P}_k$-measurable μ-a.e. on E_k. Further,

$$\int_{E_k} \left(y^{E_k} - a \right) d\mu = (1-\lambda) \int_{S_k^\Upsilon} (g - a) d\mu.$$

Thus, for each non-null sub-coalition E of S with $\mu(S \setminus E) < \delta$, one has an assignment y such that $y(t,\cdot) \in P_f(t)$ and $y(t,\cdot) - a(t,\cdot)$ is $\Upsilon_t(S)$-measurable μ-a.e. on E, and

$$\int_{E \cap S_k^\Upsilon} (y - a)\, d\mu = (1-\lambda) \int_{S_k^\Upsilon} (g - a) d\mu.$$

for all $k \in I^\Upsilon_{(S,S)}$. As a result, one obtains

$$\int_E (y - a)\, d\mu = (1-\lambda) \int_S (g - a) d\mu = 0.$$

For each $i \in \mathfrak{P}(S)$, let $\{t^i_m : m \geq 1\}$ be a sequence dense in S_i. For all $m \geq 1$, put

$$S^i_m = B\left(t^i_m,\ \frac{\delta}{2|\mathfrak{P}(S)|}\right).$$

Let

$$A^i_1 = S^i_1 \text{ and } A^i_m = S^i_m \setminus \{S^i_j : 1 \leq j < m\}$$

for all $m \geq 2$. For each $i \in \mathfrak{P}(S)$, select some m_i such that $\mu(S_i \setminus D_{m_i}) < \frac{\delta}{|\mathfrak{P}(S)|}$, where

$$D_{m_i} = \bigcup \left\{A^i_m : 1 \leq m \leq m_i\right\}.$$

So, by decomposing each A^i_m as a union of mutually disjoint non-null coalitions[5] if necessary, one can assume that there is some m_0 such that for each $i \in \mathfrak{P}(S)$, there is a family $\mathscr{R}^i = \{R^i_m : 1 \leq m \leq m_0\}$ of coalitions satisfying the diameter of each R^i_m is less than $\frac{\delta}{|\mathfrak{P}(S)|}$ and

$$\mu\left(S_i \setminus \bigcup_{m=1}^{m_0} R^i_m\right) < \frac{\delta}{|\mathfrak{P}(S)|}.$$

For all $1 \leq m \leq m_0$, define

$$R_m = \bigcup \left\{R^i_m : i \in \mathfrak{P}(S)\right\} \text{ and } R = \bigcup \left\{R_m : 1 \leq m \leq m_0\right\}.$$

Note that $\mathfrak{P}(R_m) = \mathfrak{P}(R) = \mathfrak{P}(S)$. Since $\mu(S \setminus R) < \delta$, f is Υ-blocked by R via some assignment y, and y is $\Upsilon(R_m)$-assignment for all $1 \leq m \leq m_0$. □ □

We now intend to prove an extension of Vind's theorem under the settings of an information sharing rule and the exact feasibility. Such a result is not necessarily true without some additional assumptions as the following example shows.

Example 4.4. Consider an economy with $\Omega = \{\omega_1, \omega_2, \omega_3\}$; one commodity in each state; and the space of agents is $[0, 3]$ with the Borel σ-algebra and the Lebesgue measure. Assume that

$$\mathscr{P}_t = \begin{cases} \{\{\omega_1, \omega_2\}, \{\omega_3\}\}, & \text{if } t \in [0, 1); \\ \{\{\omega_1, \omega_3\}, \{\omega_2\}\}, & \text{if } t \in [1, 2); \\ \{\omega_1, \omega_2, \omega_3\}, & \text{if } t \in [2, 3], \end{cases}$$

and the preference of each agent t is represented by a utility function U_t, where $U_t : \mathbb{R}^3_+ \to \mathbb{R}$ is defined by

$$U_t(x, y, z) = \begin{cases} x + y + z, & \text{if } t \in [0, 1); \\ x + z, & \text{if } t \in [1, 2); \\ z, & \text{if } t \in [2, 3]. \end{cases}$$

Let $\mathscr{F}_t = \bigvee\{\mathscr{P}_t : t \in [0, 3]\} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$ and $a(t, \omega_i) = 4$ for all $t \in T$ and $i = 1, 2, 3$. Suppose that Υ is the private information sharing rule. Consider an

[5] Note that the separability of T is used to get a finite collection $\{A^i_m : 1 \leq m \leq m_i\}$ of open balls for each $i \in \mathfrak{P}(S)$. Then, we apply the standard arguments of set theory to obtain a family of mutually disjoint non-null coalitions (not necessarily intervals) containing the same number of elements for each $i \in \mathfrak{P}(S)$.

allocation f defined by

$$f(t,\cdot) = \begin{cases} (11,0,0), & \text{if } t \in [0,1); \\ (1,12,0), & \text{if } t \in [1,2); \\ (0,0,12), & \text{if } t \in [2,3]. \end{cases}$$

Note that f is Υ-blocked by all non-null coalitions contained in $[0,1)$, but it cannot be Υ-blocked by any coalition whose measure is sufficiently close to 3.

Thus, to exploit the veto power of large coalitions, we now give an assumption on the informational structure $\mathscr{F}$.

$(\mathbf{P}_4)$ For all $t \in T$, $\Upsilon_t(T) \succeq \mathscr{F}_t$.

It is worthwhile to point out that the assumption $(\mathbf{P}_4)$ is standard under the fine or private information sharing rule whenever $\mathscr{F}_t = \mathscr{P}_t$ for all $t \in T$. It is also true in the case when $\Upsilon_t(T) = \mathscr{F}_t$ for all $t \in T$. As a particular case, it is true when Υ is the fine information sharing rule and $\mathscr{F}_t$ is the pooled information for all $t \in T$. However, it does not hold, in general, if (i) $\mathscr{F}_t = \mathscr{P}_t$ and Υ is the coarse information sharing rule; and (ii) $\mathscr{F}_t = \bigvee\{\mathscr{P}_t : t \in T\}$ and Υ is the private information sharing rule. Note that $(\mathbf{P}_4)$ is not satisfied in Example 4.4.

Theorem 4.5. *Suppose that the assumptions* $(\mathbf{P}_1)$*-*$(\mathbf{P}_4)$ *are satisfied for an information sharing rule* Υ *and that* $T = T_0$. *If an allocation* $f \notin \mathscr{C}^{\Upsilon}(\mathscr{E})$, *then* f *is* Υ*-blocked by a coalition* S_ε *with* $\mu(S_\varepsilon) = \varepsilon$ *for any* $\varepsilon \in (0, \mu(T))$.

Proof. Suppose that f is Υ-blocked by a coalition S via g. By Theorem 4.1, for any $\varepsilon \in (0, \mu(S))$, there is a coalition S_ε such that $\mu(S_\varepsilon) = \varepsilon$ and f is Υ-blocked by S_ε. If $\mu(S) = \mu(T)$, the proof has been completed. So, assume that $\mu(S) < \mu(T)$ and choose an $\varepsilon \in (\mu(S), \mu(T))$. Define

$$\alpha = 1 - \frac{\varepsilon - \mu(S)}{\mu(T \setminus S)}.$$

By Corollary 3.2, there are a $\lambda \in (0,1)$, a $z \in B_{++}$ and an $\Upsilon(S)$-assignment h such that $h(t) \in P_f(t)$ μ-a.e. on S and

$$\int_S (h - a) d\mu + \frac{2}{\alpha} z \mathbf{1}_\Omega = (1 - \lambda) \int_S (g - a) d\mu = 0.$$

It follows from Corollary 3.5 that there is an assignment h_ε such that $h_\varepsilon(t,\cdot) \in P_f(t)$ and $h_\varepsilon(t,\cdot) - a(t,\cdot)$ is $\Upsilon_t(T)$-measurable μ-a.e. on S, and

$$\int_S (h_\varepsilon - a) d\mu = \alpha \int_S (h - a) d\mu + (1 - \alpha) \int_S (f - a) d\mu + z\mathbf{1}_\Omega.$$

By Corollary 3.8, there are a sub-coalition R of $T \setminus S$ and an assignment $\hat{f}$ such that $\mu(R) = (1-\alpha)\mu(T \setminus S)$ and $\mathfrak{P}(R) = \mathfrak{P}(T \setminus S)$; $\hat{f}(t,\cdot) \in P_f(t)$ and $\hat{f}(t,\cdot) - a(t,\cdot)$ is $\Upsilon_t(T)$-measurable μ-a.e. on R, and

$$\int_R (\hat{f} - a) d\mu = (1 - \alpha) \int_{T \setminus S} (f - a) d\mu + z\mathbf{1}_\Omega.$$

Let $D = S \cup R$ then $\mathfrak{P}(D) = \mathfrak{P}(T)$. Consider an assignment $y : T \times \Omega \to B_+$ defined by

$$y(t,\omega) = \begin{cases} h_\varepsilon(t,\omega), & \text{if } (t,\omega) \in S \times \Omega; \\ \hat{f}(t,\omega), & \text{if } (t,\omega) \in R \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

It can be easily verified that f is Υ-blocked by the coalition D via y. □ □

Corollary 4.6. *Suppose that $\mathscr{C}_\varepsilon^\Upsilon(\mathscr{E})$ denotes the set of allocations which are not Υ-blocked by any coalition whose measure is equal to ε. Thus, it follows from Theorem 4.5 that $\mathscr{C}^\Upsilon(\mathscr{E}) = \mathscr{C}_\varepsilon^\Upsilon(\mathscr{E})$ for all $\varepsilon \in (0, \mu(T))$.*

Remark 4.7. We stress that the argument in the proof of Theorem 4.5 is very different from Hervés-Beloso et al. [21] even in the case of finitely many commodities. In particular, Lemma 3.1 plays a vital role whose proof is not straightforward. If the commodity space is an infinite dimensional space, then Lyapunov's convexity theorem does not hold. Hence, in an infinite dimensional setting, in addition to Lemma 3.1, we need other results in the previous section to prove Theorem 4.5.

4.2. **Information sharing rule for a given measure.** In this subsection, we define an information sharing rule, introduced by Hervés-Beloso et al. [21], that depend on the measure of a coalition. As a consequence, we provide a sharper characterization of core solutions.

An in Hervés-Beloso et al. [21], suppose that $\{A_k : k \in K\}$ is a partition of the interval $[0, \mu(T)]$. It can be taken as a family of thresholds in the sense that for each coalition S there is exactly one A_k such that $\mu(S) \in A_k$. Further, each A_k is associated with an information sharing rule Υ_k. If an agent t takes part in a coalition S then she has only access to the specific information prescribed by the sharing rule Υ^{k_0} if $\mu(S) \in A_{k_0}$. We assume that there is an $k_0 \in K$ such that $\Upsilon^{k_0} \succeq \Upsilon^k$ for all $k \in K$, $A_{k_0} \neq \{\mu(T)\}$ and the assumptions $(\mathbf{P}_1)$-$(\mathbf{P}_4)$ are satisfied for Υ^{k_0}. We now define the information mechanism $\widetilde{\Upsilon}$, where information that an agent t can dispose of when she becomes a member of coalition S is defined as $\widetilde{\Upsilon}_t(S) = \Upsilon_t^k(S)$ if $\mu(S) \in A_k$. The next theorem can be seen as an extension of Theorem 5.1 in Hervés-Beloso et al. [21] to an economy with an ordered Banach space whose positive cone has an interior point as the commodity space and the exact feasibility condition.

Theorem 4.8. *Assume $T = T_0$. Then $\mathscr{C}^{\widetilde{\Upsilon}}(\mathscr{E}) = \mathscr{C}^{\Upsilon^{k_0}}(\mathscr{E})$.*

Proof. Since $\mathscr{C}^{\Upsilon^{k_0}}(\mathscr{E}) \subseteq \mathscr{C}^{\widetilde{\Upsilon}}(\mathscr{E})$, it only requires to show that $\mathscr{C}^{\widetilde{\Upsilon}}(\mathscr{E}) \subseteq \mathscr{C}^{\Upsilon^{k_0}}(\mathscr{E})$. Let $f \in \mathscr{C}^{\widetilde{\Upsilon}}(\mathscr{E})$ and assume that $f \notin \mathscr{C}^{\Upsilon^{k_0}}(\mathscr{E})$. Hence, there are a coalition S and an $\Upsilon^{k_0}(S)$-assignment g such that $g(t,\cdot) \in P_f(t)$ μ-a.e. on S and

$$\int_S g(\cdot,\omega)d\mu = \int_S a(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. Pick $k \in K$ satisfying $\mu(S) \in A_k$. If $k = k_0$, we arrived at a contradiction. Assume now that $k \neq k_0$. By Theorem 4.5, there must exist some coalition $\tilde{S}$ such that $\mu(\tilde{S}) \in A_{k_0}$ and f is Υ^{k_0}-blocked by $\tilde{S}$. Thus, $f \notin \mathscr{C}^{\widetilde{\Upsilon}}(\mathscr{E})$, and this again yields a contradiction. □ □

Remark 4.9. Theorem 4.8 says that the core of $\mathscr{E}$ under the information sharing rule $\widetilde{\Upsilon}$ depends on the finest information sharing rule associated with some threshold. It is also important to note that the theorem depends neither on the number of thresholds nor no the precise thresholds.

5. Coalitional Fairness

In this section, we present an extension of Theorem 2 in Jaskold-Gabszewicz [24] to an asymmetric information economy whose commodity space is an ordered Banach space containing an interior point in its positive cone. The information that each agent can have when she becomes a member of a coalition is susceptible of being altered. It can be noted that the proof of Theorem 3.8 in Graziano and Pesce [22] or Theorem 2 in Jaskold-Gabszewicz [24] contains two parts, but similar techniques are enough to prove both parts. In contrast with them, the proofs of two parts of our result are different. Thus, we plan to decompose the result into two theorems. Since we are dealing with an asymmetric information economy with the exact feasibility condition and an infinite dimensional commodity space, techniques of our results are different from Graziano and Pesce [22] and Jaskold-Gabszewicz [24].

Definition 5.1. An allocation f is called $\mathscr{C}^{\Upsilon}_{(\mathscr{T}_1,\mathscr{T}_0)}$*-fair* if there do not exist two disjoint coalitions[6] $S_1 \in \mathscr{T}_1$, $S_2 \in \mathscr{T}_0$ and an $\Upsilon(S_1)$-assignment g such that $g(t,\cdot) \in P_t(f(t,\cdot))$ μ-a.e. on S_1 and

$$\int_{S_1} (g(\cdot,\omega) - a(\cdot,\omega))d\mu = \int_{S_2} (f(\cdot,\omega) - a(\cdot,\omega))d\mu$$

for each $\omega \in \Omega$.

Theorem 5.2. *Suppose that the assumptions* $(\mathbf{P}_1)$-$(\mathbf{P}_4)$ *are satisfied for an information sharing rule* Υ *and that* $f \in \mathscr{C}^{\Upsilon}(\mathscr{E})$*. Then* f *is* $\mathscr{C}^{\Upsilon}_{(\mathscr{T}_1,\mathscr{T}_0)}$*-fair.*

Proof. On the contrary, suppose that f is not $\mathscr{C}^{\Upsilon}_{(\mathscr{T}_1,\mathscr{T}_0)}$-fair. Thus, there must exist two disjoint coalitions $S_1 \in \mathscr{T}_1$, $S_2 \in \mathscr{T}_0$ and an $\Upsilon(S_1)$-assignment g such that $g(t,\cdot) \in P_t(f(t,\cdot))$ μ-a.e. on S_1 and

$$\int_{S_1} (g - a)d\mu = \int_{S_2} (f - a)d\mu.$$

Since $f \in \mathscr{C}^{\Upsilon}(\mathscr{E})$, one obtains $\mu(S_2) > 0$. Now, Corollary 3.2 yields a $\lambda \in (0,1)$, a $z \in B_{++}$ and an $\Upsilon(S_1)$-assignment h_1 such that $h_1(t,\cdot) \in P_f(t)$ μ-a.e. on S_1 and

$$\int_{S_1} (h_1 - a)d\mu + 7z\mathbf{1}_\Omega = (1-\lambda)\int_{S_1} (g - a)d\mu = (1-\lambda)\int_{S_2} (f - a)d\mu.$$

By Corollary 3.8, one obtains a sub-coalition R_2 of S_2 with $\mathfrak{P}(R_2) = \mathfrak{P}(S_2)$ and an assignment h_2 such that $h_2(t,\cdot) \in P_f(t)$ and $h_2(t,\cdot) - a(t,\cdot)$ is $\Upsilon_t(T)$-measurable μ-a.e. on R_2, and

$$\int_{R_2} (h_2 - a)d\mu = \lambda \int_{S_2} (f - a)d\mu + z\mathbf{1}_\Omega.$$

[6] We allow S_2 to be a null coalition.

As a result, one has

$$\int_{S_1}(h_1-a)d\mu+\int_{R_2}(h_2-a)d\mu+\int_{T\setminus S_2}(f-a)d\mu+6z\mathbf{1}_\Omega=0.$$

Applying Corollary 3.5, one has an assignment x_1 such that $x_1(t,\cdot)\in P_f(t)$ and $x_1(t,\cdot)-a(t,\cdot)$ is $\Upsilon_t(T)$-measurable μ-a.e. on S_1 and

$$\int_{S_1}(x_1-a)d\mu=\frac{1}{2}\int_{S_1}(h_1-a)d\mu+\frac{1}{2}\int_{S_1}(f-a)d\mu+z\mathbf{1}_\Omega.$$

By Corollary 3.8, one obtains a sub-coalition R_3 of R_2 with $\mathfrak{P}(R_3)=\mathfrak{P}(R_2)$ and an assignment h_3 such that $h_3(t,\cdot)\in P_f(t)$ and $h_3(t,\cdot)-a(t,\cdot)$ is $\Upsilon_t(T)$-measurable μ-a.e. on R_3, and

$$\int_{R_3}(h_3-a)d\mu=\frac{1}{2}\int_{R_2}(h_2-a)d\mu+z\mathbf{1}_\Omega.$$

The rest of the proof is decomposed into two cases.

Case 1. $\mu(T\setminus(S_1\cup S_2))=0$. Define $R_4=S_1\cup R_3$ then $\mathfrak{P}(R_4)=\mathfrak{P}(T)$. Thus, f is Υ-blocked by the coalition R_4 via the assignment h_4, defined by

$$h_4(t,\omega)=\begin{cases} x_1(t,\omega), & \text{if } (t,\omega)\in S_1\times\Omega;\\ h_3(t,\omega)+\frac{z}{\mu(R_3)}, & \text{otherwise.}\end{cases}$$

This is a contradiction.

Case 2. $\mu(T\setminus(S_1\cup S_2))\neq 0$. Since $T\setminus(S_1\cup S_2)$ is atomless, by Corollary 3.8, there exist a sub-coalition R_5 of $T\setminus(S_1\cup S_2)$ with $\mathfrak{P}(R_5)=\mathfrak{P}(T\setminus(S_1\cup S_2))$ and an assignment h_5 such that $h_5(t,\cdot)-a(t,\cdot)$ is $\Upsilon_t(T)$-measurable and $h_5(t,\cdot)\in P_f(t)$ μ-a.e. on R_5, and

$$\int_{R_5}(h_5-a)d\mu=\frac{1}{2}\int_{T\setminus(S_1\cup S_2)}(f-a)d\mu+z\mathbf{1}_\Omega.$$

Let $R_6=S_1\cup R_3\cup R_5$ then $\mathfrak{P}(R_6)=\mathfrak{P}(T)$. Define an assignment $h_6:T\times\Omega\to B_+$ by

$$h_6(t,\omega)=\begin{cases} x_1(t,\omega), & \text{if } (t,\omega)\in S_1\times\Omega;\\ h_3(t,\omega), & \text{if } (t,\omega)\in R_3\times\Omega;\\ h_5(t,\omega), & \text{otherwise.}\end{cases}$$

Note that f is Υ-blocked by the coalition R_6 via the assignment h_6, which is again a contradiction. □ □

Definition 5.3. An allocation f is called $\mathscr{C}^{\Upsilon}_{(\mathscr{T}_0,\mathscr{T}_1)}$*-fair* if there do not exist two disjoint non-null coalitions $S_1\in\mathscr{T}_0$, $S_2\in\mathscr{T}_1$ and an $\Upsilon(S_1)$-assignment g such that $g(t,\cdot)\in P_t(f(t,\cdot))$ μ-a.e. on S_1 and

$$\int_{S_1}(g(\cdot,\omega)-a(\cdot,\omega))d\mu=\int_{S_2}(f(\cdot,\omega)-a(\cdot,\omega))d\mu$$

for each $\omega\in\Omega$.

The following assumption is stronger than the assumption $(\mathbf{P}_4)$ and it plays a key role in the proof of the next theorem. It holds under the fine or private information sharing rule whenever $\mathscr{F}_t=\mathscr{P}_t$ for all $t\in T$. Moreover, it is also true if $\Upsilon_t(S)=\mathscr{F}_t$ for all $t\in S$ and $S\in\mathscr{F}_0$. However, it does not hold when $\Upsilon_t(S)$

is the private information for any agent t in some non-null coalition $S \subseteq T_0$, and $\Upsilon_t(T)$ and $\mathscr{F}_t$ are both pooled information for all $t \in T$. Note that in the last case, $(\mathbf{P}_4)$ is satisfied.

$(\mathbf{P}_5)$ For all non-null coalition $S \in \mathscr{T}_0$ and $t \in S$, $\Upsilon_t(S) \succeq \mathscr{F}_t$.

Theorem 5.4. *Suppose that* $(\mathbf{P}_1)$-$(\mathbf{P}_3)$ *and* $(\mathbf{P}_5)$ *are satisfied for an information sharing rule* Υ *and that* $f \in \mathscr{C}^{\Upsilon}(\mathscr{E})$. *Then* f *is* $\mathscr{C}^{\Upsilon}_{(\mathscr{T}_0,\mathscr{T}_1)}$*-fair.*

Proof. On the contrary, suppose that f is not $\mathscr{C}^{\Upsilon}_{(\mathscr{T}_0,\mathscr{T}_1)}$-fair. Then there exist two disjoint non-null coalitions $S_1 \in \mathscr{T}_0$, $S_2 \in \mathscr{T}_1$ and an $\Upsilon(S_1)$-assignment g such that $g(t,\cdot) \in P_t(f(t,\cdot))$ μ-a.e. on S_1 and

$$\int_{S_1} (g-a)d\mu = \int_{S_2} (f-a)d\mu.$$

By Corollary 3.2, one has a $\lambda \in (0,1)$, a $z \in B_{++}$ and an $\Upsilon(S_1)$-assignment h such that $h(t,\cdot) \in P_f(t)$ μ-a.e. on S_1 and

$$\int_{S_1} (h-a)d\mu + 19z\mathbf{1}_\Omega = (1-\lambda)\int_{S_1} (g-a)d\mu.$$

Applying Corollary 3.7, one can find a sub-coalition R_1 of S_1 and an $\Upsilon(R_1)$-assignment g_1 such that $\mu(R_1) = \lambda\mu(S_1)$, $\mathfrak{P}(R_1) = \mathfrak{P}(S_1)$, $g_1(t,\cdot) \in P_f(t)$ μ-a.e. on R_1, and

$$\int_{R_1} (g_1-a)d\mu = \lambda\int_{S_1} (g-a)d\mu + z\mathbf{1}_\Omega.$$

Combining above two equations, one has

$$\int_{S_1} (h-a)d\mu + \int_{R_1} (g_1-a)d\mu + 18z\mathbf{1}_\Omega = \int_{S_1} (g-a)d\mu.$$

Since $\mathfrak{P}(R_1) = \mathfrak{P}(S_1)$, h is an $\Upsilon(R_1)$-assignment. Thus, Corollary 3.4 implies that there must exist an $\Upsilon(R_1)$-assignment h_1 such that $h_1(t,\cdot) \in P_f(t)$ μ-a.e. on R_1 and

$$\int_{R_1} (h_1-a)d\mu = \frac{1}{2}\int_{R_1} (h-a)d\mu + \frac{1}{2}\int_{R_1} (g_1-a)d\mu + z\mathbf{1}_\Omega.$$

By Lemma 3.6, one has a sub-coalition R_2 of $S_1 \setminus R_1$ and an assignment h_2 such that $h_2(t,\cdot) \in P_f(t)$ and $h_2(t,\cdot) - a(t,\cdot)$ is $\Upsilon_t(S)$-measurable μ-a.e. on R_2, and

$$\int_{R_2} (h_2-a)d\mu = \frac{1}{2}\int_{S_1\setminus R_1} (h-a)d\mu + z\mathbf{1}_\Omega.$$

Thus, one concludes that

$$\int_{R_1} (h_1-a)d\mu + \int_{R_2} (h_2-a)d\mu + 7z\mathbf{1}_\Omega = \frac{1}{2}\int_{S_2} (f-a)d\mu.$$

Let $R_3 = R_1 \cup R_2$ and define an assignment $h_3 : T \times \Omega \to B_+$ such that

$$h_3(t,\omega) = \begin{cases} h_1(t,\omega), & \text{if } (t,\omega) \in R_1 \times \Omega; \\ h_2(t,\omega), & \text{if } (t,\omega) \in R_2 \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

Note that $\mathfrak{P}(R_3) = \mathfrak{P}(S_1)$ and h_3 is an $\Upsilon(R_3)$-assignment satisfying $h_3(t,\cdot) \in P_f(t)$ μ-a.e. on R_3. Moreover,

$$\int_{R_3} (h_3 - a)d\mu + 7z\mathbf{1}_\Omega = \frac{1}{2}\int_{S_2} (f-a)d\mu.$$

If $\int_{S_2}(f-a)d\mu = 0$ then f is Υ-blocked by the coalition R_3 via the assignment $y : T \times \Omega \to B_+$, defined by

$$y(t,\omega) = \begin{cases} h_3(t,\omega) + \frac{7z}{\mu(R_3)}, & \text{if } (t,\omega) \in R_3 \times \Omega; \\ g(t,\omega), & \text{otherwise,} \end{cases}$$

which is a contraction with the fact that $f \in \mathscr{C}^\Upsilon(\mathscr{E})$. So, $\int_{S_2}(f-a)d\mu \neq 0$ which means $\mu(T \setminus S_2) > 0$. In this case,

$$\int_{R_3} (h_3 - a)d\mu + \frac{1}{2}\int_{T\setminus S_2} (f-a)d\mu + 7z\mathbf{1}_\Omega = 0.$$

It follows from $(\mathbf{P}_5)$ that f is an $\Upsilon(T \setminus S_2)$-assignment. Applying Corollary 3.7, the above equality can be expressed as

$$\int_{R_3} (h_3 - a)d\mu + \int_{R_4} (h_4 - a)d\mu + 6z\mathbf{1}_\Omega = 0$$

for some sub-coalition R_4 of $T \setminus S_2$ with $\mathfrak{P}(R_4) = \mathfrak{P}(T \setminus S_2)$ and $\Upsilon(R_4)$-assignment h_4 satisfying $h_4(t,\cdot) \in P_f(t)$ μ-a.e. on R_4, and

$$\int_{R_4} (h_4 - a)d\mu = \frac{1}{2}\int_{T\setminus S_2} (f-a)d\mu + z\mathbf{1}_\Omega.$$

The rest of the proof is decomposed into two cases.

Case 1. $\mu(R_3 \cap R_4) = 0$. Thus, f is Υ-blocked by $R_3 \cup R_4$ via the assignment $\hat{y}$, defined as

$$\hat{y}(t,\omega) = \begin{cases} h_3(t,\omega) + \frac{6z}{\mu(R_3)}, & \text{if } (t,\omega) \in R_3 \times \Omega; \\ h_4(t,\omega), & \text{otherwise,} \end{cases}$$

which is a contradiction.

Case 2. $\mu(R_3 \cap R_4) \neq 0$. Since $R_3 \subseteq T \setminus S_2$, $h_3(t,\cdot) - a(t,\cdot)$ is $\Upsilon_t(T \setminus S_2)$-measurable μ-a.e. on R_3. In this case, there are three possibilities: (i) $\mu(R_3 \setminus R_4) \neq 0$ and $\mu(R_4 \setminus R_3) \neq 0$; (ii) exactly one of $\mu(R_3 \setminus R_4)$ and $\mu(R_4 \setminus R_3)$ is 0; and (iii) $\mu(R_3 \setminus R_4) = 0$ and $\mu(R_4 \setminus R_3) = 0$. We only work on the possibility (i) and others can be done analogously. Applying Lemma 3.3 for the coalition $R_3 \cap R_4$, Lemma 3.6 for coalitions $R_3 \setminus R_4$ and $R_4 \setminus R_3$, one can find three sub-coalitions

$$R_5 = R_3 \cap R_4,\ R_6 \subseteq R_3 \setminus R_4,\ R_7 \subseteq R_4 \setminus R_3$$

with

$$\mathfrak{P}(R_6) = \mathfrak{P}(R_3 \setminus R_4) \text{ and } \mathfrak{P}(R_7) = \mathfrak{P}(R_4 \setminus R_3)$$

and three assignments h_i for $i = 5, 6, 7$ such that $h_i(t,\cdot) - a(t,\cdot)$ is $\Upsilon_t(T \setminus S_2)$-measurable and $h_i(t,\cdot) \in P_f(t)$ μ-a.e. on R_i for $i = 5, 6, 7$ and

$$\sum_{i=5}^{7} \int_{R_i} (h_i - a)d\mu = 0.$$

Put, $R = R_5 \cup R_6 \cup R_7$ and note that $\mathfrak{P}(R) = \mathfrak{P}(T \setminus S_2)$. Thus, f is Υ-blocked by R via the assignment $y : T \times \Omega \to B_+$, defined by

$$y(t, \omega) = \begin{cases} h_i(t, \omega), & \text{if } (t, \omega) \in R_i \times \Omega,\ i = 5, 6, 7; \\ g(t, \omega), & \text{otherwise,} \end{cases}$$

which is again a contradiction. □ □

The following definition and theorem are extensions of those in Jaskold-Gabszewicz [24] to an asymmetric information economy.

Definition 5.5. An allocation f is said to be $\mathscr{C}^{\Upsilon}$*-fair* relative to $\mathscr{T}_0$ and $\mathscr{T}_1$ if it is $\mathscr{C}^{\Upsilon}_{(\mathscr{T}_0, \mathscr{T}_1)}$-fair and $\mathscr{C}^{\Upsilon}_{(\mathscr{T}_1, \mathscr{T}_0)}$-fair. The set of such allocations is denoted by $\mathscr{C}^{\Upsilon}_{\{\mathscr{T}_0, \mathscr{T}_1\}}(\mathscr{E})$.

Theorem 5.6. *Assume the assumptions* $(\mathbf{P}_1)$-$(\mathbf{P}_3)$ *and* $(\mathbf{P}_5)$ *are satisfied for an information sharing rule* Υ. *Then* $\mathscr{C}^{\Upsilon}(\mathscr{E}) \subseteq \mathscr{C}^{\Upsilon}_{\{\mathscr{T}_0, \mathscr{T}_1\}}(\mathscr{E})$.

Proof. Let $f \in \mathscr{C}^{\Upsilon}(\mathscr{E})$. Applying Theorem 5.2 and Theorem 5.4, one has f is both $\mathscr{C}^{\Upsilon}_{(\mathscr{T}_1, \mathscr{T}_0)}$-fair and $\mathscr{C}^{\Upsilon}_{(\mathscr{T}_0, \mathscr{T}_1)}$-fair. So, $f \in \mathscr{C}^{\Upsilon}_{\{\mathscr{T}_0, \mathscr{T}_1\}}(\mathscr{E})$, and this completes the proof. □ □

6. Conclusion

In this section, we compare our results to those in others and provide some open questions.

Remark 6.1. Since the assumptions $(\mathbf{P}_1)$-$(\mathbf{P}_4)$ are satisfied trivially under the fine and private information sharing rules if $\mathscr{F}_t = \mathscr{P}_t$ for all $t \in T$, Vind's theorem in the case of the fine core and the private core in Bhowmik and Cao [6] are particular cases of Theorem 4.5 in our paper. Note that Vind-type theorem for the weak fine core is also obtained as a corollary of Theorem 4.5 in our paper. In addition, Grodal's theorem in Bhowmik and Cao [6] is obtained as a special case of our Theorem 4.3. However, it is unclear to the author that whether a similar result is true in an asymmetric information economy with a Banach lattice as the commodity space and the feasibility is defined as exact. Recently, extensions of the main results in [23] and [35] to mixed economies were established in [10] and [29]. These results deal with the Aubin coalitions. Since the purpose of our paper is to analyze the standard core notion, we restrict our attention to atomless economies in Section 4. However, interested reader can consider all our results in Section 4 to the case of a mixed economy under any information sharing rule.

Remark 6.2. In an asymmetric information economy with a continuum of non-atomic agents $[0, 1]$, consider the following information sharing rules.

$$\widetilde{\Upsilon}^1_t(S) = \begin{cases} \mathscr{P}_t, & \text{if } \mu(S) < \varepsilon; \\ \bigwedge\{\mathscr{P}_t : t \in S\}, & \text{if } \mu(S) \geq \varepsilon, \end{cases}$$

$$\widehat{\Upsilon}^1_t(S) = \begin{cases} \bigwedge\{\mathscr{P}_t : t \in S\}, & \text{if } \mu(S) < \varepsilon; \\ \mathscr{P}_t, & \mu(S) \geq \varepsilon, \end{cases}$$

$$\widetilde{\Upsilon}^2_t(S) = \begin{cases} \bigvee\{\mathscr{P}_t : t \in S\}, & \text{if } \mu(S) < \varepsilon; \\ \mathscr{P}_t, & \text{if } \varepsilon \leq \mu(S) \leq \delta; \\ \bigwedge\{\mathscr{P}_t : t \in S\}, & \text{if } \mu(S) > \delta, \end{cases}$$

$$\widehat{\Upsilon}^2_t(S) = \begin{cases} \bigwedge\{\mathscr{P}_t : t \in S\}, & \text{if } \mu(S) < \varepsilon; \\ \mathscr{P}_t, & \text{if } \varepsilon \leq \mu(S) \leq \delta; \\ \bigvee\{\mathscr{P}_t : t \in S\}, & \text{if } \mu(S) > \delta, \end{cases}$$

where $0 < \varepsilon < \delta < 1$. Note that if $\mathscr{F}_t = \mathscr{P}_t$ for all $t \in T$, then the private information sharing rule satisfies $(\mathbf{P}_1)$-$(\mathbf{P}_4)$. Thus, it follows from Theorem 4.8 that $\mathscr{C}^{\widetilde{\Upsilon}^1}(\mathscr{E}) = \mathscr{C}^{\widehat{\Upsilon}^1}(\mathscr{E})$ is the private core of $\mathscr{E}$. On the other hand, if $\mathscr{F}_t = \mathscr{P}_t$ or $\bigvee\{\mathscr{P}_t : t \in T\}$, then the fine information sharing rule satisfies $(\mathbf{P}_1)$-$(\mathbf{P}_4)$. As a consequence, Theorem 4.8 claims that $\mathscr{C}^{\widetilde{\Upsilon}}(\mathscr{E}) = \mathscr{C}^{\widehat{\Upsilon}}(\mathscr{E})$ is the fine (resp. weak fine) core of $\mathscr{E}$ if $\mathscr{F}_t = \mathscr{P}_t$ (resp. $\bigvee\{\mathscr{P}_t : t \in T\}$) for all $t \in T$.

Remark 6.3. It is known that Vind's theorem or its extensions in general equilibrium theory have been employed to establish characterizations of the core in terms of non-dominated allocations and relations among several cores, refer to [7, 8, 14, 15, 19, 20]. It would be interesting to know whether those results can be obtained using Theorem 4.5 under the framework of any information sharing rule.

Remark 6.4. Comparing to Hervés-Beloso et al. [21], we additionally use the assumption $(\mathbf{P}_3)$ to obtain the main results in Section 4, which was not the case in Hervés-Beloso et al. [21]. This assumption has played vital roles in the proofs of our results in Section 4. All these results are technically different from those in Hervés-Beloso et al. [21]. In addition, we extend the main result in Grodal [23] to an asymmetric information economy where each agent's information is given by information sharing rules, which was not established in Hervés-Beloso et al. [21].

Remark 6.5. We now compare our assumptions to those in Graziano and Pesce [22]. Note that assumptions for initial endowments and utility functions in Graziano and Pesce [22] and our paper are similar. Moreover, the set of allocations of Theorem 3.8 in Graziano and Pesce [22] was required to satisfy a certain property. More precisely, for every allocation $f : T \times \Omega \to \mathbb{R}^\ell_+$ in Theorem 3.8 in their paper there is some $1 \leq j \leq \ell$ such that the j^{th}-coordinate $f^j(t, \omega) > 0$ μ-a.e. and all $\omega \in \Omega$. This restriction is not employed in our results. Further, the main result in Section 5 is technically different from that in Graziano and Pesce [22] and is valid in an asymmetric information economy whose commodity space is either the finite dimensional space or an infinite dimensional space having an interior point in its positive cone. It is also valuable to mention that an extension of Theorem 2 in Jaskold-Gabszewicz [24] to an asymmetric information economy with the exact feasibility condition first appears in our paper. Note also that Theorem 5.6 in this paper is the first extension of Theorem 2 in Jaskold-Gabszewicz [24] to an infinite dimensional framework.

Remark 6.6. Our fairness concept deals with the net trade allocation. However, some other concepts of fairness have been introduced without the notion of net trade. Firstly, Foley [17] proposed a concept of fair allocation which is efficient and satisfies the condition that each agent prefers to keep her own bundle rather than to receive bundles of other agents. In an exchange economy, such an allocation exists as shown by Varian in [34]. Differently from Jaskold-Gabszewicz [24], Varian [34] also introduced the notion of a coalitionally fair allocation. According to the definition in [34], an allocation is coalitionally fair if no coalition envies the aggregate bundle of other coalition of the same size or smaller. Besides, Zhou [38] proposed the concept of a strictly fair allocation. In this paper, we study the notion of a coalitionally fair allocation given in [24]. It would be interesting to work on other fairness notions in an asymmetric information economy under information sharing rules.

Remark 6.7. Recently, the formulation of a maximin expected utility becomes well known and it is defined as

$$\underline{U}_t(\omega, x) = \min\{U_t(\omega', x(\omega')) : \omega' \in \mathscr{P}_t(\omega)\}.$$

In this context, de Castro et al. [12] introduced the concept of the maximin core. One can analogously define the notion of a coalitionally fair allocation and verify whether results similar to our results are true in the framework of a maximin expected utility.

Remark 6.8. We conclude this section with a discussion about the coalitionally incentive compatibility of our notions. Koutsogeras and Yannelis [25] showed that the private core is weak coalitionally incentive compatible in an asymmetric information economy with finitely many agents. A similar concept of weak coalitionally incentive compatibility can be defined in a mixed economy under the information sharing rules. It can be checked that a technique similar to Theorem 4.1 in Koutsogeras and Yannelis [25] is enough to show that the core under an information sharing rule is weak coalitionally incentive compatible. In the framework of a maximin expected utility, de Castro et al. [12] introduced the notion of coalitionally incentive compatibility similar to that in Koutsogeras and Yannelis [25]. They showed that the maximin core is maximin coalitionally incentive compatible when the number of agents is finite. Note that the above concept of a maximin expected utility can be extended to the case of any information sharing rule Υ by replacing $\mathscr{P}_t(\omega)$ with the atom of $\Upsilon_t(S)$ containing ω for any non-null coalition S. Thus, by invoking the arguments of de Castro et al. [12], one can show that a variation of the maximin core under an information sharing rule is maximin coalitionally incentive compatible under the same information sharing rule in a mixed economy. However, at this stage, it is unclear to the author that whether the coalitionally incentive compatibility of any coalitionally fair allocation can be established in the above two frameworks.

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