A Theory of Coalition Formation in Constant Sum Games*

Saish Nevrekar

November, 2015

Abstract

A finite set of agents N, endowed with heterogeneous abilities, compete for a share of the reward K. Agents receive a reward share determined by their individual abilities: higher is the agent's relative position in N, in terms of ability, greater is the agent's share. If agents form coalitions, the cooperation increases the abilities of all members, over their endowed ability. Thus, forming a coalition is mutually beneficial to all its members, although they are competitors. Such coalitions are self-enforcing: every pair of members mutually agree to cooperate with each other. The optimisation problem that agents face is forming coalitions to maximise their own ability while minimising the increase in their competitors' ability. Representing the model in a n-dimensional Euclidean space and assuming perfect rationality, I analyse (1) The existence and type of equilibria attained (2) The necessary conditions for movement from the initial state (3) The effect of coalition formation on its members' abilities and payoff. *Section 4* of the paper develops an algorithmic process of coalition formation (APCF) to simulate an evolutionary mechanism of attaining equilibrium for agents lacking foresight.

1 Introduction

A set of agents N, endowed with abilities, compete for the share of a reward. The agents' abilities determine their individual reward shares or payoffs: greater is the relative position

*I am very grateful to my supervisor, P.G. Babu, for his guidance, support and encouragement. I thank Debraj Ray, Matthew Jackson, Ariel Rubenstien, Akihiko Matsui, Vijay Krishna, Rajiv Vohra, Vivek Borkar, Mallikarjuna Rao, Shubhro Sarkar and participants at the Summer School of the Econometric Society (2015) for their insightful comments. This paper is a part of my Ph.D. dissertation at Indira Gandhi Institute of Development Research (IGIDR). I acknowledge financial support from IGIDR for my research. All errors are mine.

of an agent's ability in N, more is his payoff. Cooperation between them facilitates learning that improves individual ability, over the endowed ability. Learning displays a diminishing marginal rate: Agents with higher abilities have lesser scope to learn compared to agents with lower abilities. A group of agents mutually agreeing to cooperate with each other is a *coalition*. The agreements are non-binding as agents cooperate out of mutual benefit. Agents strategically form coalitions to maximise their payoff. However, the more an agent's payoff is, lesser is the portion of the reward left for other agents. As the payoff depends on ability, the optimisation problem for agents is: forming coalitions to maximise their own ability while minimising the improvement in other agents' abilities. Assuming complete information, each agent chooses a set of agents he is willing to cooperate with. This set of agents is called a strategy and the set of strategies played by all agents is a strategy set.

Given a strategy set, I develop an algorithm to segregate agents into a partition. The partition determines the learning of agents and thus, their final abilities. The final abilities (sum of the endowed ability and total learning) determines the payoff to every agent in N. Hence, the payoff to agents is a function of the strategy set played. Therefore, the equilibrium concept is defined for the strategy set. A strategy set where no individual agent benefits by playing any other strategy is a Nash equilibrium and no group of individuals benefit by simultaneously playing any other strategy is a Strong Nash equilibrium. Perfectly rational agents always play a Strong Nash equilibrium. However, in spite of perfect rationality, it is possible that no pure strategy Nash equilibrium exists. In that case, agents play strategies depending on the beliefs they have over the strategies other agents play.

I analyse the model by representing it in an n-dimensional Euclidean Space. The initial state - a state from which agents begin game play - is the state of no coalitions: the finest refinement of N. Agents are better-off if their payoff is greater and worse-off if lesser than at the initial state. Every imputation is pareto-optimal: at least one agent is worse-off by moving to a partition that results in a different imputation. The necessary condition to attain an equilibrium that is a coarser than the initial state is that at least two agents must be better-off and both must be members of the same coalition.

Agents in this model receive their payoff directly. However, this model can also be viewed as a two-step reward apportioning mechanism. In the first step, the reward is apportioned according to the payoff function among coalitions depending on the aggregate ability of coalitions. This is the coalition's worth. The coalitional worth is then further apportioned among its members by the same payoff function with the reward being the coalition's worth. This two-step mechanism, resulting in identical payoffs from the direct mechanism, is an alternate way of looking at the model. Due to the assumption of diminishing marginal rate of learning the member with highest endowed ability learns the least and learning progressively increases with decrease in endowed ability within a coalition. If a coalition is isolated from the effects of the strategies of its non-members, the payoff decreases with increase in endowed ability. Thus, in a grand coalition, the agent with the highest endowed ability must be worse-off and the agent with the least ability must be better-off.

Next I place an assumption on the agents' rationality: agents lack foresight (I explain this concept in the next paragraph). These agents reach equilibrium by an evolutionary procedure modelled through the Algorithmic Process of Coalition Formation (APCF). In the first round, agents randomly choose a strategy that includes all agents with abilities higher than themselves and simultaneously submit it to an auctioneer. The auctioneer then runs the algorithm and declares a coalition structure. Agents that benefit from deviating from their assigned coalition resubmit new strategies in the next round. The auctioneer retains the history of strategies played and replaces the strategies of agents resubmitting new strategies. The new strategy set leads to a different coalition structure. Agents continue resubmitting strategies as long as they benefit from deviation. The APCF terminates if no agent resubmits a strategy in a round. Agents begin cooperating only after the APCF terminates. *Proposition 3* derives a condition for a *cyclic* APCF: at evey round at least one agent benefits from resubmitting a strategy. A *cyclic* APCF is non-terminating as it runs for infinite rounds without converging to any equilibrium. *Acyclic* APCFs terminate in finite rounds and converge to a Nash equilibrium. As the initial strategy set is pre-defined, a unique Nash equilibrium results if APCF is acyclic. If agents reach the Nash Equilibrium, their lack of foresight prevents them from coordinating their strategies to attain a Strong Nash equilibrium (in case a Strong Nash exists). However, depending on the game, they may happen to land at the Strong Nash equilibrium through an APCF. If the reward is discounted after every round then agents evaluate the payoff from their current strategy against the present value of the payoff from resubmitting a new strategy in the next round. Small values of the discount factor may stop a *cyclic* APCF, while for large values of the discount factor (values tending to 1) the *cyclic* APCF stops when the the reward becomes zero at infinity.

At the end of a round agents lacking foresight cannot predict strategies that other agents may resubmit, in spite of complete information. Their computational ability is limited to calculating the self-benefit of deviation, given the strategies played in that round. On the other hand, perfectly rational agents predict the final outcome in the first round and play the equilibrium strategy. However, considering the complexity of this model, assuming agents with perfect foresight is far fetched. The APCF simulates this behaviour through an evolutionary mechanism.

Among the two main approaches in coalition formation: bargaining and blocking, discussed in Ray and Vohra (1997) and (1999), the approach here is closer to the blocking approach. However, the coalition formation here does not occur through the process of coalitional deviations, but through a stable allocation mechanism that is based on the strategies agents play. This mechanism is brought about by the means of an algorithm discussed in *Section 2.* Morelli and Park (2015) study a model where agents' payoff depends on their rank in the coalition, the power of the coalition and inequality within the coalition. This idea is similar to Example 3 in Section 2 where agents' payoffs are determined by their absolute rank. Roketskiy (2014) studies a network model where the agents' utilities are dependent on their relative and absolute ranks. The competing agents' abilities in his model also improve through cooperation. However, there is no coalition formation there as agents form links based on bilateral agreements. Works by Goyal and Moranga (1990) and Goyal and Joshi (2001) have also studied similar network models. These papers have applications of the theory discussed here, with considerable variation. However, the setup of the game in this paper - formation of coalitions to maximise individual payoffs that sum to a constant - is to my knowledge not studied previously.

This paper is structured in the following manner: *Section 2* develops the model. In *Section 3*, I analyse the model in a n-dimensional Euclidean space. I discuss a process of coalition formation in *Section 4*. I discuss the possible implications of the concepts developed in *Section 5*

2 The General Model

Let $N = \{1, 2..., n\}$ be a finite set of agents. Each agent $i \in N$ is born with an endowed ability $\bar{a}_i \in \mathbb{R}_+$. By forming a coalition $S \subseteq N$ agents learn from one another and increase their ability. This learning is a function of the endowed ability of members in S, self included, given by $q : \mathbb{R}_+^{|S|} \to \mathbb{R}_+ \ \forall \ i \in S$. Agent i's learning in coalition S is additive: $q(\bar{a}_i, \bar{a}_{-S(i)}) = \sum\limits_{j \in S} q(\bar{a}_i, \bar{a}_j)$ where the term $\bar{a}_{-S(i)}$ refers to the ability of all agents in coalition S except agent i. The final ability of agent i is the sum of his endowed ability and learning: $a_i = \bar{a}_i + q(\bar{a}_i, \bar{a}_{-S(i)})$. The function $q(.)$ is a continuous and homogeneous. There is no self-learning: $q(\bar{a}_i, \bar{a}_i) = 0$. I assume that the learning function $q(.)$ decreases with increase in an agent's own endowed ability: $(\frac{\partial q(\bar{a}_i, \bar{a}_{-S(i)})}{\partial \bar{a}_i}) < 0$ and increases with increase in endowed ability of all other members of an agent's coalition: $(\frac{\partial q(\bar{a}_i, \bar{a}_{-S(i)})}{\partial \bar{a}_j}) > 0 \ \forall \ i, j \in S, i \neq j$. This implies that with an increase in ability, agents' learning from others diminishes at a marginal rate.

All agents in N compete for a share of the reward K. The payoff to any agent $i \in N$, denoted by U_i, is a fraction of the reward K such that $\sum\limits_{i=1}^{n} U_i = K$. The reward is split among some or all agents in N and are non-decreasing with increase in ability: $\frac{\partial U_i}{\partial a_i} \geq 0$. The payoff is a function of an agent's own ability and the ability of all other agents in N: $U_i = w(a_i, a_{-i})$, where $w(.)$ is a homogeneous function. I assume complete information in the model: each agent in N knows the elements in the set $(\bar{a}_i)_{i \in N}$ and the functions $q(.)$ and $w(.)$.

Note that the function $w(.)$ is not necessarily continuous. Examples of continuous and discontinuous payoff functions are given below:

Example 1: *Fractional Payoff*
A pie of size K is split into n pieces. The size of agent i's piece is directly proportional to fraction of the agent i's ability to the aggregate ability of all agents in N. The payoff to agent i is:

$$U_i = \frac{a_i K}{\sum\limits_{j=1}^{n} a_j} \; \forall \; i \in N$$

U_i is continuous and the sum of payoffs add to the size of the pie:

$$\sum_{i=1}^{n} U_i = K$$

The learning of agent $i \in S$ is:

$$q(\bar{a}_i, \bar{a}_{-S(i)}) = \sum_{j \in S, i \neq j} \frac{\bar{a}_j}{\bar{a}_i}$$

Agent i's final ability, that determines payoff, is:

$$a_i = \bar{a}_i + \sum_{j \in S, i \neq j} \frac{\bar{a}_j}{\bar{a}_i}$$

Example 2: *Relative Rank*
Agent i's payoff is the relative position of his ability in N termed as relative rank:

$$U_i = \sum_{j=1}^{n} (a_i - a_j) \; \forall \; i \in N$$

The relative rank is a continuous function that adds to zero when summed across all agents

$$\sum_{i=1}^{n} U_i = \sum_{i=1}^{n} \left(n a_i - \sum_{j=1}^{n} a_j \right) = 0$$

Note that $U_i > U_j$ if and only if $a_i > a_j$ for any $i, j \in N$. This implies that an higher payoff means a higher ability.

Example 3: *Absolute Rank*

If the rewards are apportioned by the absolute rank of an agent's ability in N, agent i's payoff is

$$U_i = \sum_{j=1}^{n} I_{\mathbb{R}^+}(a_i - a_j)$$

where $I_{\mathbb{R}^+} : \mathbb{R} \to \{0, 1\}$ is the indicator function

$$I_{\mathbb{R}^+}(a_i - a_j) = \begin{cases} 1 & \text{if } (a_i - a_j) > 0 \\ 0 & \text{if } (a_i - a_j) \leq 0 \end{cases}$$

Absolute Rank is a discontinuous payoff function and the sum of the payoffs is:

$$\sum_{i=1}^{n} U_i = \frac{n(n-1)}{2}$$

Agents play strategies x_i from a strategy space χ_i to form coalitions. A strategy $x_i \in \chi_i$ is a set of agents agent i is willing to cooperate with to form a coalition. A strategy set, $X \in \prod_{i=1}^{n} \chi_i$, is a set of strategies played by all agents in N. For example, $x_i = \{i, j, k\}$ means agent i will form a coalition with agents j, k. There is no preference ordering between choices in a strategy. Agents' strategies includes themselves. Agents cooperate only under mutual agreements. The necessary condition for agents i and j to cooperate is $i \in x_j$ and $j \in x_i$. A coalition S is formed only if all agents in S mutually agree to cooperate with each other and no super set of S satisfying this property exists. Formally, a coalition is defined as:

Definition 1: A coalition S is an element of a partition of N that satisfies
Condition 1: $i \in x_j \ \ \forall \ \ i, j \in S$
Condition 2: There exists no set S' such that $i \in x_j \ \ \forall \ \ i, j \in S'$ and $S \subset S'$

A partition π is a subset of 2^N (the power set of N) such that $\bigcup_{j=1}^{m} S_j = N$ and $\bigcap_{j=1}^{m} S_j = \emptyset$ where $S_1, S_2 ..., S_m \in 2^N$. Thus, agents cannot be part of multiple coalitions. Also, all agents within a coalition must agree to cooperate with each other to form a coalition.

The following algorithm translates a strategy set $X = \{x_1, x_2 ..., x_n\}$ into a partition π.

Algorithm 1:

Step 1: Let s be an element of the power set of N: $s \subseteq 2^N$.

Step 2: A possible coalition, $\rho(s)$, is an element of 2^N that satisfies condition 1 of *definition 1.*

$$\rho(s) = \bigcap_{i \in s} \eta_i \ \forall \ i \in s$$

where $\eta_i = x_i \cap s \ \ \forall \ \ i \in s$

Step 3: The set of possible coalitions is given by

$$P = \{\rho(s) | s \subseteq 2^N\}$$

Step 4: The elements in P satisfying conditions 2 of *Definition 1* are termed potential coalitions, given by the set

$$\varphi = \{\rho(s) \mid \textit{there exists no } \rho(s') \in P \textit{ such that } \rho(s) \subset \rho(s'), s \neq s'\}$$

Step 5:If $\rho(s) \cap \rho(s') \neq \emptyset \ \forall \ \rho(s') \in \varphi$, then $\rho(s) \in \pi$

Step 6: If $\rho(s) \cap \rho(s') \neq \emptyset$ for some $\rho(s') \in \varphi$, then $\{\rho(s), \rho(s')/(\rho(s) \cup \rho(s'))\} \in \pi$ or $\{\rho(s'), \rho(s)/(\rho(s) \cup \rho(s'))\} \in \pi$ with equal probability.

Algorithm 1 ends here. Given a strategy set, the coalitions in partition π satisfy both conditions defined in *Definition 1.* Refer Appedix A for a example of the algorithm's procedure.

This algorithm translates a given strategy set $X = \{x_1, x_2..., x_n\}$ into a partition $\pi = \{S_1, S_2..., S_m\}$. The coalitions in the partition determine each agents ability $a_i = \bar{a}_i + q(\bar{a}_i, \bar{a}_{-S(i)})$ where $i \in S$, $S \in \pi$. The resulting ability set $\{a_1, a_2..., a_n\}$ determines the payoff to every agent, $U_i = w(a_i, a_{-i})$ for all $i \in N$. Hence, the payoff to agent i is a function of the strategy set:

$$\begin{aligned} U_i &= w(a_i, a_{-i}) \\ &= w(\bar{a}_i + q(\bar{a}_i, \bar{a}_{-S(i)}), \bar{a}_{-i} + q(\bar{a}_{-i}, \bar{a}_{-S(-i)})) \\ &= u_i(x_i, x_{-i}) \\ &= u_i(X) \end{aligned} \tag{1}$$

The algorithm is introduced into this model through an auctioneer. The role of the auctioneer is to partition agents into coalitions based on the strategies they submit. Based on the strategy set, the auctioneer chooses elements from the power set of N that satisfy the first condition required for a coalition: members of a coalition must all mutually agree to cooperate with each other. The auctioneer then chooses the coarsest partition from this set in order to satisfy the second condition of a coalition: no superset of mutually agreeing members exists. If multiple partitions form, the auctioneer chooses one of those partitions with equal probability. Given the strategies agents play, no agent benefits from deviating - assuming deviation is feasible - to another coalition. Agents play strategies that are optimal to a decision rule determined by the amount of rationality they posses. I distinguish between perfectly rational agents and agents lacking foresight: perfectly rational agents posses infinite computing capacity and can foresee the action of every other agent in the economy. While, agents lacking foresight cannot foresee the actions of other agents. For a detailed explanation refer Remark 4.1.

I use two equilibrium concepts here: Nash and Strong Nash equilibrium. The equilibrium concept used depends on the rationality assumptions imposed on the agents.

A strategy set $\bar{X}$ is a Nash equilibrium if no agent benefits by a unilateral deviation. The formal definition is

Definition 2: The strategy set $\bar{X} = \{\bar{x}_1, \bar{x}_2..., \bar{x}_n\}$ is a Nash Equilibrium if

$$u_i(\bar{x}_i, \bar{x}_{-i}) \geq u_i(x_i, \bar{x}_{-i}) \ \ \forall \ x_i \in \chi_i, \ \ i \in N$$

A Strong Nash Equilibrium is a Nash equilibrium with the additional condition that no subset of agents can jointly deviate to achieve a higher payoff.

Definition 3: The strategy set $\bar{X} = \{\bar{x}_1, \bar{x}_2..., \bar{x}_n\}$ is a Strong Nash Equilibrium if

$$u_i(\bar{x}_i, \bar{x}_{-i}) \geq u_i((x_j)_{j\in S}, (\bar{x}_k)_{k\in N/S}) \ \ \forall \ S \subseteq N, \ \ i \in S, N$$

Note that the deviating coalition is not necessarily a subset of a coalition in Nash Equilibrium; agents may deviate from multiple coalitions. For example: suppose the strategy set X leads to a unique partition $\pi = \{S_1, S_2, S_3\}$ and is a Nash Equilibrium. However, if agent $i \in S_1$ and $j \in S_2$ jointly deviate to S_3, then they achieve a higher payoff. The strategy set X' corresponding to the partition $\pi' = \{\{S_1/i\}, \{S_2/j\}, \{S_3 \cup \{i, j\}\}\}$ is a strong Nash Equilibrium where agents deviate from multiple coalitions in a Nash Equilibrium.

Perfectly rational agents foresee the outcome of every conceivable strategy set and play the strategy that maximises their payoff accordingly. Thus, if an equilibrium exists, it is a Strong Nash equilibrium. The reasoning is the following: Suppose agents plan on playing strategies that are a Nash Equilibrium. If a subset of agents benefit from playing different strategies, they foresee it. As all agents in that subset benefit and all agents are aware of the perfect rationality other agents possess, they will play strategies to reach a Strong Nash Equilibrium. However, multiple Strong Nash equilibrium may exist as multiple strategy sets may translate into identical partitions and hence, identical payoff vectors.

Remark 2.1: In spite of perfectly rational agents, they may not reach an equilibrium. An equilibrium does not exist if there always exists subset of agents that can beneficially deviate. As the strategy set space χ^n is finite, agents must deviate in a manner that a cycle of strategy sets in created.

To show the non-existence of equilibrium, I show that a "cycle" occurs for a specific ordering of the payoffs to agents i and j. Suppose that x_i, x'_i are two strategies played by agent i and x_j, x'_j are played by agent j. Assuming that the strategies of the remaining agents remains unchanged irrespective of the four combinations played by agents i and j. Let the set of strategies played by the agents except i and j be represented by Y.

The payoff to agent i is as follows:

$$f_i(x_i, x_j, Y) > f_i(x'_i, x'_j, Y) > f_i(x_i, x'_j, M) > f_i(x'_i, x_j, Y)$$

The payoff to agent j is as follows:

$$f_j(x'_j, x_i, Y) > f_j(x_j, x'_i, Y) > f_j(x_j, x_i, M) > f_j(x'_j, x_i, Y)$$

For the payoff ordering we infer:
If agent i plays x_i, agent j plays x'_j
If agent j plays x'_j, agent i plays x'_i
If agent i plays x'_i, agent j plays x_j
If agent j plays x'_j, agent i plays x_i

At the last step, the cycle begins. Here the cycle is shown for preferences of two agents, but it could occur for more than two agents. Thus, in case a cycle occurs, then an Equilibrium does not exist.

If no equilibrium exists, the outcome depends on the beliefs agents have over the strategies the other agents play and reach an mixed strategy Strong Nash Equilibrium. I conclude the brief discussion for the case of non-existence of equilibrium and focus on the case where equilibrium exists. In the next section I analyse the necessary conditions for perfectly rational agents to form coalitions.

3 Analysis: Perfect Rationality

The game in strategic form is $\Big(N, (x_i)_{i\in N}, (u_i(.))_{i\in N}\Big)$. I represent this game in an n-dimensional Euclidean space and analyse for the case when equilibrium exists.

The imputation corresponding to the strategy set X is given by the function $Z : \prod_{i=1}^{n} \chi_i \rightarrow \mathbb{R}^n$

$$Z(X) = (u_1(X), u_2(X)..., u_n(X)) = (U_1, U_2..., U_n)$$

Note that the function $Z(.)$ is non-injective and non-surjective: multiple strategy vectors X can cause the same payoff vector and not all vectors in $\mathbb{R}^n$ have a corresponding strategy vector.

Remark 3.1: For every strategy set $X \in \prod_{i=1}^{n} \chi_i$ there exists a unique imputation $Z(X)$. If a strategy set leads to a unique partition, the uniqueness of the imputation is straight forward. In the case where a strategy set leads to multiple partitions, one of the partition is chosen with equal probability[1]. The imputation is the expected value of payoffs that result from each partition.

The imputation $Z(X)$ is represented in $\mathbb{R}^n$: an n-dimensional Euclidean space. Let $\Delta^n(K) \in \mathbb{R}^n$ where

$$\Delta^n(K) = \{(U_1, U_2..., U_n) \in \mathbb{R}^n | \sum_{i=1}^{n} U_i = K \text{and } U_i \geq 0 \ \forall \ i \in N\}$$

Observe that $\Delta^n(1)$ is the standard n-simplex. For values of $K > 0$, except for the linear expansion (or contraction), the properties of the space remains the same. As the payoff in the model always adds to K, $Z(X) \in \Delta^n(K)$ for all $X \in \prod_{i=1}^{n} \chi_i$. The indifference curve, $\lambda_i(K - K')$, of agent i is a set of vectors, $(U_1, U_2..., U_n) \in \Delta^n(K)$, such that the payoff to all agents other than agent i adds to a constant value K':

$$\lambda_i(K - K') = \{(U_1, U_2..., U_n) \in \Delta^n(K) \mid \sum_{j \neq i}^{n} U_j = K' \text{and } 0 \leq K' \leq K \ \forall \ i, j \in N \text{ and } i \neq j\}$$

If agent i lies on the indifference curve $\lambda_i(K - K')$, his payoff is $U_i = K - K'$ for all imputations in $\lambda_i(K - K')$.

Agents begin game play from the initial state of no coalitions: the finest refinement of N. The imputation at the initial state is $Z^I = (U_1^I, U_2^I..., U_n^I)$. Note that Z^I does not correspond to any strategy set as agents play strategies after this point. However, strategy sets $X \in \prod_{i=1}^{n} \chi^n$ exist such that $Z(X)$ equals the imputation at initial state. Let the set of these strategy sets be represented by Y such that $Z(X) = Z^I = (U_1^I, U_2^I..., U_n^I) \ \forall \ X \in Y$

Let $Z(X) \in \lambda_i(K_1)$, then agent i is better-off if $K_1 > U_i^I$ and worse-off if $K_1 < U_i^I$. The set of indifference curves where agent i is always better-off (worse-off) is called the better-off

[1]Refer step 6 in *Algorithm 1.*

set (worse-off set). The better-off and worse-off sets are respectively denoted by:

$$B_i(U_i^I) = \{\lambda_i(K_1) \mid K_1 > U_i^I\}$$

$$W_i(U_i^I) = \{\lambda_i(K_1) \mid K_1 < U_i^I\}$$

The better-off (worse-off) sets are with respect to payoff at the initial state. If an imputation is neither in the better-off or worse-off set of agent i then it must lie on the indifference curve $\lambda_i(U_i^I)$. Agents play strategies to achieve the highest possible indifference curve.

Proposition 1: For every strategy $X \in \prod_{i=1}^{n} \chi_i$ and $X \notin Y$, $Z(X) \in W_i(U_i^I)$ for at least one $i \in N$.

Proof. is by way of a contrapositive:
Suppose there exists a subset $S \subset N$ such that $Z(X) \in B_i(U_i^I)$ $\forall$ $i \in S$ and $Z(X) \in B_j(U_j^I)$ $\forall$ $j \in N/S$. This implies $Z(X) \in \bigcap_{i \in S} B_i(U_i^I)$. and $Z(X) \in \bigcap_{j \in N/S} \lambda_j(U_j^I)$
By the definition $U_i > U_i^I$ $\forall$ $i \in S$ and $U_j = U_j^I$ $\forall$ $j \in N/S$. Adding then for all agents in N,
$\sum_{i=1}^{n} U_i > \sum_{i=1}^{n} \bar{U}_i$
$\sum_{i=1}^{n} U_i > K$
However, by definition we have $\sum_{i=1}^{n} U_i = K$ and hence strategy X cannot exist. The payoff sums to K only when some agents $j \in N/S$ are worse-off such that the gain by agents in S is offset by agents in N/S. □

This implies that all imputations $Z(X) \in \Delta^n(K)$ are Pareto-Optimal. If agents move to a equilibrium $\bar{X} \notin Y$, then at least one agent is better-off and one worse-off. However, if only one agent is made better-off by this movement, then the rest would play strategies to remain at the initial state. Also, if more than one agents benefit, but are in different coalitions, then too the rest of the agents can play strategies to remain at the initial state. This leads us to the second proposition.

Proposition 2: The necessary conditions for $\bar{X} \notin Y$ to be an equilibrium are (1) $Z(\bar{X}) \in B_i(U_i) \cup B_j(U_j)$ for some $i, j \in N$ (2) $i, j \in S$ where $S \in \pi(X)$

Proof. In the first part of the proof I show that at least two agents must be better-off.
Proof is by way of a contrapositive:
Suppose that by playing the strategy set $\bar{X}$, only one agent i is better-off: $Z(\bar{X}) \in B_i(U_i^I)$.
By *Proposition 1*, there exists a non-empty subset $T \subset N$ such that $Z(\bar{X}) \in W_j(U_j^I)$ $\forall$ $j \in T$.
Assume that the rest of the agents, $N/(T \cup i)$, are neither better-off nor worse-off by playing

strategy set $\bar{X}$. I assume agents prefer to not participate if they do not benefit by participation. Thus, by playing the strategies $x_j = \{j\}\ \forall\ j \in T$ and $x_k = \{k\}\ \forall\ k \in N/(T \cup i)$ they arrive at the initial state of no coalitions irrespective of the strategy played by agent i.

The second part of the proof is based on similar reasoning. If two agents, say a and b are better-off by strategy set $\bar{X}$, but $a \notin x_b$ and $b \notin x_a$, then the other agents take the same action as above to arrive at the initial state. □

The necessary condition to attain an equilibrium coarser than the initial state, is that the resultant imputation must lie in the better-off set of at least two agents and both agents must be a part of the same coalition. If either of the two conditions is not satisfied, the rest of the agents play strategies that leads to the partition at the initial state: the finest refinement of N. In other words, agents equilibrate to a coarser partition only if the necessary conditions are satisfied. Needless to state that while equilibrating to a coarser partition more than two agents can be better-off and also lie in different coalitions.

The reward apportioning mechanism discussed till this point is direct: agents directly receive a share based on their individual ability. However, this mechanism can also be viewed as as a two-step indirect apportioning mechanism. The coalitional worth is the share a coalition receives based on its aggregate ability. The coalitional worth is then divided among the members of the coalition based on their abilities. As long as the division rule for apportioning the reward among coalitions is identical to that for apportioning the coalitional worth among it members, the model being analysed is the same. Just that this is a different perspective. Using the fractional payoff function discussed in Example 1, I use an example to explain this concept.

Example 4:
The coalitional worth is the share of reward K that coalition $S \in \pi$ receives: $v(S) = \frac{\sum_{i \in S} a_i}{\sum_{j=1}^{n} a_j} K$
where $a_i = \bar{a}_i + q(\bar{a}_i, \bar{a}_{-S(i)})\ \forall\ i \in S$
The share of agent $k \in S$ is $U_k = \frac{a_k}{\sum_{i \in S} a_i} v(S)$
Thus,

$$\begin{aligned} U_k &= \frac{a_k}{\sum\limits_{i \in S} a_i} \frac{\sum\limits_{i \in S} a_i}{\sum\limits_{j=1}^{n} a_j} K \\ &= \frac{a_k}{\sum\limits_{j=1}^{n} a_j} K \end{aligned}$$

This is the individual fractional payoff function in Example 1. A similar conclusion can be reached for any payoff function $w(.)$

I now analyse the effects of coalition formation on the payoff of its members. The assumption of diminishing marginal rate of learning causes unequal benefits to agents within a coalition. The following proposition proves that within a coalition, an agent with a higher ability always benefits lesser than agents with lower abilities.

Proposition 3: For coalition $S = \{1, 2..., m\}$, if $\bar{a}_i > \bar{a}_j$ for some $i, j \in S$, then $q(\bar{a}_j, \bar{a}_{-S(j)}) > q(\bar{a}_i, \bar{a}_{-S(i)})$.

Proof. To prove: $q(\bar{a}_j, \bar{a}_{-S(j)}) - q(\bar{a}_i, \bar{a}_{-S(i)}) > 0 \; \forall \; i, j \in S$.
Learning from others is assumed to be superadditive: $q(\bar{a}_i, \bar{a}_{-S(i)}) = \sum_{k=1}^{m} q(\bar{a}_i, \bar{a}_k)$
Note that there is no self-learning: $q(\bar{a}_i, \bar{a}_i) = 0$.
As $\frac{\partial q(\bar{a}_i, \bar{a}_{-S(i)})}{\partial \bar{a}_i} < 0$ and $\bar{a}_i > \bar{a}_j$, we have

$$\sum_{k \neq j}^{m} q(\bar{a}_i, \bar{a}_k) < \sum_{k \neq i}^{m} q(\bar{a}_j, \bar{a}_k) \tag{2}$$

We now have to compare the terms $q(\bar{a}_i, \bar{a}_j)$ and $q(\bar{a}_j, \bar{a}_i)$. Assume the abilities of the agents differ by an infinitesimally small amount h such that $\bar{a}_i = \bar{a} + h, \bar{a}_j = \bar{a}$.
Thus,

$$\begin{aligned} q(\bar{a}_j, \bar{a}_i) - q(\bar{a}_i, \bar{a}_j) &= q(\bar{a}, \bar{a} + h) - q(\bar{a} + h, \bar{a}) \\ &= h(\frac{z(\bar{a}, \bar{a} + h)}{h} - \frac{z(\bar{a} + h, \bar{a})}{h}) \end{aligned} \tag{3}$$

As $h \to 0$

$$\begin{aligned} q(\bar{a}_j, \bar{a}_i) - q(\bar{a}_i, \bar{a}_j) &= h(\lim_{h \to 0} \frac{q(\bar{a}, \bar{a} + h) - q(\bar{a}, \bar{a})}{h} - \lim_{h \to 0} \frac{q(\bar{a} + h, \bar{a}) - q_(\bar{a}, \bar{a})}{h}) \\ &= h(\frac{\partial q(\bar{a}_i, \bar{a}_j)}{\partial \bar{a}_j} - \frac{\partial q(\bar{a}_i, \bar{a}_j)}{\partial \bar{a}_i}) \end{aligned}$$

By assumption $\frac{\partial q(\bar{a}_i, \bar{a}_j)}{\partial \bar{a}_j} > 0$ and $\frac{\partial q(\bar{a}_i, \bar{a}_j)}{\partial \bar{a}_i} < 0$, therefore $q(\bar{a}_i, \bar{a}_j) < q(\bar{a}_j, \bar{a}_i)$.
Thus,

$$\begin{aligned} \sum_{k=1}^{m} q(\bar{a}_i, \bar{a}_k) &< \sum_{k=1}^{m} q(\bar{a}_j, \bar{a}_k) \\ q(\bar{a}_j, \bar{a}_{-S(j)}) &> q(\bar{a}_i, \bar{a}_{-S(i)}) \end{aligned}$$

□

Think of a coalition S in isolation; unaffected by the strategies of agents outside S. As the effects of the coalitions formed by agents outside S affect the payoff to agents in S independently, I isolate S to analyse the effects of its formation of the abilities of its members. It may be easier to understand the concept by applying the reasoning to a grand coalition N first and then extend the reasoning to a coalition $S \subset N$. For a continuous payoff function $u(.)$, the agent with the highest ability - within an isolated coalition - is always worse off and one with least ability is always better-off when compared to the initial state. The agent with the highest ability learns the least. Agents' learning progressively increases as ability decreases. As a result, the relative position of the agent with the highest ability decreases. Similarly, the relative position of the agent with the lowest ability increases.

4 The Algorithmic Process of Coalition Formation (APCF)

I develop an algorithmic process of coalition formation (APCF) to simulate an evolutionary approach of attaining equilibrium[2] for agents lacking foresight[3]. All agents begin by playing strategies

$$x_i^1 = \{j \mid \bar{a}_j \geq \bar{a}_i\}$$

Agents want to cooperate with all agents with abilities greater than themselves because by *Proposition 3* cooperating with an agent with higher ability increases payoff. Agents simultaneously submit them to an auctioneer. By *Algorithm 1*, the auctioneer chooses a coarsest partition from all possible partitions with equal probability. The process of submission of strategies and coalition formation constitutes a *round*. If every agent plays x_i^1, the partition at the end of round 1 is the finest refinement of N.

Let the strategy set played in the round $r \geq 1$ be $X^r = \{x_1^r, x_1^r ..., x_n^r\}$. At the end of round r, agent i evaluates if other strategies yield higher payoffs, given the strategies of other agents in round r. Agent i resubmits a new strategy x_i^{r+1} in the next round, $r+1$, if

$$u_i(x_i^{r+1}, x_{-i}^r) > u_i(x_i^r, x_{-i}^r) \text{ for some } i \in N \text{ where } x^{r+1} \neq x^r$$

Multiple agents may similarly change strategies in round $r+1$. In spite of complete information, agents in round r cannot predict the strategies other agents will resubmit in round $r+1$. Thus, an agent's resubmitted strategy, x_i^{r+1}, is in response to strategies played in round r: x_{-i}^r. This is a limitation imposed on the rationality of the agents.

[2] I discuss how APCF simulates an evolutionary mechanism in *Section 5*.

[3] Agents lacking foresight cannot foresee the actions of other agents and play strategies without accounting for the changes other agents would make. For a detailed discussion on the rationality assumption refer *Remark 4.1*

The auctioneer replaces the resubmitted strategies in round $r+1$ and retains the rest from round r, forming a new strategy set[4], X^{r+1}. A new partition is formed by *Algorithm 1.* The APCF *progresses* from round r (to $r+1$) if at least one agent benefits from resubmitting a different strategy. The APCF progresses for infinite rounds if at least one agent changes his strategy in each round and terminates if no agent changes his strategy in a round.

Agents cooperate (within coalitions) only after the APCF terminates. This implies that all agents evaluate their payoff based on the initial ability. The abilities improve only after agents begin cooperation.

Let the history of the strategy sets played at round r be $H^r = \{X^{r-1}, X^{r-2}..., X^1\}$.

Proposition 3: At round $r+1$, if $X^{r+1} \in H^r$, then APCF is *cyclic*: APCF progresses from every round $r \geq 1$.

Proof. Let $X^{r+1} \in H^r$

$$X^{r+1} \in \{X^{r-1}, X^{r-2}..., X^1\}$$
$$X^{r+1} = X^{r-m} \text{ for some } 1 \leq m \leq r-1$$

From H^{r+1} we know that the APCF progresses from round $r-m$ to $r+1$. Thus, the APCF will progress from $r+1$ to $r+m+2$ where

$$\begin{aligned} X^{r+1} &= X^{r-m} \\ X^{r+2} &= X^{r-m+1} \\ &\vdots \\ X^{r+m+2} &= X^{r+1} \end{aligned}$$

Thus, H^{∞} contains a recurring subset of H^{r+1}. □

The history of strategy sets, H^r, is a progression of rounds. A round progresses only if at least one agent benefits from resubmitting a strategy. If strategy set X^{r+1} is already present in H^r, then there always exists a progression to every round in the APCF. Thus, this cycle continues infinitely: the APCF is non-terminating.

If an APCF is *acyclic*, then $X^{r+1} \notin H^r$ for all rounds $r > 1$. In a round, if no agent benefits by deviating from his existing coalition, no agent resubmits a strategy. As the set $\prod_{i=1}^{n} \chi_i$ has finite strategy sets, an *acyclic* APCF terminates after a finite number of rounds r_T.

[4]Note: $X^{r+1} \neq X^r$ as at least one element $x_i^{r+1} \in X^{r+1}$ and $x^{r+1} \notin X^r$. However, it is possible for $X^r = X^{r+m}$ for all $m \neq \{-1, 1\}$.

The terminal strategy set X^{r_T} is a Coalitional Equilibrium. The coalitional equilibrium attained for an *acyclic* APCF depends on the strategy set played in the first round. For initial strategy described above, the is not necessarily a Strong Nash. As agents lacking foresight cannot coordinate strategies without communication, they may be stuck at a Coalitional equilibrium which is not a strong-Nash equilibrium, although a Strong-Nash is achievable. On the other hand, agents with perfect foresight can always predict other's strategies and coordinate without communication to attain the strong Nash equilibrium.

Remark 4.1: The APCF implicitly imposes an assumption on the rationality of the agents. Agents in this model are lack foresight. At the end of round r agents evaluate the benefit resubmitting a new strategy x_i^{r+1}, given strategies of other agents played in round r. Although they have complete information, at the end of a round they cannot predict the strategies that other agents may resubmit. Perfectly rational agents predict the strategies that other agents will resubmit and play a strategy taking it into account. Thus, perfectly rational agents play the equilibrium strategies in the first round. Considering the complexity of this model, assuming perfectly rational agents is far fetched. Agents in the real world, mostly, would have a degree of rationality. At one extreme is perfect rationality and the other is a lack of foresight. Agents with a degree of foresight between the extremes will play strategies based on some predictions. The APCF for such agents terminates in lesser rounds than that taken for agents lacking foresight. However, this paper only discusses the extremes.

In the case of *cyclic* APCF, the agents keep resubmitting strategies for infinite rounds. As agents cooperate within coalitions only after the APCF stops, the agents in a *cyclic* APCF never form coalitions at all. In the case of perfectly rational agents, agents have beliefs about the strategies of other agents. However, in an APCF beliefs play no role as strategies are resubmitted in the subsequent round. Such cycles may stop if the reward is discounted after each round by $\delta \in (0,1)$. If the APCF stops at round r, the reward available to be split is $\delta^{r-1}K$. Agents evaluate the preset value of their payoff against the the future value of payoff before resubmitting a different strategy, given the strategy of others.

If $u_i(x_i^r, x_{-i}^r) < u_i(x_i^{r+1}, x_{-i}^{r+1})$, but $\delta^r u_i(x_i^r, x_{-i}^r) > \delta^{r+1} u_i(x_i^{r+1}, x_{-i}^{r+1})$, then agent i does not resubmit strategy x_i^{r+1} in round $r+1$. However, if the discount value is large (near to 1) such that $\delta > \frac{u_i(x_i^r, x_{-i}^r)}{u_i(x_i^{r+1}, x_{-i}^{r+1})}$ $\forall\ i \in N$, then the cycle will continue till the reward becomes zero. For smaller values of δ the cycle may stop. Thus, for large discount values, agents lacking foresight land up having nothing instead of settling for a compromise in the initial rounds to receive something.

5 Discussion

I analyse a game where agents compete for a share of a reward. Their payoff - the fraction of reward received - depends on the relative position of their ability. As cooperation increases agent's endowed ability because of the learning, agents form coalitions. The game described is a direct mechanism: agents directly receive a share of the reward. However, the game can be thought as a two step mechanism. First, the reward is apportioned to coalitions based on their aggregate ability. Second, the reward received by a coalition is further apportioned among its members based on individual ability. The direct mechanism and the two-step mechanism is the same game, only viewed differently. The two-step mechanism is a more intuitive way of modelling it. For example, firms producing identical goods compete for a market share. The profits are distributed among a firm's employees based on ability.

The algorithmic process of coalition formation (APCF) introduced in *Section 4* implicitly assumes agents to lack foresight. The procedure simulates a evolutionary mechanism and could be applied without an auctioneer. Suppose agents first form coalitions based on their priors. After a period of time, they figure out which deviations are beneficial. Assuming that these deviations are feasible, agents deviate to form a new partition. They reach a coalitional equilibrium when no agent benefits from deviation. The simplifying assumption in this model is that agents cooperate only after APCF terminates. This implies that agents' ability improves from cooperation only after attaining equilibrium. All evaluations in the APCF are based on the endowed ability of the agents. If agents are allowed to cooperate after every round, their abilities will improve. Thus, the benefit of deviating must be evaluated based on improved abilities. This problem becomes non-tractable for a sizeable set N.

It is interesting to think of the cyclic APCF with respect to the real world. In a cyclic APCF, agents deviate without figuring out the cycle and continue such movement endlessly. While in the real world, agents may eventually figure out the presence of small cycles, for large cycles the agents may never become aware of the cycle. For the case where the reward is discounted, agents may stop the cycle for small values of the discount factor. However, for large values the agents stop when the reward is reduced to zero. It shows that although, compromising and not making a beneficial move makes agents better-off, their myopic vision prevents them from seeing this. In the bargain, they are left with no payoff. Also, if the effect of cooperate on abilities is accounted for at every round, the changed abilities may cause them to step out of the cycle.

The assumption placed on the rationality of agents (lacking foresight) may lead to an outcome different from what agents with perfect foresight attain. Agents lacking foresight may not reach a Strong Nash equilibrium, where agents with perfect foresight always attain it. Thus, the rationality assumption changes the outcome.

6 Appendix A

Suppose the economy consist of the following set of agents

$$N = \{a, b, c\}$$

and the strategy of each agent is

$$x_a = \{a, b\}, x_b = \{a, b\}, x_c = \{a, b, c\}$$

Step 1:

$$2^N = \{\{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}, \{abc\}\}$$

Step 2: A representative step:

$$\begin{aligned} s &= \{ac\} \\ \eta(a) &= \{a\}, \eta(b) = \{a\}, \eta(c) = \{a, c\} \\ \rho(s) &= \eta(a) \cap \eta(b) \cap \eta(c) \\ \rho(s) &= \{a\} \end{aligned} \tag{4}$$

Step 3: $P = \{\{a\}, \{b\}, \{c\}, \{ab\}\}\}$

Step 4: $\varphi = \{\{ab\}\{c\}\}\}$

Step 5: As $\rho(ab) \cap \rho(c) = \emptyset$, the partition is

$$\pi = \{\{ab\}, \{c\}\}$$

For the same example if $x(b) = \{a, b, c\}$, then Steps 1 and 2 are the same as above.

Step 3: $P = \{\{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}\}\}$

Step 4: $\varphi = \{\{ab\}\{bc\}\}\}$

Step 5: As $\rho(ab) \cap \rho(bc) = b$, agent b is indifferent to coalition $\{ab\}$ and $\{bc\}$

Step 6: The partition $\pi_1 = \{\{ab\}, \{c\}\}$ is formed with half probability and $\pi_2 = \{\{bc\}, \{a\}\}$ is formed with half probability. If the strategy set was a Coalitional Equilibrium, then we have multiplicity of equilibrium.

References

[1] **Jackson, M.** (2008).*Social and Economic Networks*, Princeton University Press

[2] **Jackson, M., and Wolinsky, A.** (1996).“A Strategic Model of Social and Economic Networks,” *Journal of Economic Theory* **71**, 44-74

[3] **Konishi, H., and Ray, D.** (2003).“Coalition formation as a dynamic process,” *Journal of Economic Theory* **110**, 1-41

[4] **Morelli, M., and Park, I.** (2015). *Internal Hierarchy and Stable Coalition Structures*, No 528, Working Papers from IGIER (Innocenzo Gasparini Institute for Economic Research), Bocconi University.

[5] **Moreno, D., and Wooders, J.** (1996). “Coalition-Proof Equilibrium,” *Games and Economic Behaviour* **70**, 80-112

[6] **Myerson, R.** (1977). “Graphs and Cooperation in Games,” *Mathematics of Operations Research* **2(3)**, 225-229

[7] **Ray, D.** (2007). *A Game-Theoretic Perspective on Coalition Formation*, OUP Oxford; 1 edition.

[8] **Ray, D., and Vohra R.** (1997).“Equilibrium Binding Agreements,” *Journal of Economic Theory* **72**(1), 30-78

[9] **Ray, D., and Vohra R.** (1999).“A Theory of Endogenous Coalition Structures,” *Games and Economic Behaviour* **26**, 286-336

[10] **Roketskiy, N.** (2014).“Competition and Networks of Collaboration,” Working Paper, New York University

[11] **Roth, A., and Sotomayor, M.** (1990).*Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, Cambridge University Press; Reprint edition

[12] **Goyal, S., and Moraga, J.** (2003). “Networks of collaboration in oligopoly”, *Games and Economic Behavior*, 43(1): 57–85, April 2003.

[13] **Goyal, S., and Joshi, M.** (2001). “RnD networks”, *RAND Journal of Economics*, 32(4):686–707, Winter 2001