Ranking distributions of an ordinal attribute*

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Abstract

This paper establishes foundational equivalences between alternative criteria for comparing distributions of an ordinally measurable attribute. . A first criterion is associated with the possibility of going from distribution to the other by a finite sequence of two elementary operations: increments of the attribute and Hammond transfers. The later transfers are like the famous Pigou-Dalton ones, but without the requirement - that would be senseless in an ordinal setting - that the "amount" transferred from the "rich" to the "poor" is fixed. A second criterion is a new easy-to-use statistical criterion associated to a specifically weighted recursion on the cumulative density of the distribution function. A third criterion is that resulting from the comparison of numerical values assigned to distributions by a large class of additively separable social evaluation functions. Dual versions of these criteria are also considered and alternative equivalence results are established. Illustrations of the criteria are also provided.

1 Introduction

When can we say that a distribution of *income* among a collection of individuals is more equal than another ? One of the greatest achievement of the modern theory of inequality measurement is the demonstration, made by Hardy, Littlewood, and Polya (1952) and popularized among economists by Kolm (1969) Atkinson (1970) Dasgupta, Sen, and Starrett (1973), Sen (1973) and Fields and Fei (1978), that the following *three* answers to this question are *equivalent*:

1) When one distribution has been obtained from the other by a finite sequence of Pigou-Dalton transfers.

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2) When one distribution would be considered better than the other by all utilitarian philosophers who assume that individuals convert income into utility by the same increasing and concave function.

3) When the Lorenz curve associated to one distribution lies nowhere below, and at least somewhere above, that of the other.

The equivalence of these three answers - often referred to as the Hardy-Littlewood-Polya (HLP) theorem - is an important result because it ties together three *a priori* distinct aspects of the process of inequality measurement.

The first one - contained in the Pigou-Dalton principle of transfers - is an *elementary transformation* of the distribution of income that captures in a crisp fashion the nature of the equalization process that is at stake. It provides a first, immediately palatable, answer to the question raised above. A distribution is more equal than another when it has been obtained from it by a finite sequence of such "clearly equalizing" Pigou-Dalton transfers.

The second aspect of the process of inequality measurement identified by the HLP theorem is the *ethical principle* underlying utilitarianism or, more generally, *additively separable* social evaluation. Attitude toward income inequality is clearly an ethical matter. It is therefore of importance to identify the ethical principles that rationalize the notion of inequality reduction underlying the Pigou-Dalton principle of transfers. While the HLP theorem points toward utilitarianism or additively separable social evaluation as a source of such rationalization, it can be shown (see e.g. Gravel and Moyes (2013)) that such a rationalization can also be obtained through a much more general class of social evaluation functions.

The third aspect of the process of inequality measurement captured by the HLP theorem is the *empirically implementable criterion* underlying Lorenz dominance. It is, indeed, immensely useful to have an implementable criterion like the Lorenz curve that enables one to check in an easy manner when one distribution dominates another. Comparisons of Lorenz curves have become a routine exercise that is performed every day by thousands of researchers worldwide. Moreover, compatibility with the Lorenz ranking of income distributions is now considered to be a minimal requirement that any numerical index of income inequality must satisfy. In this sense, the HLP theorem is, in the literal sense of the word, a *foundation* to income inequality measurement.

The current paper is concerned with establishing analogous foundations to the problem of comparing distributions of an *ordinal* or *qualitative* attribute among a collection of individuals. The last fifteen years have witnessed indeed an extensive use of data involving distributions of attributes such as access to basic services, educational achievements, health outcomes, and self-declared happiness to mention just a few. When performing normative comparisons of distributions of such attributes, it is not uncommon for researchers to disregard the ordinal measurability of the attribute and to treat it, just like income, as a variable that can be "summed", or "transferred" across individuals. Examples of those practices include Castelló-Clement and Doménech (2002) and Castelló-Clement and Doménech (2008) (who discusses inequality indices on human capital) and Pradhan, Sahn, and Younger (2003) (who decompose Theil indices applied to the heights of under 36 month children interpreted as a measure of health). Yet, following the influential contribution by Allison and Foster (2004), there has been a growing concern by researchers of duly accounting for the ordinal character of the numerical information conveyed by the indicators used in those studies. Examples of studies that have taken such a care, and have explicitly refused to use cardinal properties of the attribute when normatively evaluating its distribution, include Abul-Naga and Yalcin (2008) and Apouey (2007).

A difficulty with the normative evaluation of distributions of an ordinal attribute is that of defining an appropriate notion of inequality reduction in that context. What does it mean indeed for an ordinal attribute to be "more equally distributed" than another ? The usual notion of a Pigou-Dalton transfer used to answer this question in the case of a cardinally measurable attribute is of no clear use for that purpose. Recall indeed that a Pigou-Dalton transfer is the operation by which an individual transfers to someone with a lower quantity of the attribute a *certain quantity* of the attribute. Such a transfer is obviously meaningless if the "quantity" of the attribute is ordinal. While ordinal measurability of the attribute - provided that it is comparable across different individuals - enables one to identify which of two individuals has "more" of the attribute than the other, it does not enable one to quantify further the statement. It does not enable one to talk about a "certain quantity" of the attribute that can be transferred across individuals.

Some forty years ago, Peter J. Hammond (1976) has proposed, in the specific context of social choice theory, a so-called "minimal equity principle" that was explicitly concerned with distributions involving an ordinally measurable attribute. According to Hammond's principle, a change in the distribution that "reduces the gap" between two individuals endowed with different quantities of the ordinal attribute is a good thing, irrespective of whether or not the "gain" from the poor recipient is equal to the "loss" from the rich giver. A Pigou-Dalton transfer is clearly a particular case of a Hammond transfer, that imposes on the later the additional requirement that the amount given by the rich should be equal to the amount received by the poor. It seems to us that the purely ordinal nature of Hammond transfers qualifies them as a highly plausible instances of "ordinal inequality reduction". A distribution of an ordinal attribute is unquestionably more equal than another when it has been obtained from the later by a finite sequence of such Hammond transfers. This paper identifies a normative dominance criterion and a statistically implementable criterion that are each equivalent to the notion of equalization underlying Hammond transfers. It does so in the specific, but empirically important, case where the ordinal attribute can take only a *finite* number different values. This finite case is somewhat specific indeed for discussing Hammond equity principle. For, as is well-known in social choice theory (see e.g. D'Aspremont and Gevers (1977), D'Aspremont (1985), Sen (1977)), the Hammond equity principle is closely related to "Maximin" or "Leximin" types of criteria that rank distributions of an ordinal attribute on the basis of the smallest quantity of the attribute. Bosmans and Ooghe (2013) have even shown that any anonymous, continuous, and Pareto-inclusive transitive ranking of all vectors of $\mathbb{R}^n$ that are sensitive to Hammond transfer must be the Maximin criterion that compares such vectors on the sole basis of the size of their smallest component. As shall be seen in this paper, this apparently tight connection between Maximin or Leximin principles and Hammond transfers becomes significantly looser when attention is restricted to distributions of an attribute that can take finitely many different values.

As for the normative principle, we stick to the tradition of comparing distributions of the attribute by means of an additively separable social evaluation function. Each individual attribute quantity is thus assigned a numerical value by some function, and alternative distributions of attributes are compared on the basis of the sum - taken over all individuals - of these values. While this normative approach *can be* considered utilitarian (if the function that assigns a value to the attribute is interpreted to be a "utility" function), it *does not need to*. One could also interpret the function more generally as an *advantage function* reflecting the value assigned to each individual attribute quantity by the "social planner" or the "ethicist". It is also possible to justify axiomatically this additively separable normative evaluation in a non-welfarist setting (see e.g. Gravel, Marchant, and Sen (2011)). As the ordinal attribute (health indicator, education, access to basic services) is considered to be a good thing to the individual, it is assumed that the advantage function is increasing with respect to the attribute. We show in this paper that, in order for a ranking of distributions based on an additively separable social evaluation function to be sensitive to Hammond transfers, it is necessary and sufficient for the advantage function to satisfy a somewhat strong "concavity" property. Specifically, any increase in the quantity of the attribute obtained from some initial level must increase the advantage more than any increase in the attribute taking place at some higher level of the attribute, no matter what the later increase is. Because of this result, we therefore consider the ranking of distributions of the ordinal attribute that coincides with the unanimity of all additively separable rankings who use an advantage function that satisfies this property.

The empirical implementable criterion that we consider is, to the very best of our knowledge, a new one. Its construction is based on a curve that we call the H^+-curve, by reference to the Hammond principle of transfer to which it is, as it turns out, closely related. The H^+ curve is defined recursively as follows. It starts by assigning to the lowest possible quantity of the ordinal attribute the fraction of the population that are endowed with this quantity. For this lowest category, this number is nothing else than the relative frequency of the category or, equivalently for the lowest category, its cumulative frequency. One then proceeds recursively, for any quantity of the attribute strictly larger than this smallest quantity, by adding together the relative frequency of the population endowed with this quantity of the attribute *and* twice the value assigned by the curve to the immediately preceding category. The innovation of this curve lies precisely in its addition, to any fraction of the population falling in some category, of *twice* the value assigned by the curve to the immediately preceding category. This (doubly) larger weight given to the preceding category as compared to any given category reflects, of course, the somewhat strong redistributive flavour of the Hammond's equity principle. The criterion that we propose, and that we call H^+-dominance, is for the dominating distribution to have a H^+-curve nowhere above and somewhere below that of the dominated one. As illustrated below, the construction of these curves, and the resulting implementation of the criterion, is easy. This paper provides some justification for the use of such a H^+-curve. It does so by proving that the fact of having a distribution of an ordinal attribute that H^+-dominates another is equivalent to the possibility of going from the latter to the former by a finite sequence of Hammond transfers and/or increments of the attribute. The paper also shows that the H^+-dominance criterion coincides with the unanimity of all additively separable aggregation of advantage functions that use an advantage function that is strongly decreasingly increasing in the sense given above.

While these results justify comparing distributions of an ordinal attribute on the basis of H^+- dominance, they do not readily lead to a definition of what it means for a distribution or an ordinal attribute to be "more equal" than another. In effect, H^+ dominance combines together *both* Hammond transfers *and* increments. While the former transformation is a plausible candidate for a definition of an inequality reduction in an ordinal setting, the later is not. Could one identify an operational criterion that coincides with Hammond transfers *only* and that could serve, as a result, as an operational definition of inequality reduction in an ordinal setting ?

A parallel with the classical cardinal case - in which Pigou-Dalton transfers are commonly perceived to be synonymous with inequality reduction - can be useful to understand this point. In this classical setting, it is well-known that if two distributions of an attribute have the same sum (or the same mean), then the Lorenz domination of one distribution by the other is equivalent to the possibility of going from the dominated distribution to the dominating one by a finite sequence of Pigou-Dalton transfers. For distributions with the same mean therefore, inequality reduction in the sense of Pigou-Dalton transfers is equivalent to Lorenz domination. If there was a meaningful analogue - in the ordinal setting - to the requirement that two distributions have the "same mean", one could think at paralleling - for that analogue - the standard approach and at identifying H^+ domination with Hammond transfers only. Unfortunately there is no such meaningful analogue to the mean in the ordinal setting. However, there is another (less well-known) path taken from the theory of majorization (see e.g. Marshall, Olkin, and Arnold (2011)) that can be followed to handle that issue. From that theory (see Marshall, Olkin, and Arnold (2011) (chapter 1, p. 13, equation (14)), it can be noticed that Lorenz dominance between two distributions of a cardinal attribute with the same sum is nothing else than the *intersection* of *two* independent majorization quasi-orderings that do not assume anything about the "sum" of the attribute. The first one, called *submajorization* by Marshall, Olkin, and Arnold (2011), defines a distribution d to be better than a distribution d' if, for any rank k, the sum of income of the individuals weakly richer than k is weakly smaller in d than in d'. The second criterion, called *supermajorization* by Marshall, Olkin, and Arnold (2011), corresponds to the more usual generalized Lorenz domination criterion (see e.g. Shorrocks (1983)) according to which distribution d is better than distribution d' if the sum of the k lowest incomes is weakly larger in d than in d' no matter what is k. In this paper, we parallel the route taken by Marshall, Olkin, and Arnold (2011) by exploring a dual domination criterion based on the H^- curve of a distribution of an ordinal attribute. This curve is constructed just like the H^+ one but, contrary to the later, it starts from "above" (that is from the highest ordinal category) rather than from "below", and iteratively cumulate the (discrete) survival function rather than the (discrete) cumulative function. We then establish that this notion of H^- domination coincides with the possibility of going from the dominated distribution to the dominating one by a finite sequence of either Hammond transfers and/or decrements. As for the submajorization and supermajorization criteria in the classical cardinal setting, the ranking of distributions generated by the intersection of the two domination criteria H^+ and H^- could serve as a plausible instance of a clear inequality reduction in an ordinal setting. Indeed, as established in this paper, any finite sequence of Hammond transfers would be recorded as an improvement by this intersection ranking.

While we do not have available a proof of the converse implication, we do have results that lead somewhat toward it. One of theses results is the equivalence that we establish between the intersection of the H^+ and the H^- dominance criteria on the one hand, and the intersection of all rankings produced by additively separable social evaluation functions that use a strongly concave - in the sense alluded to above - but not necessarily monotone individual advantage function on the other. Another such a result is the examination of the two H^+ and H^- criteria that we provide when the finite grid used to define the categories is *refined*. Indeed, every criterion considered in this paper depends upon the finite grid of attribute values that is considered. What happens to the criterion when the grid thus considered is refined ? We show in this paper that there exists a level of grid refinement above which the H^+ domination criterion coincides with the Leximin ranking of the two distributions. Similarly we indicate that there exists a level of grid refinement above which the H^- domination criterion coincides with the (anti) Leximax ranking of those two distributions. Notice that these results are consistent with those obtained in classical social choice theory (see Hammond (1976), Deschamps and Gevers (1978), D'Aspremont and Gevers (1977), Sen (1977)). The Leximin criterion can be viewed as the limit of the H^+ criterion when the number of different categories of the attribute becomes arbitrarily large. Similarly the anti Leximax criterion can be viewed as the limit of the H^- criterion when the number of different categories becomes large. It follows of course from these two remarks that the intersection of the anti Leximax and the Leximin criteria is the limit of the intersection of the H^+ and the H^- criteria when the number of categories becomes large. Since, as proved in Gravel, Magdalou, and Moyes (2015), the intersection of the anti-Leximax and the Leximin criteria is the transitive closure of the binary relation "is obtained from by a Hammond transfer" applied to all vectors in a closed and convex subset of $\mathbb{R}^n$, we view this result as a suggestive evidence that the possibility of going from a distribution B to a distribution A by a finite sequence of Hammond transfers is closely related to the dominance of B by A by both the H^+ and the H^- criterion.

The plan of the rest of the paper is as follows. The next section introduces the notation, discusses the general problem of comparing distributions of an ordinal attribute, and presents the normative dominance notions, the elementary transformations (Hammond transfers and increments/decrements) and the implementation criteria (H curves) in the specific setting where the ordinal attribute can take finitely many different values. The main results identifying the elementary transformations underlying the H^+ and the H^-curves are stated and proved in the fourth section. The fourth section discusses the connection between the discrete setting in which the analysis is conducted and the classical social choice one in which the Hammond equity principle has been originally proposed and establishes the results mentioned above on the behavior of the H^+ (and the H^-) criterion when the number of different categories is enlarged. The fifth section illustrates the usefulness of the criteria for comparing distributions of Body Mass Index categories among genders in France, UK and the US. The sixth section concludes.

2 Three perspectives on comparing distributions of an ordinal attribute

2.1 Normative evaluation

We consider distributions of an ordinal attribute between a *given* number - n say - of individuals, indexed by i. As is standard in distributional analysis since at least Dalton (1920), distributions of the attribute between a *varying* number of individuals can be compared by means of the principle of *population replication* (replicating a distribution any number of times is a matter of social indifference). We assume that there are k (with $k \geq 3$) different values that the ordinal attribute can take. These values, which can be interpreted as "categories" (e.g. "being gravely ill", "being mildly ill", "being in perfect health"), are indexed by h. We denote by $\mathcal{C} = \{1, ..., k\}$ the set of these categories, assumed to be ordered from worst (e.g. being gravely ill) to the best (e.g. being in perfect health). The fact that the attribute is ordinal means that the integers $1, ..., k$ assigned to the different categories have no particular significance other than ordering the categories from the worst to the best. Hence, any comparative statement made on two distributions in which the attribute quantity is measured by the list of numbers $\{1, ..., k\}$ would be unaffected if this list was replaced by the list $f(1), ..., f(k)$ generated by any strictly increasing real valued function f admitting C as its subdomain.

A society $s = (s_1, ..., s_n) \in \mathcal{C}^n$ is a particular assignment of these categories to the n individuals, where s_i is the category assigned to i in society s. For any society s, and every category j, one can define the number n_j^s of individuals who, in s, falls in category j by:

$$n_j^s = \#\{i \in \{1, ..., n\} : s_i = j\}$$

We of course notice that $\sum_{i=1}^{k} n_i^s = n$ for every society s. If one adopts an *anonymous* perspective according to which "the names of the individuals do not matter", then the integers n_j^s (for $j = 1, ..., k$) summarize all the ethically relevant information about society s. The current paper adopts this anonymous perspective and examines, more specifically, the family of **add**itively separable rankings $\succsim^{add}$ of societies in $\mathcal{C}^n$ that can be defined, for any two societies s and s' in $\mathcal{C}^n$, by:

$$s \succsim^{add} s' \Longleftrightarrow \sum_{j=1}^{k} n_j^s \alpha_j \geq \sum_{j=1}^{k} n_j^{s\prime} \alpha_j \tag{1}$$

for some list of k real numbers α_j (for $j = 1, ..., k$). These numbers can of course be seen as numerical evaluations of the corresponding categories (in which case we could require them to satisfy $\alpha_1 < ... < \alpha_k$). These valuations may reflect subjective utility (if a utilitarian perspective is adopted) or a non-welfarist appraisal made by the social planner of the fact, for someone, to falling in the different categories to which these numbers are assigned. If such a non-welfarist perspective is adopted, the specific additive form of the numerical representation 1 of the social ordering can be axiomatically justified (see e.g. Gravel, Marchant, and Sen (2011) for such a justification in a variable population context).

The ordinal interpretation of the categories suggests that some care be taken in avoiding the normative evaluation exercise to be unduly sensitive to particular choices of the numbers α_j (for $j = 1, ..., k$). A standard way to exert such a care is to require the unanimity of evaluation over a wide class of such lists of k numbers. This underlies the following general definition of normative dominance.

Definition 1 *For any two societies s and s' in $\mathcal{C}^n$, we say that s normatively dominates s' for a family $\mathcal{A} \subset \mathbb{R}^k$ of evaluations of the k categories, denoted $s \succsim^{\mathcal{A}} s'$, if one has:*

$$\sum_{j=1}^{k} n_j^s \alpha_j \geq \sum_{j=1}^{k} n_j^{s\prime} \alpha_j$$

for all $(\alpha_1, ..., \alpha_k) \in \mathcal{A}$.

The elementary transformations and implementable criteria that will be discussed in the next two subsections will be shown to be somewhat tightly associated with specific - but somewhat large - families of evaluations of the k categories.

2.2 Elementary transformations

The definition of these transformations lies at the very heart of the problem of comparing alternative distributions of an ordinally measured attribute. Indeed, these transformations are intended to capture in a crisp and concise fashion one's intuition about the meaning of "equalizing" or "gaining in efficiency" (among others). In defining these transformation, it is important to make sure that they use only ordinal property of the attribute. In this paper, we discuss *three* such transformations.

The *first* of them - the increment - is hardly new. It captures the intuitive idea - somewhat related to efficiency - that giving to someone - up to a permutation thanks to anonymity - additional quantities of the ordinal attribute without reducing - up to a permutation - the quantity of attribute of the others is a good thing. We actually formulate this principle in the following minimalist fashion.

Definition 2 *(Increment) We will say that society s has been obtained from society s' by means of an increment, if there exist $j \in \{1, \ldots, k-1\}$ such that:*

$$n_h^s = n_h^{s'}, \ \forall \ h \neq j, j+1 \, ; \tag{2}$$

$$n_j^s = n_j^{s'} - 1 \ ; \ n_{j+1}^s = n_{j+1}^{s'} + 1 \, . \tag{3}$$

In words, society s has been obtained from society s' by an increment if the move from s' to s is the sole result of the move of one individual from a category j to the immediately superior category $j+1$. The notion of increment is the minimal formulation of the idea that increasing (up to a permutation of individuals) the quantity of the attribute for someone is a good thing if it does not reduce the attribute of others. In a somewhat reverse fashion, we can also introduce the notion of a *decrement* as follows.

Definition 3 *(Decrement) We will say that society s has been obtained from society s' by means of a decrement, if there exist $j \in \{1, \ldots, k-1\}$ such that:*

$$n_h^s = n_h^{s'}, \ \forall \ h \neq j, j+1 \, ; \tag{4}$$

$$n_j^s = n_j^{s'} + 1 \ ; \ n_{j+1}^s = n_{j+1}^{s'} - 1 \, . \tag{5}$$

In words, society s has been obtained from society s' by an decrement if the move from s' to s is the sole result of the move of one individual from a category $j+1$ to the immediately inferior category j. A decrement would be a normatively improving operation if the the categories $\{1, ..., k\}$ were decreasingly ordered or if the ordinal attribute was considered to be a "bad", rather than a "good". The following obvious, and therefore unproved, remark recalls that a decrement is nothing else than the converse of an increment.

Remark 1 *Society s has been obtained from society s' by means of a decrement if and only if society s' has been obtained from society s by means of an increment.*

The third elementary transformation considered in this paper is that underlying the *equity principle* put forth by Peter J. Hammond (1976) some forty years ago. It reflects an appealing - if not strong - notion of *aversion to inequality* in contexts where the distributed attribute - taken to be utility in Hammond (1976) - is ordinal. This principle considers that a reduction in someone's endowment of the attribute compensated by an increase in the endowment of some other person is a good thing if the looser remains, after the loss, better off than the winner. While a reduction in someone's endowment that is compensated by an increase in that of someone else may be viewed as the result of a "transfer" of attribute between the two persons, it is worth noticing that, contrary to what is the case with standard Pigou-Dalton transfers, the "quantity" of the attribute given by the donor *needs not* be equal to that received by the recipient. A Hammond transfer may involve taking "a lot" of attribute away from a "rich" person in exchange of giving "just a little bit" to a poorer one. It may, conversely, entail the transformation of a "small" quantity" taken from the rich into a "large" one given to the poor. As the comparisons of gains and loss of an ordinal attribute is meaningless, the Hammond transfer may be viewed as the natural analogue, in the ordinal setting, of the Pigou-Dalton transfer used in the cardinal one. The precise definition we give of a Hammond transfer is the following.

Definition 4 (Hammond's transfer) *We will say that society s is obtained from society s' by means of a Hammond's (progressive) transfer, if there exist four categories $1 \leq g < i \leq j < l \leq k$ such that:*

$$n_h^s = n_h^{s'}, \ \forall \ h \neq g, i, j, l \, ; \tag{6a}$$

$$n_g^s = n_g^{s'} - 1 \ ; \ n_i^s = n_i^{s'} + 1 \, ; \tag{6b}$$

$$n_j^s = n_j^{s'} + 1; \ n_l^s = n_l^{s'} - 1 \, . \tag{6c}$$

It should be noticed that a Pigou-Dalton transfer is, in the current discrete setting, nothing else than a Hammond transfer for which the indices g, i, j and l of definition 4 satisfy the additional condition that $i - g = l - j$ (a given - by $i - g$ - quantity of the attribute is transferred). This additional condition have little meaning given the ordinal nature of the attribute. But it could certainly be imposed if, as in the literature on discrete first and second order dominance (see e.g. Fishburn and Lavalle (1995) or Chakravarty and Zoli (2012)), the gap between any two adjacent categories was believed to be meaningfully constant.

2.3 Implementation criteria

Three implementation criteria are considered in this paper. The first *one - first order stochastic dominance - is standard in the literature. Its formal definition in the current setting makes use of the* cumulative distribution *function associated to a society s, that is denoted, for every $i \in \mathcal{C}$, by $F(i;s)$ and that is defined, for $i = 1, ..., k$, by:*

$$F(i;s) = \sum_{h=1}^{i} n_h^s / n \,. \tag{7}$$

Using this definition, one can define first order dominance as follows.

Definition 5 *(1st order dominance) We will say that society s first order dominates society s', which we write $s \succsim_1 s'$, if and only if:*

$$F(i;s) \leq F(i;s') \,, \quad \forall i = 1, 2, \ldots, k \,. \tag{8}$$

(remembering of course that $F(k;s) = \sum_{h=1}^{k} n_h^s / n = 1$ for any society s).

The *second* implementation criterion examined in this paper is based on the following H^+ curve (defined for any society s and any $i \in \{1, ..., k\}$):

$$H^+(i;s) = \sum_{h=1}^{i} \left(2^{i-h}\right) n_h^s / n \,. \tag{9}$$

A few remarks can be made about this curve.

First, it verifies:

$$H^+(1;s) = F(1;s) \tag{10}$$

and:

$$H^+(i;s) = \sum_{h=1}^{i-1} \left(2^{i-h-1}\right) F(h;s) + F(i;s) \,, \ \forall i = 2, 3, \ldots, k. \tag{11}$$

The different values of $H^+(\cdot;s)$ are therefore nested. Moreover, for any $i = 2, 3, \ldots, k$, we have:

$$H^+(i;s) = 2\, H^+(i-1;s) + F(i;s) - F(i-1;s) = 2\, H^+(i-1;s) + n_i^s / n. \tag{12}$$

Hence, by successive decomposition, one obtains, for all $i = 2, 3, \ldots, k$:

$$H^+(i;s) = \left(2^j\right) H^+(i-j;s) + \sum_{h=0}^{j-1} \left(2^h\right) \frac{n_{i-h}^s}{n} \,, \ \forall j = 1, 2, \ldots, i-1 \,. \tag{13}$$

This curve gives rise to the following notion of dominance - called H^+ dominance.

Definition 6 *(H^+ dominance) We will say that society s H^+-dominates society s', which we write $s \succeq_{H^+} s'$, if and only if:*

$$H^+(i;s) \leq H^+(i;s'), \quad \forall i = 1,2,\ldots,k\,. \tag{14}$$

A few additional remarks are in order on the H^+ curves and the H^+-dominance criterion to which they give rise. The definition of the H^+ curve as per expression (11) makes clear that first order stochastic dominance implies H^+ dominance. In plain English, $H^+(i;s)$ is a (specifically) weighted sum of the fractions of the population in s that are in weakly worse categories than i. The weight assigned to the fraction of the population in category h (for $h < i$) in that sum is 2^{i-h}. Hence the weights are (somewhat strongly) decreasing with respect to the categories. A nice feature of the H^+ curve - that appears strikingly in formula (12) - is its recursive construction, that is quite similar to that underlying the cumulative distribution curve. The cumulative distribution F can indeed be defined recursively by:

$$nF(1;s) = n_1^s \tag{15}$$

and, for $i = 2, ..., k$, by:

$$nF(i;s) = nF(i-1;s) + n_i^s \tag{16}$$

The recursion that defines H^+ starts just in the same way than as in (15) with:

$$nH^+(1;s) = n_1^s$$

but has the iteration formula (16) replaced by:

$$nH^+(i;s) = 2nH^+(i-1;s) + n_i^s$$

The *last* implementation criterion examined in this paper is somewhat dual to H^+ dominance. Its formal definition makes use of the *complementary cumulative distribution* function - also known as the survival function - associated to a society s, that is denoted, for every $i \in \mathcal{C}$, by $\overline{F}(i;s)$ and that is defined by:

$$\begin{aligned} \overline{F}(i;s) &= 1 - F(i,s) && (17) \\ &= \sum_{h=i+1}^{k} n_h^s/n \text{ for } i = 1, ..., k-1 \text{ and,} && (18) \\ &= 0 \text{ for } i = k && (19) \end{aligned}$$

Hence $\overline{F}(i;s)$ is the fraction of the population in s who is in a *strictly better* category than i. For technical reasons in some of the proofs below, we find useful to extend the domain of definition of $\overline{F}(.;s)$ from $\mathcal{C} = \{1, ..., k\}$ to $\{0, ..., k\}$ and to set $\overline{F}(0;s) = 1$. With this notation, one can define the last implementation criterion examined in this paper by means of the following H^- curve defined, for any society s and any $i \in \{1, ..., k\}$, by:

$$H^-(i;s) = \sum_{h=i+1}^{k} \left(2^{h-i-1}\right) n_h^s\,/n \text{ for } i = 1, ...k-1 \tag{20}$$

and:

$$H^-(k;s) = 0 \tag{21}$$

This curve is thus constructed under exactly the same recursive principle than the H^+ one, but starting with the highest category, and iterating with the complementary cumulative distribution function rather than with the standard cumulative one. The H^- curves therefore starts at category $k-1$:

$$H^-(k-1;s) = \overline{F}(k-1;s) = n_k^s/n \tag{22}$$

and satisfies:

$$H^-(i;s) = \sum_{h=i+1}^{k} \left(2^{h-i-1}\right)\overline{F}(h;s) + \overline{F}(i;s)\,,\ \forall i = 1,2,\ldots,k-1. \tag{23}$$

so that the different values of $H^-(\cdot;s)$ are nested starting from above and going below. Moreover, for any $i = 1,2,\ldots,k-2$ we have:

$$H^-(i;s) = 2\,H^-(i+1;s) + \overline{F}(i;s) - \overline{F}(i+1;s) = 2\,H^-(i+1;s) + n_i^s/n. \tag{24}$$

Hence, just as in expression (13) above, one obtains, by successive decomposition, for all $i = 1,2,\ldots,k-2$:

$$H^-(i;s) = \left(2^{k-j}\right)H^-(j-i;s) + \sum_{h=j+1}^{k}\left(2^{h-j-1}\right)\frac{n_{h-i}^s}{n}\,,\ \forall j = i+1,\ldots,k-1\,. \tag{25}$$

This curve gives rise to the following notion of dominance.

Definition 7 *(H^- dominance) We will say that society s H^--dominates society s', which we write $s \succsim_{H^-} s'$, if and only if:*

$$H^-(i;s) \leq H^-(i;s')\,,\quad \forall i = 1,2,\ldots,k\,. \tag{26}$$

As illustrated in section 6, these two curves, constructed recursively from the cumulative and complementary cumulative distribution functions thanks to formula (12) and (24), are easy to use and draw. As will also be seen in the next section, the two dominance criteria that they generate provide exact diagnostic test of the possibility of moving from the dominated distribution to the dominating one by Hammond transfers and increment (for H^+ dominance) or decrements (for H^+ dominance). Moreover, the additional criterion provided by the intersection of H^- and H^+ dominance - that we referred to as H-dominance - will turn out to appear tightly related to the notion of equalization contained in the Hammond principle of transfers. For future reference, we define formally this notion of H dominance as follows.

Definition 8 *(H dominance) We will say that society s H-dominates society s', which we write $s \succsim_H s'$, if and only if one has both $s \succsim_{H^+} s'$ and $s \succsim_{H^-} s'$.*

We end this section by providing links between some of these notions of implementable dominance. Specifically, we show that first order dominance of a society s' by a society s entails the H^+ dominance of s' by s *and* the H^- dominance of society s by s'. As for all formal results of this paper, the proof of those results are gathered in the Appendix.

Proposition 1 *Let s and $s' \in \mathcal{C}^n$. Then $s \succsim_1 s' \Longrightarrow s \succsim_{H^+} s' \wedge s' \succsim_{H^-} s$.*

3 Equivalence results

In this section, we provide a few theorems that connect together normative dominance - as per definition 1 - over specific classes of collections of numbers $\alpha_1, ..., \alpha_k$ on the one hand and, on the other, specific implementable dominance criterion as well as the possibility of going from the dominated to the dominating distribution by appropriate elementary transformations of the distributions.

We start by stating the following technical result, proved in the Appendix, that establishes an important decomposition property of the additively separable criterion underlying the notion of normative dominance favoured in this paper.

Lemma 1 *For any $s \in \mathcal{C}^n$ and any conceivable collection of k numbers $(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k$, one has:*

$$\frac{1}{n}\sum_{h=1}^{k} n_h^s \alpha_h = \alpha_k - \sum_{h=1}^{k-1} F(h; s) [\alpha_{h+1} - \alpha_h] , \tag{27}$$

or equivalently:

$$\frac{1}{n}\sum_{h=1}^{k} n_h^s \alpha_h = \alpha_1 + \sum_{h=1}^{k-1} \bar{F}(h; s) [\alpha_{h+1} - \alpha_h] . \tag{28}$$

Moreover, for all $t = 2, 3, \dots, k-1$, one has:

$$\frac{1}{n}\sum_{h=1}^{k} n_h^s \alpha_h = \alpha_t - \sum_{h=1}^{t-1} F(h; s) [\alpha_{h+1} - \alpha_h] + \sum_{h=t}^{k-1} \bar{F}(h; s) [\alpha_{h+1} - \alpha_h] . \tag{29}$$

Endowed with this result, we start with the notions of increment and decrement, as these elementary transformations can be appraised by normative dominance. Indeed, suppose that we compare alternative societies on the basis of a criterion that can be numerically represented by expression (1) for some list $\alpha_1, ..., \alpha_k$ of real numbers. A first question that arises is: what properties must these k real numbers satisfy for a criterion numerically represented as per (1) to consider an increment (a decrement) as a definite social improvement. It should not come as a surprise that the answer to this question is for the k numbers to belong to the two following sets:

$$\mathcal{A}_1^+ = \{(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k : \alpha_1 \leq ... \leq \alpha_k\}$$

and

$$\mathcal{A}_1^- = \{(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k : \alpha_1 \geq ... \geq \alpha_k\}$$

The set $\mathcal{A}_1^+$ is indeed the largest set of valuations of the k categories over which normative dominance (as per definition 1) is consistent with the idea that the "welfare" or the "advantage" associated to the categories by the social planner be increasing with respect to them. It is clear that any criterion that writes as per (1) for a list $(\alpha_1, ..., \alpha_k)$ of numbers in the set $\mathcal{A}_1^+$ will consider an increment as a normative improvement. Similarly, the set $\mathcal{A}_1^-$ is the largest collection of lists of k numbers that are consistent with the idea that the social valuation of the categories is decreasing (rather than increasing) with respect to those. Any additively separable normative judgement made by a criterion that can be represented as per (1) for a list $(\alpha_1, ..., \alpha_k)$ of numbers in $\mathcal{A}_1^-$ will consider a decrement to be a good thing (or an increment to be a bad one).

The following two propositions establish this formally.

Proposition 2 *Let s be a society that has been obtained from s' by an increment as per definition 2. Then $s \succsim^{\mathcal{A}} s'$ if and only if $\mathcal{A} = \mathcal{A}_1^+$.*

Proposition 3 *Let s be a society that has been obtained from s' by an decrement as per definition 3. Then $s \succsim^{\mathcal{A}} s'$ if and only if $\mathcal{A} = \mathcal{A}_1^-$.*

We now use these propositions to establish, with the help of Lemma 1, the following two theorems.

The first theorem has been known for quite a long time (see e.g. Lehmann (1955) or Quirk and Saposnik (1962)). We nonetheless provide a proof of part of it for completeness and for later use in the proof of the important Theorem 3 below.

Theorem 1 *For any two societies s and $s' \in \mathcal{C}^n$, the following three statements are equivalent:*
(a) *s is obtained from s' by means of a finite sequence of increments,*
(b) *$s \succsim^{\mathcal{A}_1^+} s'$,*
(c) *$s \succsim_1 s'$.*

The second theorem is dual to the previous one. It links decrements to normative dominance over the set $\mathcal{A}_1^-$ of valuations of the k categories on the one hand and (anti) stochastic dominance on the other. The formal statement of this theorem - whose proof, similar to that of theorem 1, is left to the reader - is as follows.

Theorem 2 *For any two societies s and $s' \in \mathcal{C}^n$, the following three statements are equivalent:*
(a) *s is obtained from s' by means of a finite sequence of decrements,*
(b) *$s \succsim^{\mathcal{A}_1^-} s'$,*
(c) *$s' \succsim_1 s$.*

We now turn to Hammond transfers. In a parallel fashion to what has been established before propositions 2 and 3, we first ask under what conditions on the set $\mathcal{A}$ of valuations of categories is normative dominance - as per definition 1 - sensitive to Hammond transfers (as per definition 4). It turns out that the answer to that question involves the following subset $\mathcal{A}_{\mathcal{H}}$ of $\mathbb{R}^k$

$$\mathcal{A}_{\mathcal{H}} = \left\{(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k \,|\, (\alpha_i - \alpha_g) \geq (\alpha_l - \alpha_j) \text{ for } 1 \leq g < i \leq j < l \leq k\right\} \tag{30}$$

In words, $\mathcal{A}_{\mathcal{H}}$ contains all lists of categories' evaluations that are "strongly concave" with respect to these categories in the sense that the "utility" or "advantage" gain of moving from one category to a better one is always larger when done from categories in the bottom part of the scale than when done in the upper part of it. The following proposition - proved in the Appendix - establishes that the set $\mathcal{A}_{\mathcal{H}}$ of categories' valuation is indeed the largest one over which normative dominance (definition 1) considers favorably the notion of equalization underlying Hammond transfer.

Proposition 4 *Let s, $s' \in \mathcal{C}^n$, and assume that s is obtained from s' by means of a Hammond transfer. Then $s \succsim_{\mathcal{A}} s'$ if and only if $\mathcal{A} = \mathcal{A}_{\mathcal{H}}$.*

The intuition that the set $\mathcal{A}_{\mathcal{H}}$ is capturing a "strong concavity" property is, perhaps, better seen though the following proposition, also proved in the Appendix, which establishes that $\mathcal{A}_{\mathcal{H}}$ contains all lists of categories' valuations that are "single-peaked" in the sense of admitting a largest value before which they are increasing at a (strongly) decreasing rate and after which they are decreasing at a (strongly) increasing rate.

Proposition 5 *A list $(\alpha_1, ..., \alpha_k)$ of real numbers belongs to $\mathcal{A}_{\mathcal{H}}$ if and only if there exists a $t \in \{1, ..., k\}$ such that $(\alpha_{i+1} - \alpha_i) \geq (\alpha_t - \alpha_{i+1})$ for all $i = 1, 2, \ldots, t-1$ (if any) and $(\alpha_{i'+1} - \alpha_{i'}) \leq (\alpha_{i'} - \alpha_t)$, for all $i' = t, t+1, \ldots, k-1$ (if any).*

Two particular "peaks" among those identified in Proposition 5 are of particular importance. One is when $t = k$ so that the numbers $\alpha_1, ..., \alpha_k$ are increasing (at a strongly decreasing rate) with respect to the categories. In this case, the elements of $\mathcal{A}_{\mathcal{H}}$ are also in $\mathcal{A}_1^+$, and we denote by $\mathcal{A}_{\mathcal{H}}^+ = \mathcal{A}_{\mathcal{H}} \cap \mathcal{A}_1^+$ this set of increasing and strongly concave valuations of the categories. We then have the following immediate (and therefore unproved) corollary of Proposition 5 (applied to $t = k$).

Lemma 2 *A list of k real numbers $(\alpha_1, ..., \alpha_k)$ satisfies $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1}$ for all $i \in \{1, ..., k-1\}$ if and only if it belongs to $\mathcal{A}_{\mathcal{H}}^+$.*

The other extreme of the possible peaks identified in Proposition 5 corresponds to the case where $t = 1$ so that the numbers $\alpha_1, ..., \alpha_k$ are decreasing (at a strongly increasing rate) with respect to the categories. In this case, the elements of $\mathcal{A}_{\mathcal{H}}$ are also in $\mathcal{A}_1^-$, and we denote by $\mathcal{A}_{\mathcal{H}}^- = \mathcal{A}_{\mathcal{H}} \cap \mathcal{A}_1^+$ this subset of the set of all strongly concave valuations of the categories that are also decreasing with respect to these categories. We then have the following also immediate (and unproved) corollary of Proposition 5 (applied to $t = 1$).

Lemma 3 *A list of k real numbers $(\alpha_1, ..., \alpha_k)$ satisfies $\alpha_{i+1} - \alpha_i \leq \alpha_i - \alpha_1$ for all $i \in \{1, ..., k-1\}$ if and only if it belongs to $\mathcal{A}_{\mathcal{H}}^-$.*

We now turn to the main result of this paper. This result states that H^+ dominance is *the* implementable test that enables one to check if one distribution of an ordinal attribute is obtained from another by a finite sequence of either Hammond transfers or increments. The formal statement of this result is as follows.

Theorem 3 *For any societies s and $s' \in \mathcal{C}^n$, the following three statements are equivalent:*
(a) *s is obtained from s' by means of a finite sequence of Hammond's transfers and/or increments,*
(b) *$s \succsim^{\mathcal{A}_{\mathcal{H}}^+} s'$,*
(c) *$s \succsim_{H^+} s'$.*

While a detailed proof of the equivalence of the three statements of Theorem 3 is provided in the Appendix, it may be useful to sketch here the main arguments. The fact that statement (a) implies statement (b) is an immediate consequence of Propositions 2 and 4. These propositions indicate indeed that normative dominance as per definition 1 over the set of categories' valuations$\mathcal{A}_{\mathcal{H}}^{+}$ is sensitive to Hammond transfers and increments. The proof that statement (b) implies statement (c) amounts to verifying that any list of k real numbers $(\alpha_1^i, \ldots, \alpha_k^i)$ defined, for any $i \in \{1, ..., k\}$, by:

$$\alpha_h^i = -(2^{i-h}) \text{ for } h = 1, .., i \tag{31}$$

$$\alpha_h^i = 0 \text{ for } h \in \{i+1, ..., k\} \tag{32}$$

belongs to the set $\mathcal{A}_{\mathcal{H}}^{+}$. Indeed, it is apparent from expression (9) that verifying the inequality:

$$\sum_{j=1}^{k} n_j^s \alpha_j^i \geq \sum_{j=1}^{k} n_j^{s'} \alpha_j^i$$

for any list $(\alpha_1^i, \ldots, \alpha_k^i)$ of real numbers defined as per (31) and (32) for any i is equivalent to the H^+ dominance of s' by s. As, by definition of $\succsim^{\mathcal{A}_{\mathcal{H}}^{+}}$, inequality (1) holds for all $(\alpha_1, .., \alpha_k)$ in the set $\mathcal{A}_{\mathcal{H}}^{+}$, it must hold in particular for those $(\alpha_1^i, \ldots, \alpha_k^i)$ defined as per (31) and (32) for any i. The most difficult proof that statement (c) implies statement (a) is obtained by first noting that if $s \succsim_1 s'$ holds, then the possibility of going from s' to s by a finite sequence of increments is an immediate consequence of Theorem 2. The proof is then constructed under the assumption that $s \succsim_{H^+} s'$, but that s does not dominate s' by first order dominance. Hence there must be some category at which the two cumulative density functions associated to s and s' "cross". We show that, in that case, one can make a Hammond transfer "above" the first category for which this crossing occurs in such a way that the new society obtained after the Hammond transfer remains dominated by s as per the H^+ dominance criterion. We also show that this Hammond transfer "bring to naught" at least one of the strict inequalities that distinguish $F(., s)$ from $F(., s')$. Hence if the final distribution s is not reached after this first transfer, then one can reapply the same procedure again and again as many times as required to reach s. As the number of inequalities that distinguish $F(., s)$ from $F(., s')$ is finite, this prove the implication.

We now state a theorem that is the mirror image of Theorem 3, but with increments replaced by decrements, the quasi-ordering $\succsim^{\mathcal{A}_{\mathcal{H}}^{+}}$ replaced by $\succsim^{\mathcal{A}_{\mathcal{H}}^{-}}$ and the H^+ dominance criterion replaced by the H^- dominance one. In order to state this theorem, whose proof, in the Appendix, follows an analogous line of argument as that of Theorem 3, we denote by $\mathcal{A}_{\mathcal{H}}^{-} = \mathcal{A}_{\mathcal{H}} \cap \mathcal{A}_1^{-}$ the set of decreasing and strongly concave valuations of the categories. The proof of this theorem appeals to the following immediate corollary of Proposition 5, which characterizes the set $\mathcal{A}_{\mathcal{H}}^{-}$ as the subset of $\mathcal{A}_{\mathcal{H}}$ associated to the case where the "peak" t is equal to 1.

Theorem 4 *For all societies s and $s' \in \mathcal{C}^n$, the following three statements are equivalent:*
(a) *s is obtained from s' by means of a finite sequence of Hammond's transfers and/or decrements,*

(b) $s \succsim^{\mathcal{A}_{\mathcal{H}}^{-}} s'$,
(c) $s \succeq_{H^-} s'$.

Theorems 3 and 4 show that the H^+ and H^- dominance criteria provide exact diagnostic tools of the possibility of going from the dominated society to the dominating one by a finite sequence of Hammond transfers and increments and Hammond transfers and decrements respectively. What about identifying the possibility of going from a society to another by equalizing Hammond transfers only ? It follows clearly from Theorems 3 and 4 that if this possibility exists between two societies, then the society from which these transfers originate is dominated by the society to which these transfers lead by both the H^+ and the H^- dominance criteria or, equivalently (Definition 8) by the H dominance criterion. Unfortunately, we do not have at our disposal a proof of the converse implication that the dominance of a society by another as per the H dominance criterion entails the possibility of going from the dominated society to the dominating one by a finite sequence of transfers only. Yet, it happens that that the criterion of H dominance is equivalent to additive normative dominance over the class $\mathcal{A}_{\mathcal{H}}$ of ordered lists of real numbers. The following theorem describes all the relations between Hammond transfers and the relevant normative and implementable criteria that we are aware of.

Theorem 5 *Consider any two societies s and $s' \in \mathcal{C}^n$, and the following three statements:*
(a) *s is obtained from s' by means of a finite sequence of Hammond's transfers*
(b) $s \succsim^{\mathcal{A}_{\mathcal{H}}} s'$,
(c) $s \succeq_H s'$.
Then, statement (a) implies statement (b) and statement (b) and (c) are equivalent.

4 Sensitivity of the criteria to the grid of categories.

As is well-known in classical social choice theory (see e.g. Hammond (1976), Hammond (1976), Deschamps and Gevers (1978), D'Aspremont and Gevers (1977), Sen (1977)) Hammond transfers, when combined with the "Pareto principle", is somewhat related with the *lexicographic extension of the Maximin* (*Leximin* for short) criterion for ranking various ordered lists of n numbers. For example, Theorem 4.17 in Blackorby, Bossert, and Donaldson (2005) (ch. 4; p. 123) states that the Leximin criterion is the only complete, reflexive, transitive, monotonically increasing and anonymous ranking of $\mathbb{R}^n$ that is sensitive to Hammond transfers. In the same vein, Bosmans and Ooghe (2013) and Miyagishima, Bosmans, and Ooghe (2014) have shown that the Maximin criterion is the only continuous and Pareto consistent reflexive and transitive ranking of $\mathbb{R}^n$ that is consistent with Hammond transfers. Since the H^+ dominance criterion coincides, by Theorem 3, with the possibility of going from the dominated society to the dominating one by a finite sequence of Hammond transfers and/or increments - which are nothing else than anonymous Pareto improvements - it is of interest to understand the connection between the H^+ dominance criterion and the Leximin one.

We start by defining the latter criterion in the current setting as follows.

Definition 9 *Given two societies s, $s' \in \mathcal{C}^n$, we say that s dominates s' according to the Leximin, which we write $s \succeq_L s'$, if and only if there exists $i \in \mathcal{C}$, such that $n_i < n'_i$ and $n_h = n'_h$ for all $h \in \mathcal{C}$ such that $1 \leq h < i$ (if there are any such h).*

It is clear (and well-known) that the Leximin criterion provides a complete ranking of all lists of n real numbers. The following proposition establishes that the quasi-ordering $\succeq_{H^+}$ is a strict subrelation of $s \succeq_L s'$

Proposition 6 *Assume that $n > 2$. Then, for any two societies s and $s' \in \mathcal{C}^n$, $s \succeq_{H^+} s' \implies s \succeq_L s'$, but the converse is false.*

A key difference between the current framework and that considered in classical social choice theory is of course the *discrete* nature of the former. In the framework considered here, the rankings are defined on $\mathcal{C}^n$, which can be viewed as a finite subset of $\mathbb{N}^n$. In classical social choice theory, the criteria are taken to be defined on the set $\mathbb{R}^n$ of all ordered lists of real numbers so that the attributes' quantities are free to vary continuously.

In order to connect the two frameworks, we find useful to examine what happens to the H^+ criterion when the finite grid over which it is defined is *refined*. As it turns out, for a suitably large level of grid refinement, the H^+ criterion becomes indistinguishable from the Leximin. There are obviously many ways by which a given finite grid can be refined. In this section, we consider the following notion of t-refinement of the grid $\mathcal{C} = \{1, 2, \ldots, k\}$, for $t = 0, 1, ...$

Definition 10 *The t-refinement of the grid $\mathcal{C} = \{1, 2, \ldots, k\}$ for $t = 0, 1, ...$ is the set $\mathcal{C}(t)$ defined by:*

$$\mathcal{C}(t) = \left\{ i/2^t \ : \ i = 1, 2, \ldots, (2^t)k \right\} \tag{33}$$

or, equivalently,

$$\mathcal{C}(t) = \left\{ \frac{1}{2^t}, \frac{2}{2^t}, \frac{3}{2^t}, \ldots, \frac{(2^t)\, k}{2^t} \right\}. \tag{34}$$

Notice that $\mathcal{C}(0) = \mathcal{C}$ so that the initial grid corresponds to a "zero" refinement. The grid becomes obviously finer as t increases, and it is clear that $\mathcal{C}(t) \subset \mathcal{C}(t+1)$ for all $t = 0, ...$ Hence, if the society $(s_1, s_2, \ldots, s_n) \in \mathcal{C}^n$, it also belongs to $\mathcal{C}(t)^n$ for all $t \in \mathbb{N}$. We also remark that the finite set $\mathcal{C}(t)$ tends to the continuous interval $]0, k]$ as t tends to infinity.

For any society s, and any real number x in the interval $[0, k]$, we denote by $n^s(x, t)$ the (possibly null) number of individuals in s whose attribute takes the value x in the grid $\mathcal{C}(t)$ defined by:

$$n^s(x, t) = \# \{ i \in \{1, 2, \ldots, n\} \ : \ s_i = x \}$$

We observe that, for any $t = 0, 1, ...$ and society $s \in \mathcal{C}(t)^n$, one has:

$$n^s(x, t) = 0 \text{ for all } x \notin \mathcal{C}(t) \tag{35}$$

and, since $\mathcal{C}(t) \subset \mathcal{C}(t+1)$:

$$n^s(j, t+1) = n^s(j, t) \tag{36}$$

for all $j \in \mathcal{C}(t)$. Using these numbers $n^s(.)$ and applying the definition of the H^+ curve provided by Equation (9) to the grid $\mathcal{C}(t)$ enable one to define the t-refinement of the H^+ curve, denoted H_t^+, as follows (for any $s \in \mathcal{C}(t)^n$)

$$H_t^+\left(0; s\right) = 0$$

and:

$$H_t^+\left(\frac{i}{2^t}; s\right) = \frac{1}{n}\sum_{h=1}^{i}\left(2^{i-h}\right) n^s\left(\frac{h}{2^t}, t\right), \quad \forall i = 1, 2, \ldots, (2^t)k. \tag{37}$$

We are now ready to define the notion of H^+ dominance on a $t-$refined grid as follows.

Definition 11 *Given two societies s and $s' \in \mathcal{C}^n$ and $t \in \{0, ...\}$, we say that society s H^+-dominates society s' on the grid $\mathcal{C}(t)$, which we write $s \succeq_{H^+}^t s'$, if and only if:*

$$H_t^+(x; s) \leq H_t^+(x; s'), \quad \forall x \in \mathcal{C}(t). \tag{38}$$

This definition produces a sequence of quasi-orderings $\{\succeq_{H^+}^t\}$ initiated by $\succeq_{H^+}^0 = \succeq_{H^+}$which, as it turns out, converges to the complete quasi ordering $\succeq_L$ when t becomes large enough.

The first thing to notice about such a refinement of the grid is that it reduces the incompleteness of the quasi-ordering of societies induced by the H^+ dominance criterion. Specifically, the following proposition is proved in the appendix.

Proposition 7 *For all $s, s' \in \mathcal{C}(t)^n$ and all $t = 0, 1, ...,$ $s \succeq_{H^+}^t s' \implies s \succeq_{H^+}^{t+1} s'$.*

Hence, refining the grid leads to an increase in the discriminating power of the H^+. The next theorem establishes that there exists a degree of refinement above which the H^+-dominance criterion becomes equivalent to the Leximin ordering.

Theorem 6 *For any two societies s and $s' \in \mathcal{C}^n$, the following two statements are equivalent:*

(a) *There exists a non-negative integer t such that $s \succeq_H^{t'} s'$ for all integer $t' \geq t$*

(b) $s \succeq_L s'$.

We conclude this section by noticing that a similar relation exists between the H^- dominance criterion on the one hand and the Lexicographic extension of the minimax criterion on the other. The minimax criterion is the ordering that compares a;alternative lists of n real numbers on the basis of their maximal elements, and which prefers lists with smaller maximal elements to those with larger ones. The lexicographic extension of the minimax criterion extends the principle to the second maximal element, and to the third and so on when the maximal, the second maximal and so on of two lists are identical. While the Leximin or the maximin criteria may be seen as ethically favouring the "worst off", the minimax criterion or its lexicographic extension disfavors the "best off".

The facts that H^- dominance converges to the Lexicographic extension of the Minimax criterion and H^+ dominance converges to the leximin one when the grid becomes sufficiently fine has an obvious, but important, implication for the H dominance criterion examined in the preceding section. This H dominance criterion, defined as the intersection of H^+ and H^- dominance, converges to the intersection of the Lexicographic extensions of the Minimax and the Maximin criteria when the grid becomes sufficiently fine.

5 Illustrations of the criteria

In the following section, which borrows material from Bennia, Gravel, Magdalou, and Moyes (2015) where additional discussion can be obtained, we illustrate the usefulness of our criteria for normatively comparing alternative distributions of an ordinally measurable attribute that is somewhat important for individual welfare: body weight. The adverse effect of overweight and obesity on health is indeed largely documented. Less discussed is the epidemiological evidence (see e.g. Cao, Moineddin, Urquia, Razak, and Ray (2014) or Flegal, Graubard, Williamson, and Gail (2005)) that being pathologically underweight can also be associated with significant increase in the probability of dying in the next 5 years as compared to having a "normal" weight. It is also well-known that body weight may impact individual well-being in a way that is not reducible to its health consequences, however severe these may be. Inadequate body weight may indeed affect self esteem and happiness (see e.g. Oswald and Powdthavee (2007)) and lead to social stigma (see e.g. Carr and Friedman (2005), Roberts, Kaplan, Shema, and Strawbridge (2000), Roberts, Strawbridge, Deleger, and Kaplan (2002)) or to an unfavorable image of one self when comparing with others (see e.g. Blanchflower, Landeghem, and Oswald (2010)).

We stick to the common practice of measuring body weight by the Body Mass Index (BMI), defined as the ratio of the individual weight (in kilograms) over the "surface body" (in squared meters). We use the BMI to construct various weight *categories* that are considered meaningful in terms of their impact on individual well-being from either a medical or non-medical point of view. According to the World Health Organization (WHO), these are, for the adult population above the age of 20:

Underweight (BMI below 18.5),
Normal (BMI between 18.5 and 25),
Overweight (BMI between 25 and 30)
Grade 1 obesity (BMI between 30 and 35)
Grade 2 obesity (BMI between 35 and 40)
Grade 3 obesity (BMI above 40)

In the following two tables, we report the distributions (e.g. the numbers n_j^s/n) for these six BMI categories for representative samples of the female (table 1) and the male (table 2) population in France, the US and the UK for the year 2008. We again refer to Bennia, Gravel, Magdalou, and Moyes (2015) for details on the data sources from which these tables have been constructed. Sample sizes for are as in the following figure.

	France (2008)	UK (2008-2012)	US (2007-2008)
Number of individuals	11 255	1 912	5 607
Number of men	5 369	906	2 747
Number of women	5 886	1 006	2 860

Figure 1: Sample description

category	**FRANCE**	**UK**	**US**
obesity 3	0.0114	0.0416	0.0786
obesity 2	0.0311	0.0951	0.1125
obesity 1	0.0910	0.1681	0.2045
underweight	0.0467	0.0166	0.0213
overweight	0.2490	0.3019	0.3016
normal	0.5708	0.3767	0.2815

Table 1: distributions of BMI categories among women

A difficulty with a numerical indicator like the BMI is that its relationship with the individual well-being is not monotonic. While it seems clear that individual well-being is decreasing with respect to the BMI categories when these correspond to a BMI above 25, and that being underweight is worse than having a normal weight, it is not clear how the well-being associated to the underweight category compares with that associated to any of the overweight and obese ones. Because of this ambivalence, Bennia, Gravel, Magdalou, and Moyes (2015) consider in turn each of the following rankings of BMI categories, from the worst (bottom) to the best (top):

In what follows, we illustrate the analysis by considering only the second of these rankings, that is allegedly one of the most plausible of the five. Based on this ranking, the following tables provide the cumulative distribution functions associated to Tables 1 and 2

These tables suggest the somewhat well-known fact that the distribution of body weight is more favorable in France than in the UK,and in the UK than

category	**FRANCE**	**UK**	**US**
obesity 3	0.0047	0.0121	0.0441
obesity 2	0.0211	0.0623	0.0736
obesity 1	0.1030	0.1990	0.2163
underweight	0.0132	0.0084	0.0116
overweight	0.3840	0.4536	0.3896
normal	0.4740	0.2646	0.2648

Table 2: distributions of BMI categories among men

Ranking 1	**Ranking 2**	**Ranking 3**	**Ranking 4**	**Ranking 5**
normal	normal	normal	normal	normal
underweight	overweight	overweight	overweight	overweight
overweight	underweight	obesity 1	obesity 1	obesity 1
obesity 1	obesity 1	underweight	obesity 2	obesity 2
obesity 2	obesity 2	obesity 2	underweight	obesity 3
obesity 3	obesity 3	obesity 3	obesity 3	underweight

Table 3: possible welfare rankings of BMI categories

category	**FRANCE**	**UK**	**US**
obesity 3	0.0114	0.0416	0.0786
obesity 2	0.0425	0.1364	0.1811
obesity 1	0.1335	0.3045	0.3856
underweight	0.1802	0.3211	0.4069
overweight	0.4293	0.6230	0.7095
normal	1	1	1

Table 4: distributions of BMI categories among women

category	**FRANCE**	**UK**	**US**
obesity 3	0.0047	0.0121	0.0440
obesity 2	0.0258	0.0744	0.1176
obesity 1	0.1288	0.2734	0.3339
underweight	0.1422	0.2818	0.3455
overweight	0.5263	0.7354	0.7351
normal	1	1	1

Table 5: distributions of BMI categories among men

in the US for both genders by the robust first order dominance criterion. The only exception to this concerns the US and the UK whose distribution of BMI categories for the male population is not comparable by first order dominance. Indeed, despite the fact that the fraction of obese men is larger in the US than in the UK for any level of obesity, there is a slightly larger fraction of the male population that is of normal weight in the US than in the UK. However, as indicated in Bennia, Gravel, Magdalou, and Moyes (2015), it happens that the male distribution of BMI categories in the UK dominates that of the US for the H^+ criterion.

Less well-known and discussed are the gender differences in terms of inequality in body weight. As is clear from the two tables, in no country can a first order dominance relation be observed between the male and female distributions of BMI categories. Indeed, in all three countries, women are more affected by obesity then men but women are also more likely to fall in the favorable "normal" weight category then men. However, it happens that the distribution of BMI categories is "more equally" distributed among males than among females as per the notion of equalization underlying Hammond transfers.

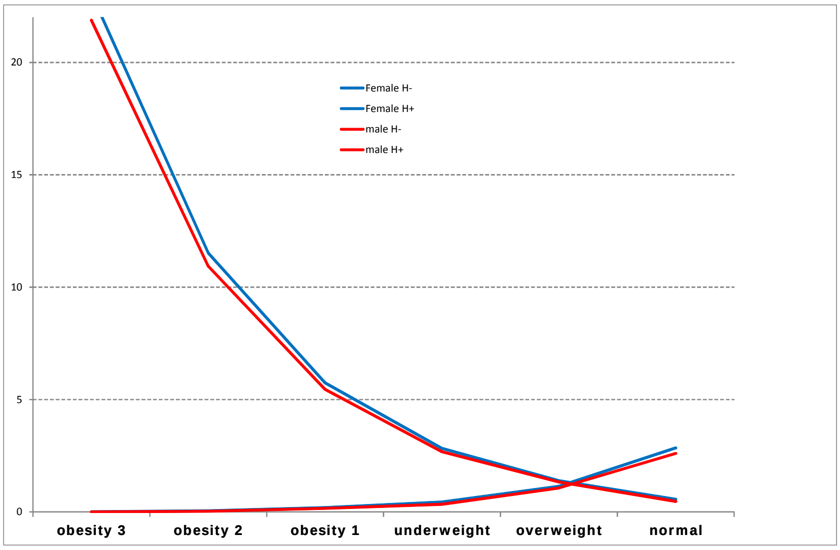

Figure 2: H+ and H- curves for men and women en France, 2008.

For instance, Figure 2 shows the H^+ and H^- curves for the female (in blue) and the male (in red) population in France. As can be seen, the males' curves lie everywhere below the females' one. Hence, the distribution of BMI categories appears to be more equally distributed among men than among women in France for this ranking of BMI categories. As it happens, the same conclusion holds for the two other countries. We believe that this male-female differential in body weight inequalities revealed by our criteria is an important issue. It is also worth mentioning that this conclusion of a better distribution of BMI categories among men than among women could not be obtained using the well-known ordinal criterion proposed by Allison and Foster (2004). In effect, the distributions of

BMI categories among males and among females do not have the same median. As a result, Allison and Foster (2004) criteria do not even apply.

6 Conclusion

The paper has provided foundations to the issue of comparing alternative distributions of an attribute ordinally measured by an indicator that takes finitely many values. The crux of the analysis has been an easy-to-use criterion, called H^+-dominance, that can be viewed as the analogue, for comparing distributions of an *ordinally measurable* attribute, of the generalized Lorenz curve used for comparing distributions of a *cardinally measurable* one. It is well-known (see e.g. Shorrocks (1983)) that a distribution of a cardinally measurable attribute dominates another for the generalized Lorenz domination criterion if and only if it is possible to go from the dominated distribution to the dominating one by a finite sequence of increments of the attribute and/or Pigou-Dalton transfers. The main result of this paper - Theorem 3 - has established an analogous result for the H^+-dominance criterion by showing that the latter criterion ranks two distributions of an attribute in the same way than would the fact of going from the dominated distribution to the dominating one by a finite sequence of increments and/or Hammond transfers of the attribute. The paper has also identified a dual H^- dominance criterion that ranks two distributions in the same way than would the fact of going from the dominated distribution to the dominating one by a finite sequence of decrements and/or Hammond transfers of the attribute. We suspect strongly that the H-dominance criterion - defined as the intersection of the H^+ and the H^- dominance one - coincides with the possibility of going from the dominated distribution to the dominated one by a finite sequence of Hammond transfers only.

As illustrated with the data provided in Bennia, Gravel, Magdalou, and Moyes (2015), we believe the H^+-dominance criterion, and the Hammond principle of transfers that justifies it along with increments, to be a useful tool for comparing distributions of an attribute that can not be meaningfully transferred *à la* Pigou-Dalton. Beside the fact of being justified by clear and meaningful elementary transformations, the H^+-dominance criterion has the advantage of being applicable to a much wider class of situations than, for instance, the widely discussed criterion proposed by Allison and Foster (2004). The later is indeed limited to distributions that have the same median. And it is not associated to clear and meaningful elementary transformations. The criterion could also be useful for comparing distributions of a cardinally measurable attribute if one is willing to accept the rather strong egalitarian ethics underlying the principle of Hammond transfer.

Among the many possible extensions of the approach developped in this paper, two strike us as particularly important. First, as the H^+ criterion (or, for that matter the H one) is incomplete, it would be interesting to obtain simple ordinal inequality indices that are compatible with Hammond transfers and, therefore, with the H criterion. We believe that obtaining an axiomatic characterization of a family of such indices would not be too difficult. Indeed, a good starting point would be to consider indices that can write as per expression (1) for some suitable choice of lists $(\alpha_1, ..., \alpha_k)$ of real numbers. A second extension, that seems clearly more difficult, would be to consider multi-dimensional attributes.

A Appendix: Proofs

A.1 Proposition 1

Let s and $s' \in \mathcal{C}^n$ be such that $s \succsim_1 s'$. By definition 5, one has $F(i;s) \leq F(i;s')$ for all $i \in \{1,...,k\}$. It follows that:

$$F(1;s) \leq F(1;s')$$

and:

$$\begin{aligned} F(1;s) + F(2;s) &\leq F(1;s') + F(2;s'), \\ 2F(1;s) + F(2;s) + F(3;s) &\leq 2F(1;s) + F(2;s) + F(3;s), \\ &... \end{aligned}$$

$$\sum_{h=1}^{i-1} \left(2^{i-h-1}\right) F(h;s) + F(i;s) \leq \sum_{h=1}^{i-1} \left(2^{i-h-1}\right) F(h;s) + F(i;s) \ \forall i = 2,3,\ldots,k$$

as required by expressions (10) and (11) that define H^+ dominance of s' by s. To establish the H^- dominance of s by s', it suffices to notice that the requirement $F(i;s) \leq F(i;s')$ for all $i \in \{1,...,k\}$ that defines $s \succsim_1 s'$ can alternatively be written (thanks to expressions (17)-(19)) as :

$$\overline{F}(i;s) \geq \overline{F}(i,s')$$

for all $i \in \{1,...,k-1\}$. This implies that:

$$\overline{F}(k-1;s) \geq \overline{F}(k-1;s')$$

and:

$$\begin{aligned} \overline{F}(k-2;s) + \overline{F}(k-1;s) &\geq \overline{F}(k-2;s') + \overline{F}(k-1;s'), \\ \overline{F}(k-3;s) + \overline{F}(k-2;s) + 2\overline{F}(k-1;s) &\geq \overline{F}(k-3;s') + \overline{F}(k-2;s') + 2\overline{F}(k-1;s') \\ &.... \end{aligned}$$

$$\sum_{h=i+1}^{k} \left(2^{h-i-1}\right) \overline{F}(h;s) + \overline{F}(i;s) \geq \sum_{h=i+1}^{k} \left(2^{h-i-1}\right) \overline{F}(h;s') + \overline{F}(i;s')\,,\ \forall i = 1,2,\ldots,k-1.$$

as required (thanks to expressions 22 and 23) by the H^- dominance of society s by society s' as per definition 7.

A.2 Lemma 1

Observe first that:

$$\sum_{h=1}^{k} n_h\,\alpha_h = \begin{cases} & n_1\,\alpha_1 \\ + & n_2\,\alpha_2 \\ + & \cdots \\ + & n_k\,\alpha_k\,, \end{cases} \tag{39}$$

or equivalently:

$$\sum_{h=1}^{k} n_h\,\alpha_h = \begin{cases} & n_1\,\alpha_1 \\ + & n_2\,\alpha_1 + n_2\,[\alpha_2-\alpha_1] \\ + & n_3\,\alpha_1 + n_3\,[\alpha_2-\alpha_1] + n_3\,[\alpha_3-\alpha_2] \\ + & \cdots \\ + & n_k\,\alpha_1 + n_k\,[\alpha_2-\alpha_1] + n_k\,[\alpha_3-\alpha_2] + \ldots n_k\,[\alpha_k-\alpha_{k-1}]\ , \end{cases} \tag{40}$$

hence:

$$\sum_{h=1}^{k} n_h\,\alpha_h = \begin{cases} & n\,\alpha_1 \\ + & (n-n_1)\,[\alpha_2-\alpha_1] \\ + & [n-(n_1+n_2)]\,[\alpha_3-\alpha_2] \\ + & \cdots \\ + & \left[n-\sum_{h=1}^{k-1} n_h\right][\alpha_k-\alpha_{k-1}]\ , \end{cases} \tag{41}$$

from which one obtains:

$$\frac{1}{n}\sum_{h=1}^{k} n_h\,\alpha_h = [\alpha_1 + (\alpha_k-\alpha_1)] - \sum_{h=1}^{k-1} F(h;s)(\alpha_{h+1}-\alpha_h) \tag{42}$$

$$= \alpha_k - \sum_{h=1}^{k-1} F(h;s)(\alpha_{h+1}-\alpha_h) \tag{43}$$

as required by Equation (27). Now, by reconsidering equation (41) and recalling that $\bar{F}(i;s) = 1 - F(i;s) = \left(n - \sum_{h=1}^{i} n_h\right)/n$ for every $i = 1,...,k$, one immediately obtains equation (28). We must now establish equation (29). For this sake, one can notice that, for any $i \in \{2,...,k-1\}$, one has:

$$\sum_{h=1}^{k} n_h\,\alpha_h = \sum_{h=1}^{i} n_h\,\alpha_h + \sum_{h=t+1}^{k} n_h\,\alpha_h\ . \tag{44}$$

If one successively decompose the two terms on the right hand of (44), one obtains for the first one:

$$\sum_{h=1}^{i} n_h\,\alpha_h = \begin{cases} & n_1\,\alpha_1 \\ + & n_2\,\alpha_1 + n_2\,[\alpha_2-\alpha_1] \\ + & n_3\,\alpha_1 + n_3\,[\alpha_2-\alpha_1] + n_3\,[\alpha_3-\alpha_2] \\ + & \cdots \\ + & n_t\,\alpha_1 + n_k\,[\alpha_2-\alpha_1] + n_i\,[\alpha_3-\alpha_2] + \ldots n_i\,[\alpha_i-\alpha_{i-1}]\ , \end{cases}$$

One has therefore:

$$\sum_{h=1}^{i} n_h\,\alpha_h = \begin{cases} & \left(\sum_{h=1}^{i} n_h\right)\alpha_1 \\ + & \left[\sum_{h=1}^{i} n_h - n_1\right][\alpha_2-\alpha_1] \\ + & \left[\sum_{h=1}^{i} n_h - (n_1+n_2)\right][\alpha_3-\alpha_2] \\ + & \cdots \\ + & \left[\sum_{h=1}^{i} n_h - \sum_{h=1}^{i-1} n_h\right][\alpha_i-\alpha_{i-1}]\ , \end{cases}$$

or equivalently:

$$\frac{1}{n}\sum_{h=1}^{i} n_h\,\alpha_h = \left(\frac{1}{n}\sum_{h=1}^{i} n_h\right)\alpha_t - \sum_{h=1}^{i-1} F(h;s)\,[\alpha_{h+1}-\alpha_h]\ . \tag{45}$$

For the second term of (44), the successive decomposition yields:

$$\sum_{h=i+1}^{k} n_h\,\alpha_h = \begin{cases} & n_{i+1}\,\alpha_{i+1} \\ + & n_{i+2}\,\alpha_{i+1} + n_{i+2}\,[\alpha_{i+2}-\alpha_{i+1}] \\ + & n_{i+3}\,\alpha_{i+1} + n_{i+3}\,[\alpha_{i+2}-\alpha_{i+1}] + n_{i+3}\,[\alpha_{i+3}-\alpha_{i+2}] \\ + & \cdots \\ + & n_k\,\alpha_{i+1} + n_k\,[\alpha_{i+2}-\alpha_{i+1}] + n_k\,[\alpha_{i+3}-\alpha_{i+2}] + \ldots n_i\,[\alpha_k-\alpha_{k-1}]\ , \end{cases}$$

This can be written as:

$$\sum_{h=i+1}^{k} n_h\,\alpha_h = \begin{cases} & \left(\sum_{h=i+1}^{k} n_h\right)\alpha_{i+1} \\ + & \left(\sum_{h=i+2}^{k} n_h\right)[\alpha_{i+2}-\alpha_{i+1}] \\ + & \left(\sum_{h=i+3}^{k} n_h\right)[\alpha_{i+3}-\alpha_{i+2}] \\ + & \cdots \\ + & n_k\,[\alpha_k-\alpha_{k-1}]\ , \end{cases}$$

or equivalently:

$$\frac{1}{n}\sum_{h=i+1}^{k} n_h\,\alpha_h = \left(\frac{1}{n}\sum_{h=i+1}^{k} n_h\right)\alpha_{i+1} + \sum_{h=i+1}^{k-1} \bar{F}(h;s)\,[\alpha_{h+1}-\alpha_h]\ . \qquad (46)$$

By summing equations (45) and (46), one concludes that:

$$\begin{aligned} \frac{1}{n}\sum_{h=1}^{k} n_h\,\alpha_h \quad &= \quad \left(\frac{1}{n}\sum_{h=1}^{i} n_h\right)\alpha_i + \left(\frac{1}{n}\sum_{h=i+1}^{k} n_h\right)\alpha_{i+1} \\ &\quad -\sum_{h=1}^{i-1} F(h;s)\,[\alpha_{h+1}-\alpha_h] + \sum_{h=i+1}^{k-1} \bar{F}(h;s)\,[\alpha_{h+1}-\alpha_h] \qquad (47) \end{aligned}$$

This equality can be further simplified, by observing that:

$$\begin{aligned} \left(\frac{1}{n}\sum_{h=1}^{i} n_h\right)\alpha_i + \left(\frac{1}{n}\sum_{h=i+1}^{k} n_h\right)\alpha_{i+1} \quad &= \quad \frac{1}{n}\left(n - \sum_{h=i+1}^{k} n_h\right)\alpha_i + \left(\frac{1}{n}\sum_{h=i+1}^{k} n_h\right)\alpha_{i+1} \\ &= \quad \alpha_i + (\alpha_{i+1}-\alpha_i)\bar{F}(i;s) \qquad (48) \end{aligned}$$

Equation (29) is then obtained from the reintroduction of (48) into (47).

A.3 Propositions 2 and 3

For proposition 2, let s be a society obtained from s' by an increment. By definition 2, there exists some $j \in \{1, ..., k-1\}$ such that:

$$n_h^s = n_h^{s'}$$

for all $h \in \{1, ...k\}$ such that $h \neq j, j+1$,

$$n_j^s = n_j^{s'} - 1$$

and,

$$n_{j+1}^s = n_j^{s'} + 1.$$

Then $s \succsim^{\mathcal{A}} s'$ if and only if: (using definition 1):

$$\sum_{j=1}^{k} n_j^s \alpha_j \geq \sum_{j=1}^{k} n_j^{s'} \alpha_j$$

$$\Longleftrightarrow$$

$$\alpha_{j+1} - \alpha_j \quad \geq \quad 0$$

by definition of an increment. As this inequality must hold for any $j \in \{1, ..., k-1\}$, this completes the proof of proposition 2.The argument for the proof of proposition 3 is similar (with definition 2 replaced by definition 3.

A.4 Theorem 1

The equivalence between statements (a) and (c) of this theorem is well-known in the literature. We therefore only prove the equivalence between statements (b) and (c). Using equation (27) of Lemma 1, one has:

$$\frac{1}{n}\left[\sum_{h=1}^{k} n_h \, \alpha_h - \sum_{h=1}^{k} n'_h \, \alpha_h\right] = \sum_{h=1}^{k-1} \left[F(h; s') - F(h; s)\right] \left[\alpha_{h+1} - \alpha_h\right] . \qquad (49)$$

Hence, if $s \succsim_1 s'$ and $(\alpha_1, ..., \alpha_k) \in \mathcal{A}_1^+$, then $\sum_{h=1}^{k} n_h \, \alpha_h \geq \sum_{h=1}^{k} n'_h \, \alpha_h$. To establish the converse implication, define, for every $i \in \{1, ..., k-1\}$ the list of k numbers $\alpha^i = (\alpha_1^i, \ldots, \alpha_k^i)$ to be such that $\alpha_h^k = 0$ for $h = 1, \ldots, i$ and $\alpha_h^i = 1$ for $h = i+1, \ldots, k$. We note that $\alpha^i \in \mathcal{A}_1^+$ for any $i \in \{1, ..., k-1\}$. Since $s \succsim^{\mathcal{A}_1^+} s'$, one must therefore have, for any $i = 1, ..., k-1$:

$$\sum_{h=1}^{k} n_h \, \alpha_h^i \quad \geq \quad \sum_{h=1}^{k} n'_h \, \alpha_h^i$$

$$\Longleftrightarrow$$

$$\sum_{h=i+1}^{k} n_h \quad \geq \quad \sum_{h=i+1}^{k} n'_h$$

$$\Longleftrightarrow$$

$$n - \sum_{h=1}^{i} n_h \geq n - \sum_{h=1}^{i} n'_h$$

$$\Longleftrightarrow$$

$$\sum_{h=1}^{i} n_h \leq \sum_{h=1}^{i} n'_h$$

as required (definition 5) for $s \succsim_1 s'$.

A.5 Proposition 4

Suppose that society s has been obtained from society s' by means of a Hammond transfer as per Definition 4. This means that there are categories $1 \leq g < i \leq j < l \leq k$ for which one has:

$$\sum_{h=1}^{k} n_h \, \alpha_h = \sum_{h=1}^{k} \alpha_h \, n'_h - \alpha_g + \alpha_i + \alpha_j - \alpha_l \,. \tag{50}$$

By definition 1, $s \succsim_{\mathcal{A}} s'$ for any s obtained from s' by means of a Hammond transfer if and only if inequality $\sum_{h=1}^{k} n_h^s \, \alpha_h - \sum_{h=1}^{k} n_h^{s'} \, \alpha_h = (\alpha_i - \alpha_g) - (\alpha_l - \alpha_j) \geq 0$ hold for all categories $1 \leq g < i \leq j < l \leq k$, which is precisely the definition of the set$\mathcal{A}_{\mathcal{H}}$.

A.6 Proposition 5

Assume that the list of numbers $(\alpha_1, ..., \alpha_k)$ belongs to$\mathcal{A}_{\mathcal{H}}$ and, therefore, satisfies $\alpha_i - \alpha_g \geq \alpha_l - \alpha_j$ for all $1 \leq g < i \leq j < l \leq k$. This implies in particular that $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1}$ for any $i \in \{1, 2, \ldots, k-2\}$. Let $t = \min\{i = 1, ..., k : \alpha_{i+1} - \alpha_i \leq 0\}$ (using the convention that $\alpha_{k+1} = \alpha_k$). Such a t clearly exists under this convention, because $k \in \{i = 1, ..., k : \alpha_{i+1} - \alpha_i \leq 0\}$. If $t = k$, then the fact that $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1}$ holds for any $i \in \{1, 2, \ldots, k-2\}$ implies that $\alpha_{i+1} - \alpha_i \geq \alpha_t - \alpha_{i+1}$ for all $i = 1, 2, \ldots, t-1$ and (trivially) that $\alpha_{i'+1} - \alpha_{i'} \leq \alpha_{i'} - \alpha_t$ holds for all $i' \in \{t, ..., k-1\} = \varnothing$. Notice that if $t = k$, then, one has $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1} > 0$ for any $i \in \{1, 2, \ldots, k-2\}$ (the alphas are increasing with respect to the categories). If $t = 1$, then the set $\{i = 1, 2, \ldots, t-1\}$ is empty so that one must simply verify that $\alpha_{i'+1} - \alpha_{i'} \leq \alpha_{i'} - \alpha_1$, for $i' = 1, ..., k-1$. But this results immediately from the definition of t (if $i' = 1$) or from applying the requirement that $\alpha_i - \alpha_g \geq \alpha_l - \alpha_j$ for all $1 \leq g < i \leq j < l \leq k$ to the particular case where $g = 1$, $i = j = i' > 1$ and $l = i' + 1$ (otherwise). Notice that if $t = 1$, then one has by definition that $0 \geq \alpha_2 - \alpha_1 \geq \alpha_j - \alpha_{j-1}$ for every $j = 3, ..., k$ so that the alphas are decreasing with the categories. Assume now that $t \in \{2, ..., k-1\}$. We must check first that $\alpha_{i+1} - \alpha_i \geq \alpha_t - \alpha_{i+1}$ for all $i = 1, 2, \ldots, t-1$. The case where $i = t-1$ is proved by observing that, by definition of t, one has $\alpha_t - \alpha_{t-1} > 0 = \alpha_t - \alpha_t$. The case where $i < t-1$ (if any) is proved by applying the statement $\alpha_i - \alpha_g \geq \alpha_l - \alpha_j$ for all $1 \leq g < i \leq j < l \leq k$ to the particular case where $g = i \in \{1, ..., t-2\}$ $i = j = i+1$ and $l = t$. To check that the inequality $\alpha_{i'+1} - \alpha_{i'} \leq \alpha_{i'} - \alpha_t$ holds for all $i' \in \{t', ..., k-1\}$, simply observe that, for $i' = t$, the inequality is obtained from the very definition of t and, for $i' > t$, it results from applying the fact that $\alpha_i - \alpha_g \geq \alpha_l - \alpha_j$ for all $1 \leq g < i \leq j < l \leq k$ to the particular case where $g = t$, $i = j = i'$ and $l = i' + 1$.
Conversely, consider any list of numbers $(\alpha_1, ..., \alpha_k)$ for which there exists a $t \in \{1, ..., k\}$ such that:

$$\alpha_{i+1} - \alpha_i \geq \alpha_t - \alpha_{i+1} \tag{51}$$

holds for all $i \in \{1, 2, \ldots, t-1\}$ (if any) and:

$$\alpha_{i'+1} - \alpha_{i'} \leq \alpha_{i'} - \alpha_t \tag{52}$$

holds for all $i' \in \{t, ..., k-1\}$ (if any). Notice that applying inequality (51) to $i = t-1$ implies that $\alpha_t - \alpha_{t-1} \geq \alpha_t - \alpha_t = 0$. Combining this recursively with inequality

(51) implies in turns that $\alpha_2 - \alpha_1 \geq \alpha_3 - \alpha_2 \geq ... \geq \alpha_t - \alpha_{t-1} \geq 0$ so that the list of numbers $(\alpha_1, ..., \alpha_k)$ is increasing from 1 up to t. Similarly, applying inequality (52) to $i' = t$ implies that $\alpha_{t+1} - \alpha_t \leq \alpha_t - \alpha_t = 0$. Combining this recursively with inequality (52) satisfied for all $i' \in \{t, ..., k-1\}$ (if any) leads to the conclusion that $\alpha_k - \alpha_{k-1} \leq \alpha_{k-1} - \alpha_{k-2} \leq ... \leq \alpha_{t+1} - \alpha_t \leq 0$ so that the list of numbers $(\alpha_1, ..., \alpha_k)$ is decreasing from t up to k. Consider then any four integers g, i, j and l satisfying $1 \leq g < i \leq j < l \leq k$. Five cases need to be distinguished:
(i) $g \geq t \geq 1$, then one has:

$$\begin{aligned}
\alpha_l - \alpha_j &= (\alpha_l - a_{l-1}) + (\alpha_{l-1} - \alpha_{l-2}) + ... + (\alpha_{j+1} - \alpha_j) \\
&\leq \alpha_{j+1} - \alpha_j \text{ (because the } \alpha_h \text{ are decreasing above } t) \\
&\leq \alpha_j - \alpha_t \text{ (by inequality (52))} \\
&= \alpha_j - \alpha_i + \alpha_i - \alpha_g + \alpha_g - \alpha_t \text{ (for any integer } g,\ i,\ j) \\
&\leq \alpha_i - \alpha_g \text{ (because the } \alpha_h \text{ are decreasing above } t)
\end{aligned}$$

(ii) $g < t \leq i \leq j < l \leq k$. Then one has:

$$\begin{aligned}
\alpha_l - \alpha_j &= (\alpha_l - a_{l-1}) + (\alpha_{l-1} - \alpha_{l-2}) + ... + (\alpha_{j+1} - \alpha_j) \\
&\leq \alpha_{j+1} - \alpha_j \text{ (because the } \alpha_h \text{ are decreasing above } t) \\
&\leq \alpha_j - \alpha_t \text{ (by inequality (52))} \\
&= \alpha_j - \alpha_i + \alpha_i - \alpha_g + \alpha_g - \alpha_t \text{ (for any integer } g,\ i,\ j) \\
&\leq \alpha_i - \alpha_g \text{ (because } \alpha_j - \alpha_i \leq 0 \text{ and } \alpha_g - \alpha_t \leq 0)
\end{aligned}$$

(iii) $g < i < t \leq j < l \leq k$. Then one has:

$$\begin{aligned}
\alpha_l - \alpha_j &= (\alpha_l - a_{l-1}) + (\alpha_{l-1} - \alpha_{l-2}) + ... + (\alpha_{j+1} - \alpha_j) \\
&\leq \alpha_{j+1} - \alpha_j \text{ (because the } \alpha_h \text{ are decreasing above } t) \\
&\leq \alpha_j - \alpha_t \text{ (by inequality (52))} \\
&\leq 0 \text{ (because the } \alpha_h \text{ are decreasing above } t) \\
&\leq \alpha_i - \alpha_g \text{ (because the } \alpha_h \text{ are increasing below } t)
\end{aligned}$$

(iv) $g < i \leq j < t \leq l \leq k$. Then one has:

$$\begin{aligned}
\alpha_i - \alpha_g &= (\alpha_i - a_{i-1}) + (\alpha_{i-1} - \alpha_{i-2}) + ... + (\alpha_{g+2} - \alpha_{g+1}) + (\alpha_{g+1} - \alpha_g) \\
&\geq \alpha_{g+1} - \alpha_g \text{ (because the } \alpha_h \text{ are increasing below } t) \\
&\geq \alpha_t - \alpha_{g+1} \text{ (by inequality (51))} \\
&= \alpha_t - \alpha_l + \alpha_l - \alpha_j + \alpha_j - \alpha_{g+1} \text{ (for any } g+1 \leq j < l \leq k) \\
&\geq \alpha_l - \alpha_j \text{ (because } \alpha_t - \alpha_l \geq 0 \text{ and } \alpha_j - \alpha_{g+1} \geq 0)
\end{aligned}$$

(v) $l < t \leq k$. In this case, one has:

$$\begin{aligned}
\alpha_i - \alpha_g &= (\alpha_i - a_{i-1}) + (\alpha_{i-1} - \alpha_{i-2}) + ... + (\alpha_{g+2} - \alpha_{g+1}) + (\alpha_{g+1} - \alpha_g) \\
&\geq \alpha_{g+1} - \alpha_g \text{ (because the } \alpha_h \text{ are increasing below } t) \\
&\geq \alpha_t - \alpha_{g+1} \text{ (by inequality (51))} \\
&= \alpha_t - \alpha_l + \alpha_l - \alpha_j + \alpha_j - \alpha_{g+1} \text{ (for any } g+1 \leq j < l \leq k) \\
&\geq \alpha_l - \alpha_j \text{ (because because the } \alpha_h \text{ are increasing below } t)
\end{aligned}$$

Hence any list of k numbers $(\alpha_1, ..., \alpha_k)$ for which there exists a $t \in \{1, ..., k\}$ such that inequalities (52) and (51) belongs to $\mathcal{A}_{\mathcal{H}}$.

A.7 Theorem 3.

A.7.1 Statement (a) implies statement (b)

Suppose s has been obtained from s' by means of an increment. It then follows from Proposition 2 that inequality 1 holds for all ordered lists of k real numbers $(\alpha_1, ..., \alpha_k)$ in the set $\mathcal{A}_1^+$. This inequality holds therefore in particular for all ordered lists of k real numbers $(\alpha_1, ..., \alpha_k)$ in the set $\mathcal{A}_{\mathcal{H}}^+ \subset \mathcal{A}_1^+$. If, on the other hand, s has been obtained from s' by means of a Hammond transfer, we know from Proposition 4 that inequality 1 holds for all ordered lists of k real numbers $(\alpha_1, ..., \alpha_k)$ in the set $\mathcal{A}_{\mathcal{H}}$ and, therefore, for all ordered list of k real numbers in the set $\mathcal{A}_{\mathcal{H}}^+ \subset \mathcal{A}_{\mathcal{H}}$. Since the binary relation $\succsim^{\mathcal{A}_{\mathcal{H}}^+}$ is transitive, the implication then follows from any finite repetition of these two elementary implications.

A.7.2 Statement (b) implies statement (c)

Assume that

$$\sum_{i=1}^{k} n_i^s \alpha_i \geq \sum_{i=1}^{k} n_i^{s'} \alpha_i \tag{53}$$

holds for all $(\alpha_1, ..., \alpha_k) \in \mathcal{A}_{\mathcal{H}}^+$. For any $i \in \{1, ..., k\}$, define the ordered list of k numbers $(\alpha_1^i, \ldots, \alpha_k^i)$ by:

$$\begin{aligned} \alpha_h^i &= -(2^{i-h}) \text{ for } h = 1, .., i \\ \alpha_h^i &= 0 \text{ for } h = i+1, .., k \end{aligned}$$

Let us first show that the ordered list $(\alpha_1^i, \ldots, \alpha_k^i)$ of real numbers thus defined belongs to $\mathcal{A}_{\mathcal{H}}^+$ for every $i \in \{1, ..., k\}$. Thanks to Lemma 2, this amounts to show that these real numbers satisfy

$$\alpha_{h+1}^i - \alpha_h^i \geq \alpha_k^i - \alpha_{h+1}^i \tag{54}$$

for every $h \in \{1, ..., k-1\}$. If $h \geq i+1$, then one has:

$$\alpha_{h+1}^i - \alpha_h^i = 0 - 0 \geq 0 - 0 = \alpha_k^i - \alpha_{h+1}^i$$

so that that inequality (54) holds for that case. If $h = i$, then

$$\alpha_{i+1}^i - \alpha_i^i = 0 + 2^0 \geq 0 - 0 = \alpha_k^i - \alpha_{i+1}^i$$

so that (54) holds also for that case. If finally $h < i$, then one has:

$$\begin{aligned} \alpha_{h+1}^i - \alpha_h^i &= -2^{i-h-1} + 2^{i-h} \\ &= 2^{i-h-1} \\ &= 0 - (-2^{i-h-1}) \\ &= \alpha_k^i - \alpha_{h+1}^i \end{aligned}$$

so that inequality (54) holds for this case as well. Since the ordered list $(\alpha_1^i, \ldots, \alpha_k^i)$ of real numbers belongs to $\mathcal{A}_{\mathcal{H}}^+$ for every $i \in \{1, ..., k\}$, Inequality (53) must hold for any such ordered list of numbers. Hence, one has, for every $i \in \{1, ..., k\}$:

$$\begin{array}{ccc} \sum_{h=1}^{k} n_h^s \alpha_h^i & \geq & \sum_{h=1}^{k} n_h^{s'} \alpha_h^i \\ & \Longleftrightarrow & \\ \sum_{h=1}^{i} 2^{i-h} n_h^s & \leq & \sum_{h=1}^{i} 2^{i-h} n_h^{s'} \end{array}$$

which is nothing else than the condition for H^+ dominance, as expressed by equation (9).

A.7.3 Statement (c) implies statement (a)

Assume that $s \succsim_{H^+} s'$ so that $H^+(i; s) \leq H^+(i; s')$ for all $i = 1, 2, \ldots, k-1$. We know from Proposition 1 that $s \succsim_1 s'$ implies $s \succsim_{H^+} s'$. Hence, if $s \succsim_1 s'$, we conclude from theorem 1 that s can be obtained from s' by means of a finite sequence of increments and the proof is done. In the following, we therefore assume that $s \succsim_{H^+} s'$ holds but that $s \succsim_1 s'$ does not hold so that there exists some $g \in \{1, 2, \ldots, k-1\}$ for which one has $F(g; s) - F(g; s') > 0$. Define then the index h by:

$$h = \min \{g \mid F(g; s) - F(g; s') > 0\} \tag{55}$$

Given that index h, one can also define the index l by:

$$l = \min \{g > h \mid F(j; s) - F(j; s') \leq 0, \forall j \in [g, k]\}. \tag{56}$$

Such a l exists because $F(k; s) - F(k; s') = 0$. Notice that, by definition of such a l, one has:

$$F(l-1; s) - F(l-1; s') > 0 \text{ and } F(l; s) - F(l; s') \leq 0, \tag{57}$$

Hence, one has (using the definition of F provided by (7), that $n_l^s < n_l^{s'}$. We now establish that there exists some $i \in \{1, 2, \ldots, h-1\}$ such that:

$$F(i; s) - F(i; s') < 0 \text{ and } F(g; s) - F(g, s') = 0, \forall g < i. \tag{58}$$

Indeed, since $H^+(g; s) \leq H^+(g; s')$ for all $g = 1, 2, \ldots, k-1$, one has either:

$$\begin{array}{c} H^+(1; s) < H^+(1; s') \\ \Longleftrightarrow \text{ (thanks to expression (10))} \\ F(1; s) < F(1; s') \end{array} \tag{59}$$

or:

$$F(1; s) = F(1; s') \tag{60}$$

If case (59) holds, then the existence of some $i \in \{1, 2, \ldots, h-1\}$ for which expression (58) holds is established (with $i = 1$). If, on the other hand, case (60) holds, then, since $H^+(2; s) \leq H^+(2; s')$ holds, we must have either:

$$H^+(2; s) < H^+(2; s')$$

$$\Longleftrightarrow \text{ (thanks to expression (11))}$$

$$2F(1; s) + F(2; s) < 2F(1; s') + F(2; s') \tag{61}$$

or:

$$2F(1; s) + F(2; s) = 2F(1; s') + F(2; s') \tag{62}$$

Again, if we are in case (61), we can conclude (since $F(1; s) = F(1; s')$) that $F(2; s) < F(2; s')$, which establishes the existence of some $i \in \{1, 2, \ldots, h-1\}$ for which expression (58) holds (with $i = 2$ in that case). If we are in case (62), we iterate in the same fashion using the definition of H^+ provided by (11). We notice that the index i for which (58) holds must be strictly smaller than h because assuming otherwise will contradict, given the definition of h and the (iterated as above) definition of H^+, the fact that $H^+(g; s) \leq H^+(g; s')$ holds for all $g = 1, 2, \ldots, k-1$. We finally note that, because of the definition of F provided by (7), the definition of the index i just provided entails that:

$$n_i^s < n_i^{s'} \tag{63}$$

and:

$$n_g^s = n_g^{s'}$$

for all $g = 1, ..., i-1$. We now proceed by defining a new society - s^1 say - that belongs to $\mathcal{C}^n$, that has been obtained from s' by means of a Hammond's transfers and that is such that $s \succsim_{H^+} s^1 \succ_{H^+} s'$. For this sake, we define the numbers δ_1 and δ_2 and δ by:

$$\delta_1 = n_i^{s'} - n_i^s \ ; \ \delta_2 = n[F(l-1; s) - F(l-1; s')] \quad \text{and} \quad \delta = \min(\delta_1, \delta_2) \tag{64}$$

We note that, by the very definition of the index i, one has $\delta_1 > 0$. We notice also that, using (57) and the definition of the index l, one has $0 < \delta_2 \leq n_l^{s'} - n_l^s$. Define then the society s^1 by:

$$n_g^{s^1} = n_g^{s'}, \ \forall \ g \neq i, i+1, l \, ;$$

$$n_i^{s^1} = n_i^{s'} - \delta \ ; \ n_{i+1}^{s^1} = n_{i+1}^{s'} + 2\delta \ ; \ n_l^{s^1} = n_l^{s'} - \delta \, ;$$

It is clear that such a society belongs to $\mathcal{C}^n$. Moreover, s^1 has been obtained from s' by δ Hammond transfers as per definition 4 where the indices g, i, j and l of this definition are, here, i, $i+1$, $i+1$ and l (respectively)). By virtue of what has been established above, this implies that $s^1 \succ_{H^+} s'$. We further notice that:

$$\begin{aligned} F(g, s^1) &= \sum_{e=1}^{g} n_e^{s^1}/n \\ &= \sum_{e=1}^{g} n_e^{s'}/n \\ &= F(g, s') \end{aligned} \tag{66}$$

for all $g = 1, ..., i-1$. Also one has:

$$\begin{aligned} F(i, s^1) &= \sum_{e=1}^{i} n_e^{s^1}/n \\ &= F(i-1, s') + n_i^{s^1}/n \\ &= F(i-1, s') + (n_i^{s'} - \delta)/n \\ &= F(i, s') - \delta/n \end{aligned} \tag{67}$$

$$\begin{aligned} F(i+1,s^1) &= F(i,s^1) + n_{i+1}^{s^1}/n \\ &= F(i,s') - \delta/n + n_{i+1}^{s^1}/n \\ &= F(i,s') - \delta/n + n_{i+1}^{s'}/n + 2\delta/n \\ &= F(i+1,s') + \delta/n \end{aligned} \tag{68}$$

Furthermore, for $g = i+2, ..., l-1$, one has:

$$\begin{aligned} F(g,s^1) &= F(i+1,s^1) + \sum_{e=i+2}^{g} n_e^{s^1}/n \\ &= F(i+1,s') + \delta/n + \sum_{e=i+2}^{g} n_e^{s^1}/n \\ &= F(i+1,s') + \delta/n + \sum_{e=i+2}^{g} n_e^{s'}/n \\ &= F(g,s') + \delta/n \end{aligned} \tag{69}$$

While finally, for $g = l, ..., k$:

$$\begin{aligned} F(g,s^1) &= F(l-1,s^1) + \sum_{e=l}^{g} n_e^{s^1}/n \\ &= F(l-1,s') + \delta/n + \sum_{e=l}^{g} n_e^{s^1}/n \\ &= F(l-1,s') + \delta/n + n_l^{s'}/n - \delta/n + \sum_{e=l+1}^{g} n_e^{s'}/n \\ &= F(g,s') \end{aligned} \tag{70}$$

We must now verify that $s \succsim_{H^+} s^1$ and, therefore, that $H^+(g;s) - H^+(g;s^1) \leq 0$ for all $g = 1, 2, \ldots, k-1$. Since $s \succsim_{H^+} s'$, we know already that $H^+(g;s) - H^+(g,s') \leq 0$ for all $h = 1, 2, \ldots, k-1$. We first observe that, by the definition just given of s^1 one has:

$$H^+(g,s^1) = \begin{cases} H^+(g,s') \text{ for } g = 1,..,i-1, \\ H^+(g,s') - \delta/n \text{ for } g = i, \\ H^+(g,s') \text{ for } g = i+1,...,l-1, \\ H^+(g,s') - 2^{g-l}\delta/n \text{ for } g = l,...,k. \end{cases} \tag{71}$$

The first line of (71) is indeed clear given expression (66) and the definition of H^+ provided by (9).The second line of (71) results from (67) and the definition of H^+

provided by (11). Consider now $g = i+1$. One has (using (11) again):

$$\begin{aligned}
H^+(i+1,s^1) &= \sum_{g=1}^{i} \left(2^{i-g}\right) F(g;s^1) + F(i+1;s^1) \\
&= \sum_{g=1}^{i-1} \left(2^{i-g}\right) F(g;s') + F(i;s') - \delta/n + F(i+1;s^1) \text{ (by (67))} \\
&= \sum_{g=1}^{i-1} \left(2^{i-g}\right) F(g;s') + F(i;s') - \delta/n + F(i+1;s') + \delta/n \text{ (by (68))} \\
&= \sum_{g=1}^{i-1} \left(2^{i-g}\right) F(g;s') + F(i;s') + F(i+1;s') \\
&= \sum_{g=1}^{i} \left(2^{i-g}\right) F(g;s') + F(i+1;s') = H^+(i+1,s') \qquad (72)
\end{aligned}$$

Combined with (12) and the fact that $n_g^{s^1} = n_g^{s'}$ for all $g = i+2, ..., l-1$, equality (72) establishes the third line of expression (71). As for the last line of (71), we start with $g = l$ and we use (12) to write:

$$\begin{aligned}
H^+(l,s^1) &= 2H^+(l-1;s^1) + n_l^{s^1}/n \\
&= 2H^+(l-1;s') + (n_l^{s'} - \delta)/n \\
&= H^+(l,s') - \delta/n \qquad (73)
\end{aligned}$$

Iterating on this expression using (12) yields:

$$\begin{aligned}
H^+(l+1,s^1) &= 2H^+(l;s^1) + n_{l+1}^{s^1} \\
&= 2(H^+(l;s') - \delta/n) + n_{l+1}^{s'} \\
&= H^+(l+1,s') - 2\delta/n \qquad (74)
\end{aligned}$$

and therefore, for any $g \in \{l, ..., k\}$:

$$H^+(g,s^1) = H^+(g,s') - 2^{g-l}\delta/n$$

as required by the last line of expression (71). We now notice that expression (71) entails that:

$$H^+(g,s) - H^+(g,s^1) = \begin{cases} H^+(g,s) - H^+(g,s') \text{ for } g = 1, .., i-1, \\ H^+(g,s) - H^+(g,s') + \delta/n \text{ for } g = i, \\ H^+(g,s) - H^+(g,s') \text{ for } g = i+1, ..., l-1, \\ H^+(g,s) - H^+(g,s') + 2^{g-l}\delta/n \text{ for } g = l, ..., k. \end{cases} \qquad (75)$$

Since by assumption $s \succsim_{H^+} s'$, this establishes that $H^+(g;s) - H^+(g;s^1) \leq 0$ for all $g \in \{1, 2, \ldots, i-1\} \cup \{i+1, ..., l-1\}$. Consider now the case $g = i$. Using (12), we know that:

$$H^+(i;s) - H^+(i,s') = 2\left(H^+(i-1;s) - H^+(i-1;s')\right) + (n_i^s - n_i^{s'})/n \qquad (76)$$

By definition of i, one has $F(h;s) - F(h;s') = 0$ for all $h < i$, so that the first term in the right hand side of equation (76) is 0. Recalling then from (64) that

$\delta_1 = n_i^{s'} - n_i^s > 0$ and that $\delta = \min(\delta_1, \delta_2)$, it follows that:

$$n_i^s - n_i^{s'} + \delta \leq 0$$

By combining equations (75) and (76), we conclude that:

$$H^+(i,s) - H^+(i,s^1) = H^+(i,s) - H^+(i,s') + \delta/n = n_i^s - n_i^{s'} + \delta \leq 0 \quad (77)$$

Consider finally the case where $g = l, \ldots, k-1$. By using equation (12) (and recalling that $\delta_2 = n[F(l-1;s) - F(l-1;s')]$), one has:

$$H^+(l;s) - H^+(l;s') = 2(H^+(l-1;s) - H^+(l-1;s')) + F(l;s) - F(l;s') - \delta_2/n. \quad (78)$$

Combining (78) with the last line of (75), and remembering that $\delta \leq \delta_2$, one obtains:

$$H^+(l,s) - H^+(l,s^1) = 2[(H^+(l-1;s) - H^+(l-1;s')] + F(l;s) - F(l;s') + (\delta - \delta_2)/n \leq 0\,. \quad (79)$$

Finally, using successive applications of equation (12), one obtains, for any $g = l+1, \ldots, k-1$:

$$\begin{aligned} H^+(g;s) - H^+(g;s') &= 2^{g-l+1}[H^+(l-1;s) - H^+(l-1;s')] \\ &\quad + \sum_{e=l}^{g-1} 2^{g-1-e}[F(e;s) - F(e;s')] \\ &\quad + F(g;s) - F(g;s') - 2^{g-l}\delta_2/n \\ &\leq 0 \end{aligned}$$

by assumption. Combined with the last line of (75) and the fact that $\delta \leq \delta_2$, this completes the proof that $s \succeq_{H^+} s^1$. Hence, we have found a society s^1 obtained from society s' by means of a Hammond's transfers that is such that $s \succsim_{H^+} s^1 \succ_{H^+} s'$. We now show that, in moving from s' to s, one has "brought to naught" at least one of the differences $|F(h;s) - F(h;s')|$ that distinguishes s from s'. That is to say, we establish the existence of some $h \in \{1, ..., k-1\}$ for which one has:

$$\left|F(h;s) - F(h;s^1)\right| = 0$$

and:

$$|F(h;s) - F(h;s')| > 0$$

This is easily seen from the fact that, in the construction of s^1, one has either:

$$\delta = \delta_1 = n_i^{s'} - n_i^s \quad (80)$$

or:

$$\delta = \delta_2 = n[F(l-1;s) - F(l-1;s')] \quad (81)$$

If we are in the case (80), one has by definition of the index i and the function F:

$$F(i;s) - F(i;s^1) = 0$$

and:

$$F(i;s) - F(i;s') < 0$$

If on the other hand we are in case (81), then, we have (using (68)):

$$
\begin{aligned}
F(l-1;s)-F(l-1;s^1) &= F(l-1,s)-[F(l-1,s')+\delta_2/n] \\
&= 0
\end{aligned}
$$

while, by definition of the index l, one has:

$$F(l-1,s)-F(l-1,s')>0$$

Now, if $s=s^1$, then the proof is complete. If $\neg(s=s^1)$ but $s\succsim_1 s^1$, then we conclude that society s can be obtained from society s' by means of a finite sequence of Hammond's transfers and increments. If $\neg(s=s^1)$ and $\neg\left\{s\succsim_1 s^1\right\}$, then we can find three categories i,h and l just as in the preceding step and construct a new distribution - say s^2 - that can be obtained from distribution s^1 by means of an (integer number of) Hammond' transfers satisfying $s\succsim_{H^+} s^2\succ_{H^+} s^1$ and so on. More generally, after a finite number - t say - of iterations, we will find a distribution s^t such that $s\succsim_{H^+} s^t\succ_{H^+} s^{t-1}$. In that case, we will have either $s=s^t$ or $s\succsim_1 s^t$. As t is finite, and there are finitely many differences of the kind $|F(h;s)-F(h;s')|$ to bring to naught, this completes the proof.

A.8 Theorem 4

A.8.1 Statement (a) implies statement (b)

Just like the (detailed) argument underlying the proof of the corresponding implication for Theorem 3, the implication results straightforwardly from Propositions 3 and 4.

A.8.2 Statement (b) implies statement (c)

Assume that the inequality:

$$\sum_{i=1}^{k} n_i^s\,\alpha_i\geq\sum_{i=1}^{k} n_i^{s'}\alpha_i \tag{82}$$

holds for all $(\alpha_1,...,\alpha_k)\in\mathcal{A}_{\mathcal{H}}^-$. For any $i=1,...,k-1$, define the ordered list of k numbers $(\alpha_1^i,\ldots,\alpha_k^i)$ by:

$$
\begin{aligned}
\alpha_h^i &= 0 \text{ for } h=1,..,i \\
\alpha_h^i &= -2^{h-i-1} \text{ for } h=i+1,..,k
\end{aligned} \tag{83}
$$

We first show that, for any $i\in\{1,...,k-1\}$, $(\alpha_1^i,\ldots,\alpha_k^i)\in\mathcal{A}_{\mathcal{H}}^-$. Thanks to Lemma 3, this amounts to show that, for any $i\in\{1,...,k-1\}$, one has:

$$\alpha_{h+1}^i-\alpha_h^i\leq\alpha_h^i-\alpha_1^i\leq 0 \tag{84}$$

for all $h\in\{1,...,k-1\}$. If $h\leq i-1$, then one has (thanks to expression (83)):

$$\alpha_{h+1}^i-\alpha_h^i=0-0\leq\alpha_h^i-\alpha_1^i=0-0=0$$

so that (84) holds for those h. If $h=i$, then one has (expression (83)):

$$\alpha_{h+1}^i-\alpha_h^i=\alpha_{i+1}^i-\alpha_i^i=-1\leq\alpha_i^i-\alpha_1^i=0-0=0$$

so that (84) holds for that case. Assume now that $h \geq i+1$. In that case, one has by expression (83):

$$\begin{aligned} \alpha^i_{h+1} - \alpha^i_h &= -2^{h-i} + 2^{h-i-1} \\ &= -2^{h-i-1} \\ &= \alpha^i_h - \alpha^i_1 \leq 0 \end{aligned}$$

so that (84) holds for those h as well. Since the ordered lists of k numbers $(\alpha^i_1, \ldots, \alpha^i_k)$ defined by (83) all belong to $\mathcal{A}^-_{\mathcal{H}}$, inequality (82) holds for any of those ordered lists. As a result, one has:

$$\begin{aligned} \sum_{h=1}^{k} n_i^s \, \alpha^i_h &\geq \sum_{i=1}^{k} n_i^{s'} \alpha^i_h \\ &\Longleftrightarrow \\ \sum_{h=i+1}^{k} n_i^s 2^{h-i-1} &\leq \sum_{h=i+1}^{k} n_i^{s'} 2^{h-i-1} \end{aligned}$$

for every $i = 1, ..., k-1$, which, by expression (20), is the definition of H^- dominance.

A.8.3 Statement (c) implies statement (a)

Assume that $s \succsim_{H^-} s'$, so that $H^-(i;s) \leq H^-(i;s')$ for all $i = 1, 2, \ldots, k-1$ (avoiding the degenerate case where s is equal to s').
<u>Step 1</u>.

From Proposition 1, $s' \succeq_1 s$ implies $s \succeq_{H^-} s'$. By Theorem 1, $s' \succeq_1 s$ if and only if s' can be obtained from s by a finite sequence of increments or, equivalently, if s can be obtained from s' by a finite sequence of decrements. Hence, if $s' \succeq_1 s$, we know that s can be obtained from s' by means of a finite sequence of decrements, and the proof is complete. In the following, we therefore assume that $s \succeq_{H^-} s'$ but that $s' \succeq_1 s$ does not hold. Hence, there exists $i \in \{1, 2, \ldots, k-1\}$ such that $F(i;s') - F(i;s) > 0$, or, equivalently, such that:

$$\begin{aligned} 1 - F(i;s) &> 1 - F(i;s') \\ &\Longleftrightarrow \\ \overline{F}(i;s) &> \overline{F}(i;s') \end{aligned}$$

Let j be defined by:

$$j = \max\left\{h \mid \bar{F}(h;s) - \bar{F}(h;s') > 0\right\} . \tag{85}$$

Given j, we let l be defined by:

$$l = \max\left\{h < j \mid \bar{F}(h-1;s) - \bar{F}(h-1;s') \leq 0\right\} . \tag{86}$$

Such a l exists because $\bar{F}(0;s) - \bar{F}(0;s') = 0$. By definition, we have:

$$\bar{F}(l-1;s) - \bar{F}(l-1;s') \leq 0 \text{ and } \bar{F}(l;s) - \bar{F}(l;s') > 0\,, \tag{87}$$

from which we deduce that $n_l^s < n_l^{s'}$.

We now prove the existence of some $m \in \{j+2, \ldots, k\}$ such that:

$$\bar{F}(m-1;s) - \bar{F}(m-1;s') < 0 \text{ and } \bar{F}(h;s) - \bar{F}(h;s') = 0\,, \forall h > m-1\,. \tag{88}$$

Note that if such a m exists, equation (88) implies that $n^s_m < n^{s\prime}_m$. To prove the existence of a m such that (88) holds, one first recall that $s \succeq_{H^-} s'$ if and only if $H^-(h;s) - H^-(h;s') \leq 0$ for all $h = 1, 2, \dots, k-1$. If $H^-(k-1;s) - H^-(k-1;s') < 0$ or, equivalently, if $\bar{F}(k-1;s) - \bar{F}(k-1;s') < 0$, one has $m = k$ and $n^s_k < n^{s\prime}_k$. If, otherwise, $H^-(k-1;s) - H^-(k-1;s') = 0$ or, equivalently, $F(k-1;s) - F(k-1;s') = 0$, one can move forward and recall that $H^-(k-2;s) - H^-(k-2;s') \leq 0$. If $H^-(k-2;s) - H^-(k-2;s') < 0$, one obtains, using (**??**), that $[\bar{F}(k-1;s) - \bar{F}(k-1;s')] + [\bar{F}(k-2;s) - \bar{F}(k-2;s')] < 0$. When combined with $\bar{F}(k-1;s) - \bar{F}(k-1;s') = 0$, this implies that $\bar{F}(k-2;s) - \bar{F}(k-2;s') < 0$ so that one can set $m = k-1$, and observe that $n^s_k < n^{s\prime}_k$ and $n^s_{k-1} < n^{s\prime}_{k-1}$. If on the other hand $H^-(k-2;s) - H^-(2;s') = 0$, then one can proceed to the next iteration (and so on until one gets eventually to $j+2$). It can not be the case that $\bar{F}(j+h;s) - \bar{F}(j+h;s') = 0$ for all $h = 1, ..., k-j-1$ because assuming so would amount, when combined with the definition of j provided by (85) and the definition of $H^-(.)$ provided by (23), to contradicting the fact that $H^-(h;s) - H^-(h;s') \leq 0$ for all $h = 1, 2, \dots, k-1$.

<u>Step 2</u>.

We now define a new society $s^1 \in \mathcal{C}^n$, obtained from s' by means of a finite sequence of Hammond transfers, and such that $s \succeq_{H^-} s^1$. We start by setting:

$$\delta_1 = n \times \big[\bar{F}(l;s) - \bar{F}(l;s')\big] \;;\; \delta_2 = n \times \big[\bar{F}(m-1;s') - \bar{F}(m-1;s)\big] \text{ and } \delta = \min\{\delta_1\,;\,\delta_2\}\,. \tag{89}$$

From equation (88), we have $\delta_2 = n^{s\prime}_m - n^s_m > 0$. Then, it is apparent from (87) that $0 < \delta_1 \leq n^{s\prime}_l - n^s_l$. Moreover, since n^s_h and $n^{s\prime}_h \in \mathbb{N}_+$ for all $h = 1, 2, \dots, k$, one also has $\delta_1, \delta_2 \in \mathbb{N}_+$, which implies that $\delta \in \mathbb{N}_+$. We then define the society s^1 as follows:

$$n^{s^1}_h = n^{s\prime}_h,\ \forall\ h \neq l, m-1, m\,;$$

$$n^{s^1}_l = n^{s\prime}_l - \delta\ ;\ n^{s^1}_{l-1} = n^{s\prime}_{m-1} + 2\delta\ ;\ n^{s^1}_m = n^{s\prime}_m - \delta\,.$$

By construction, $s^1 \in C^n$, and s^1 is obtained from s by means of exactly $\delta \in \mathbb{N}_{++}$ Hammond's transfers. Observe now that:

$$H^-(h;s) - H^-(h;s^1) = \begin{cases} H^-(h;s) - H^-(h;s') + \left(2^{l-h-1}\right)\delta/n & \text{for } h = 1, \dots, l-1, \\ H^-(h;s) - H^-(h;s') & \text{for } h = l, \dots, m-2, \\ H^-(h;s) - H^-(h;s') + \delta/n & \text{for } h = m-1, \\ H^-(h;s) - H^-(h;s') & \text{for } h = m, \dots, k-1. \end{cases} \tag{92}$$

Since $s \succeq_{H^-} s'$ or, equivalently, $H^-(h;s) - H^-(h;s') \leq 0$ for all $h = 1, 2, \dots, k-1$, if follows from (92) that $H^-(h;s) - H^-(h;s^1) \leq 0$ for all $h \in \{l, \dots, m-2\} \bigcup \{m, \dots, k-1\}$. Consider now the case where $h = m-1$. Using (**??**), one has:

$$H^-(m-1;s) - H^-(m-1;s') = 2\,\left[H^-(m;s) - H^-(m;s')\right] + \frac{1}{n}(n^s_m - n^{s\prime}_m)\,. \tag{93}$$

Remember that $\bar{H}(m;s) - \bar{H}(m;s') = 0$ by definition of m. Remember also that $\delta_2 = n^{s\prime}_m - n^s_m > 0$ and that $\delta = \min\{\delta_1\,;\,\delta_2\}$ so that $\delta \leq (n^{s\prime}_m - n^s_m)$. Hence, combining equations (92) and (93) leads to the conclusion that:

$$H^-(m-1;s) - H^-(m-1;s^1) = \frac{\delta}{n} - \frac{1}{n}(n^{s\prime}_m - n^s_m) \leq 0\,. \tag{94}$$

Consider finally the cases where $h = 1, \ldots, l-1$ Start with the case where $h = l-1$. By using equation (**??**), and recalling that $\delta_1 = n \times \left[\bar{F}(l;s) - \bar{F}(l;s')\right]$, one observes that:

$$H^-(l-1;s) - H^-(l-1;s') = 2\left[H^-(l;s) - H^-(l;s')\right] + \bar{F}(l-1;s) - \bar{F}(l-1;s') - \frac{\delta_1}{n}. \tag{95}$$

Combining (95) with (92) yields:

$$H^-(l-1;s) - H^-(i-1;s^1) = 2\left[H^-(l;s) - H^-(l;s')\right] + \bar{F}(l-1;s) - \bar{F}(l-1;s') + \frac{\delta - \delta_1}{n} \leq 0. \tag{96}$$

(since $H^-(l;s) - H^-(l;s') \leq 0$ and $(\delta - \delta_1) \leq 0$). Finally, by successive applications of equation (**??**), one obtains, for any $h = 1, \ldots, l-2$:

$$\begin{aligned} H^-(h;s) - H^-(h;s') &= (2^{l-h})\left[H^-(l;s) - H^-(l;s')\right] + \bar{F}(l-1;s) - \bar{F}(l-1;s') \\ &\quad -(2^{l-h-1})\frac{\delta_1}{n} + \sum_{i=h+1}^{l-1}(2^{i-h-1})[\bar{F}(i;s) - \bar{F}(i;s')] \end{aligned} \tag{97}$$

Combining (97) with (92) one obtains, for all $h = 1, \ldots, i-2$:

$$\begin{aligned} H^-(h;s) - H^-(h;s^1) &= (2^{l-h})\left[H^-(l;s) - H^-(l;s')\right] + \bar{F}(l-1;s) - \bar{F}(l-1;s') \\ &\quad +(2^{l-h-1})\frac{\delta - \delta_1}{n} + \sum_{i=h+1}^{l-1}(2^{i-h-1})[\bar{F}(i;s) - \bar{F}(i;s')] \\ &\leq 0 \end{aligned}$$

(since $H^-(l;s) - H^-(l;s') \leq 0$, $\delta - \delta_1$ and $\bar{F}(i;s) - \bar{F}(i;s') \leq 0$ for all $i = 1, \ldots, l-1$). Hence, $H^-(h;s) \leq H^-(h;s^1)$ for all $h = 1, ..., k-1$ and, therefore, $s \succeq_{H^-} s^1$.

<u>Step 3</u>.

Hence, we have found a society s^1 obtained from society s' by means of a sequence of Hammond's transfers that is such that $s \succsim_{H^-} s^1 \succ_{H^-} s'$. We now show that, in moving from s' to s^1, one has "brought to naught" at least one of the differences $\left|\overline{F}(h;s) - F(\overline{h};s')\right|$ that distinguishes s from s'. That is, we establish the existence of some $h \in \{1, ..., k-1\}$ for which one has:

$$\left|F(h;s) - F(h;s^1)\right| = 0$$

and:

$$|F(h;s) - F(h;s')| > 0$$

This is easily seen from the fact that, in the construction of s^1, one has either:

$$\delta = \delta_1 = n \times \left[\bar{F}(l;s) - \bar{F}(l;s')\right] \tag{98}$$

or:

$$\delta = \delta_2 = n_m^{s\prime} - n_m^s \tag{99}$$

If we are in the case (99), one has by definition of the index m and the function $\overline{F}$:

$$\overline{F}(m-1;s) - \overline{F}(m-1;s^1) = 0$$

and:

$$\begin{array}{rcl}\overline{F}(m-1;s)-\overline{F}(m-1;s') & < & 0 \\ & \Longrightarrow & \\ |F(m-1;s)-F(m-1;s')| & > & 0\end{array}$$

If, on the other hand, we are in case (98), then, we have (using (68)):

$$\begin{array}{rcl}\overline{F}(l;s)-\overline{F}(l;s^1) & = & \overline{F}(l,s)-[F(l,s')+\delta_2/n] \\ & = & 0\end{array}$$

while, by definition of the index l, one has:

$$\overline{F}(l,s)-\overline{F}(l,s')>0$$

Now, if $s = s^1$, then the proof is complete. If $s \neq s^1$ and $s^1 \succsim_1 s$, then we conclude that society s can be obtained from society s' by means of a finite sequence of Hammond's transfers and decrements. If $s \neq s^1$ and $\neg(s^1 \succsim_1 s)$, then we can find three categories j, l and m just as in the preceding step and construct a new distribution - say s^2 - that can be obtained from distribution s^1 by means of an (integer number of) Hammond' transfers satisfying $s \succsim_{H^-} s^2 \succ_{H^-} s^1$ and so on. More generally, after a finite number - t say - of iterations, we will find a distribution s^t such that $s \succsim_{H^-} s^t \succ_{H^-} s^{t-1}$. In that case, one has either $s = s^t$ or $s^t \succsim_1 s$. As t is finite, and there are finitely many differences of the kind $\left|\overline{F}(h;s)-\overline{F}(h;s')\right|$ to bring to naught, this completes the proof.

A.9 Theorem 5

A.9.1 Statement (a) implies statement (b)

This results immediately from the definition of the set $\mathcal{A}_{\mathcal{H}}$ (using Proposition 4).

A.9.2 Statement (b) implies statement (c)

Assume that the inequality $\sum_{h=1}^{k} n_h^s \alpha_h \geq \sum_{h=1}^{k} n_h^{s\prime} \alpha_h$ holds for all lists of real numbers $(\alpha_1, ..., \alpha_l) \in \mathcal{A}_H$. This implies in particular that the inequality holds for all $(\alpha_1, ..., \alpha_l) \in \mathcal{A}_H^+$ so that $s \succsim^{\mathcal{A}_{\mathcal{H}}^+} s'$ holds. It then follows from Theorem 3 that $s \succeq_{H^+} s'$. Similarly, the fact that the inequality $\sum_{h=1}^{k} n_h^s \alpha_h \geq \sum_{h=1}^{k} n_h^{s\prime} \alpha_h$ holds for all lists of real numbers $(\alpha_1, ..., \alpha_l) \in \mathcal{A}_H$ implies in particular that it holds for all $(\alpha_1, ..., \alpha_l) \in \mathcal{A}_H^-$. Hence $s \succsim^{\mathcal{A}_{\mathcal{H}}^-} s'$ holds and, thanks to Theorem 4, so does $s \succeq_{H^-} s'$. Hence one has both $s \succeq_{H^+} s'$ and $s \succeq_{H^-} s'$, as required by the definition of $\succeq_H$.

A.9.3 Statement (c) implies statement (b)

Assume that $s \succeq_H s'$. Thanks to Proposition 5, one needs to show that $\sum_{h=1}^{k} n_h^s \alpha_h \geq \sum_{h=1}^{k} n_h^{s\prime} \alpha_h$ holds for all $(\alpha_1, .., \alpha_k) \in \mathbb{R}_+^k$ for which there exists an integer $t \in \{1, ..., k\}$ such that $(\alpha_{i+1} - \alpha_i) \geq (\alpha_t - \alpha_{i+1})$, for all $i = 1, 2, \ldots, t-1$ (if any) and $(\alpha_{i'+1} - \alpha_{i'}) \leq (\alpha_{i'} - \alpha_{t'})$, for all $i \in \{t, t+1, \ldots, k-1\}$ (again if this set is non-empty). Since by definition of $\succeq_H$, one has both $s \succeq_{H^+} s'$ and $s \succeq_{H^-} s'$, we know at once from Theorems 3 and 4 that $\sum_{h=1}^{k} n_h^s \alpha_h \geq \sum_{h=1}^{k} n_h^{s\prime} \alpha_h$ holds for all list of real numbers $(\alpha_1, .., \alpha_k) \in \mathcal{A}_H^+ \cap \mathcal{A}_H^-$. These numbers are associated to an integer $t \in \{1, k\}$. Hence one only needs to prove that $\sum_{h=1}^k n_h^s \alpha_h \geq \sum_{h=1}^k n_h^{s\prime} \alpha_h$ holds for all $(\alpha_1, .., \alpha_k) \in \mathbb{R}_+^k$ for which there exists an integer $t \in \{2, ..., k-1\}$ such that $(\alpha_{i+1} - \alpha_i) \geq (\alpha_t - \alpha_{i+1})$, for all $i = 1, 2, \ldots, t-1$ and $(\alpha_{i'+1} - \alpha_{i'}) \leq (\alpha_{i'} - \alpha_{t'})$ for all $i = \{t, t+1, \ldots, k-1\}$. Using (28) in Lemma 1, we find useful to write (for any society s):

$$\frac{1}{n} \sum_{h=1}^{k} n_i^s \, \alpha_i = \alpha_k - \sum_{h=1}^{k-1} F(h; s) \left[\alpha_{h+1} - \alpha_h\right] . \tag{100}$$

or:

$$\frac{1}{n} \sum_{i=1}^{k} n_i^s \, \alpha_i = \alpha_k - \sum_{h=1}^{k-1} F(h, s) \, \theta_h \, . \tag{101}$$

with $\theta_h = \alpha_{h+1} - \alpha_h$ for every $h = 1, ..., k-1$. Letting $\vartheta_i = \theta_i - \sum_{j=i+1}^{k-1} \theta_j$ for all $i = 1, 2, \ldots, k-2$ and $\vartheta_{k-1} = \theta_{k-1}$, we rewrite $\sum_{h=1}^{k-1} F(h, s) \, \theta_h$ in (101) as follows:

For $i = 1$:

$$\begin{aligned}
F(1;s)\theta_1 &= F(1;s)\left[\theta_1 - \sum_{h=2}^{k-1} \theta_h\right] + F(1;s)\theta_2 + ... + F(1;s)\theta_{k-1} \\
&= F(1;s)\left[\theta_1 - \sum_{h=2}^{k-1} \theta_h\right] + F(1;s)\left[\theta_2 - \sum_{h=3}^{k-1} \theta_h\right] \\
&\quad +2F(1;s)\theta_3 + ... + 2F(1;s)\theta_{k-1} \\
&= F(1;s)\left[\theta_1 - \sum_{h=2}^{k-1} \theta_h\right] + F(1;s)\left[\theta_2 - \sum_{h=3}^{k-1} \theta_h\right] \\
&\quad +2F(1;s)\left[\theta_3 - \sum_{h=4}^{k-1} \theta_h\right] + 4F(1;s)\theta_4 ... + 4F(1;s)\theta_{k-1} \\
&= .... \\
&= F(1;s)\vartheta_1 + F(1;s)\vartheta_2 + 2F(1;s)\vartheta_2 + 2^2 F(1;s)\vartheta_3 + ... + 2^{k-3} F(1;s)\vartheta_{k-1}
\end{aligned} \tag{102}$$

For $i = 2$

$$
\begin{aligned}
F(2;s)\theta_2 &= F(2;s)\left[\theta_2 - \sum_{h=3}^{k-1}\theta_h\right] + F(2;s)\theta_3 + ... + F(2;s)\theta_{k-1} \\
&= F(2;s)\left[\theta_2 - \sum_{h=3}^{k-1}\theta_h\right] + F(2;s)\left[\theta_3 - \sum_{h=4}^{k-1}\theta_h\right] \\
&\quad +2F(2;s)\theta_4 + ... + 2F(2;s)\theta_{k-1} \\
&= F(2;s)\left[\theta_2 - \sum_{h=3}^{k-1}\theta_h\right] + F(2;s)\left[\theta_3 - \sum_{h=3}^{k-1}\theta_h\right] \\
&\quad +2F(2;s)\left[\theta_4 - \sum_{h=5}^{k-1}\theta_h\right] + 4F(2;s)\theta_5 ... + 4F(2;s)\theta_{k-1} \\
&= .... \\
&= F(2;s)\vartheta_2 + F(2;s)\vartheta_3 + 2F(2;s)\vartheta_4 + 2^2F(2;s)\vartheta_5 + ... + 2^{k-4}F(2;s)\vartheta_{k-1}
\end{aligned}
\tag{103}
$$

More generally, one has $F(k-1,s)\theta_{k-1} = F(k-1;s)\vartheta_{k-1}$ and, for all $i = 1, 2, \dots, k-2$:

$$
F(i;s)\theta_i = F(i;s)\,\vartheta_i + F(i;s)\sum_{h=i+1}^{k-1}\left(2^{h-i-1}\right)\vartheta_h\,. \tag{104}
$$

Hence, one can write:

$$
\begin{array}{lcl}
F(1;s)\theta_1 &=& F(1;s)\vartheta_1 + F(1;s)\vartheta_2 + 2F(1;s)\vartheta_3 + 2^2F(1;s)\vartheta_4 + ... + 2^{k-4}F(1;s)\vartheta_{k-2} + 2^{k-3}F(1;s)\vartheta_{k-1} \\
F(2;s)\theta_2 &=& F(2;s)\vartheta_2 + F(2;s)\vartheta_3 + 2F(2;s)\,\vartheta_4 + ... + 2^{k-5}F(2;s)\vartheta_{k-2} + 2^{k-4}F(2;s)\vartheta_{k-1} \\
F(3;s)\theta_3 &=& F(3;s)\vartheta_3 + F(3;s)\vartheta_4 + ... + 2^{k-6}F(3;s)\vartheta_{k-2} + 2^{k-5}F(3;s)\vartheta_{k-1} \\
F(4;s)\theta_4 &=& + F(4;s)\vartheta_4 + ... + 2^{k-7}F(4;s)\vartheta_{k-2} + 2^{k-6}F(4;s)\vartheta_{k-1} \\
\dots && \\
F(k-2;s)\theta_{k-2} &=& F(k-2;s)\vartheta_{k-2} + F(k-2;s)\vartheta_{k-1} \\
F(k-1;s)\theta_{k-1} &=& F(k-1;s)\vartheta_{k-1}
\end{array}
\tag{105}
$$

Remembering that $\vartheta_{k-1} = \theta_{k-1}$ and $\vartheta_i = \theta_i - \sum_{h=i+1}^{k-1}\theta_h$ for all $i = 1, 2, \dots, k-2$, one can use equation (11) and sum vertically the decomposition (105) to obtain:

$$
\sum_{i=1}^{k-1} F(i;s)\,\theta_h = \sum_{i=1}^{k-1} H^+(i;s)\left[\theta_i - \sum_{h=i+1}^{k-1}\theta_h\right]. \tag{106}
$$

Since $\frac{1}{n}\sum_{i=1}^{k} n_i^s\alpha_i = \alpha_k - \sum_{i=1}^{k-1} F(i;s)\,\theta_i$, one obtains finally:

$$
\frac{1}{n}\left[\sum_{i=1}^{k} n_i^s\alpha_i - \sum_{i=1}^{k} n_i^{s'}\alpha_i\right] = \sum_{i=1}^{k-1}\left[H^+(i;s') - H^+(i;s)\right]\left[\theta_i - \sum_{h=i+1}^{k-1}\theta_h\right]. \tag{107}
$$

In a symmetric fashion, one obtains from equation (28) in Lemma 1:

$$
\frac{1}{n}\sum_{i=1}^{k} n_i^s\,\alpha_i = \alpha_1 + \sum_{i=1}^{k-1}\bar{F}(i;s)\left[\alpha_{i+1} - \alpha_i\right]. \tag{108}
$$

Hence, letting $\beta_1 = \theta_1$ and $\beta_i = (\theta_i - \sum_{j=1}^{i-1}\theta_j)$ for all $i = 2, 3, \dots, k-1$, we rewrite $\sum_{i=1}^{k-1}\bar{F}(i;s)\left[\alpha_{i+1} - \alpha_i\right]$ in (108) as follows:

For $i = k-1$:

$$
\begin{aligned}
\overline{F}(k-1;s)\theta_{k-1} &= \overline{F}(k-1;s)\left[\theta_{k-1} - \sum_{h=1}^{k-2}\theta_h\right] + \overline{F}(k-1;s)\theta_{k-2} + ... + \overline{F}(k-1;s)\theta_1 \\
&= \overline{F}(k-1;s)\left[\theta_{k-1} - \sum_{h=1}^{k-2}\theta_h\right] + \overline{F}(k-1;s)\left[\theta_{k-2} - \sum_{h=1}^{k-3}\theta_h\right] \\
&\quad +2\overline{F}(k-1;s)\theta_{k-3} + ... + 2\overline{F}(k-1;s)\theta_1 \\
&= \overline{F}(k-1;s)\left[\theta_{k-1} - \sum_{h=1}^{k-2}\theta_h\right] + \overline{F}(k-1;s)\left[\theta_{k-2} - \sum_{h=1}^{k-3}\theta_h\right] \\
&\quad +2\overline{F}(k-1;s)\left[\theta_{k-3} - \sum_{h=1}^{k-4}\theta_h\right] + 4\overline{F}(k-1;s)\theta_{k-4} ... + 4\overline{F}(k-1;s)\theta_1 \\
&= .... \\
&= \overline{F}(k-1;s)\beta_{k-1} + \overline{F}(k-1;s)\beta_{k-2} + 2\overline{F}(k-1;s)\beta_{k-3} \\
&\quad +2^2\overline{F}(k-1;s)\beta_{k-4} + ... + 2^{k-3}2\overline{F}(k-1;s)\beta_1
\end{aligned} \tag{109}
$$

For $i = k-2$:

$$
\begin{aligned}
\overline{F}(k-2;s)\theta_{k-2} &= \overline{F}(k-2;s)\left[\theta_{k-2} - \sum_{h=1}^{k-3}\theta_h\right] + \overline{F}(k-2;s)\theta_{k-3} \\
&\quad +\overline{F}(k-2;s)\theta_{k-4} + ... + \overline{F}(k-2;s)\theta_1 \\
&= \overline{F}(k-2;s)\left[\theta_{k-2} - \sum_{h=1}^{k-3}\theta_h\right] + \overline{F}(k-2;s)\left[\theta_{k-3} - \sum_{h=1}^{k-4}\theta_h\right] \\
&\quad +2\overline{F}(k-2;s)\theta_{k-4} + ... + 2\overline{F}(k-2;s)\theta_1 \\
&= \overline{F}(k-2;s)\left[\theta_{k-2} - \sum_{h=1}^{k-3}\theta_h\right] + \overline{F}(k-2;s)\left[\theta_{k-3} - \sum_{h=1}^{k-4}\theta_h\right] \\
&\quad +2\overline{F}(k-2;s)\left[\theta_{k-4} - \sum_{h=1}^{k-5}\theta_h\right] + 4\overline{F}(k-2;s)\theta_{k-5} ... + 4\overline{F}(k-2;s)\theta_1 \\
&= .... \\
&= \overline{F}(k-2;s)\beta_{k-2} + \overline{F}(k-2;s)\beta_{k-3} + 2\overline{F}(k-2;s)\beta_{k-4} \\
&\quad +2^2\overline{F}(k-2;s)\beta_{k-5} + ... + 2^{k-4}2\overline{F}(k-2;s)\beta_1
\end{aligned} \tag{110}
$$

More generally, one has $\bar{F}(1;s)\theta_1 = \bar{F}(1;s)\beta_1$ and:

$$
\bar{F}(i;s)\,\theta_i = \bar{F}(i;s)\,\beta_i + \bar{F}(i;s)\sum_{h=1}^{i-1}\left(2^{i-h-1}\right)\beta_h\,, \quad \forall i = 2,3,\dots,k-1\,. \tag{111}
$$

Hence, one can conclude that:

$$\begin{array}{rcr}
\overline{F}(k-1;s)\theta_{k-1} & = & \overline{F}(k-1;s)\beta_{k-1}+\overline{F}(k-1;s)\beta_{k-2}+...+2^{k-4}\overline{F}(k-1;s)\,\beta_2+2^{k-3}\overline{F}(k-1;s)\beta_1 \\
\overline{F}(k-2;s)\theta_{k-2} & = & \overline{F}(k-2;s)\beta_{k-2}+\cdots+2^{k-5}\overline{F}(k-2;s)\beta_2+2^{k-4}\overline{F}(k-2;s)\beta_1 \\
\ldots\ldots\ldots & = & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\overline{F}(2;s)\theta_2 & = & \overline{F}(2;s)\beta_2+\overline{F}(2;s)\beta_1 \\
\overline{F}(1;s)\theta_1 & = & +\overline{F}(1;s)\theta_1
\end{array} \quad (112)$$

Using (23), and summing vertically the previous equation, one obtains:

$$\sum_{i=1}^{k-1}\overline{F}(i;s)\theta_i = H^-(1;s)\theta_1 + \sum_{i=2}^{k-1} H^-(1;s)[\theta_i - \sum_{h=1}^{i-1}\theta_h] \quad (113)$$

These decompositions being obtained, consider now an integer $t \in \{2,3,\ldots,k-1\}$ such that:

$$\theta_{t-1} \geq 0 \text{ and } (\theta_i - \sum_{h=i+1}^{t-1}\theta_h) \geq 0 \text{ for } i = 1,...,t-2 \quad (114)$$

and:

$$\theta_t \leq 0 \text{ and } (\theta_i - \sum_{h=t}^{i-1}\theta_h) \leq 0 \text{ for } i = t+1,...,k-1 \quad (115)$$

From Equation (29) in Lemma 1, one has, for any such a $t \in \{2,3,\ldots,k-1\}$:

$$\frac{1}{n}\sum_{i=1}^{k} n_i^s\,\alpha_i = \alpha_t - \sum_{h=1}^{t-1} F(i;s)\theta_i + \sum_{i=t}^{k-1}\bar{F}(i;s)\theta_i\,. \quad (116)$$

By using equations (102) - (107) and replacing category k by category t in these equations, one obtains:

$$\sum_{i=1}^{t-1} F(i;s)\theta_i = \sum_{i=1}^{t-2} H^+(i;s)\left[\theta_i - \sum_{h=i+1}^{t-1}\theta_j\right] + H^+(t-1;s)\,\theta_{t-1}\,. \quad (117)$$

Symmetrically, replacing category 1 by category t in equations (108)-(113) enables one to write:

$$\sum_{i=t}^{k-1}\overline{F}(i;s)\theta_i = H^-(t;s)\,\theta_t + \sum_{i=t+1}^{k-1} H^-(i;s)\left[\theta_i - \sum_{h=t}^{i-1}\theta_j\right]. \quad (118)$$

Combining equations (116), (117) and (118), one gets finally:

$$\begin{aligned}
\frac{1}{n}[\sum_{h=1}^{k}(n_h^s - n_h^{s'})\alpha_h] \;=\;& \sum_{h=1}^{t-2}[H^+(i;s') - H^+(i;s)]\left[\theta_i - \sum_{h=i+1}^{t-1}\theta_j\right] \\
& +[H^+(t-1;s') - H^+(t-1;s)]\,\theta_{t-1} \\
& +[H^-(t;s) - H^-(t;s')]\,\theta_t \\
& + \sum_{i=t+1}^{k-1}[H^-(i;s) - H^-(i;s')]\left[\theta_i - \sum_{h=t}^{i-1}\theta_j\right]
\end{aligned} \quad (119)$$

Since $s \succeq_H s'$, one has both $s \succeq_{H^+} s'$ and $s \succeq_{H^-} s'$. Hence $H^+(i;s) - H^+(i;s') \leq 0$ and $H^-(i;s) - H^-(i;s) \leq 0$ for all $i \in \mathcal{C}$. This, combined with equations (114) and (114), leads to the conclusion that each term of the right hand side of Equality (119) is positive, and, therefore, that $\sum_{h=1}^{k}(n_h^s - n_h^{s'})\alpha_h \geq 0$ for all lists of real numbers $(\alpha_1, ..., \alpha_k)$ in the set $\mathcal{A}_H$, as required by normative dominance.

A.10 Proposition 6

Assume that $s \succeq_{H^+} s'$ and, thanks to Theorem 3, that s can be obtained from s' by means of a finite sequence of increments and/or Hammond transfers. We immediately deduce from this information that the smallest $i \in \{1, 2, \ldots, k\}$ for which $n_i^s \neq n_i^{s\prime}$ is such that $n_i^s < n_i^{s\prime}$. But this implies that $s \succeq_L s'$. To show that the converse implication is false, one just needs to consider the following example for $k = n = 3$, and societies s and s' such that:

$$n_1^{s\prime} = 1, n_2^{s\prime} = 0,\ s_3^{s\prime} = 2$$

and:

$$n_1^s = n_3^s = 0,\ n_2^s = 3.$$

It is clear that $s \succeq_L s'$. The conclusion that $s \succeq_{H^+} s'$ does not hold follows then at once from the following table which gives the values of $H^+(j;s)$ and $H^+(j;s')$ as per expression (12) for $j = 1, 2, 3$.

	category 1	category 2	category 3
$F(;s)$	0	1	1
$H^+(:s)$	0	1	2
$F(;s\prime)$	1/3	1/3	1
$H^+(;s\prime)$	1/3	2/3	2

A.11 Proposition 7

As a preliminary of the proof, we first notice that, for any society $s \in \mathcal{C}(t)^n$, one has:

$$n^s(\frac{2\,i+1}{2^{t+1}}, t+1) = 0 \tag{120}$$

and:

$$H_{t+1}^+(\frac{2\,i+1}{2^{t+1}}; s) = 2\,H_{t+1}^+(\frac{i}{2^t}; s) \tag{121}$$

Indeed, since $\mathcal{C}(t) \subset \mathcal{C}(t+1)$, one also has $s \in \mathcal{C}(t+1)^n$. Moreover, for any $i = 1, 2, \ldots, (2^t)k$, one has:

$$\frac{i}{2^t} = \frac{2\,i}{2^{t+1}}\,, \tag{122}$$

Equation (120) then follows from the fact that $n^s(\frac{h}{2^{t+1}}, t+1) = 0$ for all $\frac{h}{2^{t+1}} \notin \mathcal{C}(t)$, while Equation (121) is an immediate consequence of Equations (120) and (122) and the fact that, thanks to Expression (12), one has:

$$H_{t+1}^+(\frac{2\,i+1}{2^{t+1}}; s) = 2\,H_{t+1}^+(\frac{2\,i}{2^{t+1}}; s) + n^s(\frac{2\,i+1}{2^{t+1}}, t+1)/n$$

for every $i = 0, 1, \ldots, (2^t)k - 1$. We also observe that:

$$H^+_{t+1}(\frac{i}{2^t}; s) = \sum_{h=1}^{i-1} \left(2^{2(i-h)-1}\right) H^+_t(\frac{h}{2^t}; s) + H^+_t(\frac{i}{2^t}; s), \tag{123}$$

for any society $s \in \mathcal{C}(t)^n$. Indeed, from Equation (37) applied to the grid $\mathcal{C}(t+1)$, we know that:

$$H^+_{t+1}(x; s) = \frac{1}{n} \sum_{h=1}^{j} \left(2^{j-h}\right) n^s(h/2^{t+1}, t+1),$$

for any $x \in \mathcal{C}(t+1)$, and $j = x2^{t+1}$). Applying this to $x = \frac{i}{2^t}$ for any $i = 1, ..., (2^t)k$ yields:

$$H^+_{t+1}(\frac{i}{2^t}; s) = \frac{1}{n} \sum_{h=1}^{2i} \left(2^{2i-h}\right) n^s(h/2^{t+1}, t+1), \quad . \tag{124}$$

for any such i. Expression (123) can then be obtained from (124) the following observations (made only for $i = 1, 2, 3$, but easily extendable to any other i). For $i = 1, 2, 3$ (124) writes:

$$\begin{aligned} H^+_{t+1}(\frac{1}{2^t}; s) &= \frac{1}{n}[2n^s(\frac{1}{2^{t+1}}, t+1) + n^s(\frac{2}{2^{t+1}}, t+1)] \\ &= \frac{1}{n} n^s(\frac{2}{2^{t+1}}, t+1) \text{ (since } n^s(\frac{1}{2^{t+1}}, t+1) = 0) \\ &= \frac{1}{n} n^s(\frac{1}{2^t}, t+1) \end{aligned} \tag{125}$$

$$\begin{aligned} H^+_{t+1}(\frac{2}{2^t}; s) &= \frac{1}{n}[8n^s(\frac{1}{2^{t+1}}, t+1) + 4n^s(\frac{2}{2^{t+1}}, t+1) + 2n^s(\frac{3}{2^{t+1}}, t+1) + n^s(\frac{4}{2^{t+1}}, t+1)] \\ &= \frac{1}{n}[4n^s(\frac{2}{2^{t+1}}, t+1) + n^s(\frac{4}{2^{t+1}}, t+1)] \\ &= \frac{1}{n}[4n^s(\frac{1}{2^t}, t+1) + n^s(\frac{2}{2^t}, t+1)] \end{aligned} \tag{126}$$

(since again $n^s(\frac{h}{2^{t+1}}, t+1) = 0$ for all $\frac{h}{2^{t+1}} \notin \mathcal{C}(t)$)

$$\begin{aligned} H^+_{t+1}(\frac{3}{2^t}; s) &= \frac{1}{n}[32n^s(\frac{1}{2^{t+1}}, t+1) + 16n^s(\frac{2}{2^{t+1}}, t+1) + 8n^s(\frac{3}{2^{t+1}}, t+1) \\ &\quad +4n^s(\frac{4}{2^{t+1}}, t+1) + 2n^s(\frac{5}{2^{t+1}}, t+1) + n^s(\frac{6}{2^{t+1}}, t+1)] \\ &= \frac{1}{n}[16n^s(\frac{2}{2^{t+1}}, t+1) + 4n^s(\frac{4}{2^{t+1}}, t+1) + n^s(\frac{6}{2^{t+1}}, t+1)] \end{aligned} \tag{127}$$

(because again $n^s(\frac{h}{2^{t+1}}, t+1) = 0$ for all $\frac{h}{2^{t+1}} \notin \mathcal{C}(t)$). Now, applying Equation (37) to the grid $\mathcal{C}(t)$, one has:

$$H^+_t(\frac{1}{2^t}; s) = \frac{1}{n} n^s(\frac{1}{2^t}, t) \tag{128}$$

$$H^+_t(\frac{2}{2^t}; s) = \frac{1}{n}[2n^s(\frac{1}{2^t}, t) + n^s(\frac{2}{2^t}, t)] \tag{129}$$

$$H^+_t(\frac{3}{2^t}; s) = \frac{1}{n}[4n^s(\frac{1}{2^t}, t) + 2n^s(\frac{2}{2^t}, t) + \frac{1}{n} n^s(\frac{3}{2^t}, t)] \tag{130}$$

so that Expression (123) for $i = 1, 2, 3$ results from combining (125)-(127) with (128)-(130).

In order to prove the result, consider two societies s and $s' \in \mathcal{C}(t)^n$ and assume that $s \succeq^t_{H^+} s'$ holds. By Definition 11:

$$H^+_t\left(\frac{i}{2^t}; s\right) \le H^+_t\left(\frac{i}{2^t}; s'\right)$$

for all $i \in \{1, ..., (2^t)k\}$. Hence, for any $h \in \{1, ..., i\}$ one has:

$$H^+_t\left(\frac{h}{2^t}; s\right) \le H^+_t\left(\frac{h}{2^t}; s'\right)$$

and, using (123):

$$H^+_{t+1}(\frac{i}{2^t}; s) \le H^+_{t+1}(\frac{i}{2^t}; s')$$

for all $i \in \{1, ..., (2^t)k\}$ which implies, thanks to (121) and definition 11 applied to the grid $\mathcal{C}(t+1)^n$ that $s \succeq^{t+1}_{H^+} s'$.

A.12 Theorem 6

The proof that statement (a) of the theorem implies statement (b) gas been established in Proposition 6 (since $\succeq_{H^+} = \succeq^0_{H^+}$). In order to prove the converse implication, consider two arbitrary societies s and s' in $\mathcal{C}^n$ such that $s \succeq_L s'$. Because of Proposition 7, we only have to show that there exists a non-negative integer t for which $s \succeq^t_H s'$ holds or, equivalently thanks to Theorem 3, that s can be obtained from s' by means of a finite sequence of increments and/or Hammond transfers on the grid $\mathcal{C}(t)$. Since $s \succeq_L s'$, there is by Definition 9 an index $i \in \{1, 2, \ldots, k\}$ such that $n^s_h = n^s(h, t) = n^{s'}(h, t) = n^{s\prime}_h$ for all $h = 1, 2, \ldots, i-1$ and $n^s_i = n^s(i, t) < n^{s'}(i, t) = n^{s\prime}_i$. Given this index i, consider a society $s'' \in \mathcal{C}^n$ such that:

$$n^{s\prime\prime}_h = n^s_h, \ \forall h = 1, \ldots, i\,;$$

$$n^{s\prime\prime}_{i+1} = \sum_{h=i+1}^{k} n^s_h\,;$$

$$n^{s\prime\prime}_h = 0, \ \forall h = i+2, \ldots, k\,.$$

Notice that $\sum_{h=1}^k n^{s\prime\prime}_h = n$ and that $F(i; s) \le F(i; s'')$ for all $i = 1, ..., k$ so that $s \succeq_1 s''$ or, equivalently by Theorem (1), that s can be obtained from s'' by means of a finite sequence of increments. We also observe that:

$$n^{s\prime}_h(h) = n^{s\prime\prime}_h, \ \forall h = 1, \ldots, i-1\,; \tag{132a}$$

$$n^{s\prime}_i - n^{s\prime\prime}_i > 0\,; \quad n^{s\prime}_{i+1} - n^{s\prime\prime}_{i+1} < 0\,; \tag{132b}$$

$$n^{s\prime}_h \ge n^{s\prime\prime}_h = 0, \ \forall h = i+2, \ldots, k\,. \tag{132c}$$

Define, for any $h \in \mathcal{C}$, the number δ_h by:

$$\delta_h = n^{s\prime}_h - n^{s\prime\prime}_h$$

It is clear that δ_h so defined is an integer (which may be positive or negative). Since $\sum_{h=i}^{k} n_h^{s'} = \sum_{h=i}^{k} n_h^{s''}$, one can write:

$$\delta_i + \delta_{i+1} = - \sum_{h=i+2}^{k} \delta_h \, . \tag{133}$$

Since, by (132c), $\delta_h \geq 0$ for all $h = i+2, \ldots, k$, one observes that $\delta_i + \delta_{i+1} \leq 0$. We consider two cases.

Case 1: $\delta_i + \delta_{i+1} = 0$. In that case, we conclude from (133) that $\sum_{h=i+2}^{k} \delta_h = 0$ and, thanks to (132c), that $n_h^{s'} = n_h^{s''}$ for all $h = i+2, \ldots, k$. Hence, one has $n_h^{s'} = n_h^{s''}$ for all $h = \{1, \ldots, i-1\} \cap \{i+2, \ldots, k\}$, and $\delta_i = n_i^{s'} - n_i^{s''} = n_{i+1}^{s''} - n_{i+1}^{s'} > 0$. Hence, s'' can be obtained from s' by means of δ_i increments from i to $i+1$ and we conclude that $s \succeq_1 s'' \succeq_1 s'$ which implies that $s \succeq_H^t s'$ for all $t = 0, ...$

Case 2: $\delta_i + \delta_{i+1} < 0$. In that case, we deduce from (133) that there is an $h \in \{i+2, \ldots, k\}$ such that $\delta_h > 0$ or, equivalently, that $n_h^{s'} > n_h^{s''} = 0$. From (132a)-(132c), one immediately observes that s'' can be obtained from s' by means of δ_i increments from category i to category $i+1$, and $(-\delta_{i+1})$ decrements (δ_{i+1} is a negative integer), from each category $h > i+1$ for which $n_h^{s'} > 0$ to category $i+1$. However more decrements than increments are required ($(-\delta_{i+1}) > \delta_i$), so that increments and decrements can not be matched one by one to produce Hammond transfers – and only Hammond transfers – in order to obtain, on the initial grid $\mathcal{C}$, s'' from s'. Yet we can match the increments with the decrements if an appropriate refinement of the grid between i and $i+1$ can be performed. First, staying on the initial grid $\mathcal{C}$, and starting from s', we can combine $(\delta_i - 1)$ increments (from i to $i+1$) to the same number of decrements starting from one or several categories h above $i+1$ and bringing the individuals from these categories to $i+1$. This generates immediately $(\delta_i - 1)$ Hammond transfers. In order complete the move from s' to s by means of Hammond transfers, we need to match the last $[\delta_i - (\delta_i - 1)] = 1$ increment from i to $i+1$ with the remaining $[(-\delta_{i+1}) - (\delta_i - 1)] > 1$ decrements that are required from each category $h > i+1$ where the number of individuals remains strictly positive to the category $i+1$. Whatever is the number $[(-\delta_{i+1}) - (\delta_i - 1)] > 1$, it is clearly possible to refine the grid $\mathcal{C}$ in such a way as to obtain at least $[(-\delta_{i+1}) - (\delta_i - 1)]$ adjacent categories between i and $i+1$. Once this refinement is obtained, one can then proceed in decomposing the last increment from i to $i+1$ into $[(-\delta_{i+1}) - (\delta_i - 1)]$ "small" increments between adjacent intermediate categories, each of which being matched with a decrement from each category $h > i+1$ for which there is a strictly positive number of individuals. Hence, it is possible to achieve s'' from s by using Hammond transfer only (provided that a suitable refinement of the grid be performed). Hence, there exists a non-negative integer t such that s'' can be obtained from s' by means of exactly $(-\delta_{i+1})$ Hammond transfers on the grid $\mathcal{C}(t)$ (recalling that a transformation on the grid $\mathcal{C}$ is also a transformation on the grid $\mathcal{C}(t)$). We conclude that $s \succeq_F s'' \succeq_H^t s'$, which completes the proof.

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