Sequential Auctions with Synergies in the Presence of More Than Two Objects

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Abstract

We examine a model of sequential English auction in which, the objects being auctioned share synergetic relationships (complementarity and substitutability) between them. We incorporate more than two objects into the model, which allows for the possibility of both types of synergies to co-occur. We divide the objects being auctioned into categories, where a category is defined as the collection of substitutable items. Additionally, inter-category objects are treated as complements. Bidders demand one unit from each category and aim to create a bundle of inter-category/complementary objects. We find out all possible outcomes of the game, and for each outcome, we find optimal bidding strategies, equilibrium selling prices, bidders' expected profits, and seller's expected revenue. We observe a decreasing price trend in one of the possible outcomes. We also discover the possibility of false bidding when certain assumptions of the model are violated.

1 Introduction

The sequential auction is the most widely used multi-object auction format as it is easy to understand and implement. A sequential auction involves selling several objects in a sequence, but only one object at a time. The objects sold in a sequential auction may not be independent and share some simplistic relationships between them. Such relationships are known as synergetic relationships in the literature. Two objects are said to be in positive synergy when they are complements, whereas negative synergy implies that the objects are substitutes. When there are only two objects to be auctioned, only one of the two synergetic relationships is possible between them. However, in the presence of more than two objects, both types of relationships can co-occur. In such an auction setup, bidders demand multiple objects and aim to create a bundle of complementary items when substitutes of each constituent of the bundle are also available for the auction. In this study, we try to understand the dynamics of such a sequential auction setup.

To understand the relevance of such an auction setting, imagine a floriculturist who sells flowers of different breeds and colors via auctions. Many florists participate in the auction and buy flowers of different kinds. They create various products like bouquets, flower baskets, etc. using the flowers, and sell them in the market. Florists consider many of these flower types as complements because they need them together in their product. They can also substitute some of the flowers in their product by others. In other words, the substitutes of each constituent of their product are also available for the auction. The florists aim to create a bundle of flowers which makes a good bouquet to be sold in the market.

A similar auction scenario is observed in various sports tournament related auctions. Indian Premier League (IPL)- a professional T20 Cricket tournament, Pro Kabaddi League (PKL) - a professional Kabaddi tournament, Premier Badminton League (PBL)- a professional badminton tournament, etc. are some of the well known examples of sports tournaments in which, players' services are auctioned via sequential English format. Sports tournaments like the ones mentioned above are usually team sports, which require a wide array of skills to win the game. Different players specialize in different skills and add different values to the team. Franchisees/bidders treat players with similar skills as substitutes, whereas those with different skills are treated as complements. The franchisees aim to assemble a team of players, which is capable of winning the tournament. In summary, the sequential auction setting analyzed in this study is observed in diverse economic situations and hence, needs to be examined thoroughly.

Sequential first and second-price auctions, in the presence of identical objects, were first analyzed by [Milgrom & Weber, 1982]. Studies in the literature which attend to the sequential auctions of non-identical objects can be classified into two major streams, namely stochastically equivalent objects and heterogeneous objects. Some notable studies associated with stochastically equivalent objects are [Sørensen, 2006, Bernhardt & Scoones, 1994, Engelbrecht-Wiggans, 1994]. Heterogenous objects were studied by [Muramoto & Sano, 2016, Kittsteiner et al., 2004, Elmaghraby, 2003]. Next, we mention the research works which evaluate the effects of synergies on the outcomes of the sequential auctions. All of these studies model synergies in mainly two different ways, namely deterministic and stochastic. A deterministic approach to model synergies evaluates the value of the bundle of objects by either multiplying or adding a constant to the sum of the values of individual objects [Branco, 1997, Menezes & Monteiro, 2003, Menezes & Monteiro, 2004, Leufkens et al., 2010]. Apart from the deterministic approaches, stochastic techniques have also been employed in the literature to model the synergies [Jeitschko & Wolfstetter, 2002, De Silva et al., 2005, Jofre-Bonet & Pesendorfer, 2014]. In such models, if a bidder has acquired an object, he derives the value of the second item from a distribution which dominates that of the first. In other words, having acquired an object, bidders are more likely to derive a higher valuation for its complement.

In summary, all the studies in the literature of sequential auctions analyze the effects of synergies in the presence of only two objects. We extend the literature by incorporating more than two objects into the model. Inclusion of more than two objects allows for both types of synergies to co-occur. Since we include this possibility, it makes our study distinct from the rest in the literature.

The model that we present in this study involves the sale of four non-identical objects. The objects are classified into two categories, where a category is defined as a collection of substitutable objects. Additionally, inter-category objects are treated as complements. Two risk-neutral bidders participate in the auction and demand one unit from each category. They aim to create a bundle of complementary (inter-category) objects, which brings them the maximum profit. The game takes place in two periods. In each period, both objects of a particular category are auctioned via a sequence of English auctions.

We examine all possible outcomes of the game. For each outcome, we find out the optimal bidding strategy, equilibrium selling prices, bidders' expected profits and the seller's expected revenue. We also discover that many items in such an auction setting are sold at their reserve prices. Some objects are sold at competitive prices, and under certain conditions, a few objects may fetch exceptionally high selling prices. We observe a decreasing trend in the selling prices of objects in one of the outcomes, as the auction progresses. We also find that, if certain assumptions of the models are violated, such an auction format can also encourage false bidding.

2 The Model

The set of players $N = \{1, 2\}$ consists of two symmetric and risk neutral bidders. The set of objects $S = \{A_1, A_2, B_1, B_2\}$ contains four indivisible and non-identical objects. We define two subsets of S, namely $A = \{A_1, A_2\}$ and $B = \{B_1, B_2\}$, which are also called *categories* of objects. We define a category as a collection of substitutable items. Additionally, items from different categories are treated as complements. Bidders demand one unit from each category and aim to create a bundle of complementary (inter-category) objects, which maximizes their total expected profits. Here, we define profit as the difference between the utility obtained from a bundle of objects and the payments made to reserve it. Throughout the paper, we use superscript i to denote the bidders and subscript k to denote objects. To put this auction setting into perspective, we proceed with an example.

Suppose a floriculturist/auctioneer wants to sell different collections of Lilies and Roses via auction. There are two potential bidders 1 and 2, who compete for these flowers. Both bidders are florists, who create various products using different flowers that they buy in the auction, such as bouquets, flower baskets, etc. Assume that auctioneer has four different collections of flowers, which come from two breeds, namely roses and lilies. These collections of flowers as follows: Red Roses (RR), White Roses (WR), Pink Lilies (PL) and Yellow Lilies (YL). Hence, the set S is described as $S = \{RR, WR, PL, YL\}$. We assume that each of the four collections of flowers is indivisible and hence cannot be divided into sub-parts and sold separately. In other words, a collection of flowers is treated as a unit throughout the auction. Both bidders consider flowers from the two breeds, Roses and Lilies, to be essential for a bouquet and treat them as complementary. Additionally, bidders consider flowers from the same breed as substitutes. Hence, they treat RR and WR as substitutes. Similarly, PL and YL are also considered substitutable. Therefore, the categories are defined as follows Rose $= \{RR, WR\}$ and Lily $= \{PL, YL\}$. Bidders aim to create a bouquet using one kind of flowers from each breed. In other words, they demand only one collection from each category. The unit demand from each category can be interpreted as follows. Since flowers are perishable goods, bidders intend to buy only a limited amount, which will be used to make the desired products [1].

The game takes place in two periods i.e. $t \in \{1, 2\}$. In each period, all objects from a particular category are sold via a sequence of English auctions. At $t = 1$, both objects from category A are auctioned, followed by category B objects at $t = 2$. Before the game begins (at $t = 0$), the seller releases an announcement which notifies the potential buyers about all the objects to be auctioned. In the context of our example, at $t = 0$, the auctioneer informs the two florists about the breed and color of the flowers to be auctioned at $t = 1, 2$. Both types of roses are auctioned at $t = 1$, whereas both types of lilies are auctioned at $t = 2$. In both periods, sequential English is employed as the format of the auction.

[1] The rational is not the perishableness the but the composition of different flowers (breed and color) in a bouquet

After the announcement, bidders independently decide their relative rankings over the substitutable objects. Given two substitutable objects, we say that a bidder ranks one of them higher than the other when he derives greater utility from it than its substitute. For example, if bidder i derives greater utility from object A_1 than its substitute, i.e., A_2, we say that he ranks A_1 higher than A_2. Bidders' relative rankings are common knowledge throughout the game.

Given the relative rankings of the bidders, the sequence of sale O is determined as follows. In a particular category, if both bidders rank the same object higher, then that object is presented first. Otherwise, the order of sale remains (A_1, A_2) at $t = 1$ and (B_1, B_2) at $t = 2$. In any case, we denote the object being auctioned first and second at $t = 1$ by A_f and A_s respectively. Similarly, the first and second objects to be auctioned at $t = 2$ are denoted by B_f and B_s. The order of sale $O = (A_f, A_s, B_f, B_s)$ is also common knowledge throughout the game.

We denote the relative rankings of bidders i for category A and B objects by $r^i_A = (r^i_{A_f}, r^i_{A_s})$ and $r^i_B = (r^i_{B_f}, r^i_{B_s})$ respectively. Both r^i_A and r^i_B are two-dimensional binary vectors. A bidder's relative rank for a particular category reflects his preferences over the objects in that category. A bidder's rankings for both categories are collectively represented by $r^i = (r^i_A, r^i_B) = (r^i_{A_f}, r^i_{A_s}, r^i_{B_f}, r^i_{B_s})$. For example, if bidder 1 ranks object A_f higher than A_s in category A, then $r^1_A = (r^1_{A_f}, r^1_{A_s}) = (1, 0)$. Similarly, if he ranks B_s higher than B_f in category B, then $r^1_B = (0, 1)$. Collectively, the relative ranking vector for bidder 1, denoted by r^1, is as $r^1 = (1, 0, 0, 1)$.

The relative rankings r^1 and r^2 are assumed to be common knowledge. In the context of our example, the relative rankings being common knowledge can be interpreted as follows. Both bidders have been functioning in separate markets with distinct customers for a long time. Assume that bidders derive their rankings from their customers' preferences. Hence, each bidder has perfect information regarding his opponent's rankings. Relative rankings are not actual valuations. They are only indicators of preference relations over same category objects.

In each time period, objects belonging to a particular category are sold via a sequence of English auctions. At the beginning of each period, both bidders independently inspect the objects being auctioned in that period and receive private signals about their quality. These signals are different from the public announcement that was made at $t = 0$. We denote the signals received by bidder i at $t = 1$ and at $t = 2$ by x^i_A and x^i_B respectively, where $x^i_A, x^i_B \in [\delta_0, \delta_1]$. Here, δ_0 and δ_1 are common knowledge and $\delta_0, \delta_1 \in \mathbb{R}_+$, $\delta_1 > \delta_0$.

After observing the quality signal regarding the category A objects, i.e., x^i_A, bidder i calculates his valuation of category A objects as follows:

$$v^i_A = (v^i_k)_{k \in \{A_f, A_s\}} = ((1 - r^i_k)\delta_0 + r^i_k x^i_A)_{k \in \{A_f, A_s\}} \ \forall i \in N \tag{1}$$

In words, a bidder's valuation of his lower ranked object remains fixed at δ_0, whereas that of the higher ranked object is $x^i_A \in [\delta_0, \delta_1]$, known privately to him. Mathematically, $r^i_k = 1$ implies that $v^i_k > \delta_0$, whereas $r^i_k = 0$ implies $v^i_k = \delta_0$. Bidder i's signal x^i_A can also be interpreted as his *type* at $t = 1$. History at the beginning of $t = 1$, H_A can be described as $H_A = \{r^1, r^2, O\}$. In words, bidders are aware of their own rankings, their opponent's rankings, and the order of sale. Actions taken by bidder i are nothing but the prices at which he decides to leave the auctions of A_f and A_s. We denote the actions taken by bidder i at $t = 1$ by $a^i_A = (a^i_{A_f}, a^i_{A_s})$. Since all valuations are at least δ_0, which is common knowledge, the auctioneer starts the auction of every object at price δ_0 (reserve price). Therefore, a bidder has to pay at least δ_0 in order to buy an object. However, a bidder is allowed to stay inactive during an auction or bid zero. Hence, the set of actions can mathematically be described as:

$$a_A^i = (a_{A_f}^i, a_{A_s}^i) \in (\{0\} \cup [\delta_0, \infty)) \times (\{0\} \cup [\delta_0, \infty)) \;\; \forall i \in N \tag{2}$$

We say that bidder i participates in the auction of object k, when his bid is greater than or equal to δ_0, i.e. $a_k^i \geq \delta_0$, $\forall k \in \{A_f, A_s\}$. Outcome of the game at $t = 1$ depends on the actions taken by both bidders. It can be described with the help of three variables $w_A^1 = (w_k^1)_{k\in\{A_f,A_s\}}$, $w_A^2 = (w_k^2)_{k\in\{A_f,A_s\}}$ and P_A. We define the outcome for bidder i as:

$$w_A^i = (w_k^i)_{k\in\{A_f,A_s\}} = (I(a_k^i > a_k^{-i}))_{k\in\{A_f,A_s\}} \; \forall i \in N \tag{3}$$

where $I(c)$ is an indicator random variable which takes value 1, when condition c holds. In words, $w_k^i = 1$ denotes that bidder i won object k. The other aspect of the outcome is the selling prices of the two objects. Since the objects are sold via a sequence of English auctions, a bidder pays only when he wins the object. Also, the price paid by the winner is the bid/action taken by his opponent. Mathematically:

$$P_A(a_A^1, a_A^2) = (P_k)_{k\in\{A_f,A_s\}} = (w_k^1 a_k^2 + w_k^2 a_k^1)_{k\in\{A_f,A_s\}} \tag{4}$$

At the beginning of $t = 2$, bidder i observes the second quality signal, i.e. x_B^i. He calculates his valuations of category B objects as follows:

$$v_B^i = (v_k^i)_{k\in\{B_f,B_s\}} = ((1 - r_k^i)\delta_0 + r_k^i x_B^i)_{k\in\{B_f,B_s\}} \; \forall i \in N \tag{5}$$

At $t = 2$, bidder i's type is defined as a tuple, which consists of both of his private signals, i.e., $x^i = (x_A^i, x_B^i)$. At the beginning of second period, the history H_B can be described as $H_B = \{r^1, r^2, a_A^1, a_A^2, w_A^1, w_A^2, P_A, O\}$. Just as before, bidder i's actions are the prices at which he leaves the auctions of objects B_f and B_s. Actions taken by bidder i at $t = 2$ are denoted by a_B^i, which are defined as follows:

$$a_B^i = (a_{B_f}^i, a_{B_s}^i) \in (\{0\} \cup [\delta_0, \infty)) \times (\{0\} \cup [\delta_0, \infty)) \;\; \forall i \in N \tag{6}$$

Outcome of the game at the end of $t = 2$ can be described in a similar fashion as that at $t = 1$. Explicitly, the outcome for bidder i at $t = 2$ is as follows

$$w_B^i = (w_k^i)_{k\in\{B_f,B_s\}} = (I(a_k^i > a_k^{-i}))_{k\in\{B_f,B_s\}} \; \forall i \in N \tag{7}$$

whereas selling prices can be described as

$$P_B(a_B^1, a_B^2) = (P_k)_{k\in\{B_f,B_s\}} = (w_k^1 a_k^2 + w_k^2 a_k^1)_{k\in\{B_f,B_s\}} \tag{8}$$

We collectively denote the outcome for bidder i from both periods by w^i, where $w^i = (w_A^i, w_B^i)$.

A strategy for bidder i is defined as a function from the set of types to the set of actions. We assume that bidding strategies are symmetric and invertible. A bidder is required to bid in the auctions of all four objects i.e. two objects in each time period. Hence, the overall strategy of a bidder consists of four bidding functions, i.e. $b(x_A^i, x_B^i) = \big(b_{A_f}(x_A^i), b_{A_s}(x_A^i), b_{B_f}(x_A^i, x_B^i), b_{B_s}(x_A^i, x_B^i)\big)$. It should be noted that bidding functions of first period (b_{A_f} and b_{A_s}) do not depend on the signal of the second period x_B^i. Mathematically, bidding functions of first time period are described as:

$$b_k : [\delta_0, \delta_1] \rightarrow \{0\} \cup [\delta_0, \infty) \; \forall k \in \{A_f, A_s\} \tag{9}$$

$w^i = (w^i_{A_f}, w^i_{A_s}, w^i_{B_f}, w^i_{B_s})$	$u^i(w^i)$ if $r^i = (1,0,1,0)$
$(1,0,1,0)$	$\alpha(x^1_A + x^1_B)$
$(1,0,0,1)$	$\beta(x^1_A + \delta_0)$
$(0,1,1,0)$	$\beta(\delta_0 + x^1_B)$
$(0,1,0,1)$	$2\delta_0$

Table 1: Utilities derived by bidder i from various outcomes (w^i) given his rankings (r^i)

In contrast, bidding functions of the second period depend on both signals. Mathematically, they can be described as:

$$b_k : [\delta_0, \delta_1] \times [\delta_0, \delta_1] \to \{0\} \cup [\delta_0, \infty) \ \forall k \in \{B_f, B_s\} \tag{10}$$

Next, we explain our approach to model substitutability between intra-category objects. As mentioned previously, a bidder demands only one unit from each category. In other words, if a bidder wins both items from a given category, he doesn't utilize his lower ranked object at all. Therefore, having obtained the higher ranked object, a bidder's utility from a lower ranked object is zero. This assumption is enough to account for the substitutable relationship between intra-category objects. We model complementary relationships between inter-category objects using multiplicative factors. Specifically, a bidder's total utility from winning both of his higher ranked objects is obtained by multiplying the sum of the individual values by a constant α, where $\alpha > 1$. If a bidder wins his higher ranked object from only one of the two categories, his utility from this bundle is determined by multiplying the sum of individual values by a constant β where $1 < \beta < \alpha$. Bidders don't realize any synergy if both of their objects are lower ranked ones and hence, the value of the bundle is just the summation of individual values. Table 1 summarizes the utilities realized by bidder i, denoted by $u^i(\cdot)$, from various bundles, given that bidder i ranks A_f higher in category A, and B_f in category B, i.e. $r^i = (r^i_{A_f}, r^i_{A_s}, r^i_{B_f}, r^i_{B_s}) = (1,0,1,0)$.

The utility of player i, depends on the outcome of the game i.e. w^i, which again depends on the actions of the players as shown in equations (3) and (7). Let $a^i = (a^i_{A_f}, a^i_{A_s}, a^i_{B_f}, a^i_{B_s})$ denote the actions/bids of bidder i in the auctions of all four objects. The gross utility structure of player i described in Table 1, can mathematically be expressed as:

$$\begin{aligned} u^i(a^i, a^{-i}) &= (r^i_A \cdot w^i_A)(r^i_B \cdot w^i_B)\left[\alpha(x^i_A + x^i_B)\right] + (r^i_A \cdot w^i_A)(1 - r^i_B \cdot w^i_B)\left[\beta(x^i_A + \delta_0)\right] \\ &+ (1 - r^i_A \cdot w^i_A)(r^i_B \cdot w^i_B)\left[\beta(\delta_0 + x^i_B)\right] + (1 - r^i_A \cdot w^i_A)(1 - r^i_B \cdot w^i_B)\left[2\delta_0\right] \ \forall i \in N \end{aligned} \tag{11}$$

Since the format of auctions is sequential English, bidder i pays only when he wins an object. In such cases, bidder i pays the amount equal to his opponent's bid. Therefore, payments of player i, denoted by ϕ^i, can mathematically be expressed as:

$$\phi^i(a^i, a^{-i}) = \sum_{k \in S} w^i_k a^{-i}_k \ \forall i \in N \tag{12}$$

Finally, payoff of bidder i is obtained by subtracting his total payment from his total utility. We denote the payoff of bidder i by π^i, which can be written as:

$$\pi^i(a^i, a^{-i}) = u^i(a^i, a^{-i}) - \phi^i(a^i, a^{-i}) \ \forall i \in N \tag{13}$$

Next, we describe the belief structure of bidder i in both periods. We assume that the quality signals are independently and identically distributed in $[\delta_0, \delta_1]$ according to the density $f(\cdot)$ and distribution $F(\cdot)$. Consequently, the common prior p at $t = 0$ is given as $p = \prod_{i\in N, k\in\{A,B\}} f(x_k^i)$. Bidders *consistently* derive their beliefs regarding their opponent's signals from p using Bayes rule. We assume that $f(\cdot)$ is continuous and differentiable on its support $[\delta_0, \delta_1]$. At $t = 1$, bidder i knows his type x_A^i but believes that x_B^i, x_A^{-i} and x_B^{-i} are independently and identically distributed according to the density $f(\cdot)$. At $t = 2$, bidder i is aware of both x_A^i and x_B^i, but believes that x_B^{-i} is distributed according to the density $f(\cdot)$. Bidder i's belief about his opponent's signal at $t = 1$, i.e. x_A^{-i} has to be updated. This update in bidder i's belief about x_A^{-i} takes into account the information released by bidder $-i$'s actions at $t = 1$. Bidder i's beliefs about x_A^{-i} are updated according to the Bayes rule as follows:

$$\tilde{f}(x_A^{-i}) = r_{A_f}^{-i}\left\{ w_{A_f}^{-i} \cdot \frac{f(x_A^{-i}).I\left[x_A^{-i} > b_{A_f}^{-1}\left(P_{A_f}\right)\right]}{\int_{b_{A_f}^{-1}\left(P_{A_f}\right)}^{\delta_1} f(x_A^{-i})dx_A^{-i}} + (1 - w_{A_f}^{-i}) \cdot b_{A_f}^{-1}\left(P_{A_f}\right)\right\} +$$
$$r_{A_s}^{-i}\left\{ w_{A_s}^{-i} \cdot \frac{f(x_A^{-i}).I\left[x_A^{-i} > b_{A_s}^{-1}\left(P_{A_s}\right)\right]}{\int_{b_{A_s}^{-1}\left(P_{A_s}\right)}^{\delta_1} f(x_A^{-i})dx_A^{-i}} + (1 - w_{A_s}^{-i}) \cdot b_{A_s}^{-1}\left(P_{A_s}\right)\right\} \quad (14)$$

The above expression consists of two terms, one of which is always zero, because bidder $-i$ ranks only one of A_f and A_s higher. The first expression becomes relevant when bidder $-i$ ranks A_f higher than A_s i.e. $r_{A_f}^{-i} = 1$ and $r_{A_s}^{-i} = 0$. Bidder $-i$'s valuation of his lower ranked object is always δ_0, which is independent of his type x_A^{-i}. Therefore, bidder $-i$'s overall profit from obtaining a lower ranked object from category A does not depend on x_A^{-i}. Subsequently, bidder $-i$'s bid for his lower ranked object is also independent of x_A^{-i}. Therefore, any information regarding bidder $-i$'s type, i.e. x_A^{-i}, can only come from his bid of his higher ranked object. Hence the first expression, only depends on $b_{A_f}(\cdot)$ as A_f is the higher ranked object. If bidder $-i$ wins A_f i.e. $w_{A_f}^{-i} = 1$, bidder i can only infer that $x_{A_f}^{-i}$ is higher than $b_{A_f}^{-1}\left(P_{A_f}\right)$, where P_{A_f} is the selling price of the object A_f. (Eq (4)). Consequently, bidder i updates his beliefs accordingly using Bayes rule as in the above equation. If bidder $-i$ loses his higher ranked object i.e. $w_{A_f}^{-i} = 0$, then bidder i can exactly find out his opponent's signal to be equal to $b_{A_f}^{-1}\left(P_{A_f}\right)$ as described in the equation above. Similarly, the second term becomes relevant when bidder $-i$ ranks A_s higher than A_f.

We define that, a strategy profile $\mathbf{b}^* = (b^*(x^1), b^*(x^2))$ constitutes a subgame perfect Bayesian equilibrium, if for each player, each type x^i and each possible action $b(x^i)$:

$$\mathbb{E}\left[\pi^i\left(\mathbf{b}^*\right)\right] \geq \mathbb{E}\left[\pi^i\left((b(x^i), b^*(x^{-i}))\right)\right] \quad (15)$$

3 Results

In this section, we examine all possible situations which can arise as a result of the game described above. For each of these possible scenarios, we find out the optimal bidding strategy, equilibrium selling prices along with the bidders' expected profits and seller's expected revenue.

Notation	Explanation
$N = \{1, 2\}$	Set of players
$S = \{A_1, A_2, B_1, B_2\}$	Set of objects
$A = \{A_1, A_2\}$	Set of category A objects
$B = \{B_1, B_2\}$	Set of category B objects
$t \in \{1, 2\}$	Time period
$O = \{\{A_f, A_s\}, \{B_f, B_S\}\}$	Order of sale
$r^i_A = (r^i_{A_f}, r^i_{A_s})$	Relative rankings of player i for category A objects
$r^i_B = (r^i_{B_f}, r^i_{B_s})$	Relative rankings of player i for category B objects
$r^i = (r^i_A, r^i_B)$	Relative rankings of player i for both categories
$\delta_0, \delta_1 \in \mathbb{R}_+, \delta_0 < \delta_1$	Real numbers in $\mathbb{R}_+$
$x^i_A \in (\delta_0, \delta_1)$	Signal and type of player i at $t = 1$
$x^i_B \in (\delta_0, \delta_1)$	Signal of player i at $t = 2$
$x^i = (x^i_A, x^i_B)$	Type of player i at $t = 2$
$f(\cdot), F(\cdot)$	Density and distribution of signals x^i_A and x^i_B
$v^i_A = (v^i_{A_f}, v^i_{A_s})$	Valuation of bidder i for category A objects
$v^i_B = (v^i_{B_f}, v^i_{B_s})$	Valuation of bidder i for category B objects
$v^i = (v^i_A, v^i_B)$	Valuation of bidder i for both categories
$H_A = \{r^1, r^2, O\}$	History at $t = 1$
$H_B = \{r^1, r^2, a^1_A, a^2_A, w^1_A, w^2_A, P_A, O\}$	History at $t = 2$
$a^i_A = (a^i_{A_f}, a^i_{A_s})$	Actions by player i at $t = 1$
$a^i_B = (a^i_{B_f}, a^i_{B_s})$	Actions by player i at $t = 2$
$a^i = (a^i_A, a^i_B)$	Actions by player i for both categories
$P_A = (P_{A_f}, P_{A_s})$	Selling prices of category A objects
$P_B = (P_{B_f}, P_{B_s})$	Selling prices of category B objects
$P = (P_A, P_B)$	Selling prices of objects of both categories
$w^i_A = (w^i_{A_f}, w^i_{A_s})$	Indicates objects won by bidder i in category A
$w^i_B = (w^i_{B_f}, w^i_{B_s})$	Indicates objects won by bidder i in category B
$w^i = (w^i_A, w^i_B)$	Indicates objects won by bidder i from both categories
$b_A = (b_{A_f}(x^i_A), b_{A_s}(x^i_A))$	Bidding functions for category A objects
$b_B = (b_{B_f}(x^i_A, x^i_B), b_{B_s}(x^i_A, x^i_B))$	Bidding functions for category B objects
$b = (b_A, b_B)$	Overall strategy of a bidder for the game
$\mathbf{b}^* = (b^*(x^1), b^*(x^2))$	Strategy profile
$u^i(a^i, a^{-i})$	Total utility of bidder i
$\phi^i(a^i, a^{-i})$	Sum of all the payments made by bidder i
$\pi^i(a^i, a^{-i})$	Payoff/Profit of bidder i

Table 2: Notations and Explanations

3.1 Possible Classes of Games

Bidder i's relative ranking for all four objects is collectively represented by r^i, i.e. $r^i = (r^i_{A_f}, r^i_{A_s}, r^i_{B_f}, r^i_{B_s})$. It is a 4-dimensional binary vector and can take four different values. Mathematically:

$$r^i \in \{(0,1,0,1),(0,1,1,0),(1,0,0,1),(1,0,1,0)\}\ \forall i \in \{1,2\}$$

Let (r^1, r^2) denote the tuple of relative rankings of all four objects of the two bidders. For example, if $r^1 = (1,0,1,0)$ and $r^2 = (0,1,0,1)$, then, $(r^1, r^2) = ((1,0,1,0),(0,1,0,1))$. The pair (r^1, r^2) collectively represents the relative rankings of both bidders. Since bidders' relative rankings are independent of each other, the pair (r^1, r^2) can take 16 different values. But, out of the 16 possible values of (r^1, r^2), some values are not possible. This is because, if both bidders rank an object higher in a particular category, the auctioneer sells that object first. Therefore, it is not possible to have a situation in which, both bidders rank a particular object higher in any category, and it is presented second. For example, it is not possible to have $r^1 = r^2 = (0,1,0,1)$. This is because, having such a pair means that both bidders rank objects A_s and B_s higher, both of which are presented second at $t = 1$ and 2. The list of all values of (r^1, r^2), which cannot occur can be found in Table 4. We denote the set of possible values of (r^1, r^2) by R. Bidders' relative rankings influence their strategies and hence, also affect the outcome of the game. Therefore, each possible value of (r^1, r^2) can lead to a different outcome of the game. However, it is possible to cluster the elements of R, into a few subsets, to facilitate the further analysis of the game. Following definitions are useful to understand these subsets.

- Competition (C): "Competition" in a particular category occurs, when bidders' higher ranked objects are the same in that category. For example, if both bidders rank object A_f higher, then we say that there is competition in category A.

- No competition (N): "No competition" in a particular category occurs, when bidders' higher ranked objects are different in that category. For example, if bidder 1 ranks B_f higher, but bidder 2's higher ranked object is B_s, then we say that there is no competition in category B.

Based on the definitions above, we define four subsets of R, namely NN, NC, CN, and CC. A brief description of these subsets can be found in Table 3. All possible values of (r^1, r^2) are classified into these four subsets, depending on the presence/absence of competition in categories A and B. For example, if $r^1 = (1,0,1,0)$ and $r^2 = (0,1,1,0)$, then bidders' higher ranked objects are different in category A, but the same in category B. This value of (r^1, r^2) falls into the NC class of games, when there is no competition in category A but competition in category B. Others values can also be classified in the same way. Table 3 describes all possible values of (r^1, r^2) and how they can be classified into different subsets.

Next, we argue that optimal strategies, equilibrium selling prices, and outcome of the game for one value of (r^1, r^2) will be the same for all other values of (r^1, r^2) belonging to the same subset. Please note that all values of (r^1, r^2) within a subset differ from one another only in categories with no competition. For example, $((1,0,1,0),(0,1,1,0)) \in NC$ and also, $((0,1,1,0),(1,0,1,0)) \in NC$. These two values of (r^1, r^2) differ from each other only in bidders' preferences in category A. In the former, bidder 1 ranks A_f higher but bidder 2 ranks A_s higher, while in the latter, bidder 1 and 2's higher ranked objects

Class of the Game/Subset of R	Category A	Category B
NN	No Competition	No Competition
NC	No Competition	Competition
CN	Competition	No Competition
CC	Competition	Competition

Table 3: Possible classes of games

r^1 \ r^2	(0,1,0,1)	(0,1,1,0)	(1,0,0,1)	(1,0,1,0)
(0,1,0,1)	-	-	-	NN
(0,1,1,0)	-	-	NN	NC
(1,0,0,1)	-	NN	-	CN
(1,0,1,0)	NN	NC	CN	CC

Table 4: Relation between classes of games and players' relative rankings

are A_s and A_f respectively. In both cases, bidder's don't compete in category A, and the only difference in the relative rankings of bidders in the two situations is the identities of the objects. Since objects from the same category are stochastically equivalent (their valuations are derived from the same distribution), their identities don't matter. And the two situations are equivalent. Hence, optimal strategy for one value of (r^1, r^2) can be extended to the other values of (r^1, r^2) from the same subset. Because of this equivalence, we say that all values of (r^1, r^2) belonging to the same subset define a *class* of the game. We explore each of these classes separately in the next section.

3.2 Bidding Strategies and Equilibrium Selling Prices

3.2.1 NN class of games

In this section, we find optimal bidding strategies and equilibrium selling prices for all NN class games. In all realizations of the game of NN class, bidders' higher ranked objects differ in both categories. Consequently, there is no competition in any of the two categories. The following lemma describes the optimal bidding functions in the NN class of games at $t = 2$.

Lemma 1. *In all realizations of the game of NN class, the optimal bidding functions of bidder i at t = 2, i.e.* $b^*_B\left(x^i_A, x^i_B\right) = (b^*_k(x^i_A, x^i_B))_{k\in(B_f,B_s)}$ *are given as follows:*

$$b^*_k\left(x^i_A, x^i_B|(r^1, r^2) \in NN\right) = r^i_k\delta_0 \;\forall k \in \{B_f, B_s\} \tag{16}$$

Proof. As mentioned previously, bidders demand only one unit from each category. Hence, the utility realized by a bidder from a lower ranked object is zero, if he has already obtained the higher ranked object from that category. Since auction of B_s is a simple English auction, the optimal bid is the same as the utility from the object. Suppose bidder i ranks object B_s higher and bidder $-i$'s higher ranked object is B_f. Bidder i knows that if $-i$ wins object B_f, $-i$'s utility for object B_s will be zero. Consequently, bidder $-i$ will bid zero or will not participate in the auction of B_s. This event would enable bidder i to win B_s at minimum price δ_0. Hence, bidder i does not participate in the auction of B_f and ensures that $-i$ wins it at minimum price δ_0. Subsequently, bidder i wins his higher ranked object B_s at δ_0. In summary, bidders bid δ_0 for their higher ranked and zero for their lower ranked object from category B as mentioned in the lemma. □

Corollary 2. *In all realizations of the game of NN class, the equilibrium selling prices at* $t = 2$*, i.e.,* $P_B = (P_{B_f}, P_{B_s})$ *are given as follows:*

$$P_k\left(b_k^*(x_A^1, x_B^1), b_k^*(x_A^2, x_B^2)|(r^1, r^2) \in NN\right) = \delta_0 \ \forall k \in \{B_f, B_s\} \tag{17}$$

Lemma 3. *In all realizations of the game of NN class, the optimal bidding functions of bidder* i *at* $t = 1$*, i.e.,* $b_A^*(x_A^i) = (b_k^*(x_A^i))_{k \in (A_f, A_s)}$ *are given as follows:*

$$b_k^*\left(x_A^i|(r^1, r^2) \in NN\right) = r_k^i \delta_0 \ \forall k \in \{A_f, A_s\} \tag{18}$$

Proof. At $t = 1$, both bidders know that they are going to win their higher ranked objects from category B, irrespective of the results of category A auctions. Hence, category A and B auctions are independent in NN class of games. Therefore, the arguments given in the proof of Lemma 1 can be used to arrive at a similar result as that for $t = 1$. As given in the lemma, both bidders bid zero for their lower ranked and δ_0 for their higher ranked item. □

Corollary 4. *In all realizations of the game of NN class, selling prices at* $t = 1$*, i.e.,* $P_A = (P_{A_f}, P_{A_s})$ *are given as follows:*

$$P_k\left(b_k^*(x_A^1), b_k^*(x_A^2)|(r^1, r^2) \in NN\right) = \delta_0 \ \forall k \in \{A_f, A_s\} \tag{19}$$

In an NN class game, all four objects are sold at reserve price δ_0. The following corollary gives the payoffs of each bidder for all realizations of the game of NN class.

Corollary 5. *In all realizations of the game of NN class, the overall payoff of bidder* i *is given as follows:*

$$\pi^i(\mathbf{b}^*) = \alpha(x_A^i + x_B^i) - 2\delta_0 \ \forall i \in N \tag{20}$$

3.2.2 NC class of games

In this section, we determine a bidder's optimal bidding strategy in the NC class of games. In any realization of the game of NC class, bidders rank different objects higher in category A but rank the same object higher in category B. As a result, bidders do not compete in category A but contest for their higher ranked object in category B. We characterize the equilibria for such a game below.

Lemma 6. *In all realizations of the game of NC class, the optimal bidding functions of bidder* i *at* $t = 2$*, i.e.* $b_B^*(x_A^i, x_B^i) = (b_k^*(x_A^i, x_B^i))_{k \in (B_f, B_s)}$ *are given as follows:*

$$b_{B_f}^*\left(x_A^i, x_B^i|r_A^i \cdot w_A^i = 1, (r^1, r^2) \in NC\right) = \delta_0 + \alpha(x_A^i + x_B^i) - \beta(x_A^i + \delta_0) \tag{21}$$

$$b_{B_f}^*\left(x_A^i, x_B^i|r_A^i \cdot w_A^i = 0, (r^1, r^2) \in NC\right) = \beta(\delta_0 + x_B^i) - \delta_0 \tag{22}$$

$$b_{B_s}^*\left(x_A^i, x_B^i|w_{B_f}^i = 0, (r^1, r^2) \in NC\right) = \delta_0 \tag{23}$$

$$b_{B_s}^*\left(x_A^i, x_B^i|w_{B_f}^i = 1, (r^1, r^2) \in NC\right) = 0 \tag{24}$$

Proof. At $t = 2$, a bidder can either be a winner or a loser of his higher ranked object from category A i.e. $r^i_A.w^i_A = 1$ or 0. Since each bidder's demand is only one unit, a bidder gets his lower ranked object in case he loses his higher ranked one in category A. The event that bidder i won his higher ranked object from category A can be represented by the equation $r^i_A \cdot w^i_A = 1$. In case he loses his higher ranked object from category A, $r^i_A \cdot w^i_A = 0$.

At $t = 2$, object B_f is auctioned first, as it is ranked higher by both bidders. Using backward induction, we consider the auction of object B_s first. When object B_s is auctioned, both bidders are aware of the outcome of object B_f's auction, i.e. $w^1_{B_f}$ and $w^2_{B_f}$ are common knowledge. Suppose bidder i won object B_f i.e. $w^i_{B_f} = 1$ and $w^{-i}_{B_f} = 0$. Since bidder i's need of category B objects is satisfied, he bids zero in the auction of object B_s. Mathematically,

$$b^*_{B_s}\left(x^i_A, x^i_B | w^i_{B_f} = 1, (r^1, r^2) \in NC\right) = 0$$

consequently, bidder $-i$ need not go beyond δ_0 in order to win object B_s, i.e.

$$b^*_{B_s}\left(x^{-i}_A, x^{-i}_B | w^{-i}_{B_f} = 0, (r^1, r^2) \in NC\right) = \delta_0$$

We now consider the auction of object B_f. Since both bidders rank B_f higher, they compete with each other in order to buy it. Optimal bidding function in the auction of B_f depends on the result of category A auctions. In other words, a bidder's bidding function depends on whether or not he won his higher ranked object from category A. Hence, we find optimal bidding function for both such cases.

CaseI (Winner: $r^i_A \cdot w^i_A = 1$): Assume that bidder i wins his higher ranked objects from category A, i.e. $r^i_A \cdot w^i_A = 1$, and pays p^i_A for it. If bidder i also wins his higher ranked object from category B i.e. B_f at price $p^i_{B_f}$, his profit $\pi^i(\cdot)$ will be given as:

$$\pi^i(\cdot | r^i_A \cdot w^i_A = 1, w^i_{B_f} = 1) = \alpha(x^i_A + x^i_B) - p^i_{B_f} - p^i_A$$

where $\alpha(x^i_A + x^i_B)$ is his total utility and p^i_A and $p^i_{B_f}$ are the prices paid for higher ranked objects from category A and B respectively. On the other hand, if he loses object B_f, he goes on to win object B_s and pays δ_0, hence his payoff is given as:

$$\pi^i(\cdot | r^i_A \cdot w^i_A = 1, w^i_{B_f} = 0) = \beta(x^i_A + \delta_0) - \delta_0 - p^i_A$$

where $\beta(x^i_A + \delta_0)$ is his total utility and $\delta_0 + p^i_A$ is the total price paid by him. Bidder i would stay in the auction of object B_f, until he is indifferent between the two payoffs, i.e.

$$\pi^i_{B_f}(\cdot | r^i_A \cdot w^i_A = 1, w^i_{B_f} = 1) = \pi^i_{B_f}(\cdot | r^i_A \cdot w^i_A = 1, w^i_{B_f} = 0)$$

solving the above equation gives:

$$p^i_{B_f} = b^*_{B_f}\left(x^i_A, x^i_B | r^i_A \cdot w^i_A = 1, (r^1, r^2) \in NC\right) = \delta_0 + \alpha(x^i_A + x^i_B) - \beta(x^i_A + \delta_0)$$

Case II (Loser: $r^i_A \cdot w^i_A = 0$): Suppose bidder i won his lower ranked object from category A, i.e. $r^i_A \cdot w^i_A = 0$ and paid p_A. If bidder i wins his higher ranked object from category B, i.e. B_f at price p_{B_f}, his profit is given as:

$$\pi^i(\cdot | r^i_A \cdot w^i_A = 0, w^i_{B_f} = 1) = \beta(\delta_0 + x^i_B) - p_A - p_{B_f}$$

where $\beta(\delta_0 + x_B^i)$ is the total utility and $p_A + p_{B_f}$ is the total payment made by bidder i. If bidder i loses B_f, his total profit is given as follows:

$$\pi^i(\cdot|r_A^i \cdot w_A^i = 0, w_{B_f}^i = 0) = 2\delta_0 - p_A - \delta_0$$

He will leave the auction of B_f when he becomes indifferent between the two situations i.e.

$$\pi_{B_f}^i(\cdot|r_A^i \cdot w_A^i = 0, w_{B_f}^i = 1) = \pi_{B_f}^i(\cdot|r_A^i \cdot w_A^i = 0, w_{B_f}^i = 0)$$

solving the above equation gives

$$p_{B_f} = b_{B_f}^*\left(x_A^i, x_B^i | r_A^i \cdot w_A^i = 0, (r^1, r^2) \in NC\right) = \beta(\delta_0 + x_B^i) - \delta_0$$

□

Corollary 7. *In all realizations of the game of NC class, selling prices at $t = 2$, i.e., $P_B = (P_{B_f}, P_{B_s})$ are given as follows:*

$$P_{B_f}\left(b_{B_f}^*(x_A^1, x_B^1), b_{B_f}^*(x_A^2, x_B^2)|(r^1, r^2) \in NC\right) = \min\left\{\delta_0 + \alpha(x_A^1 + x_B^1) - \beta(x_A^1 + \delta_0), \delta_0 + \alpha(x_A^2 + x_B^2) - \beta(x_A^2 + \delta_0)\right\} \quad (25)$$

$$P_{B_s}\left(b_{B_s}^*(x_A^1, x_B^1), b_{B_s}^*(x_A^2, x_B^2)|(r^1, r^2) \in NC\right) = \delta_0 \quad (26)$$

Lemma 8. *In all realizations of the game of NC class, the optimal bidding functions of bidder i at $t = 1$, i.e. $b_A^*(x_A^i) = (b_k^*(x_A^i))_{k \in (A_f, A_s)}$ are given as follows:*

$$b_k^*\left(x_A^i | (r^1, r^2) \in NC\right) = r_k^i \delta_0 \ \forall k \in \{A_f, A_s\} \quad (27)$$

Proof. Suppose bidder i's higher ranked object is A_s and $-i$'s higher ranked object is A_f. If $-i$ wins A_f, his utility for A_s will be zero. We show that, $-i$'s optimal bid in the auction of A_s will also be zero. A bidder makes a positive profit, only if his total payment for both the objects is lower than the value of the bundle. Hence, if a bidder pays more to reserve the first object, he would be spend less for the second. One can follow this line of argument to claim that bidder $-i$ may bid non-zero amount in the auction of A_s, in order to sabotage bidder i's purchasing ability at $t = 2$ (false bidding). However, we show that it is not feasible in the present model.

Suppose bidder $-i$ decides to engage in false bidding and takes up the price to p_{A_s}. At this price, if $-i$ wins A_s, his loss would be p_{A_s}. On the other hand, if $-i$ loses it, he successfully decreases the purchasing ability of bidder i by $p_{A_s} - \delta_0$, which would help $-i$ at $t = 2$. Suppose that the probability of bidder $-i$ losing A_s is q, when the price is p_{A_s}. Then, his expected profit from engaging in false bidding is $q(p_{A_s} - \delta_0) + (1 - q)(-\delta_0)$. At the beginning of the auction, i.e., when $p_{A_s} = \delta_0$, the expected profit is $(1 - q)(-\delta_0)$. If this expected profit is to be greater than zero, then $q > 1$, which is not possible. Hence, it is not profitable for bidder $-i$ to engage in false bidding. Bidder i knows this fact and lets bidder $-i$ win his higher ranked object A_f. Bidder i achieves this by bidding zero in the auction of A_f. Consequently, bidder $-i$ wins A_f at price δ_0. Bidder $-i$ bids zero in the auction of A_s and bidder i wins A_s at price δ_0. In summary, both bidder bid δ_0 for their higher ranked objects and zero for their lower ranked one as described in the lemma. □

Corollary 9. *In all realizations of the game of NC class, selling prices at $t = 1$, $P_A = (P_{A_f}, P_{A_s})$ are given as follows:*

$$P_k\left(b_k^*(x_A^1), b_k^*(x_A^2)|(r^1, r^2) \in NN\right) = \delta_0 \ \forall k \in \{A_f, A_s\} \tag{28}$$

In an NC class game, both objects from category A are sold at reserve price δ_0. Whereas in category B, the object B_f is sold at a price higher than δ_0, but B_s's selling price remains δ_0. The following corollary describes the pay-off of each bidder in an NC game.

Corollary 10. *In all realizations of the game of NC class, the overall payoff of bidder i is given as follows:*

$$\pi^i\left(\mathbf{b}^*|w_{B_f}^i = 1, (r^1, r^2) \in NC\right) = \alpha(x_A^i + x_B^i) - \alpha(x_A^{-i} + x_B^{-i}) + \beta(x_A^{-i} + \delta_0) - 2\delta_0 \tag{29}$$

$$\pi^i\left(\mathbf{b}^*|w_{B_f}^i = 0, (r^1, r^2) \in NC\right) = \beta(x_A^i + \delta_0) - 2\delta_0 \tag{30}$$

3.2.3 CN class of games

In this section, we examine realizations of the game of CN class and determine the optimal bidding strategy. In CN class of games, bidders rank the same object higher in category A but different objects in category B. Hence, category B is devoid of any competition, whereas bidders compete in category A. We describe the optimal bidding functions at $t = 2$ in the following lemma.

Lemma 11. *In all realizations of the game of CN class, the optimal bidding functions of bidder i at $t = 2$, i.e. $b_B^*(x_A^i, x_B^i) = (b_k^*(x_A^i, x_B^i))_{k\in(B_f,B_s)}$ are given as follows:*

$$b_k^*\left(x_A^i, x_B^i|(r^1, r^2) \in CN\right) = r_k^i\delta_0 \ \forall k \in \{B_f, B_s\} \tag{31}$$

Proof. Since there is no competition, the proof is exactly same as that in Lemma 1. □

Corollary 12. *In all realizations of the game of CN class, selling prices at $t = 2$, $P_B = (P_{B_f}, P_{B_s})$ are given as follows:*

$$P_k\left(b_k^*(x_A^i, x_B^i), b_k^*(x_A^i, x_B^i)|(r^1, r^2) \in CN\right) = \delta_0 \ \forall k \in \{B_f, B_s\} \tag{32}$$

Lemma 13. *In all realizations of the game of CN class, the optimal bidding functions of bidder i at $t = 1$, i.e. $b_A^*(x_A^i) = (b_k^*(x_A^i))_{k\in(A_f,A_s)}$ are given as follows:*

$$b_{A_f}^*\left(x_A^i|r_{A_f}^i = 1, (r^1, r^2) \in CN\right) = \delta_0 + \alpha(x_A^i + \mathbb{E}\left[x_B^i\right]) - \beta(\mathbb{E}\left[x_B^i\right] + \delta_0) \tag{33}$$

$$b_{A_s}^*\left(x_A^i|w_{A_f}^i = 0, (r^1, r^2) \in CN\right) = \delta_0 \tag{34}$$

$$b_{A_s}^*\left(x_A^i|w_{A_f}^i = 1, (r^1, r^2) \in CN\right) = 0 \tag{35}$$

Proof. At $t = 1$, both bidders know that their higher ranked objects from category B are different and they are going to win their higher ranked objects at $t = 2$. Since both bidders win their higher ranked objects from category B, irrespective of category A auctions (Lemma 11), outcomes of category A and B auctions are independent. In category A, object A_f is auctioned first as it is ranked higher by both bidders. However, using backward induction, we consider the auction of object A_s first. When object A_s is auctioned, both bidders are aware of the outcome of object A_f's auction, i.e. $w^1_{A_f}$ and $w^2_{A_f}$ are public knowledge. Suppose bidder i won object A_f i.e. $w^i_{A_f} = 1$ and $w^{-i}_{A_f} = 0$. Since bidder i's need of category A objects is satisfied, he bids zero in the auction of object A_s. Mathematically,

$$b^*_{A_s}\left(x^i_A | w^i_{A_f} = 1, (r^1, r^2) \in NC\right) = 0$$

consequently, bidder $-i$ need not go beyond δ_0 in order to win object A_s, i.e.

$$b^*_{A_s}\left(x^{-i}_A | w^{-i}_{A_f} = 0, (r^1, r^2) \in NC\right) = \delta_0$$

Next, we find out the optimal bidding function for the object A_f. According to our assumption made previously, bidders don't know their valuations of category B objects at $t = 1$. But they know that, in this case, they will win their higher ranked object from category B at price δ_0. Suppose bidder i wins the auction of object A_f at price p_{A_f}, his expected payoff will be given as:

$$\mathbb{E}[\pi^i(\cdot|w^i_{A_f} = 1)] = \int_{\delta_0}^{\delta_1} \left(\alpha(x^i_A + x^i_B) - p_{A_f} - \delta_0\right) f(x^i_B) dx^i_B = \alpha(x^i_A + \mathbb{E}[x^i_B]) - p_{A_f} - \delta_0$$

On the other hand, if he loses the auction of A_f and goes on to win A_s, his expected profit will be given as:

$$\mathbb{E}[\pi^i(\cdot|w^i_{A_f} = 0)] = \int_{\delta_0}^{\delta_1} \left(\beta(x^i_B + \delta_0) - 2\delta_0\right) f(x^i_B) dx^i_B = \beta(\mathbb{E}[x^i_B] + \delta_0) - 2\delta_0$$

A bidder would stay in the auction of object A_f, until he becomes indifferent between these two situations. Mathematically,

$$\mathbb{E}[\pi^i(\cdot|w^i_{A_f} = 1)] = \mathbb{E}[\pi^i(\cdot|w^i_{A_f} = 0)]$$

solving the above equation gives:

$$p_{A_f} = b^*_{A_f}\left(x^i_A | r^i_{A_f} = 1, (r^1, r^2) \in CN\right) = \delta_0 + \alpha(x^i_A + \mathbb{E}\left[x^i_B\right]) - \beta(\mathbb{E}\left[x^i_B\right] + \delta_0)$$

□

Corollary 14. *In all realizations of the game of CN class, selling prices at $t = 1$, i.e., $P_A = (P_{A_f}, P_{A_s})$ are given as follows:*

$$\begin{aligned} &P_{A_f}\left(b^*_{A_f}(x^1_A), b^*_{A_f}(x^2_A) | (r^1, r^2) \in CN\right) = \\ &\quad \min\left\{\delta_0 + \alpha(x^1_A + \mathbb{E}\left[x^1_B\right]) - \beta(\mathbb{E}\left[x^1_B\right] + \delta_0), \delta_0 + \alpha(x^2_A + \mathbb{E}\left[x^2_B\right]) - \beta(\mathbb{E}\left[x^2_B\right] + \delta_0)\right\} \end{aligned} \tag{36}$$

$$P_{A_s}\left(b^*_{A_s}(x^1_A), b^*_{A_s}(x^2_A) | (r^1, r^2) \in CN\right) = \delta_0 \tag{37}$$

In a CN class game, both objects from category B are sold at reserve price δ_0. Whereas in category A, the first object is sold at a price higher than δ_0 but the selling price of the second object remains δ_0. The following corollary describes bidders' payoffs in a CN class game.

Corollary 15. *In all realizations of the game of CN class, the overall payoff of bidder i is given as follows:*

$$\pi^i\left(\mathbf{b}^*|w^i_{A_f}=1,(r^1,r^2)\in CN\right)=\alpha(x^i_A+x^i_B)-\alpha(x^{-i}_A+\mathbb{E}\left[x^{-i}_B\right])+\beta(\mathbb{E}\left[x^{-i}_B\right]+\delta_0)-2\delta_0 \tag{38}$$

$$\pi^i\left(\mathbf{b}^*|w^i_{A_f}=0,(r^1,r^2)\in CN\right)=\beta(\delta_0+x^i_B)-2\delta_0 \tag{39}$$

3.2.4 CC class of games

In this section, we explore the optimal bidding strategy and equilibrium selling prices in a CC class game. In this class of the game, bidders' relative rankings are the same in both categories. Therefore, higher ranked objects are auctioned first in both categories, i.e., A_f in category A and B_f in category B. Using the backward induction, we first characterize the equilibrium bidding strategy for the category B auctions.

Lemma 16. *In all realizations of the game of CC class, the optimal bidding functions of bidder i at $t=2$, i.e., $b^*_B(x^i_A,x^i_B)=(b^*_k(x^i_A,x^i_B))_{k\in(B_f,B_s)}$ are given as follows:*

$$b^*_{B_f}\left(x^i_A,x^i_B|w^i_{A_f}=1,(r^1,r^2)\in CC\right)=\delta_0+\alpha(x^i_A+x^i_B)-\beta(x^i_B+\delta_o) \tag{40}$$

$$b^*_{B_f}\left(x^i_A,x^i_B|w^i_{A_f}=0,(r^1,r^2)\in CC\right)=\beta(\delta_0+x^i_B)-\delta_0 \tag{41}$$

$$b^*_{B_s}\left(x^i_A,x^i_B|w^i_{B_f}=0,(r^1,r^2)\in CC\right)=\delta_0 \tag{42}$$

$$b^*_{B_s}\left(x^i_A,x^i_B|w^i_{B_f}=1,(r^1,r^2)\in CC\right)=0 \tag{43}$$

Proof. First we point out that at $t=2$, bidders are asymmetric in the CC class of game. This happens because only one of the two bidders wins the higher ranked object from category A. This induces an asymmetry between them when they compete for category B objects. Therefore, the optimal strategies of the bidders depend upon the outcome in the first period. We consider the category B auctions first.

We derive the optimal bidding strategy for the object B_s first. Since it is the second object to be auctioned at $t=2$, outcome of the auction of object B_f is common knowledge. Suppose bidder i does not win object B_f, i.e. $w^i_{B_f}=0$ and $w^{-i}_{B_f}=1$. Since bidder $-i$'s need of category B object has been satisfied, he bids zero in the auction of B_s. Please note that this argument holds irrespective of the outcome of the category A auctions. Hence, the following equation holds whether or not a bidder wins his higher ranked object from category A, i.e., $w^{-i}_{A_f}=0$ or 1.

$$b^*_{B_s}\left(x^{-i}_A,x^{-i}_B|w^{-i}_{B_f}=1,(r^1,r^2)\in CC\right)=0$$

Consequently, bidder i need not go beyond δ_0 in order to win object B_s. Here also, the argument is valid irrespective of the results of category A auctions. Hence,

$$b^*_{B_s}\left(x^i_A,x^i_B|w^i_{B_f}=0,(r^1,r^2)\in CC\right)=\delta_0$$

We now find the optimal bidding strategies in the auction of object B_f. Results of category A auctions influence the bidding behavior in the auction of B_f as it is ranked higher by both bidders. We find the optimal bidding functions for the auction of B_f for the winner as well as the loser of the higher ranked object from category A.

Case I (Winner of A_f): Suppose bidder i won object A_f at $t = 1$, i.e. $w^i_{A_f} = 1$ and paid p_{A_f}. If he also wins object B_f, i.e. $w^i_{B_f} = 1$ at price p_{B_f}, his total profit will be given as:

$$\pi^i\left(\cdot|w^i_{A_f} = 1, w^i_{B_f} = 1, (r^1, r^2) \in CC\right) = \alpha(x^i_A + x^i_B) - p_{A_f} - p_{B_f}$$

On the other hand, if he loses B_f, he wins object B_s, and pays δ_0. His payoff is given as:

$$\pi^i\left(\cdot|w^i_{A_f} = 1, w^i_{B_f} = 0, (r^1, r^2) \in CC\right) = \beta(x^i_A + \delta_0) - p_{A_f} - \delta_0$$

Bidder i would stay in the auction of object B_f, until he becomes indifferent between winning and losing, i.e.

$$\pi^i\left(\cdot|w^i_{A_f} = 1, w^i_{B_f} = 1, (r^1, r^2) \in CC\right) = \pi^i\left(\cdot|w^i_{A_f} = 1, w^i_{B_f} = 0, (r^1, r^2) \in CC\right)$$

Solving the above equation gives:

$$p_{B_f} = b^*_{B_f}\left(x^i_A, x^i_B|w^i_{A_f} = 1, (r^1, r^2) \in CC\right) = \delta_0 + \alpha(x^i_A + x^i_B) - \beta(x^i_B + \delta_o)$$

Case II (Loser of A_f): Suppose bidder i loses A_f, i.e., $w^i_{A_f} = 0$. Since there is unit demand from each category, he wins object A_s. Suppose he pays p_{A_s} for it. If he wins B_f at a price p_{B_f}, his profit will be:

$$\pi^i\left(\cdot|w^i_{A_f} = 0, w^i_{B_f} = 1, (r^1, r^2) \in CC\right) = \beta(\delta_0 + x^i_B) - p_{A_s} - p_{B_f}$$

On the other hand, if he loses B_f, he wins his lower ranked object B_s and his payoff is given as:

$$\pi^i\left(\cdot|w^i_{A_f} = 0, w^i_{B_f} = 0, (r^1, r^2) \in CC\right) = 2\delta_0 - p_{A_s} - \delta_0$$

Bidder i will stay in the auction of B_f, until he becomes indifferent between the two situations, i.e.

$$\pi^i\left(\cdot|w^i_{A_f} = 0, w^i_{B_f} = 1, (r^1, r^2) \in CC\right) = \pi^i\left(\cdot|w^i_{A_f} = 0, w^i_{B_f} = 0, (r^1, r^2) \in CC\right)$$

Solving the above equation gives:

$$p_{B_f} = b^*_{B_f}\left(x^i_A, x^i_B|w^i_{A_f} = 0, (r^1, r^2) \in CC\right) = \beta(\delta_0 + x^i_B) - \delta_0$$

□

Corollary 17. *In all realizations of the game of CC class, selling prices at $t = 2$, $P_B = (P_{B_f}, P_{B_s})$ are given as follows:*

$$P_{B_f}\left(b^*_{B_f}(x^1_A, x^1_B), b^*_{B_f}(x^2_A, x^2_B)|w^i_{A_f} = 1, (r^1, r^2) \in CC\right) = \min\left\{\delta_0 + \alpha(x^i_A + x^i_B) - \beta(x^i_B + \delta_o), \beta(\delta_0 + x^{-i}_B) - \delta_0\right\} \quad (44)$$

$$P_{B_s}\left(b^*_{B_s}(v^1), b^*_{B_s}(v^2)|(r^1, r^2) \in CC\right) = \delta_0 \quad (45)$$

Now we direct our attention to category A auctions. While bidding for category A objects, bidders evaluate not only their immediate payoffs from category A but also the effects of the outcome of category A auction on category B. The following lemma describes the optimal bidding strategy for the higher ranked object from category A.

Lemma 18. *In all realizations of the game of CC class, the optimal bidding functions of bidder i at $t = 1$, i.e. $b^*_A(x^i_A) = (b^*_A(x^i_A))_{k\in(A_f,A_s)}$ are given as follows:*

$$b^*_{A_f}\left(x^i_A|(r^1,r^2) \in CC\right) = \Gamma_1(x^i_A) + \Gamma_2(x^i_A) - \frac{\Gamma_3(x^i_A)}{1 - F\left(x^i_A\right)} \tag{46}$$

$$b^*_{A_s}\left(x^i_A|w^i_{A_f} = 0, (r^1,r^2) \in CC\right) = \delta_0 \tag{47}$$

$$b^*_{A_s}\left(x^i_A|w^i_{A_f} = 1, (r^1,r^2) \in CC\right) = 0 \tag{48}$$

where

$$\Gamma_1(x^i_A) = \iint_{C1} \left[\alpha(x^i_A + x^i_B) - \beta(\delta_0 + x^{-i}_B) + \delta_0\right] f(x^i_B) f(x^{-i}_B) dx^i_B dx^{-i}_B \tag{49}$$

$$\Gamma_2(x^i_A) = \iint_{C2} \left[\beta(x^i_A + \delta_0) - \delta_0\right] f(x^i_B) f(x^{-i}_B) dx^i_B dx^{-i}_B \tag{50}$$

$$\Gamma_3(x^i_A) = \iiint_{C3} \left[\beta(\delta_0 + x^i_B) - \alpha(x^{-i}_A + x^{-i}_B) + \beta(x^{-i}_A + \delta_0) - 2\delta_0\right] f(x^{-i}_A) f(x^i_B) f(x^{-i}_B) dx^{-i}_A dx^i_B dx^{-i}_B \tag{51}$$

$$C_{WW} : \alpha(x^i_A + x^i_B) - \beta(x^i_A + \delta_0) + \delta_0 > \beta(\delta_0 + x^{-i}_B) - \delta_0 \tag{52}$$

$$C_{WL} : \alpha(x^i_A + x^i_B) - \beta(x^i_A + \delta_0) + \delta_0 < \beta(\delta_0 + x^{-i}_B) - \delta_0 \tag{53}$$

$$C_{LW} : x^{-i}_A > x^i_A; \beta(\delta_0 + x^{-i}_A) - \delta_0 > \delta_0 + \alpha(x^{-i}_A + x^{-i}_B) - \beta(x^{-i}_A + \delta_0) \tag{54}$$

Proof. At $t = 1$, object A_f is auctioned first as it is ranked higher by both bidders. Using backward induction, we consider the auction of object A_s first. When object A_s is auctioned, both bidders are aware of the outcome of object A_f's auction, i.e., $w^1_{A_f}$ and $w^2_{A_f}$ are common knowledge. Suppose bidder i won object A_f i.e. $w^i_{A_f} = 1$ and $w^{-i}_{A_f} = 0$. Since bidder i's need of category A object is satisfied, he bids zero in the auction of object A_s. He does not engage in false bidding as it is not favorable as shown in Lemma 8. Mathematically,

$$b^*_{A_s}\left(x^i_A|w^i_{A_f} = 1, (r^1,r^2) \in CC\right) = 0$$

Consequently, bidder $-i$ need not go beyond δ_0 in order to win A_s. Therefore:

$$b^*_{A_s}\left(x^{-i}_A|w^{-i}_{A_f} = 0, (r^1,r^2) \in CC\right) = \delta_0$$

Next, we consider the auction of object A_f. Assume that bidder i wins the higher ranked object from category A and pays p_{A_f}. There are two possibilities at $t = 2$. Bidder i can either win or lose the higher ranked object from category B. If he also wins higher ranked object from category B, his payoff is given as:

$$\pi^i\left(\cdot|w^i_{A_f} = 1, w^i_{B_f} = 1\right) = \alpha(x^i_A + x^i_B) - p_{A_f} - \beta(\delta_0 + x^{-i}_B) + \delta_0$$

In this equation, $\alpha(x_A^i + x_B^i)$ is the total utility of bidder i by winning both of his higher ranked objects. And, $\beta(\delta_0 + x_B^{-i}) - \delta_0$ is the price paid by bidder i for B_f (Eq. 40) as it is the losing bid. Bidder i wins the higher ranked object from category B when he bids higher than his opponent in the auction of B_f, i.e,

$$\alpha(x_A^i + x_B^i) - \beta(x_A^i + \delta_0) + \delta_0 > \beta(\delta_0 + x_B^{-i}) - \delta_0$$

We denote this condition by C_{WW} and it can be obtained from Eq 39 and 40. Therefore, the expected pay-off from winning the higher ranked objects from both categories can be described as:

$$\mathbb{E}\left[\pi^i\left(\cdot|w_{A_f}^i = 1, w_{B_f}^i = 1\right)\right] = \frac{\iint_{C_{WW}}\left[\alpha(x_A^i + x_B^i) - p_{A_f} - \beta(\delta_0 + x_B^{-i}) + \delta_0\right] f(x_B^i) f(x_B^{-i}) dx_B^i dx_B^{-i}}{\iint_{C_{WW}} f(x_B^i) f(x_B^{-i}) dx_B^i dx_B^{-i}}$$

Where denominator of the above equation denotes the probability of the bidder i winning the higher ranked objects from both categories. Let this probability be denoted by P_{WW}.

Consider the other situation in which, bidder i wins the higher ranked object only from category A. In such a situation, he wins the lower ranked object from category B and pays δ_0 (Eq. 41) for it. Therefore, his pay-off is given by the following expression:

$$\pi^i\left(\cdot|w_{A_f}^i = 1, w_{B_f}^i = 0\right) = \beta(x_A^i + \delta_0) - p_{A_f} - \delta_0$$

Here $\beta(x_A^i + \delta_0)$ is the total utility obtained by bidder i and p_{A_f} and δ_0 are the total payments made by him. Bidder i loses B_f when his bid is lower than that of his opponent, i.e., the following condition holds:

$$\alpha(x_A^i + x_B^i) - \beta(x_A^i + \delta_0) + \delta_0 < \beta(\delta_0 + x_B^{-i}) - \delta_0$$

We denote this condition by C_{WL} and it can be obtained from Eq 39 and 40. Therefore, the expected pay-off of bidder i, when he wins the higher ranked object only from category A, is given by the following expression:

$$\mathbb{E}\left[\pi^i\left(\cdot|w_{A_f}^i = 1, w_{B_f}^i = 0\right)\right] = \frac{\iint_{C_{WL}}\left[\beta(x_A^i + \delta_0) - p_{A_f}^i - \delta_0\right] f(x_B^i) f(x_B^{-i}) dx_B^i dx_B^{-i}}{\iint_{C_{WL}} f(x_B^i) f(x_B^{-i}) dx_B^i dx_B^{-i}}$$

Where denominator of the above equation denotes the probability of bidder i winning the higher ranked object from category A but not from B. Let this probability be denoted by P_{WL}.

Next, we consider the two possibilities that arise when bidder i loses A_f. Here again, bidder i can either win or lose his higher ranked object from category B. We first consider the situation in which he loses the higher ranked object from category A but wins from category B. In such a scenario, his expected pay-off is given by the following expression:

$$\pi^i\left(\cdot|w_{A_f}^i = 0, w_{B_f}^i = 1\right) = \beta(\delta_0 + x_B^i) - \alpha(x_A^{-i} + x_B^{-i}) + \beta(x_A^{-i} + \delta_0) - 2\delta_0$$

Here $\beta(\delta_0 + x_B^i)$ is the total utility obtained by him. The total payments made by bidder i for A_s and B_f are δ_0 and $\delta_0 + \alpha(x_A^{-i} + x_B^{-i}) - \beta(x_A^{-i} + \delta_0)$ respectively. Bidder i finds himself in this situation, when the following two conditions hold:

$$x_A^{-i} > x_A^i$$

$$\beta(\delta_0 + x_A^{-i}) - \delta_0 > \delta_0 + \alpha(x_A^{-i} + x_B^{-i}) - \beta(x_A^{-i} + \delta_0)$$

We collectively represent these two conditions as C_{LW}. Given C_{LW}, the expected profit of bidder i from winning the higher ranked object only from category B can be represented by the following expression:

$$\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=0, w^i_{B_f}=1\right)\right] = \frac{\iiint_{C_{LW}} \left[\beta(\delta_0 + x_B^i) - \alpha(x_A^{-i} + x_B^{-i}) + \beta(x_A^{-i} + \delta_0) - 2\delta_0\right] \tilde{f}(x_A^{-i}) f(x_B^i) f(x_B^{-i}) dx_A^{-i} dx_B^i dx_B^{-i}}{\iiint_{C_{LW}} \tilde{f}(x_A^{-i}) f(x_B^i) f(x_B^{-i}) dx_A^{-i} dx_B^i dx_B^{-i}}$$

The denominator of the above equation denotes the probability of bidder i losing the higher ranked object from category A but winning that from category B. Let this probability be denoted by P_{LW}. Please note that the density of x_A^{-i} used in the above equation is the updated according to the Baye's rule as described in Eq 14.

When bidder i loses both of his higher ranked objects, his payoff remains zero, as his overall utility is the same as the total payments made by him, i.e.

$$\pi^i\left(\cdot|w^i_{A_f}=0, w^i_{B_f}=0\right) = 0$$

Therefore, his expected profit also remains zero, i.e.

$$\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=0, w^i_{B_f}=0\right)\right] = 0$$

Let the probability of bidder i losing both of his higher ranked object be denoted by P_{LL}.

Bidder i would leave the auction of A_f, when he becomes indifferent between winning and losing:

$$\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=1, w^i_{B_f}=1\right)\right] \cdot P_{WW} + \mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=1, w^i_{B_f}=0\right)\right] \cdot P_{WL} = \mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=0, w^i_{B_f}=1\right)\right] \cdot P_{LW} + \mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=0, w^i_{B_f}=0\right)\right] \cdot P_{LL}$$

Using the respective values in the above expression:

$$\iint_{C_{WW}} \left[\alpha(x_A^i + x_B^i) - p_{A_f} - \beta(\delta_0 + x_B^{-i}) + \delta_0\right] f(x_B^i) f(x_B^{-i}) dx_B^i dx_B^{-i} + \iint_{C_{WL}} \left[\beta(x_A^i + \delta_0) - p_{A_f} - \delta_0\right] f(x_B^i) f(x_B^{-i}) dx_B^i dx_B^{-i} = \iiint_{C_{LW}} \left[\beta(\delta_0 + x_B^i) - \alpha(x_A^{-i} + x_B^{-i}) + \beta(x_A^{-i} + \delta_0) - 2\delta_0\right] \tilde{f}(x_A^{-i}) f(x_B^i) f(x_B^{-i}) dx_A^{-i} dx_B^i dx_B^{-i}$$

where $\tilde{f}(\cdot)$ is the updated density of x_A^{-i} i.e.

$$\tilde{f}(x_A^{-i}) = \frac{f(x_A^{-i})}{\int_{x_A^i}^{\delta_1} f(x_A^{-i}) dx_A^{-i}}$$

hence, the previous expression can be written as

$$\iint_{C_{WW}} \left[\alpha(x_A^i + x_B^i) - \beta(\delta_0 + x_B^{-i}) + \delta_0\right] f(x_B^i) f(x_B^{-i}) dx_B^i dx_B^{-i} + \iint_{C_{WL}} \left[\beta(x_A^i + \delta_0) - \delta_0\right] f(x_B^i) f(x_B^{-i}) dx_B^i dx_B^{-i} - p_{A_f} \iint_{C_{WW}+C_{WL}} f(x_B^i) f(x_B^{-i}) dx_B^i dx_B^{-i} = \frac{1}{1 - F\left(x_A^i\right)} \left[\iiint_{C_{LW}} \left[\beta(\delta_0 + x_B^i) - \alpha(x_A^{-i} + x_B^{-i}) + \beta(x_A^{-i} + \delta_0) - 2\delta_0\right] f(x_A^{-i}) f(x_B^i) f(x_B^{-i}) dx_A^{-i} dx_B^i dx_B^{-i}\right]$$

Since

$$\iint_{C_{WW}+C_{WL}} f(x_B^i) f(x_B^{-i}) dx_B^i dx_B^{-i} = 1$$

p_{A_f} can be found out as

$$p_{A_f} = \Gamma_1(x_A^i) + \Gamma_2(x_A^i) - \frac{\Gamma_3(x_A^i)}{1 - F(x_A^i)}$$

where $\Gamma_1(x_A^i)$, $\Gamma_2(x_A^i)$ and $\Gamma_3(x_A^i)$ are as described in Eq 48, 49 and 50. □

Corollary 19. *In all realizations of the game of CC class, selling prices at $t = 1$, $P_A = (P_{A_f}, P_{A_s})$ are given as follows:*

$$P_{A_f}\left(b^*_{A_f}(x_A^1), b^*_{A_f}(x_A^2) | (r^1, r^2) \in CC\right) = \min\left\{\Gamma_1(x_A^1) + \Gamma_2(x_A^1) - \frac{\Gamma_3(x_A^1)}{1 - F(x_A^1)}, \Gamma_1(x_A^2) + \Gamma_2(x_A^2) - \frac{\Gamma_3(x_A^2)}{1 - F(x_A^2)}\right\} \quad (55)$$

$$P_{A_s}\left(b^*_{A_s}(x_A^1), b^*_{A_s}(x_A^2) | (r^1, r^2) \in CC\right) = \delta_0 \quad (56)$$

In the realization of the game of CC class, higher ranked objects from both categories are sold at prices higher than δ_0, whereas lower ranked objects are sold at δ_0. The following corollary describes the expected payoffs of each bidder in a CC class game.

Corollary 20. *In all realizations of the games of CC class, the overall payoff of bidder i is given as follows:*

$$\pi^i\left(\mathbf{b}^* | w^i_{A_f} = 1, w^i_{B_f} = 1, (r^1, r^2) \in CN\right) = \alpha(x_A^i + x_B^i) - \Gamma_1(x_A^{-i}) - \Gamma_2(x_A^{-i}) + \frac{\Gamma_3(x_A^{-i})}{1 - F\left(x_A^{-i}\right)} - \beta(\delta_0 + x_B^{-i}) + \delta_0 \quad (57)$$

$$\pi^i\left(\mathbf{b}^* | w^i_{A_f} = 1, w^i_{B_f} = 0, (r^1, r^2) \in CN\right) = \beta(x_A^i + \delta_0) - \Gamma_1(x_A^{-i}) - \Gamma_2(x_A^{-i}) + \frac{\Gamma_3(x_A^{-i})}{1 - F\left(x_A^{-i}\right)} - \delta_0 \quad (58)$$

$$\pi^i\left(\mathbf{b}^* | w^i_{A_f} = 0, w^i_{B_f} = 1, (r^1, r^2) \in CN\right) = \beta(\delta_0 + x_B^i) - \alpha(x_A^{-i} + x_B^{-i}) + \beta(x_B^{-i} + \delta_o) - 2\delta_0 \quad (59)$$

$$\pi^i\left(\mathbf{b}^* | w^i_{A_f} = 0, w^i_{B_f} = 0, (r^1, r^2) \in CN\right) = 0 \quad (60)$$

3.3 Impact of the Order of Sale

In our model, the seller decides the order of sale based on the bidders' relative rankings. As mentioned previously, if an object is ranked higher by both bidders, the seller presents it first. Next, we argue that this indeed is the revenue maximizing order in which, the seller will present the objects.

In any category, if the seller first presents an object which is ranked lower by both bidders, it may remain unsold. This is because buying a lower ranked object always gives zero profit to a bidder. Hence, a rational bidder will always want to buy his higher ranked object first. A bidder participates in the auction of his lower ranked object, only when he fails to reserve the higher ranked one. The auctioneer anticipates this and always presents the higher ranked object first. Hence, the seller is already optimizing his revenue as far as the order of sale within a category is concerned. Next we explore the possibility of optimizing the seller's revenue by interchanging the order in which categories are presented.

In the model, we assume that the bidders derive their signals from a distribution $F(\cdot)$ at both time periods. Therefore, from the seller's point of view, bidders' valuations for their higher ranked objects from both categories are independent and identically distributed random variables. Hence, the seller's expected revenue is not affected by the order in which categories are presented during the auction.

4 Discussion

In this section, we discuss our results obtained previously and their implications to get more insights into the problem. We start with interpreting the bidding strategies.

4.1 Interpretations of Bidding Strategies

In this subsection, we try to interpret and give intuitive explanations of the optimal bidding strategies of all 4 classes of games. The NN class games don't involve any competition in any of the categories. Hence, the optimal bidding strategy requires bidders to bid minimum which is δ_0. Since bidders rank different objects higher, they never compete. As a result, both bidders win their higher ranked objects from both categories.

In an NC class game, bidders compete in category B but not in A. Hence, the optimal bidding strategy of bidder i requires him to bid minimum, i.e. δ_0, in the auction of higher ranked object from category A. Because of the absence of any competition, both bidders win their higher ranked objects from category A. During the auction of higher ranked object from category B, bidder i's optimal strategy also depends on the valuation of the higher ranked object from category A. This is because of the inter-category complementary relationships between the objects. The prescribed strategy is to stay in the auction of higher ranked object from category B, until its price reaches $\delta_0+\alpha(x^i_A+x^i_B)-\beta(x^i_A+\delta_0)$. This expression can be perceived as a summation of two components. The first component is δ_0, which is the minimum amount every bidder has to pay in order to reserve any object (higher or lower ranked). The second component, which is $\alpha(x^i_A+x^i_B)-\beta(x^i_A+\delta_0)$, can be recognized as the difference between the utilities. $\alpha(x^i_A+x^i_B)$ is the utility that bidders i realizes when he wins his higher ranked object from both categories. Likewise, $\beta(x^i_A+\delta_0)$ is the utility realized by bidder i, when he loses his higher ranked object from category B. Hence, if this difference in utilities is taken away from bidder i, he becomes indifferent between the two situations. Therefore, in the presence of competition, bidder i is willing to sacrifice this extra utility in order to get his higher ranked object from category B.

In a CN class game, bidders compete in category A but not in B. While bidding for category A objects, both bidders are aware that they would win their higher ranked objects from category B. However, bidders are not aware of their actual valuations for category B objects at $t=1$. Therefore, they use expected values of x^i_B's while bidding in category A auctions. Optimal bidding strategies in a CN class game can be interpreted in exactly the same manner as those in NC class games.

In a CC type game, bidders compete in both categories. Consequently, when bidders get to category B auctions, they are asymmetric. This asymmetry arises owing to the fact that only one of the two bidders gets the higher ranked object from category A. As a result, bidders use different strategies in category B auctions. The optimal bidding strategy of a bidder who lost his higher ranked object from category A (let bidder i), is to stay in the auction of object B_f till its price reaches $\beta(\delta_0 + x_B^i) - \delta_0$. We rewrite this expression as $\delta_0 + \beta(\delta_0 + x_B^i) - 2\delta_0$ in order to be able to provide a better interpretation. This expression can be broken down to two components with first component being δ_0. This is the base price all bidders have to pay in order to reserve any object. The second component, i.e. $\beta(\delta_0+x_B^i)-2\delta_0$ can be seen as a difference between the utilities realized in two situations. The utility realized by bidder i is $\beta(\delta_0 + x_B^i)$, if he wins his higher ranked object from category B, whereas he realizes the utility of $2\delta_0$ when he loses the higher ranked object from category B. Bidder i is willing to sacrifice this difference in utilities in order to reserve the higher ranked object from category B. In other words, loss of this extra utility makes bidder i indifferent between the two situations. A distinguishing feature of this expression is that it is independent of x_A^i. This demonstrates that bidder i's strategy in category B auction is independent of his values x_A^i, because he is not able to explore the complementarities between the objects. The optimal strategy of a bidder (let bidder $-i$) who won his higher ranked object from category A, stays in the auction of the higher ranked object from category B, until the price reaches $\delta_0+\alpha(x_A^{-i}+x_B^{-i})-\beta(x_A^{-i}+\delta_0)$. This is exactly the same expression as that of an NC type game, and hence can be interpreted in the same way as before.

Bidder i's optimal bidding strategy for the higher ranked object from category A requires him to stay in the auction until the price reaches $\Gamma_1(x_A^i) + \Gamma_2(x_A^i) - \frac{\Gamma_3(x_A^i)}{1-F\left(\frac{x_A^i-\delta_0}{\delta_1}\right)}$. This expression is very complicated and hence difficult to interpret intuitively.

4.2 Comparison Between Bids Across Game Types

In this section, we compare the optimal strategies for higher ranked objects in category A and B across game types. We assume that both x_A^i and x_B^i are uniformly distributed between $\delta_0 = 1$ and $\delta_1 = 2$. Synergy parameters α and β were chosen to be 1.25 and 1.1 respectively. Figure 1 compares the optimal bidding functions of bidder i for the higher ranked object from category A as function of x_A^i in different classes of the game. As shown in the figure, category A bids are constant at $\delta_0 = 1$ in NN and NC class games for all values of x_A^i (no competition). Optimal bidding strategy increases linearly with x_A^i in a CN type game. The optimal bidding strategy for the higher ranked object in category A is given as follows:

$$b^*_{A_f}\left(x_A^i|(r^1,r^2)\in CC\right) = 0.264682+1.22061\cdot x_A^i+0.00538636\cdot\left(x_A^i\right)^2-0.000306818\cdot\left(x_A^i\right)^3$$

Although, it is a cubic equation in x_A^i, the coefficients of square and cube terms are very small. Therefore, its graph looks like a straight line in Figure 1. However, as visible from the figure, a bidder bids highest for his higher ranked object from category A in a CC game for a given value of x_A^i. This indicates the presence of intense competition in a CC class game. Intuitively, while bidding for the higher ranked object from category A in a CC class game, bidders not only consider the outcome of category A auctions,

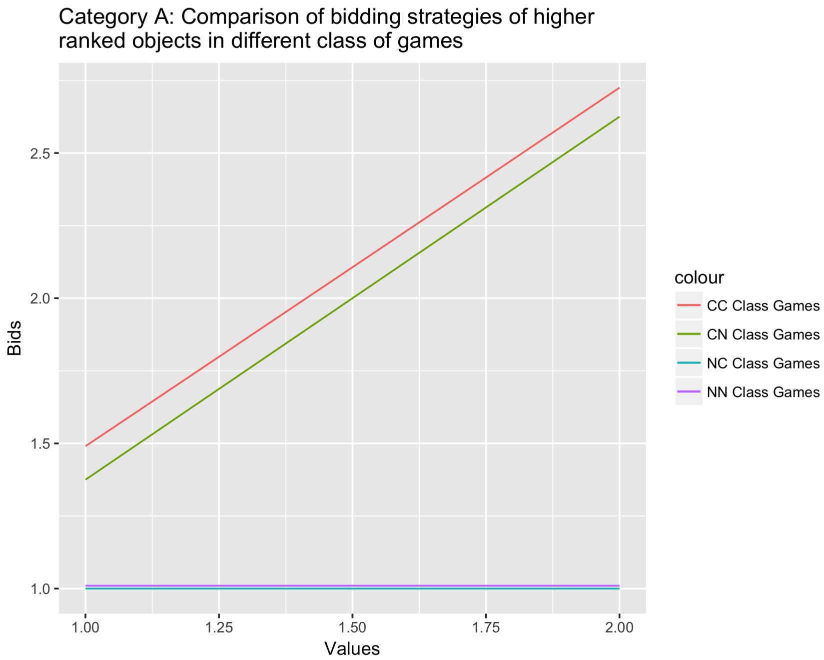

Figure 1: Category A Bids

but also the effects it will have on the outcome of category B auctions. In other words, while bidding for the higher ranked object from category A, they also compete for the advantage they get in category B auctions by winning in category A.

Now, we turn our attention to the comparison of the optimal bidding strategies for the higher ranked object from category B across different game types. Figure 2 illustrates how the optimal bidding functions of bidder i for the higher ranked object from category B varies with x_B^i across different classes of the game. Optimal strategies in the NN and CN class games are constant at $\delta_0 = 1$ (no competition). The optimal bidding function of the winner of the higher ranked object from category A in a CC class game, coincides with that of both bidders in an NC class game. This is because in an NC class game, each bidder enters into category B auctions as a winner owing to the absence of any competition in category A. These two strategies increase linearly with x_B^i. The optimal bidding function of a bidder who lost the higher ranked object in category A, in a CC class game, also increases linearly with x_B^i. But it is always lesser than that of the bidder who won the higher ranked object, for any value of x_B^i. This shows that winning the higher ranked object from category A gives an advantage to a bidder, while bidding for the higher ranked object from category B.

4.3 Ranking of Bidders' Expected Profits and Seller's Expected Revenue

In this section, we analyze the bidders' ex-ante expected profits and the seller's ex-ante expected revenue. Different classes of the game have different values of these two variables depending on the level of competition present. In an NN class game, bidders don't compete at all, and hence, they get their higher ranked objects from both categories at the minimum price. This also leads to the lowest revenue to the seller. Therefore, the expected bidders' profit from an NN class game is the highest, whereas the expected

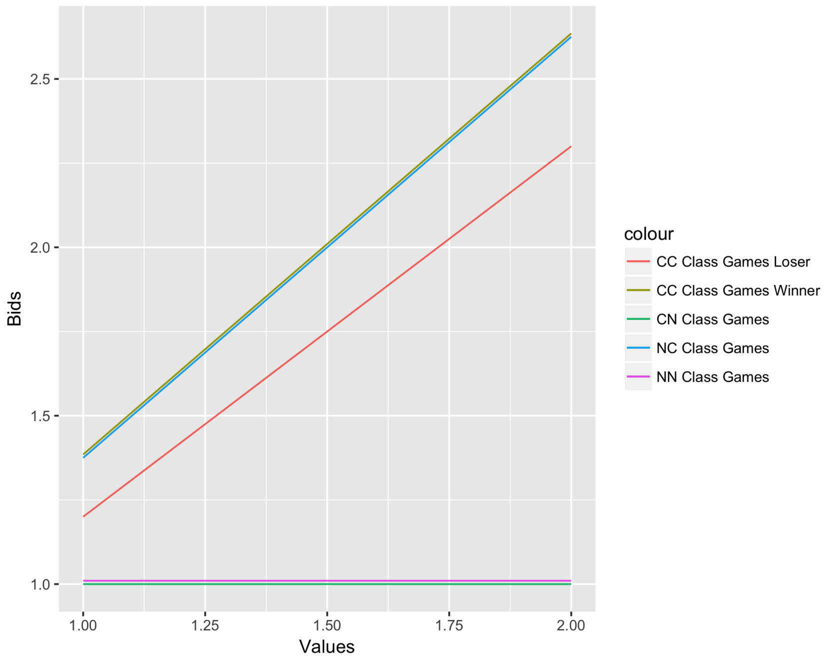

Figure 2: Category B Bids

revenue of the seller is the lowest. In a CC class game, bidders' higher ranked objects are the same in both categories. This leads to intense competition between them, which subsequently leads to the their lowest expected profit. The presence of intense competition benefits the seller and his expected revenue is the maximum in a CC class game. In the NC and CN classes of the game, bidders compete in only one of the two categories. Since the valuations for both categories' objects are stochastically equivalent, bidders' expected profits are equal for these two categories. This stochastic equivalency also leads to the equal expected revenue of the seller from both of these classes of the game. The level of competition in a CN or NC class of the game is neither as low as that of an NN class game nor as high as that of a CC class game. Because of this, a bidder's expected profit is higher than that of a CC but lower than that from an NN class of the game. The expected revenue of the seller follows the opposite order, i.e., it is higher than that of an NN class game but lower than that of a CC class game. In summary, a bidder's expected profits from all four classes of the game can be ranked as follows:

$$P_{N,N} > P_{N,C} = P_{C,N} > P_{C,C}$$

Correspondingly, the seller's expected revenue across the four classes of the game can be ranked as follows:

$$R_{C,C} > R_{C,N} = R_{N,C} > R_{N,N}$$

4.4 Possibility of Price Trends

Next, we explore the possibility of any price trend in the model, keeping the order of sale fixed. We say that a price trend is observed in an auction format, if an object gets a higher/lower selling price purely because of its position in the order of sale.

Within a category, selling price of one object is always δ_0, while that of the other object is either equal or more than δ_0. This entirely depends on whether bidders compete in that category or not, and is unaffected by order of sale within a category. In other words, no intra-category price trends are observed.

While comparing inter-category price trends, it is sufficient to only compare the prices of higher ranked objects, since lower ranked objects are always sold at δ_0. In an NN class game, selling prices of all the objects are δ_0, and we see no price trends. In an NC class game, selling prices of both objects from category A remain δ_0, but the higher ranked object from category B gets a competitive price. However, this increase in the selling price is purely because of the competition in category B. Similarly, in a CN type game, selling prices decrease, but again, this happens due to the competition in category A. In all three of these game types, the order in which categories are presented, does not affect the selling prices. However, in a CC class game, the higher ranked object from category A gets a higher selling price than that from category B. Here, the competition is present in both categories, bidders draw values from the same distribution yet the higher ranked object from category A fetches a higher selling price. This leads to a decrease in selling prices resulting purely because of the order in which categories are presented. If category B had been presented before category A, the higher ranked object from category B would have yielded a higher selling price. Therefore, the order of sale affects the selling prices only in a CC class game which yields decreasing prices.

4.5 Possibility of False Bidding

We say that a bidder engages in false bidding, when he chooses to bid up the prices of the objects to harm the other bidders. In our model, we show that there is no possibility of false bidding (Lemma 8). However, our proof relies on the assumption that a bidder's value for his lower ranked object is always the minimum. If this assumption doesn't hold, a bidder will have an incentive to sabotage the other bidder's purchasing ability or engage in false bidding.

5 Conclusions

In this study, we attempted to model a variant of a sequential English auction, which involves selling four objects with synergetic relationships. We introduced the idea of categories of objects, which are nothing but the collection of substitutable items, to better approach the problem. We found out that in the presence of two bidders, this auction can have several different outcomes which can be grouped into four classes. These classes of the game are characterized by the presence or absence of competition in the two categories, which subsequently influences the optimal strategies of bidders. We found out the optimal bidding strategies and equilibrium selling prices of all the objects for all four classes of the game. We discovered that in such an auction format, many objects are always sold at their reserve price, some objects are sold at competitive prices, while some objects can get exceptionally high selling prices. Our analysis reveals that, when bidders compete in both categories, the expected selling price of the higher ranked object from category A is higher than that from category B. Hence, decreasing selling prices are observed in a CC class game. We also observed that the total revenue of the seller doesn't change with the change in the order of sale. We discovered that there is a possibility of false bidding in such an auction format, if certain assumptions of the model are violated.

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