Preferences and pollution cycles*

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Abstract

In a recent empirical work, Hanna and Oliva (2011) have found a negative impact of pollution on labor supply (leisure effect). In a theoretical work, Bosi, Desmarchelier and Ragot (2013) have shown that the leisure effect may promote macroeconomic volatility. Previous theoretical literature focused only on the effects of pollution on consumption demand (Michel and Rotillon, 1995) neglecting those on labor supply. In a continuous-time Ramsey model, we study the interplay between the effects on consumption demand and labor supply. We introduce nonseparable preferences either between pollution and consumption or between pollution and labor supply, and we show that a compensation effect (positive impact of pollution on consumption demand) jointly with a a leisure effect promotes local indeterminacy through a Hopf bifurcation.

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1 Introduction

An important stream of literature applies the Ramsey model to pollution issues. Pollution affects the fundamentals and in particular preferences.

Keeler et al. (1972) and Forster (1973) pioneered this literature by introducing the pollution in Ramsey agents' preferences. In Forster (1973), pollution is a flow which reduces the utility, but preferences are separable: pollution does not affect the marginal utility of consumption. In Keeler et al. (1972), pollution is a stock and consumption and environmental quality are nonseparable goods. Normality ensures the uniqueness and the saddle-path stability of the steady state as in Forster (1973). The same holds in Van der Ploeg and Withagen (1991) with a negative cross derivatives (that is a marginal utility of consumption decreasing in the pollution level).

The interplay between consumption and pollution in a Ramsey model is fully characterized by Heal (1982).[1] Heal studies the growth path of a Ramsey model where the marginal utility of consumption is affected by the stock of pollution without imposing any restrictive assumption on this interplay.[2] When the stock of pollution increases the marginal utility of current consumption (the so-called *adjacent complementarity*), a limit cycle arises near the steady state through a Hopf bifurcation.

Heal's conclusions cast some doubts about the robustness of saddle-path stability in the dominant literature. The effects of pollution on growth through the consumption channel were also studied by Michel and Rotillon (1995) in an endogenous growth model with learning-by-doing. They considered different effects of pollution on consumption. On the one hand, pollution can stimulate the consumption demand through what they call a *compensation effect* (the household consumes more to compensate the drop in utility due to a higher pollution). On the other hand, if the household likes to consume in a pleasant environment, a rise in the pollution level reduces the consumption demand (they call this phenomenon *distaste effec*t). The presence of positive and negative externalities (learning-by-doing and pollution respectively) makes endogenous growth dynamics richer and more complicated in their model: the social optimum converges to a zero growth rate in presence of distaste or weak compensation effects, while a long-run positive endogenous growth rate arises under large compensation effects.

More recently, Fernandez, Perez and Ruiz (2012) have studied a discrete-time Ramsey economy with endogenous labor supply where pollution comes from the use of capital and reduces the household's utility. They focus on local indeterminacy and find that separability between consumption and pollution in the utility function prevents equilibrium multiplicity. Beyond the sustainability issue, they raise also the convergence question under pollution, especially when households' preferences are non-separable.

[1] Heal (1982) revisits Ryder and Heal (1973) by interpreting the stock of past consumption in terms of pollution instead of habit formation.

[2] The flows of pollution come from households' consumption.

Some scholars have addressed the stability issue under the effects of pollution on consumption demand and labor supply in the overlapping generations literature (see Zhang (1999) and Seegmuller and Verchère (2004) among others). Richer and possibly chaotic dynamics arise in their framework.

The literature on the effects of pollution on consumption demand is now well-established. Conversely, few papers have considered the impact of pollution on labor supply. Seegmuller and Verchère (2007) raise the question of stability in a OG model where pollution influences labor supply.

Even if the hypothesis of pollution effects on the marginal utility of consumption is convenient from a theoretical point of view, there is no evidence to support this assumption. The empirical literature points out instead a negative effect of pollution on labor supply. For instance, Hanna and Oliva (2011) have considered the effect on the labor supply in a neighborhood of a polluting refinery in Mexico City and found that a one percent increase in air pollution results in a 0.61 percent decrease in the hours worked. A close conclusions are also found by Graff, Zivin and Neidell (2010) and Carson, Koundouri and Nauges (2011). Following these evidences, Bosi, Desmarchelier and Ragot (2013) build a discrete-time Ramsey economy where the stock of pollution has no direct effects on the marginal utility of consumption but affects the marginal disutility of labor supply. In their model, positive or negative pollution effects on labor supply may arise (what they call respectively in the spirit of Michel and Rotillon (1996) *disenchantment* or *leisure effects*). The ambiguity rests on the following mechanism. In the case of *disenchantment effect*, the larger pollution decreases the utility of leisure and provides an incentive to increase the worked hours. Conversely, in the case of *leisure effect*, an increase in pollution deteriorates the working conditions and urge households to work less. In Bosi, Desmarchelier and Ragot (2013), the steady state is unique and a large *leisure effect* leads to persistent cycles through a flip bifurcation near the steady state.

Even if Bosi, Desmarchelier and Ragot (2013) fit the evidence, their simplified framework exclude any direct effect of pollution on marginal utility of consumption and, in turn, on consumption demand. The present paper aims to develop an unified framework to take into account both the effect of pollution on consumption demand (Michel and Rotillon (1996)) and on labor supply (Bosi, Desmarchelier and Ragot (2013)). Our model allows to encompass the changes in the consumer's behavior in presence of a deterioration of environmental quality and their effects on dynamics in the short and long run.

To that purpose, we develop a continuous-time Ramsey economy with separable preferences in consumption and labor but nonseparable either in consumption and pollution or in labor and pollution. Throughout this framework, we find that a *distaste effect* jointly with a *leisure effect* always implies the existence of unique steady state. The analysis of local dynamics allows us to find richer dynamics such as the emergence of local indeterminacy through a Hopf bifurcation when a sufficiently large *compensation effect* combine with a strong enough *leisure effect*.

The rest of the paper is organized as follows. In the second and third section, we introduce the model and we derive the dynamic system. In Section 4, we provide sufficient conditions for the existence and uniqueness of a steady state. Section 5 introduces general conditions for local bifurcations and indeterminacy for three-dimensional dynamic systems with two predetermined variables. The separable isoelastic case is addressed in Section 6. Section 7 provides simulations. Section 8 concludes.

2 Fundamentals

We consider a continuous-time Ramsey economy with pollution and capital accumulation. A representative household faces a consumption-leisure arbitrage by supplying a labor force to a sector of perfectly competitive firms. These firms produce a single commodity which plays the role of capital or consumption good. Because of the constant returns to scale, firms can be represented by a single aggregate firm. Pollution is a by-product of production activities and affects the individual welfare as a negative externality by distorting the consumption-leisure arbitrage.

2.1 Preferences

The household earns a capital income rh and a labor income wl where $h = h(t)$ and $l = l(t)$ denote the individual wealth and labor supply at time t. For notational simplicity, we will omit the time argument in the following. Income is consumed and saved/invested according to the budget constraint:

$$\dot{h} \leq (r - \delta) h + wl - c \tag{1}$$

The gross investment includes the capital depreciation at the rate δ.

For simplicity, the population of consumers-workers is constant over time: $N = 1$. Such normalization implies $L = Nl = l$, $K = Nh = h$ and $h = K/N = kl$.

Assumption 1 *Preferences are separable in consumption and labor:*

$$U(c, l, P) \equiv u(c, P) - v(l, P) \tag{2}$$

with $u_c > 0$*,* $u_P \leq 0$*,* $v_l > 0$*,* $v_P \geq 0$ *as first-order restrictions,* $u_{cc} < 0$*,* $v_{ll} > 0$ *as second-order restrictions, and* $\lim_{c \to 0^+} u_c = \infty$*,* $\lim_{l \to 0^+} v_l = 0$ *as a limit conditions.*

We do not impose any restriction on the sign of the cross-derivatives u_{cP} and v_{lP}. Even if preferences are separable in consumption and labor supply, pollution affects the marginal utilities of both of them and, hence, the consumption-labor arbitrage through a general equilibrium effect.

According to Michel and Rotillon (1995), pollution has a *distaste effect* on consumption if $U_{cP} < 0$: an increase in pollution reduces the marginal utility of consumption and, thereby, household's propensity to consume. The opposite effect ($U_{cP} > 0$) is called *compensation effect*: an increase in pollution raises the propensity to consume. This terminology has been extended by Bosi, Desmarchelier and Ragot (2013) to the effects of pollution on labor supply. They say that pollution has a *leisure effect* in the case of positive effect of pollution on labor disutility ($U_{lP} < 0$): an increase in pollution decreases labor supply and raises in turn the leisure demand. Pollution worsens working conditions (for example, the negative impact of global warming rests on a positive correlation between heat and work painfulness) and gives an incentive to substitute leisure to working time. Conversely, the opposite effect ($U_{lP} > 0$) is called *disenchantment effect*. In this case, leisure time decreases with pollution. Households like to enjoy leisure in a healthy and pleasant environment (for instance, pollution may dissuade people from going outdoor and encourage them to work more).

The agent maximizes the intertemporal utility function $\int_0^\infty e^{-\rho t} U(c, l, P)\, dt$ under the budget constraint (1) where $\rho > 0$ is the rate of time preference. This program is correctly defined under Assumption 1.

Proposition 1 *The first-order conditions result in a static consumption-leisure arbitrage*

$$U_c = \lambda = -U_l / w \tag{3}$$

a dynamic Euler equation $\dot{\lambda} = \lambda(\rho + \delta - r)$ *and the budget constraint (1) now binding* $\dot{h} = (r - \delta) h + wl - c$ *jointly with* $\lim_{t \to \infty} e^{-\rho t} \lambda(t) h(t) = 0$*, the transversality condition.*

Proof. See the Appendix. ■

2.2 Technology

At time t representative firm produces a single output $Y(t)$. Technology is represented by a constant returns to scale production function: $Y(t) = F(K(t), L(t))$, where $K(t)$ and $L(t)$ are the demands for capital and labor at time t.

Assumption 2 *The production function* $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is* C^1*, homogeneous of degree one, strictly increasing and concave. Inada conditions hold:* $f(0) = 0$*,* $f'(0^+) = +\infty$*,* $f'(+\infty) = 0$*, where* $f(k) \equiv F(k, 1)$ *is the average productivity and* $k \equiv K/L$ *denotes the capital intensity.*

The firm chooses the amount of capital and labor to maximize the profit taking as given the real interest rate $r(t)$ and the real wage $w(t)$. In the following, for notational simplicity, we will omit the time argument t.

The program $\max_{K,L} [F(K, L) - rK - wL]$ is correctly defined under Assumption 2 and the first-order conditions write:

$$\begin{aligned} r &= f'(k) \equiv r(k) \\ w &= f(k) - kf'(k) \equiv w(k) \end{aligned}$$

We introduce the capital share in total income α and the elasticity of capital-labor substitution σ:

$$\begin{aligned} \alpha(k) &\equiv \frac{kf'(k)}{f(k)} \\ \sigma(k) &= \alpha(k) \frac{w(k)}{kw'(k)} \end{aligned} \tag{4}$$

In addition, we determine the elasticities of factor prices:

$$\frac{kr'(k)}{r(k)} = -\frac{1-\alpha(k)}{\sigma(k)} \tag{5}$$

$$\frac{kw'(k)}{w(k)} = \frac{\alpha(k)}{\sigma(k)} \tag{6}$$

2.3 Pollution

The aggregate stock of pollution P is a pure negative externality. Technology is dirty and pollution persists. We assume a simple linear accumulation process:

$$\dot{P} = -aP + bY \tag{7}$$

where $a \geq 0$ captures the natural rate of pollution absorption and $b \geq 0$ the environmental impact of production. Since, under Assumption 2, $Y = Lf(k) = lf(k)$, the process of pollution accumulation (7) writes:

$$\dot{P} = -aP + blf(k)$$

3 Equilibrium

At equilibrium, good and labor markets clear. Applying the Implicit Function Theorem to the consumption-labor arbitrage (3), we obtain (c, l) as a function of (λ, k, P), that is $c = c(\lambda, k, P)$ and $l = l(\lambda, k, P)$. Let us introduce the following second-order elasticities of the utility function $U(c, l, P)$:[3]

$$E \equiv \begin{bmatrix} \varepsilon_{cc} & \varepsilon_{cl} & \varepsilon_{cP} \\ \varepsilon_{lc} & \varepsilon_{ll} & \varepsilon_{lP} \\ \varepsilon_{Pc} & \varepsilon_{Pl} & \varepsilon_{PP} \end{bmatrix} \equiv \begin{bmatrix} \frac{cU_{cc}}{U_c} & \frac{cU_{cl}}{U_l} & \frac{cU_{cP}}{U_P} \\ \frac{lU_{lc}}{U_c} & \frac{lU_{ll}}{U_l} & \frac{lU_{lP}}{U_P} \\ \frac{PU_{Pc}}{U_c} & \frac{PU_{Pl}}{U_l} & \frac{PU_{PP}}{UP} \end{bmatrix}$$

The various effects of pollution on preferences can be reinterpreted in terms of elasticities. Pollution has a *distaste effect* on consumption if $\varepsilon_{Pc} < 0$ and a *compensation effect* on consumption if $\varepsilon_{Pc} > 0$. According to Assumption 1, $U_l < 0$ and, thus, pollution has a *leisure effect* if $\varepsilon_{Pl} > 0$ and a *disenchantment effect* if $\varepsilon_{Pl} < 0$.

Proposition 2 *The matrix of partial elasticities is given by*

$$\begin{bmatrix} \frac{\lambda}{c}\frac{\partial c}{\partial \lambda} & \frac{k}{c}\frac{\partial c}{\partial k} & \frac{P}{c}\frac{\partial c}{\partial P} \\ \frac{\lambda}{l}\frac{\partial l}{\partial \lambda} & \frac{k}{l}\frac{\partial l}{\partial k} & \frac{P}{l}\frac{\partial l}{\partial P} \end{bmatrix} = \frac{M}{\varepsilon_{cc}\varepsilon_{ll} - \varepsilon_{lc}\varepsilon_{cl}} \tag{8}$$

where

$$M \equiv \begin{bmatrix} \varepsilon_{ll} - \varepsilon_{lc} & -\frac{\alpha}{\sigma}\varepsilon_{lc} & \varepsilon_{lc}\varepsilon_{Pl} - \varepsilon_{ll}\varepsilon_{Pc} \\ \varepsilon_{cc} - \varepsilon_{cl} & \frac{\alpha}{\sigma}\varepsilon_{cc} & \varepsilon_{cl}\varepsilon_{Pc} - \varepsilon_{cc}\varepsilon_{Pl} \end{bmatrix} \tag{9}$$

[3] In the case of explicit utility functions, the first and second-order elasticities are related when the same fundamental parameters appear in both of them.

Proof. See the Appendix. ■

In the separable case (2), the elasticities matrix simplifies

$$E \equiv \begin{bmatrix} \varepsilon_{cc} & 0 & \varepsilon_{cP} \\ 0 & \varepsilon_{ll} & \varepsilon_{lP} \\ \varepsilon_{Pc} & \varepsilon_{Pl} & \varepsilon_{PP} \end{bmatrix} \tag{10}$$

and we get

$$\begin{bmatrix} \frac{\lambda}{c}\frac{\partial c}{\partial \lambda} & \frac{k}{c}\frac{\partial c}{\partial k} & \frac{P}{c}\frac{\partial c}{\partial P} \\ \frac{\lambda}{l}\frac{\partial l}{\partial \lambda} & \frac{k}{l}\frac{\partial l}{\partial k} & \frac{P}{l}\frac{\partial l}{\partial P} \end{bmatrix} = \frac{M}{\varepsilon_{cc}\varepsilon_{ll}} = \begin{bmatrix} \frac{1}{\varepsilon_{cc}} & 0 & -\frac{\varepsilon_{Pc}}{\varepsilon_{cc}} \\ \frac{1}{\varepsilon_{ll}} & \frac{\alpha}{\sigma}\frac{1}{\varepsilon_{ll}} & -\frac{\varepsilon_{Pl}}{\varepsilon_{ll}} \end{bmatrix} \tag{11}$$

In our model, dynamics are represented by a three-dimensional system with two predetermined variables (k and P) and one non-predetemined (λ).

Proposition 3 *The equilibrium transition is driven by the following dynamic system:*

$$\begin{aligned} \frac{\dot{\lambda}}{\lambda} &= \rho + \delta - r(k) \\ \frac{\dot{k}}{k} &= \frac{r(k) - \delta + \frac{w(k)}{k} - \frac{c(\lambda,k,P)}{kl(\lambda,k,P)} - \frac{\lambda}{l}\frac{\partial l}{\partial \lambda}\left[\rho + \delta - r(k)\right] - \frac{P}{l}\frac{\partial l}{\partial P}\left[b\frac{l(\lambda,k,P)f(k)}{P} - a\right]}{1 + \frac{k}{l}\frac{\partial l}{\partial k}} \\ \frac{\dot{P}}{P} &= b\frac{l(\lambda,k,P)\,f(k)}{P} - a \end{aligned} \tag{12}$$

Proof. See the Appendix. ■

4 Steady state

At the steady state, $\dot{\lambda} = \dot{k} = \dot{P} = 0$ and system (12) becomes

$$r(k) = \rho + \delta \tag{13}$$

$$c(\lambda,k,P) = [\rho k + w(k)]\, l(\lambda,k,P) = \frac{a}{b}P - \delta k l(\lambda,k,P) \tag{14}$$

$$l(\lambda,k,P)\, f(k) = \frac{a}{b}P \tag{15}$$

because $f(k) = kr(k) + w(k)$.

We observe that the capital intensity $k = r^{-1}(\rho + \delta)$ remains that of Modified Golden Rule (MGR) and pollution does not affect it.

Consider the system

$$\frac{c(\lambda,k,P)}{l(\lambda,k,P)} = \rho k + w(k) \tag{16}$$

$$l(\lambda,k,P)\, f(k) = \frac{a}{b}P \tag{17}$$

Let

$$\varsigma(\lambda) \equiv \frac{c(\lambda, k, P(\lambda))}{l(\lambda, k, P(\lambda))} > 0 \text{ and } \varepsilon_\varsigma(\lambda) \equiv \frac{\lambda \varsigma'(\lambda)}{\varsigma(\lambda)}$$

where $P(\lambda)$ is implicitly defined by (17).

Replacing k from (13) in $\varsigma(\lambda) = \rho k + w(k)$, we compute the leisure λ and eventually the pollution $P(\lambda)$ of steady state.

Assumption 3

$$\frac{1 + \frac{\varepsilon_{Pc}}{\varepsilon_{cc}}}{1 + \frac{\varepsilon_{Pl}}{\varepsilon_{ll}}} > \frac{\varepsilon_{ll}}{\varepsilon_{cc}} \tag{18}$$

where the elasticities are evaluated at the steady state.

This inequality holds for instance if $\varepsilon_{Pc} < -\varepsilon_{cc}$ (*distaste effect* ($\varepsilon_{Pc} < 0$) or weak *compensation effect* ($0 < \varepsilon_{Pc} < -\varepsilon_{cc}$)) jointly with $\varepsilon_{Pl} > -\varepsilon_{ll}$ (*leisure effect* ($\varepsilon_{Pl} > 0$) or weak *disenchantment effect* ($-\varepsilon_{ll} < \varepsilon_{Pl} < 0$)). We will provide an explicit inequality in terms of the exogenous parameters in the case of separable and isoelastic preferences.

Proposition 4 *(uniqueness of the steady state) Let Assumptions 1 and 2 hold. The stationary capital intensity k is always unique. In addition, under Assumption 3, the steady state (λ, k, P) is unique (sufficient condition).*

Proof. See the Appendix. ■

5 Local dynamics, bifurcations and indeterminacy

In order to study the local dynamics, we linearize the three-dimensional dynamic system (12):

$$\begin{aligned} \dot{\lambda} &= f_1(\lambda, k, P) \\ \dot{k} &= f_2(\lambda, k, P) \\ \dot{P} &= f_3(\lambda, k, P) \end{aligned}$$

around the steady state and we obtain a Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial \lambda} & \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial \lambda} & \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial P} \\ \frac{\partial f_3}{\partial \lambda} & \frac{\partial f_3}{\partial k} & \frac{\partial f_3}{\partial P} \end{bmatrix} \tag{19}$$

In the following, we provide general conditions for local bifurcations and, in the case of a three-dimensional system with two predetermined variables, local indeterminacy. In Section 6, we will apply these results to an isoelastic case (isoelastic separable preferences and Cobb-Douglas production function).

5.1 Bifurcations

In continuous time, a local bifurcation generically arises when the real part of an eigenvalue $\lambda(p)$ of the Jacobian matrix crosses zero in response to a change of parameter p. Denoting by p^* the critical parameter value of bifurcation, we get generically two cases: (1) saddle-node bifurcation when a real eigenvalue crosses zero: $\lambda(p^*) = 0$, (2) Hopf bifurcation when the real part of two complex and conjugate eigenvalues $\lambda(p) = a(p) \pm ib(p)$ crosses zero. More precisely, we require $a(p^*) = 0$ and $b(p) \neq 0$ in a neighborhood of p^* (see Bosi and Ragot (2011, p. 76)).

System (12) is three-dimensional with two predetermined variables (k and P) and one jump variable (λ). Thus, multiple equilibria (local indeterminacy) arise when the three eigenvalues of the Jacobian matrix (19) evaluated at the steady state have negative real parts: either $\lambda_1, \lambda_2, \lambda_3 < 0$ or $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 < 0$ and $\lambda_3 < 0$.

Proposition 5 *Under Assumption 3, saddle-node bifurcations are ruled out.*

Proof. Under Assumption 3, the steady state is unique. The class of saddle-node bifurcations (elementary saddle node, transcritical and pitchfork) always involves multiple steady states (Bosi and Ragot, 2011). ■

A Hopf bifurcation occurs when the real part of two complex and conjugate eigenvalues $\lambda(p) = a(p) \pm ib(p)$ crosses zero. More precisely, we require $a(0) = 0$ and $b(p) \neq 0$ in a neighborhood of $p = 0$, where $p = 0$ is the normalized bifurcation value of parameter.

Consider the Jacobian matrix J and focus on the expressions of determinant, sum of minors of order two and trace in terms of eigenvalues:

$$\begin{aligned} D &= \lambda_1\lambda_2\lambda_3 \\ S &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \sum_{i=1}^{3} \det M_{ii} \\ T &= \lambda_1 + \lambda_2 + \lambda_3 \end{aligned}$$

Proposition 6 *(Hopf characterization) In the case of a three-dimensional system, a Hopf bifurcation generically arises if and only if $D = ST$ and $S > 0$.*

Proof. See the Appendix. ■

We will show later the occurrence of a Hopf bifurcation in a standard case (explicit conditions and simulation).

5.2 Local indeterminacy

In our model, dynamics involves two predetermined variables (k and P) and a jump variable (λ). As seen above, indeterminacy requires the three eigenvalues with negative real parts: either $\lambda_1, \lambda_2, \lambda_3 < 0$ or $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 < 0$ and $\lambda_3 < 0$.

Proposition 7 *(local indeterminacy) In the case of system (12), if all the eigenvalues are real, the equilibrium is locally indeterminate if and only if* $D, T < 0$ *and* $S > 0$.

Proof. See the Appendix. ■

Consider the possibility of local indeterminacy through a Hopf bifurcation. Unfortunately, Proposition 7 is of little use because, it is difficult to know whether the eigenvalues are real. In the nonreal case, the necessary condition of Proposition 7 still holds. Indeed, indeterminacy ($\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 < 0$ and $\lambda_3 < 0$) implies

$$\begin{aligned} D &= \lambda_1\lambda_2\lambda_3 = \left[(\operatorname{Re}\lambda_1)^2 + (\operatorname{Im}\lambda_1)^2\right]\lambda_3 < 0 \\ S &= \lambda_1\lambda_2 + (\lambda_1 + \lambda_2)\lambda_3 = (\operatorname{Re}\lambda_1)^2 + (\operatorname{Im}\lambda_1)^2 + 2\operatorname{Re}\lambda_1\lambda_3 > 0 \\ T &= \lambda_1 + \lambda_2 + \lambda_3 = 2\operatorname{Re}\lambda_1 + \lambda_3 < 0 \end{aligned}$$

However, the sufficient condition fails: even if

$$D = \lambda_1\lambda_2\lambda_3 = \left[(\operatorname{Re}\lambda_1)^2 + (\operatorname{Im}\lambda_1)^2\right]\lambda_3 < 0$$

still implies $\lambda_3 < 0$, conditions $D, T < 0$ and $S > 0$ do not rule out the unpleasant case $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 > 0$.

We provide instead another sufficient condition for local indeterminacy, that is more restrictive.

Proposition 8 *(local indeterminacy through a Hopf bifurcation) Let* p_H *the Hopf bifurcation value of a parameter* p *such that* $D(p_H) = S(p_H)T(p_H)$ *and* $S(p_H) > 0$. *If* $D(p_H) < 0$, *the equilibrium is locally indeterminate for some value of* p *around* p_H.

Proof. See the Appendix. ■

6 Isoelastic case

In order to provide conditions for local bifurcations and indeterminacy in terms of fundamental parameters and relevant economic interpretations, we focus on standard functional forms.

The separable case (Assumption 1) is suitable for our local analysis because of the lack of direct cross effects between the marginal utility of consumption and labor. However, we need to introduce more structure for the purpose of economic analysis. In the isoelastic case, the elasticities of matrix (11) are constant and have an easy economic interpretation. Thus, we consider isoelastic separable preferences:

$$u(c, P) \equiv \frac{\left(cP^{-\eta}\right)^{1-\varepsilon}}{1-\varepsilon} \text{ and } v(l, P) \equiv \omega\frac{\left(lP^{\psi}\right)^{1+\varphi}}{1+\varphi} \quad (20)$$

where $1/\varepsilon \geq 0$ is the consumption elasticity of intertemporal substitution, $1/\varphi \geq 0$ is the Frisch elasticity of intertemporal substitution and $\omega > 0$ is the weight of disutility of labor in total utility. In addition, $\eta, \psi \geq 0$ (Assumption 1).

In addition, we focus on a Cobb-Douglas production function giving the following intensive output:

$$f(k) = Ak^{\alpha} \tag{21}$$

We observe that, in this case, α becomes constant and $\sigma = 1$.

The elasticities on the RHS of matrix (11) appear only in the first two columns of the elasticities matrix E (see (10)):

$$\tilde{E} \equiv \begin{bmatrix} \varepsilon_{cc} & \varepsilon_{cl} \\ \varepsilon_{lc} & \varepsilon_{ll} \\ \varepsilon_{Pc} & \varepsilon_{Pl} \end{bmatrix} = \begin{bmatrix} \frac{cu_{cc}}{u_c} & 0 \\ 0 & \frac{lv_{ll}}{v_l} \\ \frac{Pu_{Pc}}{u_c} & \frac{Pv_{Pl}}{v_l} \end{bmatrix} = \begin{bmatrix} -\varepsilon & 0 \\ 0 & \varphi \\ (\varepsilon - 1)\eta & (1+\varphi)\psi \end{bmatrix}$$

and depends directly on the fundamental parameters.

The elasticities in the third column of E (see (10)) are more complicated because they are not directly parametric but involve the endogenous variables: λ, k, P. Fortunately, we no longer need them in the following. Hence, matrix (11) simplifies:

$$\begin{bmatrix} \frac{\lambda c_\lambda}{c} & \frac{kc_k}{c} & \frac{Pc_P}{c} \\ \frac{\lambda l_\lambda}{l} & \frac{kl_k}{l} & \frac{Pl_P}{l} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\varepsilon} & 0 & \eta\frac{\varepsilon-1}{\varepsilon} \\ \frac{1}{\varphi} & \frac{\alpha}{\varphi} & -\psi\frac{1+\varphi}{\varphi} \end{bmatrix} \tag{22}$$

where now the more compact expression y_x denotes the derivative $\partial y/\partial x$.

We observe that, if $\eta > 0$, a *distaste effect* holds when $0 < \varepsilon < 1$, while a *compensation effect* arises when $\varepsilon > 1$. Our isoelastic specification rules out any *disenchantment effect* if $\psi > 0$, but captures the *leisure effect* empirically found by Hanna and Oliva (2011).

The dynamic system (12) writes:

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta - r(k) \tag{23}$$

$$\frac{\dot{k}}{k} = \frac{\rho + \frac{w(k)}{k} - \frac{c(\lambda,k,P)}{kl(\lambda,k,P)} - \frac{1+\varphi}{\varphi}\left(\rho + \delta - r(k) + \psi\left[a - b\frac{l(\lambda,k,P)f(k)}{P}\right]\right)}{1 + \frac{\alpha}{\varphi}} \tag{24}$$

$$\frac{\dot{P}}{P} = b\frac{l(\lambda,k,P)f(k)}{P} - a \tag{25}$$

6.1 Steady state

In the isoelastic case, the steady state values depend explicitly on the fundamental parameters and the comparative statics leads to unambiguous results.

Proposition 9 *In the isoelastic case, there exists a unique steady state:*

$$\lambda = \left(\frac{B}{\rho k + w}\right)^{\frac{(\varphi+\psi+\varphi\psi)\varepsilon}{\varphi+\psi+\varphi\psi+\varepsilon+(1-\varepsilon)\eta}} \tag{26}$$

$$k = \left(\frac{\alpha A}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \tag{27}$$

$$P = C\lambda^{\frac{1}{\varphi+\psi+\varphi\psi}} \tag{28}$$

where $w = (1-\alpha) Ak^{\alpha}$ *and*

$$B \equiv \frac{C^{\frac{\varepsilon-1}{\varepsilon}\eta+\frac{1+\varphi}{\varphi}\psi}}{(w/\omega)^{\frac{1}{\varphi}}}$$

$$C \equiv \left[Ak^{\alpha}\frac{b}{a}\left(\frac{w}{\omega}\right)^{\frac{1}{\varphi}}\right]^{\frac{\varphi}{\varphi+\psi+\varphi\psi}}$$

Proof. See the Appendix. ■

It is interesting to note that we do not need to impose restriction (18) to ensure the uniqueness of steady state. (18) represents a sufficient condition for uniqueness (Proposition 4). The only restriction we need in the isoelastic case is $\varphi + \psi + \varphi\psi + \varepsilon + (1-\varepsilon)\eta \neq 0$. Otherwise, the steady state (λ) fails to exist (see (26)).

Focus now on the comparative statics.

As seen in the general case, the capital intensity remains that of MGR as in the Ramsey model and the basic parameters α, δ, ρ and A plays the usual role on k. The preferences parameters ε, η, φ, ψ and ω play no direct role on k (but affect the consumption demand and the labor supply through λ and P). This result is not surprising: indeed, we are focusing on a market economy where, differently from the planner's solution, the pollution externality is not internalized and has no marginal effect on the Euler equation $\dot{\lambda}/\lambda = \rho+\delta-r(k)$.

Consider

$$\ln\lambda = \frac{(\varphi+\psi+\varphi\psi)\varepsilon}{\varphi+\psi+\varphi\psi+\varepsilon+(1-\varepsilon)\eta}\left(z-\frac{1}{\varphi}y+\left(\frac{\varepsilon-1}{\varepsilon}\eta+\frac{1+\varphi}{\varphi}\psi\right)\frac{\varphi x+y}{\varphi+\psi+\varphi\psi}\right) \tag{29}$$

$$\ln P = \frac{\varphi x + y + \ln\lambda}{\varphi+\psi+\varphi\psi} \tag{30}$$

where

$$x \equiv \ln\left(Ak^{\alpha}\frac{b}{a}\right)$$

$$y \equiv \ln\frac{w}{\omega}$$

$$z \equiv -\ln(\rho k + w)$$

that is

$$\begin{aligned}
x &= \frac{\alpha}{1-\alpha}\ln\alpha + \frac{1}{1-\alpha}\ln A - \frac{\alpha}{1-\alpha}\ln(\rho+\delta) - \ln\frac{a}{b} \\
y &= \frac{\alpha}{1-\alpha}\ln\alpha + \frac{1}{1-\alpha}\ln A - \frac{\alpha}{1-\alpha}\ln(\rho+\delta) + \ln(1-\alpha) - \ln\omega \\
z &= -\frac{\alpha}{1-\alpha}\ln\alpha - \frac{1}{1-\alpha}\ln A + \frac{1}{1-\alpha}\ln(\rho+\delta) - \ln[\rho+(1-\alpha)\delta]
\end{aligned}$$

Computations give

$$\begin{aligned}
\frac{\partial P/\partial\varepsilon}{P/\varepsilon} &= \frac{\ln\lambda + \eta\ln P}{\varphi+\psi+\varphi\psi+\varepsilon+\eta-\varepsilon\eta} \\
\frac{\partial\lambda/\partial\varepsilon}{\lambda/\varepsilon} &= \frac{\partial P/\partial\varepsilon}{P/\varepsilon}(\varphi+\psi+\varphi\psi) \qquad (31)\\
\frac{\partial P/\partial\eta}{P/\eta} &= \frac{(\varepsilon-1)\eta\ln P}{\varphi+\psi+\varphi\psi+\varepsilon+\eta-\varepsilon\eta} \qquad (32)\\
\frac{\partial\lambda/\partial\eta}{\lambda/\eta} &= \frac{\partial P/\partial\eta}{P/\eta}(\varphi+\psi+\varphi\psi) \qquad (33)
\end{aligned}$$

with $\ln(\lambda P^{\eta}) = \ln(cP^{-\eta})^{-\varepsilon}$.

Using (29) and (30) we get more explicitly

$$\begin{aligned}
\frac{\partial P/\partial\varepsilon}{P/\varepsilon} &= \frac{(1+\varphi)(\eta+\psi)(x+z)+(\eta-1)(y-\varphi z)}{(\varphi+\psi+\varphi\psi+\varepsilon+\eta-\varepsilon\eta)^2}\varepsilon \qquad (34)\\
\frac{\partial P/\partial\eta}{P/\eta} &= \frac{\varphi x+y+\varepsilon(x+z)}{(\varphi+\psi+\varphi\psi+\varepsilon+\eta-\varepsilon\eta)^2}(\varepsilon-1)\eta
\end{aligned}$$

Proposition 10 *The qualitative impacts of ε on P and λ are the same. In particular, if $\eta = 1$, then*

$$\frac{\partial\lambda/\partial\varepsilon}{\lambda/\varepsilon}, \frac{\partial P/\partial\varepsilon}{P/\varepsilon} > 0$$

iff

$$a < b\frac{\rho+\delta}{\rho+(1-\alpha)\delta}$$

Proof. The qualitative impacts are the same because of (31). If $\eta = 1$, (34) becomes

$$\frac{\partial P/\partial\varepsilon}{P/\varepsilon} = \frac{\varepsilon(x+z)}{(1+\varphi)(1+\psi)}$$

with

$$x+z = \ln\left[\frac{b}{a}\frac{\rho+\delta}{\rho+(1-\alpha)\delta}\right]$$

The proposition follows. ■

Assumption 4 *$\varepsilon\eta < \varepsilon+\eta+\varphi+\psi+\varphi\psi$.*

Assumption 4 writes

$$\frac{P}{c}\frac{\partial c}{\partial P} < 1 + \frac{\varphi}{\varepsilon}\left(1 - \frac{P}{l}\frac{\partial l}{\partial P}\right)$$

and is satisfied under the joint assumption of distaste and leisure effect:

$$\frac{P}{c}\frac{\partial c}{\partial P} < 0 \text{ and } \frac{P}{l}\frac{\partial l}{\partial P} < 0$$

Proposition 11 *The qualitative impacts of η on λ and P are the same because of (33). Under Assumption 4, they are positive under dominant income effects ($\varepsilon > 1$) and high pollution ($P > 1$) or dominant substitution effects ($\varepsilon < 1$) and low pollution ($P < 1$).*

Proof. We observe that, under Assumption 4,

$$\frac{\partial P/\partial \eta}{P/\eta} > 0 \text{ iff } (\varepsilon - 1)\ln P > 0$$

■

Focus for instance on the second case (substitution effects and low pollution). According to 22, a higher η implies a stronger distaste effect. Then, for a given pollution level, individuals consume less and save more which increases the production level and the pollution stock in turn.

For φ and ψ, focus on the simplified case $\varepsilon = 1$ (logarithmic felicity). We have

$$\begin{aligned}
\ln \lambda &= \frac{x\psi + x\psi\varphi - y + z(\varphi + \psi + \varphi\psi)}{(1+\varphi)(1+\psi)} \\
\ln P &= \frac{\varphi x + y + \ln \lambda}{\varphi + \psi + \varphi\psi}
\end{aligned}$$

and, thus,

$$\begin{aligned}
\frac{\partial P/\partial \varphi}{P/\varphi} &= -\frac{\varphi}{1+\varphi}\frac{y+z}{(1+\varphi)(1+\psi)} \\
\frac{\partial \lambda/\partial \varphi}{\lambda/\varphi} &= -\frac{\partial P/\partial \varphi}{P/\varphi} \\
\frac{\partial P/\partial \psi}{P/\psi} &= -\frac{\psi}{1+\psi}\frac{(1+\varphi)x + y + z}{(1+\varphi)(1+\psi)} \\
\frac{\partial \lambda/\partial \psi}{\lambda/\psi} &= -\frac{\partial P/\partial \psi}{P/\psi}
\end{aligned}$$

Proposition 12 *The qualitative impacts of φ on P and λ are opposite. More explicitly,*

$$\frac{\partial P/\partial \varphi}{P/\varphi} > 0 \text{ (and } \frac{\partial \lambda/\partial \varphi}{\lambda/\varphi} < 0\text{) iff } \omega > \frac{(1-\alpha)(\rho+\delta)}{\rho + (1-\alpha)\delta} \; (< 1)$$

Corollary 13 *The impact of φ on pollution is positive if $\omega = 1$.*

Proposition 14 *The qualitative impacts of ψ on P and λ are opposite. More explicitly,*

$$\frac{\partial P/\partial\psi}{P/\psi} < 0 \text{ (and } \frac{\partial\lambda/\partial\psi}{\lambda/\psi} > 0\text{) iff } A\left(\frac{\alpha}{\rho+\delta}\right)^{\alpha} > \left(\frac{a}{b}\right)^{1-\alpha}\left[\omega\frac{\rho+(1-\alpha)\delta}{(1-\alpha)(\rho+\delta)}\right]^{\frac{1-\alpha}{1+\varphi}}$$

This equivalence is interpreted as follows. A higher ψ implies a stronger leisure effect and, thus, a lower labor supply, which reduces the production level and the pollution level in turn. Such a relation is magnified under a large environmental effect of production (b).

Following the MGR, such variations of ψ, l and P have no effect on the stationary value of capital intensity (k). In addition, at the steady state, $c = \gamma kl$ (see the proof of Lemma 18): the decrease of labor supply (l) induced by a higher ψ entails a lower consumption level (c) and a higher marginal utility of consumption, that is λ (see 3).

Corollary 15 *The impact of φ on pollution is positive if the TFP (A) is low, the natural rate of pollution absorption (a) is high or the environmental impact of production (b) is low.*

A higher φ means a lower leisure effect. Then, for a given pollution level, the representative household works more which enhances the production level and the pollution stock in turn. Under a *distaste effect*, a higher pollution level implies that the household reduces his consumption demand. This increases the marginal utility of consumption and, according to 3, λ as well.

In addition,

$$\begin{aligned}
\frac{\partial\lambda/\partial p}{\lambda/p} &= \frac{(1+\varphi)\,\psi p x_p - p y_p + (\varphi+\psi+\varphi\psi)\,p z_p}{(1+\varphi)(1+\psi)} \\
\frac{\partial P/\partial p}{P/p} &= \frac{(1+\varphi)\,p x_p + p y_p + p z_p}{(1+\varphi)(1+\psi)}
\end{aligned}$$

with $p = A, a/b, \rho, \delta, \omega$.

$$\begin{aligned}
(Ax_A, Ay_A, Az_A) &= \left(\frac{1}{1-\alpha}, \frac{1}{1-\alpha}, -\frac{1}{1-\alpha}\right) \\
\left(\frac{a}{b}x_{\frac{a}{b}}, \frac{a}{b}y_{\frac{a}{b}}, \frac{a}{b}z_{\frac{a}{b}}\right) &= (-1, 0, 0) \\
(\rho x_\rho, \rho y_\rho, \rho z_\rho) &= \left(-\frac{\alpha}{1-\alpha}\frac{\rho}{\rho+\delta}, -\frac{\alpha}{1-\alpha}\frac{\rho}{\rho+\delta}, \frac{1}{1-\alpha}\frac{\rho}{\rho+\delta} - \frac{\rho}{\rho+(1-\alpha)\delta}\right) \\
(\delta x_\delta, \delta y_\delta, \delta z_\delta) &= \left(-\frac{\alpha}{1-\alpha}\frac{\delta}{\rho+\delta}, -\frac{\alpha}{1-\alpha}\frac{\delta}{\rho+\delta}, \frac{1}{1-\alpha}\frac{\delta}{\rho+\delta} - \frac{(1-\alpha)\delta}{\rho+(1-\alpha)\delta}\right) \\
(\omega x_\omega, \omega y_\omega, \omega z_\omega) &= (0, -1, 0)
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial\lambda/\partial A}{\lambda/A} &= -\frac{1}{1-\alpha}\frac{1}{1+\psi} \\
\frac{\partial P/\partial A}{P/A} &= \frac{1}{1-\alpha}\frac{1}{1+\psi}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\lambda/\partial\frac{a}{b}}{\lambda/\frac{a}{b}} &= -\frac{\psi}{1+\psi} \\
\frac{\partial P/\partial\frac{a}{b}}{P/\frac{a}{b}} &= -\frac{1}{1+\psi}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\lambda/\partial\rho}{\lambda/\rho} &= \frac{\frac{\rho}{\rho+\delta}\frac{\alpha+\varphi+(1-\alpha)\psi+(1-\alpha)\psi\varphi}{1-\alpha}-\frac{\rho}{\rho+(1-\alpha)\delta}\left(\varphi+\psi+\varphi\psi\right)}{(1+\varphi)(1+\psi)} \\
\frac{\partial P/\partial\rho}{P/\rho} &= \frac{\frac{\rho}{\rho+\delta}\frac{1-2\alpha-\alpha\varphi}{1-\alpha}-\frac{\rho}{\rho+(1-\alpha)\delta}}{(1+\varphi)(1+\psi)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\lambda/\partial\delta}{\lambda/\delta} &= \frac{\frac{\delta}{\rho+\delta}\frac{\alpha+\varphi+(1-\alpha)\psi+(1-\alpha)\psi\varphi}{1-\alpha}-\frac{(1-\alpha)\delta}{\rho+(1-\alpha)\delta}\left(\varphi+\psi+\varphi\psi\right)}{(1+\varphi)(1+\psi)} \\
\frac{\partial P/\partial\delta}{P/\delta} &= \frac{\frac{\delta}{\rho+\delta}\frac{1-2\alpha-\alpha\varphi}{1-\alpha}-\frac{(1-\alpha)\delta}{\rho+(1-\alpha)\delta}}{(1+\varphi)(1+\psi)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\lambda/\partial\omega}{\lambda/\omega} &= \frac{1}{(1+\varphi)(1+\psi)} \\
\frac{\partial P/\partial\omega}{P/\omega} &= -\frac{1}{(1+\varphi)(1+\psi)}
\end{aligned}$$

The impact of α remains too ambiguous and complicate.

Proposition 16 *The effects of A, a/b, ρ and ω on P and λ are the following*

$$\begin{aligned}
\frac{\partial P/\partial A}{P/A} &= -\frac{\partial\lambda/\partial A}{\lambda/A}=\frac{1}{1-\alpha}\frac{1}{1+\psi}>0 \\
\frac{\partial P/\partial\frac{a}{b}}{P/\frac{a}{b}} &= \frac{1}{\psi}\frac{\partial\lambda/\partial\frac{a}{b}}{\lambda/\frac{a}{b}}=-\frac{1}{1+\psi}<0 \\
\frac{\partial P/\partial\omega}{P/\omega} &= -\frac{\partial\lambda/\partial\omega}{\lambda/\omega}=-\frac{1}{(1+\varphi)(1+\psi)}<0
\end{aligned}$$

These elasticities deserve an economic interpretation.

A higher A implies a higher production level and in turn a higher pollution level. Such an increase of the pollution stock lowers the labor supply (leisure effect) and, thus, the marginal disutility of labor supply (Assumption 1) and λ in turn (see 3). In addition, From (14), for a given l and P, a higher a/b implies a higher consumption level. Since $\varepsilon = 1$, the LHS of equation (3) writes $1/c = \lambda$, that is, the increase of c is followed by a drop of λ. According to equation (17), for a given k, a higher a/b induces a lower pollution level. Finally, according to the functional forms (20) and (21), the RHS of equation (3) writes

$$\lambda = \omega \frac{\left(lP^{\psi}\right)^{\varphi} P^{\psi}}{(1-\alpha) k^{\alpha}}$$

with an underlying positive relation between ω an P: a higher ω increases the marginal disutility of labor supply, the household reduces his labor supply which reduces the production level and the pollution stock in turn.

Proposition 17 *The effects of ρ and δ on P and λ are the following*

$$\begin{aligned}
\frac{\partial P/\partial \rho}{P/\rho} &= -M\rho \left[\rho (1+\varphi) + \delta (1-\alpha)(2+\varphi)\right] < 0 \\
\frac{\partial \lambda/\partial \rho}{\lambda/\rho} &= M\rho \left(\rho (1+\varphi) + \delta (1-\alpha)\left[1 - \psi (1+\varphi)\right]\right) > 0 \textit{ iff } \rho > \left(\psi - \frac{1}{1+\varphi}\right)(1-\alpha)\delta \\
\frac{\partial P/\partial \delta}{P/\delta} &= -M\delta \left[\rho (\alpha+\varphi) + \delta (1-\alpha)(1+\varphi)\right] < 0 \\
\frac{\partial \lambda/\partial \delta}{\lambda/\delta} &= M\delta \left(\rho (1+\varphi) + (1-\alpha)\left[\rho\varphi + (\rho\psi + \delta)(1+\varphi)\right]\right) > 0
\end{aligned}$$

where $M \equiv \alpha / \left[(1-\alpha)(\rho+\delta)(\rho + (1-\alpha)\delta)(1+\varphi)(1+\psi)\right] > 0$.

These elasticities can be easily interpreted. Because of the MGR, we notice that a higher ρ implies lower capital stock, production level and pollution stock at the end. Conversely, since a higher ρ induces a lower pollution level, the representative household increases his labor supply (leisure effect), the marginal disutility of labor supply and, eventually, λ. Similar arguments explain the effects of δ on P and λ.

6.2 Local dynamics

System (23)-(25) writes:

$$\begin{aligned}
\dot{\lambda} &= f_1(\lambda, k, P) \equiv \lambda \left[\rho + \delta - r(k)\right] \\
\dot{k} &= f_2(\lambda, k, P) \equiv \frac{\varphi}{\alpha+\varphi} \\
&\quad \left[\rho k + w(k) - \frac{c(\lambda,k,P)}{l(\lambda,k,P)} - k\frac{1+\varphi}{\varphi}\left(\rho + \delta - r(k) + \psi\left[a - b\frac{l(\lambda,k,P) f(k)}{P}\right]\right)\right] \\
\dot{P} &= f_3(\lambda, k, P) \equiv bl(\lambda, k, P) f(k) - aP
\end{aligned}$$

We linearize it around the steady state through the computation of the Jacobian.

In the following, let

$$\begin{aligned}
\theta &\equiv \alpha\frac{1+\varphi}{\varphi} \text{ and } \tau \equiv \frac{\alpha+\varphi}{\varphi} > \theta \\
\mu &\equiv \psi\frac{1+\varphi}{\varphi},\ \gamma \equiv \frac{r}{\alpha} - \delta,\ s \equiv (1-\alpha)\,r \\
n &\equiv \mu\frac{a}{\varphi} + \gamma\left(\frac{1}{\varepsilon} + \frac{1}{\varphi}\right) \text{ and } \xi \equiv \gamma\left(\mu + \eta\frac{\varepsilon-1}{\varepsilon}\right)
\end{aligned} \tag{35}$$

with $r = \rho + \delta$.

Lemma 18 *Let D, S and T be the determinant, the sum of diagonal minors of order two and the trace of the Jacobian matrix evaluated at the steady state. Thus,*

$$\begin{aligned}
D &= \frac{as}{\tau}\left[(1+\mu)\left(n - \frac{a\mu}{\varphi}\right) - \frac{\xi}{\varphi}\right] \\
S &= \frac{a\theta\xi - ns}{\tau} - a\rho\,(1+\mu) \\
T &= \rho - a + a\mu\frac{\theta-\tau}{\tau}
\end{aligned} \tag{36}$$

Proof. See the Appendix. ■

6.3 Bifurcations and indeterminacy

In this subsection we provide general conditions under which the system undergoes a Hopf bifurcation and local indeterminacy arises. At the end, we characterize the particular cases with no pollution effects on (1) consumption and leisure, (2) leisure, (3) consumption.

6.3.1 Limit cycles

Let

$$\eta_H \equiv \frac{\varepsilon}{\varepsilon-1}\left(\frac{\xi_H}{\gamma} - \mu\right) \tag{37}$$

with

$$\xi_H \equiv \frac{s\,(1+\mu)\left(n - \frac{a\mu}{\varphi}\right) + \left[\rho\tau\,(1+\mu) + \frac{ns}{a}\right]\left(\rho - a - a\mu\frac{\tau-\theta}{\tau}\right)}{\frac{s}{\varphi} + \theta\left(\rho - a - a\mu\frac{\tau-\theta}{\tau}\right)} \tag{38}$$

and $S > 0$, that is

$$\xi_H > \frac{ns + a\rho\tau\,(1+\mu)}{a\theta} \tag{39}$$

Proposition 19 *(limit cycles) There exists a parameter region such that, when η goes through η_H, the system undergoes a Hopf bifurcation.*

Proof. See the Appendix. ■

It is interesting to see that $\lim_{\alpha \to 1} \eta_H = \varepsilon / (\varepsilon - 1)$. Then, $\eta_H > 0$ iff $\varepsilon > 1$, that is the occurrence of a Hopf bifurcation requires dominant income effects. Recall that

$$\frac{P}{c}\frac{\partial c}{\partial P} = -\eta \frac{1-\varepsilon}{\varepsilon}$$

that is, in this limit case, a Hopf bifurcation occurs only under a compensation effect ($\partial c/\partial P > 0$ or $\varepsilon_{Pc} > 0$) as in Michel and Rotillon (1995).

Assume a rise of P near the steady state. Since $\partial c/\partial P > 0$ and $\partial l/\partial P < 0$ (matrix (22)), this entails an increase of c jointly with a decrease of k and a decrease of l. These two effects imply a fall in the production level and, in turn, a decrease of pollution. By this channel, deterministic endogenous fluctuations occur near the steady state.

Proposition 20 *Let $\varepsilon > 1$ and $\rho > \alpha a$. Then, with no capital depreciation ($\delta = 0$), we have*

$$\frac{\partial \eta_H}{\partial \psi} < 0 \text{ iff } \varepsilon > \alpha \frac{1 + \varphi(2+\varphi)}{1 - \alpha(2-\alpha)}$$

Proof. Using (38), we compute the derivative with $\delta = 0$.

$$\frac{\partial \eta_H}{\partial \psi} = \frac{1+\varphi}{\varphi} \frac{\rho\varphi(\rho - a\alpha)(1-\alpha)(\alpha+\varphi)(\alpha - \varepsilon + \alpha\varepsilon(2-\alpha) + \alpha\varphi(2+\varphi))}{(\varepsilon - 1)(\rho(\alpha\varphi+1)(\alpha+\varphi) - a\alpha(1+\varphi)((\alpha+\varphi) + \mu\varphi(1-\alpha)))^2}$$

■

Proposition 20 means that, in the case of strong income effects (high ε) and weak natural rate of pollution absorption ($\rho > \alpha a$), the greater is the sensitivity of labor supply to pollution (ψ), the lower is the critical sensitivity of consumption demand to pollution (η_H) for which a limit cycle occurs.

6.3.2 Local indeterminacy

Proposition 21 *(local indeterminacy through a Hopf bifurcation) If*

$$(1+\mu)\left(n - \frac{a\mu}{\varphi}\right) - \frac{\xi_H}{\varphi} < 0 \qquad (40)$$

then there exists a parameter region where indeterminacy occurs.

Proof. See the Appendix. ■

Corollary 22 *In the case of compensation effects ($\varepsilon > 1$), local indeterminacy through a Hopf bifurcation arises if*

$$\eta_H > \frac{\varepsilon + \varphi(1+\mu)}{\varepsilon - 1}$$

Proof. Replace n and ξ_H from (37) in (40), and solve the inequality for η_H. ■

The possibility of self-fulfilling prophecies rests on equilibrium indeterminacy. Let us provide an intuition for them in our economy.

Let the economy be at the steady state and assume that any consumer expect today an increase in the pollution level tomorrow. Since $\partial c/\partial P > 0$ and $\partial l/\partial P < 0$, she wants a higher consumption demand tomorrow jointly with a lower labor supply. She needs to save more today to finance a larger consumption tomorrow under a lower labor income. The increase in capital intensity will enhance the production level and promote an increase in the pollution stock. Hence, the expectation of higher pollution tomorrow turns out to be self-fulfilling.

Focus on relation (40):

$$\lim_{a\to 0^+}\left[(1+\mu)\left(n-\frac{a\mu}{\varphi}\right)-\frac{\xi_H}{\varphi}\right]=-\infty \tag{41}$$

From (41), it appears that local indeterminacy is more likely when the natural rate of pollution absorption (a) is low, that is pollution is more persistent and the negative effects of production as well. This result contrasts with many contributions. Indeed, using a Ramsey framework displaying endogenous growth, Fernandez, Perez and Ruiz (2012) and Itaya (2008) have studied indeterminacy with pollution as a flow. A same result holds also in the OLG literature (see Seegmuller and Verchère (2007)).

6.3.3 Particular cases

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(1) $\eta = \psi = 0$ (no effects).

(2) $\psi = 0$ and $\eta > 0$ (distaste effect ($0 < \varepsilon < 1$) or compensation effect ($\varepsilon > 1$)).

(3) $\eta = 0$ and $\psi > 0$ (leisure effect).

7 Simulation

We have provided general conditions for the occurrence of limit cycles (Proposition 6) and more explicit conditions in the isoelastic case (Proposition 19). The following section provide a graphical example of the existence of stable limit cycles through a calibrated simulation.

In this section, we propose a numerical investigation of local dynamics using the MATCONT package for MATLAB. We calibrate the values according to quarterly data:

Parameter	A	ω	a	b	α	δ	ε	φ	ψ	ρ
Value	1	1	0.01	0.01	0.33	0.025	2	0.5	3.78	0.01

$\varepsilon = 2$ ensures the existence of a compensation effect while φ and ψ are set in order to satisfy (39):

$$\xi_H = 1.998\,3 > \frac{ns + a\rho\tau\,(1+\mu)}{a\theta} = 1.224\,1$$

According to this calibration, we find $\eta_H \approx 26.625$. MATCONT detects a local bifurcation when the bifurcation parameter varies in a convenient range. In our case, we consider the range around the critical value $(26, 27) \ni \eta_H$ and MATCONT finds independently the critical value $\eta_H = 26.624837$ which is very

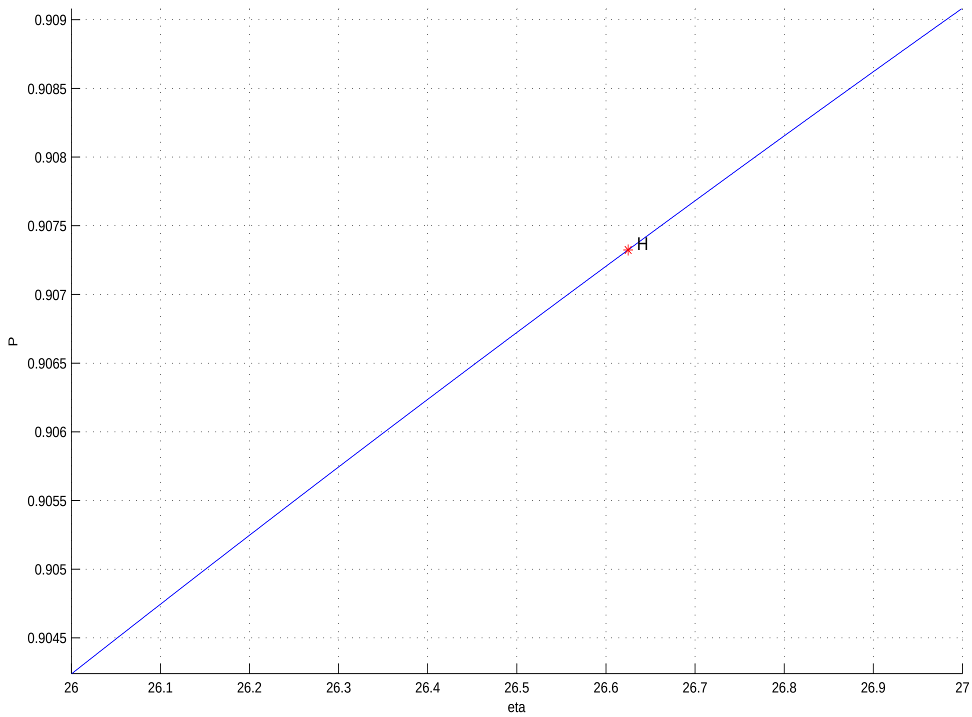

Figure 1. Equilibrium continuation.

The first Lyapunov coefficient associated with the Hopf boundary is $l_1 = -1.201350 * 10^{-5}$ (See Kuznetsov (1998), p. 99), which means that we are considering a supercritical Hopf bifurcation at $\eta_H = 26.624837$. Figure 2 represents the stable limit cycle in the (k, λ, P)-plane.

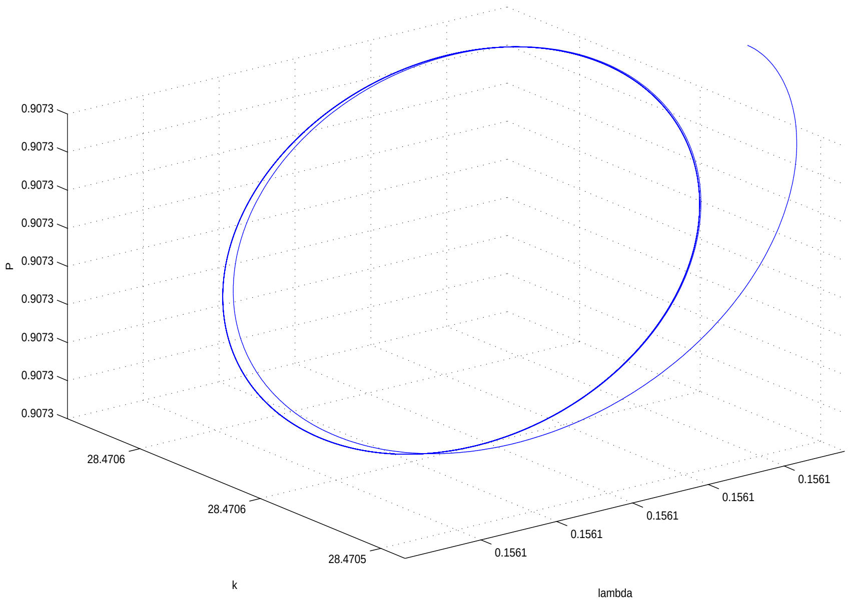

Figure 2. Stable limit cycle.

At the Hopf bifurcation point ($\eta = \eta_H$), the steady state becomes:

$$(k, \lambda, P) = (28.470616, 0.15609615, 0.90732381)$$

The corresponding eigenvalues become:

$$\begin{aligned}
\lambda_1 &= -0.0457699 \\
\lambda_2 &= 0.0679497i \\
\lambda_3 &= -0.0679497i
\end{aligned}$$

8 Conclusion

Within a unified framework, we have studied together the pollution effects on consumption demand and labor supply. We have provided sufficient conditions to ensure the uniqueness of the steady state and introduced a general method to address the issue of local bifurcations and indeterminacy for three-dimensional dynamic systems in presence of two predetermined variables. Applying these general results to the case of separable isoelastic preferences, we have found that a *compensation effect* coupled with a *leisure effect* leads to local indeterminacy through a Hopf bifurcation. These results add value to the existing literature on the stability issue under pollution in Ramsey economies.

9 Appendix

Proof of Proposition 1

The Hamiltonian writes $\tilde{H} = e^{-\rho t} U(c,l,P) + \tilde{\lambda}\left[(r-\delta)h + wl - c\right]$ and the first-order conditions

$$\begin{aligned} \partial \tilde{H}/\partial \tilde{\lambda} &= (r-\delta)h + wl - c = \dot{h} \\ \partial \tilde{H}/\partial h &= \tilde{\lambda}(r-\delta) = -\tilde{\lambda}' \\ \partial \tilde{H}/\partial c &= e^{-\rho t} U_c - \tilde{\lambda} = 0 \\ \partial \tilde{H}/\partial l &= e^{-\rho t} U_l + \tilde{\lambda} w = 0 \end{aligned}$$

jointly with the transversality condition $\lim_{t\to\infty} \tilde{\lambda}(t)\, h(t) = 0$. Setting $\lambda \equiv e^{\rho t}\tilde{\lambda}$, we find $\dot{\lambda} - \rho\lambda = e^{\rho t}\tilde{\lambda}'$ and equations in Proposition 1. The discounted Hamiltonian $H \equiv e^{\rho t}\tilde{H}$ becomes $H = U(c,l,P) + \lambda\left[(r-\delta)h + wl - c\right]$. ■

Proof of Proposition 2

Differentiating the system

$$\begin{aligned} \lambda - U_c(c,l,P) &= 0 \\ \lambda w(k) + U_l(c,l,P) &= 0 \end{aligned}$$

we get

$$\begin{aligned} \varepsilon_{cc}\frac{dc}{c} + \varepsilon_{lc}\frac{dl}{l} &= \frac{d\lambda}{\lambda} - \varepsilon_{Pc}\frac{dP}{P} \\ \varepsilon_{cl}\frac{dc}{c} + \varepsilon_{ll}\frac{dl}{l} &= \frac{d\lambda}{\lambda} + \alpha\frac{dk}{k} - \varepsilon_{Pl}\frac{dP}{P} \end{aligned}$$

that is

$$\begin{bmatrix} \frac{dc}{c} \\ \frac{dl}{l} \end{bmatrix} = \frac{M}{\varepsilon_{cc}\varepsilon_{ll} - \varepsilon_{lc}\varepsilon_{cl}} \begin{bmatrix} \frac{d\lambda}{\lambda} \\ \frac{dk}{k} \\ \frac{dP}{P} \end{bmatrix}$$

where M is given by (9). Thus, we obtain the following matrix of partial elasticities (8). ■

Proof of Proposition 3

Let us reconsider the dynamic system:

$$\begin{aligned} \dot{\lambda} &= \lambda\left[\rho + \delta - r(k)\right] \\ \dot{h} &= (r-\delta)h + wl - c \\ \dot{P} &= -aP + blf(k) \end{aligned}$$

We observe that $h = kl$ and, thus,

$$\frac{\dot{h}}{h} = \frac{\dot{k}}{k} + \frac{\dot{l}}{l}$$

In addition, $l = l(\lambda, k, P)$ and, thus,

$$\frac{\dot{l}}{l} = \frac{\lambda}{l}\frac{\partial l}{\partial \lambda}\frac{\dot{\lambda}}{\lambda} + \frac{k}{l}\frac{\partial l}{\partial k}\frac{\dot{k}}{k} + \frac{P}{l}\frac{\partial l}{\partial P}\frac{\dot{P}}{P}$$

where the elasticities

$$\frac{\lambda}{l}\frac{\partial l}{\partial \lambda}, \frac{k}{l}\frac{\partial l}{\partial k}, \frac{P}{l}\frac{\partial l}{\partial P}$$

are given by (8).

We obtain the following three-dimensional dynamic system

$$\begin{aligned}
\frac{\dot{\lambda}}{\lambda} &= \rho + \delta - r(k) \\
\frac{\dot{k}}{k} &= r(k) - \delta + \frac{w(k)}{k} - \frac{c(\lambda, k, P)}{kl(\lambda, k, P)} - \frac{\dot{l}}{l} \\
&= r(k) - \delta + \frac{w(k)}{k} - \frac{c(\lambda, k, P)}{kl(\lambda, k, P)} - \frac{\lambda}{l}\frac{\partial l}{\partial \lambda}\frac{\dot{\lambda}}{\lambda} - \frac{k}{l}\frac{\partial l}{\partial k}\frac{\dot{k}}{k} - \frac{P}{l}\frac{\partial l}{\partial P}\frac{\dot{P}}{P} \\
\frac{\dot{P}}{P} &= b\frac{l(\lambda, k, P) f(k)}{P} - a
\end{aligned}$$

that is system (12). ■

Proof of Proposition 4

Assumption 2 ensures that a stationary level of capital k exists according to equation (13). The concavity of f ensures also that there is a unique stationary level of capital.

We apply first the Implicit Function Theorem to equation (17) to obtain a function $P(\lambda)$ with

$$P'(\lambda) = \frac{\frac{\lambda l_\lambda}{l}\frac{l}{\lambda}}{\frac{a}{bf(k)} - \frac{Pl_P}{l}\frac{l}{P}}$$

Noticing that, at the steady state, $l/P = a/(bf)$, we get the multiplier elasticity of pollution:

$$\zeta \equiv \frac{\lambda P'(\lambda)}{P(\lambda)} = \frac{\frac{\lambda l_\lambda}{l}}{1 - \frac{Pl_P}{l}}$$

Replacing $P = P(\lambda)$ into equation (16), we find

$$\varsigma(\lambda) \equiv \frac{c(\lambda, k, P(\lambda))}{l(\lambda, k, P(\lambda))} = \rho k + w(k) > 0$$

with

$$\begin{aligned}
\varepsilon_\varsigma(\lambda) &\equiv \frac{\lambda \varsigma'(\lambda)}{\varsigma(\lambda)} = \frac{\lambda c_\lambda}{c} - \frac{\lambda l_\lambda}{l} + \zeta\left(\frac{Pc_P}{c} - \frac{Pl_P}{l}\right) \\
&= \frac{\lambda c_\lambda}{c} - \frac{\lambda l_\lambda}{l}\frac{1 - \frac{Pc_P}{c}}{1 - \frac{Pl_P}{l}}
\end{aligned} \tag{42}$$

The continuity of ς implies that, if there are multiple steady state, the slope $\varsigma'(\lambda)$ changes its sign from a steady state to another. Conversely, if $\varsigma'(\lambda)$ is always negative at the steady state λ, then the steady state is unique.

Under Assumption 1 (separability), expression (42) writes

$$\varepsilon_{\varsigma}(\lambda) = \frac{\lambda c_{\lambda}}{c} - \frac{\lambda l_{\lambda}}{l} \frac{1 - \frac{P c_P}{c}}{1 - \frac{P l_P}{l}} = \frac{1}{\varepsilon_{cc}} - \frac{1}{\varepsilon_{ll}} \frac{1 + \frac{\varepsilon_{Pc}}{\varepsilon_{cc}}}{1 + \frac{\varepsilon_{Pl}}{\varepsilon_{ll}}} \quad (43)$$

(see elasticities (11)) with $\varepsilon_{cc} < 0$ and $\varepsilon_{ll} > 0$. Thus $\varepsilon_{\varsigma}(\lambda) < 0$ if and only if (18) holds. ■

Proof of Proposition 6

Necessity. In a three-dimensional dynamic system, we require at the bifurcation value: $\lambda_1 = ib = -\lambda_2$ with no generic restriction on λ_3 (see Bosi and Ragot (2011) or Kuznetsov (1998) among others). The characteristic polynomial of J is given by: $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - T\lambda^2 + S\lambda - D$. Using $\lambda_1 = ib = -\lambda_2$, we find $D = b^2\lambda_3$, $S = b^2$, $T = \lambda_3$. Thus, $D = ST$ and $S > 0$.

Sufficiency. In the case of a three-dimensional system, one eigenvalue is always real, the others two are either real or nonreal and conjugated. Let us show that, if $D = ST$ and $S > 0$, these eigenvalues are nonreal with zero real part and, hence, a Hopf bifurcation generically occurs.

We observe that $D = ST$ implies

$$\lambda_1\lambda_2\lambda_3 = (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)$$

or, equivalently,

$$(\lambda_1 + \lambda_2)\left[\lambda_3^2 + (\lambda_1 + \lambda_2)\lambda_3 + \lambda_1\lambda_2\right] = 0 \quad (44)$$

This equation holds if and only if $\lambda_1 + \lambda_2 = 0$ or $\lambda_3^2 + (\lambda_1 + \lambda_2)\lambda_3 + \lambda_1\lambda_2 = 0$. Solving this second-degree equation for λ_3, we find $\lambda_3 = -\lambda_1$ or $-\lambda_2$. Thus, (44) holds if and only if $\lambda_1 + \lambda_2 = 0$ or $\lambda_1 + \lambda_3 = 0$ or $\lambda_2 + \lambda_3 = 0$. Without loss of generality, let $\lambda_1 + \lambda_2 = 0$ with, generically, $\lambda_3 \neq 0$ a real eigenvalue. Since $S > 0$, we have also $\lambda_1 = -\lambda_2 \neq 0$. We obtain $T = \lambda_3 \neq 0$ and $S = D/T = \lambda_1 \lambda_2 = -\lambda_1^2 > 0$. This is possible only if λ_1 is nonreal. If λ_1 is nonreal, λ_2 is conjugated, and, since $\lambda_1 = -\lambda_2$, they have a zero real part. ■

Proof of Proposition 7

Necessity. In the real case, we obtain $D = \lambda_1\lambda_2\lambda_3 < 0$, $S = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$ and $T = \lambda_1 + \lambda_2 + \lambda_3 < 0$.

Sufficiency. We want to prove that, if $D, T < 0$ and $S > 0$, then $\lambda_1, \lambda_2, \lambda_3 < 0$. Notice that $D < 0$ implies $\lambda_1, \lambda_2, \lambda_3 \neq 0$.

$D < 0$ implies that at least one eigenvalue is negative. Let, without loss of generality, $\lambda_3 < 0$. Since $\lambda_3 < 0$ and $D = \lambda_1\lambda_2\lambda_3 < 0$, we have $\lambda_1\lambda_2 > 0$. Thus, there are two subcases: (1) $\lambda_1, \lambda_2 < 0$, (2) $\lambda_1, \lambda_2 > 0$. If $\lambda_1, \lambda_2 > 0$, $T < 0$ implies $\lambda_3 < -(\lambda_1 + \lambda_2)$ and, hence,

$$S = \lambda_1\lambda_2 + (\lambda_1 + \lambda_2)\lambda_3 < \lambda_1\lambda_2 - (\lambda_1 + \lambda_2)^2 = -\lambda_1^2 - \lambda_2^2 - \lambda_1\lambda_2 < 0$$

a contradiction. Then, $\lambda_1, \lambda_2 < 0$. ■

Proof of Proposition 8

By Proposition 6, we have $\operatorname{Re}\lambda_1(p_H) = \operatorname{Re}\lambda_2(p_H) = 0$. $\lambda_3(p_H) < 0$ is implied by $D(p_H) = [\operatorname{Im}\lambda_1(p_H)]^2 \lambda_3(p_H) < 0$. Thus, there exists $\varepsilon > 0$ such that, generically, we have $\operatorname{Re}\lambda_1(p), \operatorname{Re}\lambda_2(p), \lambda_3(p) < 0$ (local indeterminacy) for any $p \in (p_H - \varepsilon, p_H)$ or, alternatively, for any $p \in (p_H, p_H + \varepsilon)$. ■

Proof of Proposition 9

From (2) and (20), (3) writes

$$c = \left[\lambda P^{\eta(1-\varepsilon)}\right]^{-1/\varepsilon} \text{ and } l = \left[w\lambda P^{-\psi(1+\varphi)}/\omega\right]^{1/\varphi} \tag{45}$$

(13) gives (27). Equation (17) yields (28). Replacing (27) and (28) in (45) and (45) in (16), we find (26). ■

Proof of Lemma 18

The Jacobian matrix (19) becomes:

$$J = \begin{bmatrix} 0 & s\frac{\lambda}{k} & 0 \\ \frac{\partial f_2}{\partial \lambda} & \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial P} \\ a\frac{\lambda l_\lambda}{l}\frac{P}{\lambda} & a\left(\alpha + \frac{kl_k}{l}\right)\frac{P}{k} & a\left(\frac{Pl_P}{l} - 1\right) \end{bmatrix}$$

with

$$\begin{aligned} \frac{\partial f_2}{\partial \lambda} &= \frac{1}{\tau}\frac{k}{\lambda}\left[a\mu\frac{\lambda l_\lambda}{l} + \gamma\left(\frac{\lambda l_\lambda}{l} - \frac{\lambda c_\lambda}{c}\right)\right] \\ \frac{\partial f_2}{\partial k} &= \frac{1}{\tau}\left[a\mu\left(\alpha + \frac{kl_k}{l}\right) + \gamma\left(\frac{kl_k}{l} - \frac{kc_k}{c}\right) + \rho - \frac{s}{\varphi}\right] \\ \frac{\partial f_2}{\partial P} &= \frac{1}{\tau}\frac{k}{P}\left[a\mu\left(\frac{Pl_P}{l} - 1\right) + \gamma\left(\frac{Pl_P}{l} - \frac{Pc_P}{c}\right)\right] \end{aligned}$$

because, at the steady state,

$$\frac{c}{kl} = \gamma > 0,\ \frac{w}{k} = r\frac{1-\alpha}{\alpha} \text{ and } b\frac{lf(k)}{P} = a$$

Using (5), (6) and (22), we find

$$J = (m_{ij}) = \begin{bmatrix} 0 & s\frac{\lambda}{k} & 0 \\ \frac{n}{\tau}\frac{k}{\lambda} & \rho + a\mu\frac{\theta}{\tau} & -\frac{\xi + a\mu(1+\mu)}{\tau}\frac{k}{P} \\ \frac{a}{\varphi}\frac{P}{\lambda} & a\theta\frac{P}{k} & -a(1+\mu) \end{bmatrix}$$

■

Proof of Proposition 19

Focus on Proposition 6 and expressions (36) for D, S and T. We know that a Hopf bifurcation arises if and only if $D = ST$ and $S > 0$, that is if and only if

$$\begin{aligned}
\frac{as}{\tau}\left[(1+\mu)\left(n-\frac{a\mu}{\varphi}\right)-\frac{\xi}{\varphi}\right] &= \left[\frac{a\theta\xi-ns}{\tau}-a\rho(1+\mu)\right]\left(\rho-a+a\mu\frac{\theta-\tau}{\tau}\right)\\
\frac{a\theta\xi-ns}{\tau}-a\rho(1+\mu) &> 0
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
\xi_H &\equiv \frac{s(1+\mu)\left(n-\frac{a\mu}{\varphi}\right)+\left[\rho\tau(1+\mu)+\frac{ns}{a}\right]\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}{\frac{s}{\varphi}+\theta\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}\\
\xi_H &> \frac{ns+a\rho\tau(1+\mu)}{a\theta}\ (>0)
\end{aligned}$$

A Hopf bifurcation generically occurs if the following restriction is satisfied:

$$\frac{s(1+\mu)\left(n-\frac{a\mu}{\varphi}\right)+\left[\rho\tau(1+\mu)+\frac{ns}{a}\right]\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}{\frac{s}{\varphi}+\theta\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)} > \frac{ns+a\rho\tau(1+\mu)}{a\theta} \tag{46}$$

If

$$\frac{s}{\varphi}+\theta\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)>0 \tag{47}$$

(46) becomes equivalent to

$$a(1+\mu)(\varphi n\theta-\tau\rho-a\mu\theta)-ns>0 \tag{48}$$

Let us show that inequalities (47) and (48) are satisfied for some parametric values. Consider the case $a < \rho$ and $\alpha \approx 1$. Inequalities (47) and (48) become

$$\lim_{\alpha\to 1}\left[\frac{s}{\varphi}+\theta\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)\right]=\theta(\rho-a)>0$$

and

$$\lim_{\alpha\to 1}\left[a(1+\mu)(\varphi n\theta-\tau\rho-a\mu\theta)-ns\right]=a\rho\tau(1+\mu)\frac{\varphi}{\varepsilon}>0$$

because

$$\lim_{\alpha\to 1} n \equiv \mu\frac{a}{\varphi}+\rho\left(\frac{1}{\varepsilon}+\frac{1}{\varphi}\right)$$

■

Proof of Proposition 21

Notice that

$$D(p_H)=\frac{as}{\tau}\left[(1+\mu)\left(n-\frac{a\mu}{\varphi}\right)-\frac{\xi_H}{\varphi}\right]<0$$

and apply Proposition 8. ■

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* *

10 Calibration

Parameters.

$\theta = \alpha \frac{1+\varphi}{\varphi}$

$\tau = \frac{\alpha+\varphi}{\varphi}$

$\mu = \psi \frac{1+\varphi}{\varphi}$

$r = \rho + \delta$

$\gamma = \frac{r}{\alpha} - \delta$

$s = (1-\alpha) r$

$n = \mu \frac{a}{\varphi} + \gamma \left(\frac{1}{\varepsilon} + \frac{1}{\varphi} \right)$

$A = 1$

$\omega = 1$

$a = 0.01$

$b = 0.01$

$\alpha = 0.33$

$\delta = 0.025$

$\varepsilon = 2$

$\varphi = 0.5$

$\psi = 3.78$

$\rho = 0.01$

$$\eta_H \equiv \frac{\varepsilon}{\varepsilon-1} \left(\frac{\frac{s(1+\mu)\left(n-\frac{a\mu}{\varphi}\right)+\left(\rho\tau(1+\mu)+\frac{ns}{a}\right)\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}{\frac{s}{\varphi}+\theta\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}}{\gamma} - \mu \right)$$

$$\xi_H \equiv \frac{s(1+\mu)\left(n-\frac{a\mu}{\varphi}\right)+\left(\rho\tau(1+\mu)+\frac{ns}{a}\right)\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}{\frac{s}{\varphi}+\theta\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}$$

Steady state:

$$\eta = \frac{\varepsilon}{\varepsilon-1} \left(\frac{\frac{s(1+\mu)\left(n-\frac{a\mu}{\varphi}\right)+\left(\rho\tau(1+\mu)+\frac{ns}{a}\right)\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}{\frac{s}{\varphi}+\theta\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}}{\gamma} - \mu \right)$$

$$\xi = \frac{s(1+\mu)\left(n-\frac{a\mu}{\varphi}\right)+\left(\rho\tau(1+\mu)+\frac{ns}{a}\right)\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}{\frac{s}{\varphi}+\theta\left(\rho-a-a\mu\frac{\tau-\theta}{\tau}\right)}$$

$$\lambda = \left(\frac{B}{\rho k + (1-\alpha) A k^{\alpha}} \right)^{\frac{(\varphi+\psi+\varphi\psi)\varepsilon}{\varphi+\psi+\varphi\psi+\varepsilon+(1-\varepsilon)\eta}} = 0.156\,10$$

$$k = \left(\frac{\alpha A}{\rho+\delta} \right)^{\frac{1}{1-\alpha}} = 28.471$$

$P = C\lambda^{\frac{1}{\varphi+\psi+\varphi\psi}} = 0.907\,33$
$w = (1-\alpha)\,A\left(\left(\frac{\alpha A}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha}$
$B = \frac{C^{\frac{\varepsilon-1}{\varepsilon}\eta+\frac{1+\varphi}{\varphi}\psi}}{(w/\omega)^{\frac{1}{\varphi}}}$
$C = \left(Ak^{\alpha}\frac{b}{a}\left(\frac{w}{\omega}\right)^{\frac{1}{\varphi}}\right)^{\frac{\varphi}{\varphi+\psi+\varphi\psi}}$
$P = 0.907\,33$
Eigenvalues.

$$J = (m_{ij}) = \begin{bmatrix} 0 & s\frac{\lambda}{k} & 0 \\ \frac{n}{\tau}\frac{k}{\lambda} & \rho + a\mu\frac{\theta}{\tau} & -\frac{\xi+a\mu(1+\mu)}{\tau}\frac{k}{P} \\ \frac{a}{\varphi}\frac{P}{\lambda} & a\theta\frac{P}{k} & -a\,(1+\mu) \end{bmatrix}$$

$$\begin{aligned} \lambda_1 &= -0.0457699 \\ \lambda_2 &= 0.0679497i \\ \lambda_3 &= -0.0679497i \end{aligned}$$