Equivalence Nucleolus for Partition Function Games

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Abstract

In coalitional game theory, the issue of stability in presence of externalities has been given very little attention. Moreover, there is no solution concept in the literature which guarantees non-emptiness of the set of stable outcomes under this environment. Using the partition function form representation, we propose a new solution concept which is unique and always non-empty. It is also proved that if payoff distribution rule is an equivalence relation, non-emptiness is always guaranteed.

1 Introduction

In coalitional game theory, *stability* is the answer often associated with one of the fundamental questions-*what coalitions will form*? In a vast majority of literature, it is assumed that the payoff of a coalition is independent of the structure of outside coalitions existing in the system and there are many famous solution concepts to analyze the stability e.g. the core, the nucleolus, the bargaining set and the kernel. This assumption does not seem to be realistic in many socio-economic applications and therefore it is important to consider externalities. To capture such situations, the *partition function games* (Thrall and Lucas, 1963), in which each coalition is assigned a payoff depending on the entire coalition structure, are widely used to develop solution concepts. Extensions of the core to partition function games have been proposed in the literature to analyze the stability of coalitional games with externalities. In such games, a coalition can have more than one value depending on how the outside players partition themselves, and while

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testing the coalitional deviations in the core, certain behavioral assumptions like optimism and pessimism about the reaction of outside players are made. Such assumptions lead to different outcomes and lack of consistency. This issue is well taken in the γ-core (Chander and Tulkens, 1997) which is based on the individual's best strategy residual players, and further improved in the r-core (Huang and Sjostrom, 2003) and the recursive core (Koczy, 2007) which allow arbitrary reactions. Nevertheless, it is well known that the core of coalitional games in presence of externalities can be empty ((Koczy, 2007);(Funaki and Yamato, 1999)) and attempts are being made to develop a solution concept which can guarantee non-emptiness of the stable outcomes (McCain, 2009). Other classical solution concepts like the bargaining set (Aumann and Maschler, 1961) and the nucleolus (Schmeidler, 1969), which are always nonempty for games without externalities, have also begun to receive attention for the extension to the games with externalities. This is the basic premise of our work. In our model, we introduce a concept, called *bargaining power* and define a payoff division rule, called *equality of satisfaction values* to obtain stable outcome. The payoff division rule is motivated from the egalitarian solution of a two person bargaining problem by (Myerson, 2013), which is guided by the *equal gain principle*. Here the players with equal satisfaction value come together to form a coalition but two players with different satisfaction values can never be a part of the same coalition. The division rule is first proved to be an equivalence relation and using the fundamental theorem of equivalence relation, it is proved that the stable outcome is always nonempty. It is also shown that any division rule, if it is an equivalence relation, always gives a non-empty stable outcome.

In the subsequent sections, we discuss preliminaries and some key definitions in Section 2, and present the model in Section 3. We illustrate the model through numerical examples in Section 4 and show some important results in Section 5.

2 Preliminaries

A finite set of players N is given. A coalition C is a subset of N. Structuring of N into a set of disjoint coalitions is called partition of N, denoted by P. An embedded coalition over N is a pair of the form (C, P) where $C \in P$. E_N the set of all embedded coalitions over N. A *characteristic function* $v : 2^N \mapsto \mathbb{R}$, associates with each coalition $C \subseteq N$, a real valued payoff $v(C)$ that the coalition's members can distribute among themselves. Also $v(\emptyset) = 0$. A *partition function* $u : E_N \mapsto \mathbb{R}$ is a mapping that assigns a real number $u(C, P)$ to each embedded coalition (C, P). A *characteristic function game (CFG)* is represented as a pair $(N, v(C))$ and a *partition function game (PFG)* is represented as a pair $(N, u(C, P))$.

2.1 Key Definitions

If $P = \{C_1, C_2, \cdots, C_r, \phi\}$ and $Q = \{B_1, B_2, \cdots, B_s, \phi\}$ are two partitions and $\forall i = 1, 2, \cdots, s$ $\exists k \in \{1, 2, \cdots, r\}$ such that $C_k \neq \phi$, then Q is called the *refinement* of partition P, if $B_i \subset C_k \in P$. For any partition P and a coalition $S \notin P$, *residual partition* of P with respect to coalition S, denoted by P'_S, is given by $P'_S = \{C | \exists B \in P \text{ such that } C = B - S\} \cup \{S\}$. A vector of payments $x = (x_1, x_2, \cdots, x_N)$ to the players of a game $G = \{N, u(C, P)\}$ is *admissible* to partition P, if $\forall S \in P, \sum_{i \in S} x_i = x_S = u(S, P)$. A payoff vector $x \in \mathbb{R}^n$ is *individually rational* [1] payoff vector, if $\forall i \in N, x_i \geq \min u(\{i\}, P')$ where $P' \in P_{N-\{i\}} \cup \{i\}$. *Imputation set* $I(G, x)$ is a set of payment vectors $x \in \mathbb{R}^n$ of a game $G = \{N, u(C, P)\}$, if x is admissible and $x \geqslant_{lex} y$,[2] where y is the individually rational payoff vector to the game G, with its elements sorted in increasing order. A *payoff configuration* to a game G, is a pair (P, x) where P is a partition and x is an imputation corresponding to P. An *outcome* of a game (N, u) is a payoff configuration (P, x) to that game.

3 The Solution Concept

3.1 Bargaining Power

Consider a partition function game $(N, u(C, P))$ and a payoff configuration (P, x) associated with it. Any deviation of a set of players $S \notin P$ leads to the residual partition P'_S. If $\sum_{i \in S} x_i < u(S, P'_S)$, then the players constituting S have incentive to deviate from their affiliations in P and form P'_S. P'_S puts externalities on residual coalitions, thereby changing their potential payoffs which they

[1] Individual rationality vs Participation rationality (Koczy, 2007): Participation rationality assumes $x_i \geq 0 \forall i \in N$. We define individual rationality for PFF games which is along the line of its classical definition. However, the main purpose of individually rational payoff vector is to provide the greatest lower bound of imputations

[2] $\geqslant_{lex}$ denotes the lexicographical ordering. If $(x, y) \in \mathbb{R}^m$, then $x \geqslant_{lex} y$ iff, $x = y$ or $\exists t s.t. 1 \leq t \leq m$ and $\forall i s.t. 1 \leq i \leq t, x_i = y_i$ and $x_t \geq y_t$.

can generate now. This provokes residual players to restructure themselves which may not be good for S. Therefore, the players in S would not deviate, if there is a scope of loosing due to residual players' actions. In short, a deviation is not credible if it can be nullified or countered. We consider that credible deviation of a player reflects its influence in a game. Hence, for every $i \in N$ in a game $(N, u(C, P))$, *bargaining power* of player i, denoted by B_i, is defined as a real number which a player assigns itself as a measure of its influence in the game. It is an intrinsic value of each player in the game in a sense that, it does not depend on partitions.

Definition 1 (Objection). *Let (P, x) be a payoff configuration to a game G. Also $C_P(i) = C_j$ such that $i \in C_j$ and $C_j \in P$ where all C_j(s) are disjoint sets. An objection of i against $C_j - \{i\}$ in the first refinement of P with respect to i, will lead to a payoff configuration (P'_S, y) consists of the residual partition of P with respect to S such that $i \in S$, $S \notin P, S \in P'_S$ and a payoff vector y admissible to it. For which, $\sum_{i \in S} y_i \leq u(S, P'_S)$. Also $\forall k \in S, y_k \geq x_k$ and $y_i > x_i$.*

Definition 2 (Counter-objection). *Let (P, x) be a payoff configuration for a game G, and (P'_S, y) be an objection of i against $C_P(i) - \{i\}$ in the residual partition of P with respect to S, a counter-objection of any coalition $T \subset N - S$, where $T \notin P'_S$ and $T \in R$, against S is a payoff configuration (R, z), such that $\sum_{i \in T} z_i \leq u(T, R)$. Also $\forall k \in T, z_k \geq y_k$ and $\exists k \in S$ such that $z_k < x_k$.*

Following steps are followed to compute the bargaining power of a player:

1. **Step 1:** Choose a payoff configuration (P, x) randomly. Check for any objection (P'_S, y) to it, such that $i \in S, S \notin P, S \in P'_S$ and $\sum_{i \in S} y_i \leq u(S, P'_S)$, $y \in \mathbb{R}^n$. Also $\forall i \in S, y_i \geq x_i$ and $\exists k \in S$ such that $y_k > x_k$. If there is no objection, go to step 3.

2. **Step 2:** If (P'_S, y) is an objection to (P, x), check if there exists a counter-objection (R, z) such that, $\exists T \subset N - S, T \notin P'_S$ and $\sum_{i \in T} z_i \leq u(T, R)$, $z \in \mathbb{R}^n$. Also $\forall i \in T, z_i \geq y_i$ and $\exists k \in S$ such that $z_k < x_k$. If counter-objection exists, neglect the objection as it is not a credible objection.

3. **Step 3:** Repeat the above steps for all the given partitions, unless all the possibilities are exhausted and there is no credible objection.

4. **Step 4:**If (P, x) is the only payoff configuration with no credible objection, then the lower limit of each element of the payoff vector x is equal to the bargaining power of the corresponding player. In case of multiple such payoff configurations, choose the one with the most refined partition.

3.2 Equivalence Nucleolus

For a given partition function game $(N, u(C, P))$ and a payoff configuration (P, x) associated with it, we introduce some terms as follows. *Satisfaction value* of player i, denoted by s_i, is defined as $s_i = x_i - B_i$. Due to admissibility constraint, x_i changes with change in partition. Hence the same player can have different satisfaction values under different partitions. For a given partition P_j of N, a sequence of satisfaction values of all players, denoted by $e(P_j)$, is defined as $e(P_j) =< s_1, s_2, \cdots, s_n >$. The *most preferred sequence* of satisfaction values is the one which is lexicographically maximal among all such sequences when their elements are sorted in non-decreasing order.

Definition 3 (Equality of Satisfaction Values)**.** *According to the rule of equality of satisfaction values -"the satisfaction values of all the players within the same coalition should be same".*

For illustration,let there be a coalition consists of two players - 1 and 2. The bargaining powers of 1 and 2 are B_1 and B_2 respectively.x_1 and x_2 are the payoffs received by the players 1 and 2 respectively. According to the rule of equality of satisfaction values, $(x_1 - B_1) = (x_2 - B_2)$. Here we like to mention that for a two players coalition, our proposed division rule becomes very similar to the egalitarian solution guided by the *equal gain principle* for a two person bargaining problem Myerson (2013). The rule of equality of satisfaction values can be considered as a generalization of the equal gain principle for more than two players.

Definition 4 (Justifiable Outcome)**.** *For a given partition function game $(N, u(C, P))$, a payoff configuration (P, x) is said to be justifiable, if $\forall C \in P$ and $\forall i, j \in C$, where $i \neq j$, $s_i = s_j$; Also $\forall i, j \in N$ such that $i \in C$ and $j \notin C$, $s_i \neq s_j$.*

The rule essentially says that in a justifiable outcome, the payoff should be distributed in such a manner that the satisfaction values of each player within a coalition is equal and no two players across different coalitions have the same satisfaction values. Thus the players with same satisfaction value come together, segregating themselves in various coalitions and thereby forming a partition.

Definition 5 (Equivalence Nucleolus)**.** *A payoff configuration (P, x) is called equivalence nucleolus of a game $(N, u(C, P))$, if it is justifiable and the payoff vector associated with it constitutes the most preferred sequence.*

Definition 6 (Stable Outcome)**.** *An outcome (P, x) of a game $(N, u(C, P))$ is stable if it coincides with the equivalence nucleolus.*

Following steps are followed to compute the equivalence nucleolus of a game:

1. **Step 1:** Get the bargaining power B_i, $\forall i \in N$ as the initial solution.

2. **Step 2:** Choose any partition P_k and let x be a payoff vector associated with it. Divide $u(S_r, P_k)$ $\forall r$ where $\bigcup_r S_r = P_k$, in such a way that, $(x_i - B_i) = (x_j - B_j)$, $\forall i, j \in S_r$, where $i \neq j$ and $(x_i - B_i) \neq (x_j - B_j)$, if either of i or j does not belong to S_r, along with the admissibility constraint, $\sum_{i \in S_r} x_i = u(S_r, P_k)$, compute x. Repeat the above steps for all the given partitions of N.

3. **Step 3:** Write the sequence of satisfaction values $(x_i - B_i)$ for all players in all partitions. For every sequence, sort the satisfaction values in non-decreasing order. Choose the sequence which is lexicographically maximal. This is the most preferred sequence.

4. **Step 4:** Select the payoff configuration corresponding to the most preferred sequence. This is the equivalence nucleolus.

4 Numerical Examples

Example 1

Let $N = \{1, 2, 3, 4\}$ be a set of 4 players and u be the partition function such that:

$$u(123, 4) = (7, 0)$$

$$u(12, 34) = (4.4, 4.4)$$

$$u(12, 3, 4) = (3, 3, 3)$$

$$u(1, 2, 34) = (3, 3, 3)$$

$$u(1, 2, 3, 4) = (2, 2, 2, 2)$$

Payoffs not indicated here are all zero [3]. For the sake of simplicity in representation, here we use abbreviated notations [4] for coalitions, partitions and payoffs.

Solution:Equivalence nucleolus is computed in two stages - (i) The bargaining power of every player participating in the game is calculated, and (ii) The division rule is applied to divide the payoffs. For the given game, the bargaining power of the players are found to be $B_1 = 2, B_2 = 2, B_3 = 2.6$ and $B_4 = 1.4$. Now we divide the payoffs of coalitions in such a way that the satisfaction values all players within a coalition are equal, Then we find the sequences of satisfaction values and choose the one which is lexicographically maximal. The following table describes the computation.

Table 1: Finding Equivalence Nucleolus

Partition	Collective payoffs	Payoff vector	Satisfaction sequence
$(123, 4)$	$(7, 0)$	$(32/15, 32/15, 41/15, 0)$	$(0.13, 0.13, 0.13, -1.4)$
$(12, 34)$	$(4.4, 4.4)$	$(2.2, 2.2, 2.8, 1.6)$	$(0.2, 0.2, 0.2, 0.2)$
$(12, 3, 4)$	$(3, 3, 3)$	$(1.5, 1.5, 3, 3)$	$(-0.5, -0.5, 0.4, 1.6)$
$(1, 2, 34)$	$(3, 3, 3)$	$(3, 3, 2.1, 0.9)$	$(1, 1, -0.5, -0.5)$
$(1, 2, 3, 4)$	$(2, 2, 2, 2)$	$(2, 2, 2, 2)$	$(0, 0, -0.6, 0.6)$

It is trivial that a sequence containing all non-negative satisfaction values will be preferred over the one which contains at least one negative satisfaction value.In this example, there is only one sequence which contains all non-negative satisfaction values and that is corresponding to the partition $(12, 34)$ with the payoff vector $(2.2, 2.2, 2.8, 1.6)$. Hence equivalence nucleolus of

[3]The data is taken from Koczy (2003) for comparison purpose.

[4]For example: $u(12, 34) = (4.4, 4.4)$ represents the following: The partition is $(\{1, 2\}, \{3, 4\})$, which contains two coalitions $\{1, 2\}$ and $\{3, 4\}$. u is the partition function such that $u(\{1, 2\}, (\{1, 2\}, \{3, 4\})) = 4.4$ and $u(\{3, 4\}, (\{1, 2\}, \{3, 4\})) = 4.4$

the game is the payoff configuration $((12,34),(x_1 = 2.2, x_2 = 2.2, x_3 = 2.8, x_4 = 1.6))$. The approach followed by Koczy (2003) gives the core outcome of the game as $((12,34),x)$ where $x = (x_1, x_2, x_3, x_4)$ satisfies $x_1, x_2, x_4 \geq 2$ and $x_3 \geq 2.6$. However, we notice that the core outcome of this game may not exist. For instance $x_3 = 2.6$ and $x_4 = 2$ lies in the core but it violates admissibility constraint, because in this case $x_3 + x_4 = 4.6$ which cannot exceed $u(\{3,4\},(\{1,2\},\{3,4\})) = 4.4$. Classically, it is similar to efficiency condition which is one of the reasonable requirements of a solution concept.

Example 2

Consider a Cournot oligopoly market in which the firms produce output at unit cost of c and face a linear demand function $p = A - bx$. The firms are free to form coalitions among themselves and the profit each coalition accruing is given by, $D/(m+1)^2$, where $D = (A-c)^2/b$ and m is the number of coalitions into which the firms existing in the market partition themselves.[5]

Representing the game in the partition function form: Let $N = \{1,2,3\}$ be a set of 3 firms existing in the market and u be the partition function, then the payoffs, in the abbreviated form, could be written as

$$u(123) = (D/4)$$

$$u(1,2,3) = (D/16, D/16, D/16)$$

$$u(i,jk) = (D/9, D/9) \forall i,j,k \in N$$

Solution: Equivalence nucleolus is computed in two stages - (i) The bargaining power of every firm is calculated, and (ii) The division rule is applied to divide the payoffs. The bargaining power of the players are found to be $B_1 = D/12, B_2 = D/12$ and $B_3 = D/12$ (Calculation method: In $(1,2,3)$ each player gets $D/16$ which has an objection as (123) because each player in the objection can get a better value of $D/12$. Further (123) could have an objection in the form of (i,jk), in which i better off with a value of $D/9$, but this partition has a counter-objection in the form of $(1,2,3)$, which already had a credible objection in the form of (123) with $x_1 = D/12, x_2 = D/12, x_3 = D/12$). Table 2 describes the computation of payoff vectors

[5] This example is taken from Ray (2007)

Table 2: Finding Equivalence Nucleolus

Partition	Collective payoffs	Payoff vector	Satisfaction sequence
(123)	$(D/4)$	$(D/12, D/12, D/12)$	$(0,0,0)$
$(1,2,3)$	$(D/16, D/16, D/16)$	$(D/16, D/16, D/16)$	$(-D/48, -D/48, -D/48)$
(i, jk)	$(D/9, D/9)$	$(D/9, D/18, D/18)$	$(D/36, -D/36, -D/36)$

and satisfaction sequences. Here $(0,0,0)$ is the only sequence with all non-negative satisfaction values, hence this is lexicographically maximal among all the sequences. Hence equivalence nucleolus of the game is the payoff configuration corresponding to the sequence $(0,0,0)$, which is $((123),(x_1 = D/12, x_2 = D/12, x_3 = D/12))$. Our solution coincides with the solution of Ray (2007).

5 Results

Lemma 1. *A relation R described as "equality of satisfaction values", on a set N, is an equivalence relation.*

Proof. Let R be the relation on the set N, defined as aRb, if a has same satisfaction value as b. *Reflexive*: Suppose $a \in N$, then $(x_a - B_a) = (x_a - B_a)$. Hence R is reflexive. *Symmetric*: Suppose $a, b \in N$ and aRb, then $(x_a - B_a) = (x_b - B_b) \Rightarrow (x_b - B_b) = (x_a - B_a) \Rightarrow bRa$. Hence R is symmetric. *Transitive*: Suppose $a, b, c \in N$, aRb and bRc, then $aRb \Rightarrow (x_a - B_a) = (x_b - B_b)$ and $bRc \Rightarrow (x_b - B_b) = (x_c - B_c)$. It implies that $(x_a - B_a) = (x_c - B_c) \Rightarrow aRc$. Hence R is transitive. Since R is reflexive, symmetric and transitive, it is an equivalence relation. □

Theorem 1. *Equivalence nucleolus always exists.*

Proof. According to the fundamental theorem of equivalence relation, an equivalence relation on a set N induces a partition of N. Therefore if the function generating the sequence of satisfaction values is an equivalence relation (which is always the case according to Lemma 1), there exists at least one partition.It is trivial that this property would not disappear due to lexicographical ordering of sequences and choosing the maximal sequence, which constitutes the equivalence nucleolus. □

Corollary 1. *Any division rule, if it is an equivalence relation, always gives a non-empty stable outcome. However, uniqueness may not be guaranteed.*

Theorem 2. *Equivalence nucleolus is unique.*

Proof. Suppose equivalence nucleolus is not unique, then we have two different partitions $P1$ and $P2$, such that $e(P1) = e(P2) \Rightarrow \langle s(i) \rangle_{P1} = \langle s(j) \rangle_{P2} \Rightarrow (s(i))_{P1} = (s(j))_{P2}\ \forall i, j \in N \Rightarrow i, j \in S$, where $S \in P1, P2$, because S is an equivalence class. This implies that $P1 = P2$. Two sets are equal if they both have the same members. It contradicts our initial assumption that $P1$ and $P2$ are different partitions. Hence the equivalence nucleolus is unique. $\square$

6 Conclusion

This paper considers the issue of stability of coalitional games with externalities, using the partition function form representation of such games. We first define some terminologies and then propose a solution concept called equivalence nucleolus which draws its motivation from the classical nucleolus given by Schmeidler (1969). We prove that the equivalence nucleolus is unique and always non-empty. We like to mention that we do not consider the optimality of coalition structure e.g. one which maximizes social welfare, rather our focus is to find a point at which no player has incentive to deviate. The proposed solution concept can be used to analyze issues such as the strategic actions of cartels, environmental and other public goods agreements, international relations, high vote share - seats dis-proportionality in political game setting, research and development collaborations,etc. Our future work includes the geometrical characterization of the equivalence nucleolus and development of a computational method for the same.

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