Group Contributions in TU games : The k-lateral value

Surajit Borkotokey[*] Dhrubajit Choudhury[†] Loyimee Gogoi[‡] Rajnish Kumar[§]

Abstract

In this paper we introduce the notion of group contributions in TU games and propose a new value which we call the k-lateral value. The Shapley like values implicitly assume that players are independent in deciding to leave or join a coalition. However, in many real life situations players are bound by the decisions taken by their peers. This leads to the idea of group contributions where we consider the marginality of groups upto a certain size. We show that group contributions can play an important role in determining players' shares in the total resource they generate. The proposed value considers both egalitarianism and marginalism and thus is a member of the class of solidarity values. We provide two characterizations of our value.

Keywords: TU Cooperative game; the Shapley value; Group contributions; the k-lateral value.

Mathematics Subject Classification (2010) : 91A12

JEL Classification: C71, D63, D81

1 Introduction

Most of the values in TU games revolve around the notion of marginalism or egalitarianism or both, depending upon the problem domains. In this paper we introduce the idea of group contributions of players and propose a new value– the k-lateral value– that is attributed to both marginalism and egalitarianism. We provide two axiomatic characterizations of this value and compare it to other values of TU games in the literature.

[*]Department of Mathematics, Dibrugarh University, Dibrugarh, Assam-786004, India; Email: surajit-bor@yahoo.com

[†]Department of Mathematics, Dibrugarh University, Dibrugarh, Assam-786004, India; Email:dhrubajit@gmail.com

[‡]Department of Mathematics, Dibrugarh University, Dibrugarh, Assam-786004, India; Email: loyimee-gogoi@gmail.com

[§]Economics Group, Queen's Management School, Queen's University, Belfast, UK;Email: rajnish.kumar@qub.ac.uk

Perhaps the most widely used allocation rule to share the joint costs or the joint surplus in smaller coalitions is the Equal division rule. This rule seems prevalent even when there are obvious differences in the individual contributions by the members in a coalition. Examples include the profit sharing of law firms where the member lawyers of the firm get equal shares of the profit irrespective of how they differ in their abilities in various dimensions. Another example is that of sharing the resources in a family. In deciding the family laws, e.g., the Hindu Undivided Family (HUF) inheritance law, the equality principle is the main underlying idea. All the siblings in an HUF, which can include up to several generations, have equal inheritance rights on the property of the family. Their rights do not depend on their individual contributions in the family wealth. An interesting example discussed in [11] is that of the sharing of the profits by the salmon fishermen in the Pacific Northwest. There are fishing groups who share the information on the whereabouts of the hunts within the group. It is a common knowledge within the group about who is good at finding the schools of Salmon, but there is no provision of side payments. Many times the coalitions of limited size tend to be formed amongst homogeneous agents who are similar in some attributes viz., their abilities (see e.g., [11]). In other words, there is an ordering of the agents based on, say, their productivity. Coalitions are formed as intervals[1] on that ordering. However, when there are complementarities among the agents, which is inherently the case in characteristic function form TU games, such orderings cannot be made.

Therefore, in our paper we propose a value for TU games which considers all the possible coalitions of certain size as equally probable. We focus on the new notion of *group contributions* of players within a coalition. Group contributions of players within a coalition must be considered when players are unable to take independent decisions to join or leave a coalition alone. There could be no way to identify who is responsible the most and who is the least among the contributors. Thus it is reasonable to split such group contributions equally amongst the members of the coalitions and to add it to the payoff to each player along with the shares from her own individual contributions. In some other situations, it so happens that adding the shares from the group contributions to a player's payoffs brings about her solidarity to her peers and integrity to the organization. Such gestures are important in organizations having employees of almost similar capacities and efficiencies. Our value includes both individual marginal contributions of the players which is standard in all Shapley like solutions and the equal shares from their group contributions.

The implicit idea of group contributions is, however, not new in the literature of TU games. Grabisch [12] proposed a model where the players in a coalition interact with each other to form groups based on the similar interests. Alternatively, in TU games with coalition structures (see for example [1, 13, 15, 19] etc.) the grand coalition is partitioned into groups or union structures. The value is then computed in two stages: first, among the groups of the coalition structure and next, among the coalition members. All such models however, assume

[1] By an interval we mean a subset of the agents who are consecutive in the ordering.

that the coalition structure is given endogeneously and therefore the group sizes are also fixed *a priori*. In our proposed model, we allow all possible coalition structures where the group sizes can vary over a range of possible values. Motivated by the original Shapley's rule of counting and also the procedure proposed by Owen[19] and Kamijo [16], we allow the players to enter a room following an order to form the grand coalition and find their contributions in groups. The size of such groups ranges from 1 to some pre-defined index k.[2]

Next we count the number of the group contributions so obtained. Our value then divides these contributions equally among the group members from all possible formations of groups and all possible orders of entrance. It follows that under the present framework, the Shapley value considers the contributions of all groups of size 1 and therefore, our value recovers the Shapley value under the special case $k = 1$. Consequently the interactions among the players responsible for generating group contributions of group size 1 can be termed as the individual interactions. Thus, the Shapley value builds on this notion of individual interactions. A $k \geq 1$ signifies the maximum allowable level of group interactions within a coalition: call it the k-lateral interaction. We call our value the k-lateral value to highlight this interaction. It is worth noting that when $k = n$, the k-lateral value is the average of the Shapley value and the Equal Division. This is indeed the α-Egalitarian Shapley value for $\alpha = \frac{1}{2}$ introduced by Joosten [14] and latter discussed in details by van den Brink et al. [26]. Thus our value takes the Shapley value on one extreme ($k = 1$) and the $\frac{1}{2}$-Egalitarian Shapley value ($k = n$) on the other extreme.

Recall that the Shapley value is characterized by efficiency, symmetry, linearity and the null player property. The difference among the Shapley like values is commonly explained from the viewpoint of who obtains a zero payoff, see [16, 25, 21]. In [24], the null player axiom, where players with zero productivity get zero payoff is replaced by the nullifying player axiom. According to this axiom, players having the property that their inclusion in a coalition makes the coalition non-productive, get zero payoff. The nullifying player axiom leads to the characterization of the Equal division. Similarly in the characterization of the solidarity value in [18], the null player axiom is replaced by the A-null player axiom where players show solidarity to the non-productive players in the game by sharing some of their marginal contributions. Alternative characterizations of the Shapley and solidarity values that follow similar arguments can be found in [2, 6, 23] etc.

In our characterizations, we consider two types of null players, we call them the k-null players of type I and type II or simply the k^1 and k^2-null players. Both these k-null players contribute nothing in groups on an average and our value awards them zero payoffs. The axioms on these two types of k-null players are less extreme than both the null player and the nullifying player. Consequently, our value is less marginalistic than the Shapley value and also less egalitarian than the Equal Division.

The rest of the paper proceeds as follows. In Section 2 we present the preliminary concepts

[2]In section 3 we give a more formal definition.

pertaining to the development of the paper. Section 3 describes a procedure to compute the k-lateral value followed by its characterization using some standard axioms in Section 4. Section 5 details an example and finally Section6 concludes.

2 Preliminaries

Let $N \subset \mathbb{N}$ be a finite set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ a characteristic function. A pair (N, v) is a cooperative game with transferable utility, or simply, a TU-game. Subsets of N are called coalitions. Thus for $S \in 2^N$, $v(S)$ denotes the worth generated by the players in S under some binding agreement. With some abuse of notations we denote the singleton sets without braces. Thus we write $S \cup i$ for $S \cup \{i\}$, $S \setminus i$ for $S \setminus \{i\}$ etc. The size (cardinality) of coalition S is denoted by the corresponding lower case letter s. Let $\mathcal{G}(N)$ denote the class of all TU games with player set N. $\mathcal{G}(N)$ forms a vector space of dimension $2^n - 1$ under the standard addition and scalar multiplication of set functions. If no ambiguity about N arises, we denote the TU game (N, v) simply by v.

The increase or decrease in worth when player $i \in S \subseteq N$ leaves coalition S is called the marginal contribution of player i in the coalition S which is denoted by $m_i^v(S)$ and is given by

$$m_i^v(S) = v(S) - v(S \setminus i). \tag{2.1}$$

The unanimity games $u_T : 2^N \to \mathbb{R}$, $T \subseteq N$ is defined as follows.

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

The class of unanimity games is a basis for the linear space $\mathcal{G}(N)$.

A value on $\mathcal{G}(N)$ is a function that assigns a single payoff vector $\Phi(v) = (\Phi_i(v))_{i \in N} \in \mathbb{R}^n$ to every game $v \in \mathcal{G}(N)$. Different values have been proposed in the literature since the introduction of the Shapley value (see, e.g., [2, 5, 6, 7, 17]). Here we mention briefly about the Shapley value, the Equal Division and the α-egalitarian Shapley value as they are closely related to our proposed value. Recall Shapley's interpretation of the Shapley value from Section 1 (also see [4]) that says that suppose the "grand coalition" $N = \{1, 2, ..., n\}$ forms in a way such that the players enter the coalition one by one. This order of entrance can be expressed by a permutation $\pi : N \to N$ of the players. Let the collection of all permutations on N be denoted by $\Pi(N)$. For every $\pi \in \Pi(N)$, let $P(\pi, i) = \{j \in N | \pi(j) < \pi(i)\}$ be the set of players that enter before player i in the order π. The Shapley value [22] is the solution $\Phi^{Sh} : \mathcal{G}(N) \to \mathbb{R}^n$ that assigns to every player i her expected marginal contribution in $P(\pi, i) \cup i$, given that every order of entrance π has equal probability of $\frac{1}{n!}$ to occur and is given by,

$$\Phi_i^{Sh}(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} [v(P(\pi, i) \cup i) - v(P(\pi, i))] \tag{2.3}$$

After simplifications Eq.(2.3) becomes,

$$\Phi_i^{Sh}(v) = \sum_{S \subseteq N \,:\, i \in S} \frac{(s-1)!\,(n-s)!}{n!} [v(S) - v(S \setminus i)] \tag{2.4}$$

or

$$\Phi_i^{Sh}(v) = \sum_{S \subseteq N} \frac{s!\,(n-s-1)!}{n!} [v(S \cup i) - v(S)], \quad \forall v \in \mathcal{G}(N). \tag{2.5}$$

The Equal division rule is a solution $\Phi^{ED} : \mathcal{G}(N) \to \mathbb{R}^n$ that distributes the worth $v(N)$ of the grand coalition equally among all players in any games, i.e.,

$$\Phi_i^{ED}(v) = \frac{v(N)}{n}, \quad \forall v \in \mathcal{G}(N). \tag{2.6}$$

For $\alpha \in [0, 1]$, the α-egalitarian Shapley value $\Phi^{\alpha-ES}$ due to [14] is a convex combination of Φ^{ED} and Φ^{Sh} which has the following form.

$$\Phi_i^{\alpha-ES}(v) = \alpha \Phi_i^{ED}(v) + (1-\alpha)\Phi_i^{Sh}(v), \quad \forall v \in \mathcal{G}(N). \tag{2.7}$$

It follows from Eq.(2.7), that the parameter α in $\Phi^{\alpha-ES}$ determines the amount of solidarity that is shown among the players in sharing the wealth.

For the game $v \in \mathcal{G}(N)$, a player $i \in N$ is called a null player if for every coalition $S \subseteq N$, we have $v(S) = v(S \setminus i)$. A player $i \in N$ is called a nullifying player if $v(S) = 0$ for all coalitions S such that $i \in S$. There has been a number of characterizations of the Shapley value, the Equal division rule and the α-egalitarian Shapley value in the literature (see, e.g., [7, 8, 9, 10, 29, 30]). Following four axioms are standard to characterize the Shapley value.

Axiom 1. Efficiency (*Eff*): A value $\Phi : \mathcal{G}(N) \to \mathbb{R}^n$ is efficient if for each game $v \in \mathcal{G}(N)$:

$$\sum_{i \in N} \Phi_i(v) = v(N)$$

Axiom 2. Null Player (*NP*): A value $\Phi : \mathcal{G}(N) \to \mathbb{R}^n$ satisfies the null player axiom if for every game $v \in \mathcal{G}(N)$ it holds that $\Phi_i(v) = 0$ for every null player $i \in N$.

Axiom 3. Symmetry (*Sym*): A value $\Phi : \mathcal{G}(N) \to \mathbb{R}^n$ satisfies Symmetry if for $i, j \in N$ such that $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, then $\Phi_i(v) = \Phi_j(v)$.

Axiom 4. Linearity (*Lin*): A value $\Phi : \mathcal{G}(N) \to \mathbb{R}^n$ is linear if for all games $u, w \in \mathcal{G}(N)$ every pair of $\alpha, \beta \in \mathbb{R}$ and every player $i \in N$:

$$\Phi_i(\alpha u + \beta w) = \alpha \Phi_i(u) + \beta \Phi_i(w).$$

Replacing the null player axiom *NP* by the axiom of nullifying player namely, the nullifying player gets zero payoff, the Equal division rule can be characterized [24]. The axiom of null player in a productive environment (*NPE*) states that for all $v \in \mathcal{G}(N)$ and $i \in N$ such that i is a null player in v and $v(N) \geq 0$ then $\Phi_i(v) \geq 0$. The *NPE* along with *Eff*, *Sym* and *Lin* characterize the α-egalitarian Shapley value [7].

A value that satisfies *Eff*, *Sym* and *Lin* is called an *ESL* value [21]. We will use the following proposition from [21] for characterization of our k-lateral value at a latter stage.

Proposition 1. *(Proposition 2 in [21], pp 184) A value Φ^{ESL} on $\mathcal{G}(N)$ is an* ESL *value if and only if there exists a unique collection of real constants $B = \{b_s : s \in \{0, 1, 2, 3, ..., n\}\}$ with $b_0 = 0$ and $b_n = 1$ such that for every $v \in \mathcal{G}(N)$,*

$$\Phi_i^{ESL}(v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} \Big\{ b_{s+1} v(S \cup i) - b_s v(S) \Big\} \tag{2.8}$$

or equivalently,

$$\Phi_i^{ESL}(v) = \Phi_i^{Sh}(Bv) \tag{2.9}$$

where $(Bv)(S) = b_s v(S)$ for each coalition of size s.

3 The k-lateral value

In this section we introduce our new value for TU Cooperative games : the k-lateral value. As mentioned in Section 1, our approach resembles with Shapley's [22] approach where the players are allowed to enter into a coalition prescribed by a particular order assuming that all possible orders of entrance have equal probabilities. Motivated by the procedure of counting adopted originally by Shapley[22], Owen [19] and Kamijo [16], we compute the group contributions of the players over all orders of entrance into forming the grand coalition, and allow each member in this group to receive equal shares from their group contributions. Let $N = \{1, 2, 3, ..., n\}$ be given. In Shapley's procedure, the marginal contributions of each player are computed immediately after she joins the other players who have entered before her. In Owen's and Kamijo's procedure, the players join a coalition one by one following an order but their contributions are computed from the components of the fixed coalition structure. In our counting process also, the players are allowed to enter according to the same order π one by one but we wait till they form groups of a particular size. Let $\emptyset = S_0 \subset S_1 \subset S_2 \subset S_3 \subset ... \subset S_{m-1} \subset S_m = N$ be one such sequence of coalition formation. Fix a k: $1 \leq k \leq n$, such that k is the maximum allowable size of these groups i.e., $|S_j \setminus S_{j-1}| \leq k$ for $1 \leq j \leq m$. The group contribution of $S_j \setminus S_{j-1}$, for $1 \leq j \leq m$ is given by $v(S_j) - v(S_{j-1})$. Thus the equal share of each player $i \in S_j \setminus S_{j-1}$ from this group contribution denoted by $A_i^v(S_j)$ is given by

$$A_i^v(S_j) = \frac{v(S_j) - v(S_{j-1})}{|S_j - S_{j-1}|}, \quad \forall i \in S_j \setminus S_{j-1} \tag{3.1}$$

Note that in particular, when $|S_j \setminus S_{j-1}| \leq 1$ then we have $m = n$ and obtain the standard marginal contributions of the individual players given in Eq.(2.1) i.e., $m_i^v(S_j) = A_i^v(S_j)$. Now each sequence of coalitions of the form $\emptyset = S_0 \subset S_1 \subset S_2 \subset S_3 \subset ... \subset S_{m-1} \subset S_m = N$ with $|S_j \setminus S_{j-1}| \leq k$ for $1 \leq j \leq m$ where the players follow a particular order π (say) of entrance gives rise to a sequence of pairwise disjoint groups $C^\pi = \{C_1^\pi, C_2^\pi, ..., C_m^\pi\}$ of N such that $S_j = \cup_{r=1}^{j} C_r^\pi$, $1 \leq j \leq m$ (equivalently $C_j^\pi = S_j \setminus S_{j-1}$) with $\max_{r=1}^{m} c_r^\pi \leq k$. Let us call such a $C^\pi = \{C_1^\pi, C_2^\pi, ..., C_m^\pi\}$ a partition of N prescribed by π. Conversely, given a partition $C = \{C_1, C_2, ..., C_m\}$ prescribed by an order π, there is always a sequence $\emptyset = S_0 \subset S_1 \subset S_2 \subset S_3 \subset ... \subset S_{m-1} \subset S_m = N$ with $S_j = \cup_{r=1}^{j} C_r$, $1 \leq j \leq m$. Define by $\text{index}(C^\pi) = \max_{j=1}^{m} c_j^\pi$ the index of a partition C^π prescribed by an order π. Let $\Pi(N,k) = \Big\{ C^\pi = \{C_1, C_2, ..., C_m\} \mid \pi \in \Pi(N) \Big\}$ be the set of all partitions on N with $\text{index}(C) \leq k$ prescribed by each partition $\pi \in \Pi(N)$. It follows that for each $C = \{C_1, C_2, ..., C_m\} \in \Pi(N,k)$, there exists

(a) a unique order π_C such that if $C_i = \{i_1, i_2, ..., i_{c_i}\}$ for $1 \leq i \leq m$ then $\pi_C(i_j) = \sum_{p=1}^{i-1} c_p + j$ for all $1 \leq j \leq c_i$.

(b) a unique sequence $c = \{c_1, c_2, ..., c_m\}$ of positive integers containing at most n terms such that $\sum_{p=1}^{m} c_p = n$. Thus the members of the sequence c represents the cardinalities of the groups of players within N.

Conversely for a permutation π on N and a sequence $\{c_1, c_2, ..., c_m\}$ of positive numbers which sums upto n determines a unique partition $C = \{C_1, C_2, ..., C_m\}$ on N such that $C_i = \{\pi^{-1}(\sum_{q=1}^{i-1} c_q + 1), ..., \pi^{-1}(\sum_{q=1}^{i-1} c_q + c_i)\}$ for $1 \leq i \leq m$.

Let $\mathcal{B}(n,k)$ be the set of all finite sequences $\{c_1, c_2, ..., c_m\}$ of positive integers with $\sum_{i=1}^{m} c_i = n$ and $1 \leq c_i \leq k$ for $1 \leq i \leq m$. It is obvious that $1 \leq m \leq n$. Clearly there is a bijection $\Pi(N,k) \leftrightarrow \Pi(N) \times \mathcal{B}(n,k)$ such that $C \leftrightarrow (\pi_C, c)$. Let $\alpha(n,k) = |\mathcal{B}(n,k)|$. Thus $\alpha(n,k)$ denotes the number of partitions with $\text{index}(C) \leq k$ that can form with n players. This idea can be easily extended to any arbitrary coalition $S \subseteq N$ and we can define $\alpha(s,k)$ exactly in the same manner. Now, observe that $|\Pi(N,k)| = |\Pi(N)|\alpha(n,k) = n!\alpha(n,k)$. For each $C \in \Pi(N,k)$ and $i \in N$, there exists some p with $1 \leq p \leq m$ such that $i \in C_p$. Define the following set.

$$P(C,i) = \{j \in N : \pi_C(j) < \min_{r \in C_p} \pi_C(r)\}.$$

Following Eq.(3.1), the equal share of player i from her group contribution in $P(C,i) \cup C_p$ when she is in $C_p \in C \in \Pi(N,k)$ is given by,

$$A_i^v(P(C,i) \cup C_p) = \frac{1}{c_p}\Big\{ v(P(C,i) \cup C_p) - v(P(C,i)) \Big\} \tag{3.2}$$

We call $A_i^v(P(C,i) \cup C_p)$ the group contribution of i from C_p with respect to C to make it short. Now we define the k-lateral value as follows.

Definition 1. The k-lateral value $\Phi^k : \mathcal{G}(N) \to \mathbb{R}^n$ is a value that assigns to every player $i \in N$ her average group contributions from each member C_p with respect to all the partitions $C \in \Pi(N,k)$, following all possible orders of entrance with the assumption that occurrence of each order of entrance has equal probability $\frac{1}{|\Pi(N,k)|}$. Formally we have,

$$\Phi_i^k(v) = \frac{1}{n!\alpha(n,k)} \sum_{\substack{C \in \Pi(N,k) \\ C_p \in C : i \in C_p}} A_i^v(P(C,i) \cup C_p) \tag{3.3}$$

Remark 1. Note that the contributions of the groups within a coalition described in Eq.(3.3) include the individual contributions of the player given by Eq.(2.1). This addresses the marginal prospects of $\Phi^k(v)$. Adding equal shares from the group contributions to the final payoff of a player prescribed by Φ^k gives an egalitarian flavour to the solution. Thus Φ^k brings a kind of solidarity into the model.

For our convenience, we take $\alpha(0,k) = 1$. Following standard derivations of $\alpha(s,k)$ for different combinations of the parameters s and k are important for the rest of the paper. The proofs of these results have been relegated to the appendix.

Proposition 2. For $S \subseteq N$, the quantity $\alpha(s,k)$ satisfies the following.

(a) For $s \geq k \geq 1$,

$$\alpha(s,k) = \sum_{r=1}^{s} \left\{ \binom{s-1}{r-1} + \sum_{i=1}^{\left[\frac{s-r}{k}\right]} (-1)^i \binom{r}{i} \binom{s-ik-1}{r-1} \right\} \tag{3.4}$$

(b) For $k = 1$ and all $s \geq 1$, $\alpha(s,k) = 1$.

(c) For $s \leq k$, $\alpha(s,k) = 2^{s-1}$.

(d) For $s > k$, we have

$$\sum_{t=1}^{k} \alpha(s-t,k) = \alpha(s,k) \tag{3.5}$$

(e) For $s \leq k$, $\sum_{t=1}^{s} \alpha(s-t,k) = \alpha(s,k)$.

Example 1. Let us take an example to illustrate the computational procedure of the k-lateral value described above. Take $N = \{1,2,3,4\}$ and $k = 2$. In view of Proposition 2, we have $\alpha(4,2) = 5$. Therefore there will be 5 different sequences of positive integers 1 and 2 (since $k = 2$ here) for each order. They are : $c^1 = \{1,1,1,1\}$, $c^2 = \{1,2,1\}$, $c^3 = \{1,1,2\}$, $c^4 = \{2,1,1\}$ and $c^5 = \{2,2\}$. There will be $n! = 4! = 24$ orders in which the players enter the room and form groups within coalitions. Consider in particular, the order given by $\pi_1 = \{1,2,3,4\}$. Then the pair (π_1, c^1) uniquely determines the partition $C^1_{\pi_1} = \{\{1\},\{2\},\{3\},\{4\}\}$. Similarly we get the other partitions with respect to the pairs (π_1,c^2), (π_1,c^3), (π_1,c^4) and (π_1,c^5) as $C^2_{\pi_1} = \{\{1\},\{2,3\},\{4\}\}$, $C^3_{\pi_1} = \{\{1\},\{2\},\{3,4\}\}$, $C^4_{\pi_1} = \{\{1,2\},\{3\},\{4\}\}$ and $C^4_{\pi_1} =$

$\{\{1,2\},\{3,4\}\}$ respectively. In Table 1, we identify the worths of the coalitions required for computing the group contributions with regard to each of the four partitions prescribed by order π_1.

	S_j/C_i	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
π_1	$C^1_{\pi_1}$	$v(\emptyset)$	$v(1)$	$\times$	$\times$	$\times$	$v(1,2)$	$\times$	$\times$	$\times$	$\times$	$\times$	$v(1,2,3)$	$\times$	$\times$	$\times$	$v(N)$
	$C^2_{\pi_1}$	$v(\emptyset)$	$v(1)$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$v(1,2,3)$	$\times$	$\times$	$\times$	$v(N)$
	$C^3_{\pi_1}$	$v(\emptyset)$	$v(1)$	$\times$	$\times$	$\times$	$v(1,2)$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$v(N)$
	$C^4_{\pi_1}$	$v(\emptyset)$	$\times$	$\times$	$\times$	$\times$	$v(1,2)$	$\times$	$\times$	$\times$	$\times$	$\times$	$v(1,2,3)$	$\times$	$\times$	$\times$	$v(N)$
	$C^5_{\pi_1}$	$v(\emptyset)$	$\times$	$\times$	$\times$	$\times$	$v(1,2)$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$v(N)$

Table 1: Coalitional worths required for the group contributions according to π_1

Table 2 refers to the shares from each of the group contributions made by the players prescribed by the partition π_1. Shares due to other orders can be obtained in a similar way.

		1	2	3	4
π_1	$C^1_{\pi_1}$	$v(1)-v(\emptyset)$	$v(1,2)-v(1)$	$v(1,2,3)-v(1,2)$	$v(1,2,3,4)-v(1,2,3)$
	$C^2_{\pi_1}$	$v(1)-v(\emptyset)$	$\frac{1}{2}[v(1,2,3)-v(1)]$	$\frac{1}{2}[v(1,2,3)-v(1)]$	$v(1,2,3,4)-v(1,2,3)$
	$C^3_{\pi_1}$	$v(1)-v(\emptyset)$	$v(1,2)-v(1)$	$\frac{1}{2}[v(1,2,3,4)-v(1,2)]$	$\frac{1}{2}[v(1,2,3,4)-v(1,2)]$
	$C^4_{\pi_1}$	$\frac{1}{2}[v(1,2)-v(\emptyset)]$	$\frac{1}{2}[v(1,2)-v(\emptyset)]$	$v(1,2,3)-v(1,2)$	$v(1,2,3,4)-v(1,2,3)$
	$C^4_{\pi_1}$	$\frac{1}{2}[v(1,2,3,4)-v(3,4)]$	$\frac{1}{2}[v(1,2,3,4)-v(3,4)]$	$\frac{1}{2}[v(1,2,3,4)-v(1,2)]$	$\frac{1}{2}[v(1,2,3,4)-v(1,2)]$

Table 2: Share of group contributions from $C^i_{\pi_1}$, $i \in N$.

Recall that in the computation of the Shapley value, each order π gives one set of marginal contributions of the players when they form the grand coalition according to π. Here we have 5 $(=\alpha(n,k))$ sets of alternative group contributions.

After using standard rules of combinatorics and Proposition 2, an equivalent expression of Eq.(3.3) is obtained as follows.

$$\Phi_i^k(v) = \sum_{S\subseteq N: i\in S} \sum_{\substack{T\subseteq S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} \{v(S)-v(S\setminus T)\} \tag{3.6}$$

Remark 2. Note that using the standard rules mentioned in Proposition 2 and following the counting procedure described above we observe the following.
Given $T \subseteq N$ such that $t \le k$, the probability of forming a coalition S such that $T \subseteq S$ is given by $\dfrac{(n-s)!(s-t)!t!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)}$. The average group contribution of player i from T is therefore given by $\frac{v(S)-v(S\setminus T)}{t}$. Now the expectation $E_i(v)$ of the average group contributions of $i \in N$ over all the coalitions S and all $T \subseteq S$ such that $i \in T$, $1 \le t \le k$ is

given by

$$E_i(v) = \sum_{S\subseteq N: i\in S} \sum_{\substack{T\subseteq S\,:\, i\in T\\ 1\le t\le k}} \frac{(n-s)!(s-t)!(t)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} \left\{\frac{v(S)-v(S\setminus T)}{t}\right\}$$

$$= \sum_{S\subseteq N: i\in S} \sum_{\substack{T\subseteq S\,:\, i\in T\\ 1\le t\le k}} \frac{(n-s)!(s-t)!(t-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} \left\{v(S)-v(S\setminus T)\right\},$$

It follows from Eq.(3.6) that $E_i(v) = \Phi_i^k(v)$. Thus Φ^k is the expectation of the average group contributions of player i due to game v.

4 Characterization

In this section, we follow the standard Shapley procedure to characterize our k-lateral value. First we show that the value satisfies *Eff*, *Sym* and *Lin*. We define two types of null players and accordingly define two alternative axioms on these null players, namely (kNP_1 and kNP_2). They replace the standard null player axiom *NP* of the Shapley value. We show that our value satisfies both these two null player axioms. For the converse part, i.e., to show that a value that satisfies *Eff*, *Sym*, *Lin* and kNP_1 or kNP_2 must be the k-lateral value, we adopt the following procedure. Due to *Lin* it is sufficient to define a basis for the class of games. Due to Symmetry, the k-lateral value gives equal shares to the members of the coalition on which the basis is defined and all the other players outside this coalition get zero payoffs following either of kNP_1 or kNP_2. It is then not hard to show that the k-lateral value is the unique value satisfying the aforementioned axioms.

Proposition 3. *The k-lateral value Φ^k with $k \geq 1$ satisfies Eff, Sym and Lin.*

Proof. We have from Eq.(3.6) the following.

$$\Phi_i^k(v) = \sum_{S\subseteq N: i\in S} \sum_{\substack{T\subseteq S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!)(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} \left\{v(S)-v(S\setminus T)\right\} \tag{4.1}$$

Rewrite Eq. (4.1) as follows.

$$\Phi_i^k(v) = \sum_{S\subseteq N: i\in N\setminus S} \sum_{\substack{T\subseteq N\setminus S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!)(n-s-t)!(s)!\alpha(s+t,k)\alpha(n-s-t,k)}{n!\alpha(n,k)} \{v(S\cup T)-v(S)\}$$

Therefore,

$$\sum_{i=1}^{n} \Phi_i^k(v) = \sum_{S \subseteq N} \sum_{i \in S} \sum_{\substack{T \subseteq S \,:\, i \in T \\ 1 \le t \le k}} \frac{(t-1)!(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} \{v(S) - v(S \setminus T)$$
$$- \sum_{S \subseteq N} \sum_{i \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S \,:\, i \in T \\ 1 \le t \le k}} \frac{(t-1)!)(n-s-t)!(s)!\alpha(s,k)\alpha(n-s-t,k)}{n!\alpha(n,k)} \{v(S \cup T) - v(S)\} \tag{4.2}$$

The coefficient of $v(N)$ in Eq. (4.2) is

$$\sum_{i \in N} \sum_{1 \le t \le k} \frac{(t-1)!(n-n)!(n-t)!\alpha(n-t,k)}{n!\alpha(n,k)} \binom{n-1}{t-1} = \sum_{i \in N} \frac{1}{n!\alpha(n,k)} \sum_{1 \le t \le k} \alpha(n-t,k)$$
$$= \sum_{i \in N} \frac{1}{n!\alpha(n,k)} \alpha(n,k) = 1$$

Suppose that $S \subsetneq N$. Then the coefficient of $v(S)$ in Eq.(4.2) is given by,

$$\sum_{i \in S} \sum_{\substack{T \subseteq S \,:\, i \in T \\ 1 \le t \le k}} \frac{(t-1)!(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} \binom{s-1}{t-1}$$
$$- \sum_{i \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S \,:\, i \in T \\ 1 \le t \le k}} \frac{(t-1)!)(n-s-t)!(s)!\alpha(s,k)\alpha(n-s-t,k)}{n!\alpha(n,k)} \binom{n-s-1}{t-1}$$
$$= \sum_{i \in S} \sum_{1 \le t \le \min\{k,s\}} \frac{(n-s)!(s-1)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)}$$
$$- \sum_{i \in N \setminus S} \sum_{1 \le t \le \min\{k,n-s\}} \frac{(n-s-1)!(s)!\alpha(s,k)\alpha(n-s-t,k)}{n!\alpha(n,k)}$$
$$= \sum_{i \in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)}{n!\alpha(n,k)} \sum_{1 \le t \le \min\{k,s\}} \alpha(s-t,k)$$
$$- \sum_{i \in N \setminus S} \frac{(n-s-1)!(s)!\alpha(s,k)}{n!\alpha(n,k)} \sum_{1 \le t \le \min\{k,n-s\}} \alpha(n-s-t,k)$$
$$= \sum_{i \in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)}{n!\alpha(n,k)} \alpha(s,k) - \sum_{i \in N \setminus S} \frac{(n-s-1)!(s)!\alpha(s,k)}{n!\alpha(n,k)} \alpha(n-s,k)$$
$$= s \frac{(n-s)!(s-1)!\alpha(n-s,k)}{n!\alpha(n,k)} \alpha(s,k) - (n-s) \frac{(n-s-1)!(s)!\alpha(s,k)}{n!\alpha(n,k)} \alpha(n-s,k)$$
$$= \frac{(n-s)!(s)!\alpha(n-s,k)}{n!\alpha(n,k)} \alpha(s,k) - (n-s) \frac{(n-s)!(s)!\alpha(s,k)}{n!\alpha(n,k)} \alpha(n-s,k)$$
$$= 0$$

It follows that $\sum_{i=1}^{n} \Phi_i^k(v) = v(N)$. □

The proof for showing that the k-lateral value is *Lin* and *Sym* goes exactly in the same way as that in the standard Shapley value characterization [22] and therefore it is omitted.

In view of Proposition 3, Φ^k is an ESL value. Therefore by Proposition 1, there exists a unique collection of real constants $B = \{b_s : s \in \{0, 1, 2, 3, ..., n\}\}$ with $b_0 = 0$ and $b_n = 1$ such that for every $v \in \mathcal{G}(N)$,

$$\Phi_i^k(v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} \Big\{ b_{s+1} v(S \cup i) - b_s v(S) \Big\} \tag{4.3}$$

Proposition 4. *The k-lateral value Φ^k with $k \geq 1$ is in the form Eq.(4.3) with the sequence of non negative real numbers $B = \{b_s : s \in 0, 1, 2, ..., n\}$ where $b_s = \dfrac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n,k)}$ for $s \geq 1$ and $b_0 = 0$.*

Proof. Rearranging the terms in Eq.(3.6) we obtain

$$\begin{aligned}
\Phi_i^k(v) &= \sum_{S \subseteq N : i \in S} \sum_{\substack{T \subseteq S \,:\, i \in T \\ 1 \leq t \leq k}} \frac{(n-s)!(s-t)!(t-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} \Big\{ v(S) - v(S \setminus T) \Big\} \\
&= \sum_{S \subseteq N : i \in S} \sum_{1 \leq t \leq \min\{k,s\}} \binom{s-1}{t-1} \frac{(n-s)!(s-t)!(t-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} v(S) \\
&\quad - \sum_{S \subseteq N : i \in S} \sum_{\substack{T \subseteq S \,:\, i \in T \\ 1 \leq t \leq k}} \frac{(n-s)!(s-t)!(t-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} v(S \setminus T)
\end{aligned} \tag{4.4}$$

Let $P = T \setminus i$, $Q = (S \setminus T) \cup i$. Then $i \in Q$, $Q \setminus i = S \setminus T$. Therefore $T \subset (N \setminus Q) \cup i \implies T \setminus i \subset N \setminus Q \implies P \subset N \setminus Q$. Again

$$\begin{aligned}
T &= P \cup i \\
S &= Q \cup T \cup i = Q \cup T = Q \cup P \cup i = Q \cup P \\
Q \cap P &= \{(S \setminus T) \cup i\} \cap (T \setminus i) = \emptyset \\
N \setminus S &= N \setminus (Q \cup P)
\end{aligned}$$

Therefore $p = t-1, s = q+p, n-s = n-q-p, s-t = q-1$. We have from Eq.(4.4) and

Proposition 2(d) and (e),

$$\Phi_i^k(v) = \sum_{S\subseteq N: i\in S} \sum_{1\le t\le \min\{k,s\}} \frac{(n-s)!(s-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} v(S)$$

$$- \sum_{Q\subseteq N: i\in Q} \sum_{\substack{P\subseteq N\setminus Q \\ 0\le p\le \min\{k-1,n-s\}}} \frac{(n-q-p)!(q-1)!p!\alpha(n-q-p,k)\alpha(q-1,k)}{n!\alpha(n,k)} v(Q\setminus i)$$

$$= \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)}{n!\alpha(n,k)} \sum_{1\le t\le \min\{k,s\}} \alpha(s-t,k)v(S)$$

$$- \sum_{Q\subseteq N: i\in Q} \sum_{0\le p\le \min\{k-1,n-q\}} \binom{n-q}{p} \frac{(n-q-p)!(q-1)!p!\alpha(n-q-p,k)\alpha(q-1,k)}{n!\alpha(n,k)} v(Q\setminus i)$$

$$= \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)\alpha(s,k)}{n!\alpha(n,k)} v(S)$$

$$- \sum_{Q\subseteq N: i\in Q} \sum_{0\le p\le \min\{k-1,n-q\}} \frac{(n-q)!(q-1)!\alpha(n-q-p,k)\alpha(q-1,k)}{n!\alpha(n,k)} v(Q\setminus i)$$

$$= \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)\alpha(s,k)}{n!\alpha(n,k)} v(S)$$

$$- \sum_{Q\subseteq N: i\in Q} \frac{(n-q)!(q-1)!\alpha(q-1,k)}{n!\alpha(n,k)} \sum_{1\le p+1\le \min\{k,n-q+1\}} \alpha(n-q-p,k)v(Q\setminus i)$$

$$= \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)\alpha(s,k)}{n!\alpha(n,k)} v(S)$$

$$- \sum_{Q\subseteq N: i\in Q} \frac{(n-q)!(q-1)!\alpha(q-1,k)\alpha(n-q+1,k)}{n!\alpha(n,k)} v(Q\setminus i)$$

$$= \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!}{n!} \left\{ \frac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n,k)} v(S) - \frac{\alpha(n-s+1,k)\alpha(s-1,k)}{\alpha(n,k)} v(S\setminus i) \right\} \tag{4.5}$$

Let $b_s = \frac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n,k)}$ for $s \ge 1$ and $b_0 = 0$. Then $B = \{b_s : s \in 0,1,2,...,n\}$ is a real sequence of non negative real numbers with $b_0 = 0$, $b_n = 1$ and the result follows. □

Remark 3. Eq.(4.5) provides an alternative representation of Φ^k. For $k = n$, $\alpha(s,n) = 2^{s-1}$ where $1 \le s \le n$ and $\alpha(0,n) = 1$. Let $b_s = \dfrac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n,k)}$ for $1 \le s \le n$. Therefore

$b_s = \dfrac{1}{2}$ for $k = n$, $1 \le s < n$ and $b_n = 1$. Take $b_0 = 0$. Then the n-lateral value becomes,

$$\begin{aligned}
\Phi_i^n(v) &= \sum_{S \subseteq N : i \in S} \frac{(n-s)!(s-1)!}{n!} \Big\{ b_s v(S) - b_{s-1} v(S \setminus i) \Big\} \\
&= \sum_{S \subsetneq N : i \in S} \frac{(n-s)!(s-1)!}{n!} \Big\{ \frac{1}{2} v(S) - \frac{1}{2} v(S \setminus i) \Big\} + \Big\{ \frac{v(N)}{n} - \frac{v(N \setminus i)}{2n} \Big\} \\
&= \sum_{S \subsetneq N : i \in S} \frac{(n-s)!(s-1)!}{n!} \Big\{ \frac{1}{2} v(S) - \frac{1}{2} v(S \setminus i) \Big\} + \Big\{ \frac{v(N)}{2n} - \frac{v(N \setminus i)}{2n} \Big\} + \frac{v(N)}{2n}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{S \subseteq N : i \in S} \frac{(n-s)!(s-1)!}{n!} \Big\{ v(S) - v(S \setminus i) \Big\} + \frac{v(N)}{2n} \\
&= \frac{1}{2} \Big\{ \Phi_i^{Sh}(v) + \Phi_i^{ED}(v) \Big\}
\end{aligned}$$

Therefore $\Phi^n = \dfrac{1}{2}\{\Phi^{Sh} + \Phi^{ED}\}$. It follows that when $k = n$, the value Φ^n divides the half of the worth of the grand coalition $v(N)$ equally among the players and the other half is divided among the players as par the Shapley value. This is indeed the α-egalitarian Shapley value first proposed by Joosten [14] and latter discussed in details by van den Brink et al., [26], with $\alpha = \frac{1}{2}$.

Remark 4. In view of Remark 3, we explore now, for any $k \in \{1, 2, ..., n\}$ if there exists a constant $\alpha_k \in [0, 1]$ such that $\alpha_k \Phi^{Sh} + (1 - \alpha_k)\Phi^{ED} = \Phi^k$. Note that for $S \subsetneq N$, $\Phi^{ED}(b_S) = 0$. Therefore $\alpha_k \Phi_i^{Sh}(b_S) = \Phi_i^k(b_S)$ for all $i \in N$ and $S \subsetneq N$. For any non empty subset $S \subsetneq N$ we have,

$$\Phi_i^{Sh}(b_S) = \begin{cases} \dfrac{(s-1)!(n-s)!}{n!} & \text{if } i \in S \\ -\dfrac{s!(n-s-1)!}{n!} & \text{otherwise} \end{cases}$$

and

$$\Phi_i^k(b_S) = \begin{cases} \dfrac{(s-1)!(n-s)!\alpha(s,k)\alpha(n-s,k)}{n!\alpha(n,k)} & \text{if } i \in S \\ -\dfrac{s!(n-s-1)!\alpha(s,k)\alpha(n-s)}{n!\alpha(n,k)} & \text{otherwise} \end{cases}$$

It follows that $\alpha_k = \dfrac{\alpha(s,k)\alpha(n-s)}{\alpha(n,k)}$ which depends on the size s of coalition S for $n > 3$. Therefore α_k cannot be a constant for $n > 3$. It follows that Φ^k is not a convex combination of the Shapley value and the Equal division rule for $1 < k < n$ and $n > 3$.

4.1 Two types of null-players

Recall that in our counting process, we allowed the players to enter one by one following a particular order but waited till a group of size no more than k had formed. We computed the contribution of this group which is then divided equally among the players. In this way we allowed the players to finally form the grand coalition. Based on the counting of the groups formed henceforth, we define two types of null players and their respective null player axioms as follows.

The k-null-player of Type-I

Let S be a coalition with $i \in S$. Consider the partition, $C = \{C_1, C_2, ..., C_{p-1}, C_p, C_{p+1}, ..., C_m\}$ of N with index$(C) \leq k$ such that $S = \cup_{j=1}^{p} C_j$. It follows that there is an order π_C of N such that the players in C enter according to π_C. Consider the subset $C_S = \{C_1, C_2, ..., C_{p-1}, C_p\}$ of C. C_S is a partition of S prescribed by π_C. The number of such partitions of S prescribed by any order is $\alpha(s, k)$. The total number of partitions of N prescribed by any order in which S is the union of first p members, $1 \leq p \leq s$ is therefore given by $\alpha(s, k)\alpha(n - s, k)$. Then following Eq.(3.2), the total group contribution of all the players in the group C_p when player i enters $S = \cup_{j=1}^{p} C_j$ in the last i.e., $\pi_C(s) = i$ with respect to C is given by $c_p A_i^v(P(C, i) \cup C_p)$.

Now define the k-lateral group contribution $M_i^{(S,k)}(v)$ of player i from the coalition S, by the average of the total group contributions of all the players in the group C_p when player i enters $S = \cup_{j=1}^{p} C_j$ in the last i.e., $\pi_C(s) = i$ with respect to all $C \in \Pi(N, k)$.
Formally we have,

$$M_i^{(S,k)}(v) = \frac{1}{(s-1)!\alpha(s,k)\alpha(n-s,k)} \sum_{\substack{C \in \Pi(N,k): \\ S = \cup_{q=1}^{p} C_q, \pi_C(s) = i}} c_p A_i^v(P(C, i) \cup C_p)$$

$$= \sum_{\substack{T \subset S: i \in T \\ 1 \leq t \leq k}} \frac{(s-t)!\alpha(s-t,k)(t-1)!}{(s-1)!\alpha(s,k)} \Big\{ v(S) - v(S \setminus T) \Big\}$$

Definition 2. Given $v \in \mathcal{G}(N)$, a player $i \in N$ is called a k-null player of type I or a k^1-null player in short, if her k-lateral group contributions $M_i^{(S,k)}(v) = 0$ for all coalitions S such that $i \in S$.

Observe that when $k = 1$, the 1^1-null player is the standard null player characterizing the Shapley value. Further we note that the k^1-null player and her group members with respect to each partition, on an average makes no contribution to the corresponding coalition. Thus the k^1-null player not only contributes nothing of her own on an average but also she forces her group members to keep their average contributions zero. Therefore, it is justified to award her zero payoff under the k-lateral value. In the following, We have the k^1-null player axiom or the kNP_1 in short.

Axiom 5. k^1-Null Player (kNP_1): For $v \in \mathcal{G}(N)$ and for any k^1-null player $i \in N$ of v, $\Phi_i^k(v) = 0$.

Proposition 5. *The k-lateral value Φ^k, $k \geq 1$ satisfies kNP_1.*

Proof. The proof is immediate from Eq.(3.6). □

The k-null player of Type-II

Recall that every sequence of positive integers $c = \{c_1, ..., c_p\}$ with $1 \leq p \leq s$ and $0 < c_p \leq k$ and the order π determine a partition $C_S^\pi = \{C_1^\pi, ..., C_p^\pi\}$ so that $S = \cup_{j=1}^p C_j^\pi$. Then $\alpha(s, k)$ is the number of such partitions of S according to the particular order π. The total number of partitions of N prescribed by π in which S is the union of first p members, $1 \leq p \leq s$ is therefore given by $\alpha(s, k)\alpha(n - s, k)$. Thus the probability that S is chosen from N with this property prescribed by a particular order π is given by $\frac{\alpha(s,k)\alpha(n-s,k)}{\alpha(n,k)}$. Let a random variable take the value $v(S) > 0$ when S is formed such that for some p with $1 \leq p \leq s$, S is the union of first p members of the partitions of index $\leq k$, prescribed by π and $v(S) = 0$ otherwise. Then the expectation that the random variable takes $v(S)$ when S is chosen from N according to the above rule is given by $\frac{\alpha(s,k)\alpha(n-s,k)}{\alpha(n,k)}v(S)$. Let us call it the expected k-lateral worth of S. Now fix an i from N. Find those S's of N in which $\pi(s) = i$, i.e., i is the last member to enter in S. Then the expected k-lateral worth of $S \setminus i$ is found to be $\frac{\alpha(s-1,k)\alpha(n-s+1,k)}{\alpha(n,k)}v(S \setminus i)$. Based on this formulation, we now define the k-null player of type II or in short the k^2-null player and the corresponding k^2-null player axiom or kNP_2 in short.

Definition 3. Given $v \in \mathcal{G}(N)$, a player $i \in N$ is said to be a k-null player of type II or a k^2-null player in short of v if for all coalitions S such that $i \in S$, the expected k-lateral worths of S and $S \setminus i$ are identical. Thus formally, $i \in N$ is a k^2-null player if for each $S \subseteq N$,

$$\frac{\alpha(s, k)\alpha(n - s, k)}{\alpha(n, k)}v(S) = \frac{\alpha(s - 1, k)\alpha(n - s + 1, k)}{\alpha(n, k)}v(S \setminus i) \tag{4.6}$$

It follows from Eq.(4.6) that for $k = 1$, the 1^2-null player becomes the null player. When $k > 1$, the k-null player contributes nothing to the coalitions on an average when both her individual (groups of size 1) and group contributions are measured. Therefore, it is justified to award the k^2-null player zero payoff under the k-lateral value. We have the following k^2-null player axiom or the kNP_2 in short.

Axiom 6. k^2-Null Player (kNP_2): For $v \in \mathcal{G}(N)$ and for any k^2-null player $i \in N$ of v, $\Phi_i^k(v) = 0$.

Proposition 6. *The k-lateral value Φ^k, $k \geq 1$ satisfies kNP_2.*

Proof. Immediately follows from the definition. □

Remark 5. Define a new game $\bar{v}$ on N as follows: $\bar{v} = \frac{\alpha(s,k)\alpha(n-s,k)}{\alpha(n,k)} v(S)$ for all $S \subseteq N$. Call it the associate game of v with respect to group contributions. Then we have $\Phi_i^k(v) = \Phi_i^{Sh}(\bar{v})$ for all $i \in N$. Therefore the k-lateral value over the game v is the Shapley value over its associate game $\bar{v}$. Furthermore, player i is a k^2-null player of game v if and only if i is a null player of $\bar{v}$.

Remark 6. It is interesting to note that the two types of null players defined above are neither equivalent nor they imply each other. Take for example, a game v on $N = \{1, 2, 3\}$ as follows. $v(\{1\}) = 0$, $v(\{2\}) = v(\{3\}) = 2$, $v(\{1, 2\}) = v(\{1, 3\}) = 1$, $v(\{2, 3\}) = 2$ and $v(\{1, 2, 3\}) = 2$. Here player 1 is a 2^1-null player of v but not a 2^2-null player of v. Consider another game w on $N = \{1, 2, 3\}$ as follows. $w(1) = 0$, $w(2) = w(3) = w(\{1, 2\}) = w(\{1, 3\}) = 1$, $w(\{2, 3\}) = 3$ and $w(\{1, 2, 3\}) = 4$. Here, player 1 is a 2^2-null player of w but not a 2^1-null player of w.

The Characterization Theorem

In this section, we prove a couple of characterization theorems for the k-lateral value using the axioms *Eff, Lin, Sym* and either of kNP_1 or kNP_2. In view of propositions 3 and 5, the characterization only requires to show that if a value satisfies *Eff, Lin, Sym* and either of kNP_1 or kNP_2 it must be given by Eq.(3.3) or equivalently Eq.(3.6). Both the proofs are constructive and we start with the introduction of a couple of new bases for the class $\mathcal{G}(N)$ of games. Every $v \in \mathcal{G}(N)$ is then expressed as a linear combination of these bases. Therefore, following *Lin* it will suffice to obtain the expression of the k-lateral value for these bases. Let us start with the axioms *Eff, Lin, Sym* or ESL in short and the axiom kNP_1. To complete the proofs following propositions are needed. For a non empty coalition $T \subseteq N$, define $D_T : 2^N \to \mathbb{R}$ such that

$$D_T(S) = \begin{cases} f(s,t), & \text{if } T \subsetneq S \\ 1, & \text{if } T = S \\ 0, & \text{otherwise} \end{cases} \tag{4.7}$$

where the value $f(s,t)$ of the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be so obtained that each $i \notin T$ is a k^1-null player. Thus under this assumption, we must have $M_i^{(S,k)}(D_T) = 0$ for all $S \subseteq N$, $i \in S$ and for each non empty coalition $i \notin T \subseteq N$. Therefore,

$$\sum_{\substack{M \subseteq S \,:\, i \in M \\ 1 \le m \le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} \{D_T(S) - D_T(S \setminus M)\} = 0 \tag{4.8}$$

Eq.(4.8) implies,

$$\sum_{\substack{M \subseteq S \,:\, i \in M \\ 1 \le m \le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} \{D_T(S)\}$$

$$= \sum_{\substack{M \subseteq S \,:\, i \in M \\ 1 \le m \le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} \{D_T(S \setminus M)\}$$

It follows that,

$$\sum_{1\le m\le k} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} \binom{s-1}{m-1} D_T(S)$$
$$= \sum_{\substack{M\subseteq S\,:\,i\in M\\ 1\le m\le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} D_T(S\setminus M)$$

Thus we have,

$$\begin{aligned} D_T(S) &= \sum_{\substack{M\subseteq S\,:\,i\in M\\ 1\le m\le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} D_T(S\setminus M) \\ f(s,t) &= \sum_{\substack{M\subseteq S\setminus T\,:\,i\in M\\ 1\le m\le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} D_T(S\setminus M) \\ &= \sum_{1\le m\le \min\{k,s-t\}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} \binom{s-t-1}{m-1} f(s-m,t) \\ &= \sum_{1\le m\le \min\{k,s-t\}} \frac{\alpha(s-m,k)}{\alpha(s,k)} \frac{\binom{s-t-1}{m-1} f(s-m,t)}{\binom{s-1}{m-1}} \end{aligned}$$

Using the above recursive relation of $f(s,t)$ with $f(s,s) = 1$ for all $1 \le s \le n$, we can find all the values of $f(s,t)$. It is easy to show that D_T so defined is a TU game. The proofs of Propositions 7 and 8 presented in the following are kept in the Appendix.

Proposition 7. *The set of games $\{D_T : T \subseteq N, T \neq \emptyset\}$ is a basis for $\mathcal{G}(N)$ and every player $i \notin T$ is a k^1-null player.*

Proposition 8. *For an ESL value Φ having kNP_1 and $T \subset N$, $\Phi(D_T)$ is uniquely determined by*

$$\Phi_i(D_T) = \begin{cases} \dfrac{f(n,t)}{t}, & \text{if } i \in T \\ 0, & \text{otherwise} \end{cases} \tag{4.9}$$

Theorem 1. *The k-lateral value Φ^k is the unique value that satisfies Eff, Lin, Sym and kNP_1.*

Proof. Since $\{D_T : T \subset N, T \neq \emptyset\}$ is a basis for $\mathcal{G}(N)$ therefore any game $v \in \mathcal{G}(N)$ can be expressed uniquely as $v = \sum_{T\subset N:T\neq\emptyset} \gamma_T^v D_T$ where $\gamma_T^v \in \mathbb{R} : T \subseteq N$. Since Φ^k is linear therefore $\Phi_i^k(v) = \sum_{T\subset N:T\neq\emptyset} \gamma_T^v \Phi_i^k(D_T)$. By Proposition (8), $\Phi_i^k(D_T)$ is uniquely determined by Eq.(7.3). This completes the proof. □

For the second characterization, we define the k-unanimity game denoted by $W_T : 2^N \to R$ for each non empty coalition $T \subseteq N$ in explicit form as follows.

$$W_T(S) = \begin{cases} \dfrac{\alpha(n-t,k)\alpha(t,k)}{\alpha(n-s,k)\alpha(s,k)}, \text{ if } T \subseteq S \\ \quad 0, \quad \text{otherwise} \end{cases} \tag{4.10}$$

Note that W_T is identical with the unanimity game for $k = 1$. For $T = S$, $W_T(T) = 1$.

Remark 7. For $T \neq \emptyset$, the game W_T possesses the following properties.

(a) $W_T(T) = 1$ for $T \subset N$.

(b) $W_T(S) = 0$ for $T \not\subset S$.

(c) $W_T(S) = \dfrac{\alpha(n-s+1,k)\alpha(s-1,k)}{\alpha(n-s,k)\alpha(s,k)} W_T(S \setminus i)$ for $T \subset S \setminus i$.

Proposition 9. *For an ESL value Φ having kNP_2 and $T \subset N$, $\Phi(W_T)$ is uniquely determined by*

$$\Phi_i(W_T) = \begin{cases} \dfrac{\alpha(n-t,k)\alpha(t,k)}{t\alpha(n,k)}, & \textit{if } i \in T \\ 0, & \textit{otherwise} \end{cases} \tag{4.11}$$

The proof of Proposition 9 is kept in the Appendix.

Theorem 2. *The k-lateral value Φ^k is the unique value that satisfies Eff, Lin, Sym and kNP_2.*

Proof. Consider the set $\{W_T | T \subseteq N, T \neq \emptyset\}$. By similar procedure as in Prop. 7, the set $\{W_T | T \subseteq N, T \neq \emptyset\}$ forms a basis of $\mathcal{G}(N)$. Any game $v \in \mathcal{G}(N)$ can be expressed uniquely as $v = \sum_{T \subset N : T \neq \emptyset} \gamma_T^v W_T$ where $\gamma_T^v = \sum_{S \subset T : S \neq \emptyset} (-1)^{t-s} \dfrac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n-t,k)\alpha(t,k)} v(S)$. Using the expression of γ_T^v we derive the following.

$$\begin{aligned}
\sum_{T \subset N : T \neq \emptyset} \gamma_T^v W_T(S) &= \sum_{T \subset S : T \neq \emptyset} \gamma_T^v \frac{\alpha(n-t,k)\alpha(t,k)}{\alpha(n-s,k)\alpha(s,k)} \\
&= \sum_{T \subset S : T \neq \emptyset} \frac{\alpha(n-t,k)\alpha(t,k)}{\alpha(n-s,k)\alpha(s,k)} \sum_{R \subset T : R \neq \emptyset} (-1)^{t-r} \frac{\alpha(n-r,k)\alpha(r,k)}{\alpha(n-t,k)\alpha(t,k)} v(R) \\
&= \sum_{T \subset S : T \neq \emptyset} \sum_{R \subset T : R \neq \emptyset} (-1)^{t-r} \frac{\alpha(n-r,k)\alpha(r,k)}{\alpha(n-s,k)\alpha(s,k)} v(R) \\
&= \sum_{R \subset S : R \neq \emptyset} \sum_{T \subset S : R \subset T} (-1)^{t-r} \frac{\alpha(n-r,k)\alpha(r,k)}{\alpha(n-s,k)\alpha(s,k)} v(R) \\
&= \sum_{R \subset S : R \neq \emptyset} \left\{ \sum_{T \subset S : R \subset T} (-1)^{t-r} \right\} \frac{\alpha(n-r,k)\alpha(r,k)}{\alpha(n-s,k)\alpha(s,k)} v(R) \\
&= v(S) + \sum_{R \subset S : R \neq \emptyset} \left\{ \sum_{t = r : s \neq r}^{s} (-1)^{t-r} \binom{s-r}{t-r} \right\} \frac{\alpha(n-r,k)\alpha(r,k)}{\alpha(n-s,k)\alpha(s,k)} v(R)
\end{aligned}$$

Since $\sum_{t=r:s\neq r}^{s}(-1)^{t-r}\binom{s-r}{t-r}=0$ therefore $\sum_{T\subset N:T\neq\emptyset}\gamma_T^v W_T(S)=v(S)$.

Since Φ^k is linear therefore $\Phi_i^k(v)=\sum_{T\subset N:T\neq\emptyset}\gamma_T^v\Phi_i^k(W_T)$. By Proposition (9), $\Phi_i^k(W_T)$ is uniquely determined by Eq.(4.11). Therefore $\Phi_i^k(v)$ is unique and determined by

$$\Phi_i^k(v)=\sum_{T\subset N:i\in T}\gamma_T^v\frac{\alpha(n-t,k)\alpha(t,k)}{t\alpha(n,k)}$$

□

Remark 8. Note that in the proof of Theorem 1, the basis D_T is obtained recursively while in Theorem 2, the basis W_T is expressed in a closed form to illustrate the procedure of obtaining the k-lateral value in an explicit way, however to show only the existence and uniqueness, such explicit forms are seemingly redundant.

4.2 Logical Independence

Logical independence of the axioms of Theorem 1 can be illustrated by the following examples.

(a) The value $\beta^k:\mathcal{G}(N)\to R^N$ given by

$$\beta_i^k(v)=\frac{1}{2^{|N|-1}}\sum_{\substack{S\subseteq N\\ :\,i\in S}}\sum_{\substack{T\subseteq S:\,i\in T\\ 1\leq t\leq k,}}\frac{(s-t)!(t-1)!\alpha(s-t,k)}{(s-1)!\alpha(s,k)}\Big\{v(S)-v(S\setminus T)\Big\}$$

for all $i\in N$ satisfies kNP_1, *Lin* and *Sym* but does not satisfy *Eff*.

(b) The value $\gamma^k:\mathcal{G}(N)\to R^N$ given by

$$\gamma_i^k(v)=\frac{1}{2^{|N|-1}}\sum_{\substack{S\subseteq N\\ :\,i\in S}}\left\{\frac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n,k)}v(S)-\frac{\alpha(n-s+1,k)\alpha(s-1,k)}{\alpha(n,k)}v(S\setminus i)\right\}$$

for all $i\in N$ satisfies kNP_2, *Lin* and *Sym* but does not satisfy *Eff*.

(c) The equal division rule Φ^{ED} satisfies *Eff*, *Sym* and *Lin* but it does neither satisfy kNP_1 nor kNP_2.

(d) The value $\bar{\Phi}^k:\mathcal{G}(N)\to R^N$ given by $\bar{\Phi}_i^k(v)=\dfrac{\beta_i^k(v)}{\sum_{j\in N}\beta_j^k(v)}v(N)$ $\left(\text{or } \bar{\Phi}_i^k(v)=\dfrac{\gamma_i^k(v)}{\sum_{j\in N}\gamma_j^k(v)}v(N)\right)$ for all $i\in N$ satisfies kNP_1 (or kNP_2), *Eff* and *Sym* but does not satisfy *Lin* if $\sum_{j\in N}\beta_j^k(v)\neq 0$ $\big(\text{or } \sum_{j\in N}\gamma_j^k(v)\neq 0\,\big)$.

(e) Consider the basis $\{D_T : T \subseteq N, T \neq \emptyset\}$ of the games defined by the Eq. (7.3) for the class $\mathcal{G}(N)$. Suppose that $i = \min_{j \in T} j$. Let β be a value such that $\beta_i(D_T) = D_T(N)$ and $\beta_j(D_T) = 0$ for $j \in N, j \neq i$. Extend β linearly for all games in $\mathcal{G}(N)$. β satisfies kNP_1, *Lin* and *Eff* but does not satisfy *Sym*.

(f) Consider the basis $\{W_T : T \subseteq N, T \neq \emptyset\}$ of the games defined by the Eq. (4.10) for the class $\mathcal{G}(N)$. Suppose that $i = \min_{j \in T} j$. Let γ be a value such that $\gamma_i(W_T) = W_T(N)$ and $\gamma_j(W_T) = 0$ for $j \in N, j \neq i$. Extend γ linearly for all games in $\mathcal{G}(N)$. γ satisfies kNP_2, *Lin* and *Eff* but does not satisfy *Sym*.

5 Examples

Let us take a numerical example to highlight the k-lateral interactions in a TU game and how they influence the k-lateral value for different choices of k. Take $N = \{1, 2, 3, 4\}$ and $v \in \mathcal{G}(N)$ as follows. $v(S) = 0$ if $\{1, 2\} \not\subseteq S$, $v(1, 2) = 2$, $v(1, 2, 3) = 4$, $v(1, 2, 4) = 6$ and $v(1, 2, 3, 4) = 8$. The k-lateral value $\Phi^k(v)$ for different choices of k including the Shapley value where $k = 1$ are given below.

$$\begin{aligned}
\Phi^1(v) &= (3.0, 3.0, 0.6, 1.4) = \Phi^{Sh}(v)\\
\Phi^2(v) &= (2.6, 2.6, 1.2, 1.6)\\
\Phi^3(v) &= (2.57, 2.57, 1.24, 1.62)\\
\Phi^4(v) &= (2.5, 2.5, 1.33, 1.67)
\end{aligned}$$

Observe that this is a special example where all the players are individually non-productive. Also neither player 1 nor 2 alone can have a non-zero contribution to a coalition. They are productive only when they are together in a coalition. Thus in this stylized example, we want to see how and why the marginal productivities of player 1 and 2 can be compensated by solidarity towards 3 and 4. The Shapley value considers the individual and unilateral contributions of 1 and 2, even though they are dependent on each other in generating the worth of the grand coalition. Such dependence among players in deciding to join or leave a coalition is not explicitly seen in Shapley formulations. Thus under the Shapley value their productivity is the highest. However, when we consider players' group contributions, with an increase of the size of the groups, sharing is more egalitarian. Therefore, more solidarity for player 3 and 4 is ensured as the group contributions are shared equally among all the players.

6 Conclusion

This paper proposes a new value for TU games – the k-lateral value – that considers group contributions of players within a coalition. All the Shapley like marginalistic values implicitly assume that players individually and independently decide to join or leave a coalition of their own. However, there are instances where players within a coalition are influenced by each other on making such decisions and finally they make collective decisions. Since a marginalistic value awards payoffs to the players based on their own contributions, their reliance on the others in generating the worth should be given due consideration. This led us to define the notion of group contributions. Our value computes the average of all such individual and group contributions of the players. By an example we have explained the difference between the Shapley value and the k-lateral values for different levels of interactions. The characterization of the new value is done using standard axioms of Efficiency, Symmetry, and Linearity along with two new axioms: the k-null player axioms of type I and type II. Similar formulations can be put forward to other Shapley like values which we keep for our future studies.

Acknowledgement

The work is supported by UKIERI [184-15/2017(IC)].

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7 Appendix

Proof of Proposition 2.

(a) By definition of $\mathcal{B}(n,k)$,

$$\mathcal{B}(s,k) = \{(x_1, x_2, ..., x_r) : x_1 + x_2 + ... + x_r = s, 1 \leq x_i \leq k, 1 \leq i \leq r \leq s\}$$

Therefore

$$\alpha(s,k) = \sum_{r=1}^{s} |\{(x_1, x_2, ..., x_r) : x_1 + x_2 + ... + x_r = s, 1 \leq x_i \leq k, 1 \leq i \leq r\}| \tag{7.1}$$

The number of positive integer solutions of $x_1+x_2+...+x_r=s$ is $\binom{s-1}{r-1}$. Suppose that i be the least number of variables with $x_j>k$. In this case, the number of positive solutions of $x_1+x_2+...+x_r=s$ is $\binom{r}{i}\binom{s-ik-1}{r-1}$.
Therefore the number of solutions of $x_1+x_2+...+x_r=s$ with $1\le x_i\le k, 1\le i\le r$ is

$$\binom{s-1}{r-1}+\sum_{i=1}^{\left[\frac{s-r}{k}\right]}(-1)^i\binom{r}{i}\binom{s-ik-1}{r-1}$$

Thus

$$\alpha(s,k)=\sum_{r=1}^{s}\left\{\binom{s-1}{r-1}+\sum_{i=1}^{\left[\frac{s-r}{k}\right]}(-1)^i\binom{r}{i}\binom{s-ik-1}{r-1}\right\}$$

□

(b) For $k=1$ and $r=s$, the only solution of $x_1+x_2+...+x_r=s$ is $(1,1,1,...,1)$. For $r<s$, there is no solution in $x_1+x_2+...+x_r=s$ with $k=1$. The result follows. □

(c) By (a),

$$\alpha(s,k)=\sum_{r=1}^{s}\{(x_1,x_2,...,x_r):x_1+x_2+...+x_r=s, 1\le x_i\le k, 1\le i\le r\}$$

Since $x_1+x_2+...+x_r=s$ therefore each $x_i\le s$ for all $1\le r\le s$.
Since $s\le k$, therefore for all $i\in\{1,2,...,s\}$ we must have $x_i\le k$.
It follows that,

$$\begin{aligned}\alpha(s,k)&=\sum_{r=1}^{s}\Big\{(x_1,x_2,...,x_r):x_1+x_2+...+x_r=s, 1\le x_i\le k, 1\le i\le r\Big\}\\&=\sum_{r=1}^{s}\Big\{(x_1,x_2,...,x_r):x_1+x_2+...+x_r=s, 1\le i\le r\Big\}\\&=\sum_{r=1}^{s}\binom{s-1}{r-1}\\&=2^{s-1}\end{aligned}$$

□

(d) Following Eq.(7.1) we have,

$$\alpha(s,k)=\sum_{r=1}^{s}\{(x_1,x_2,...,x_r):x_1+x_2+...+x_r=s, 1\le x_i\le k, 1\le i\le r\}.$$

Therefore,

$$\sum_{t=1}^{k}\alpha(s-t,k)=\sum_{i=1}^{k}\sum_{r=1}^{s-i}[(x_1,x_2,...,x_r):x_1+x_2+...+x_r=s-i, 1\le x_i\le k, 1\le i\le r] \tag{7.2}$$

Since $s > k$, we must have $r \geq 2$. Then

$$\begin{aligned}
\alpha(s,k) &= \sum_{r=2}^{s} \Big\{(x_1, x_2, ..., x_r) : x_1 + x_2 + ... + x_r = s, 1 \leq x_i \leq k, 1 \leq i \leq r\Big\} \\
&= \sum_{r=2}^{s} \Big\{(x_1, x_2, ..., x_r) : x_1 + x_2 + ... + x_{r-1} = s - x_r, 1 \leq x_i \leq k, 1 \leq i \leq r\Big\} \\
&= \sum_{r=2}^{s} \sum_{x_r=1}^{k} \Big\{(x_1, x_2, ..., x_{r-1}) : x_1 + x_2 + ... + x_r = s - x_r, 1 \leq x_i \leq k, 1 \leq i \leq r\Big\} \\
&= \sum_{r=1}^{s-1} \sum_{t=1}^{k} \Big\{(x_1, x_2, ..., x_r) : x_1 + x_2 + ... + x_r = s - t, 1 \leq x_i \leq k, 1 \leq i \leq r\Big\} \\
&= \sum_{t=1}^{k} \sum_{r=1}^{s-t} \Big\{(x_1, x_2, ..., x_r) : x_1 + x_2 + ... + x_r = s - t, 1 \leq x_i \leq k, 1 \leq i \leq r\Big\}
\end{aligned}$$

Note that, if $r > s$ then the equation $x_1 + x_2 + ... + x_r = s$ has no solution. Therefore,

$$\begin{aligned}
\alpha(s,k) &= \sum_{t=1}^{k} \sum_{r=1}^{s-t} \Big\{(x_1, x_2, ..., x_r) : x_1 + x_2 + ... + x_r = s - i, 1 \leq x_i \leq k, 1 \leq i \leq r\Big\} \\
&= \sum_{t=1}^{k} \alpha(s-t, k)
\end{aligned}$$

□

(e) Observe that $\alpha(0,k) = 1$ when $k > 0$ and for $1 \leq s \leq k$, $\alpha(s,k) = 2^{s-1}$. Therefore,

$$\begin{aligned}
\sum_{1 \leq t \leq s} \alpha(s-t, k) &= \sum_{1 \leq t \leq s-1} 2^{s-t-1} + \alpha(0,k) \\
&= \sum_{0 \leq t \leq s-2} 2^t + 1 \\
&= 2^{s-1} - 1 + 1 \\
&= 2^{s-1} \\
&= \alpha(s,k)
\end{aligned}$$

□

Proof of Proposition 7

By definition of D_T, each $i \in N \setminus T$ is a k^1-null player in the games D_T as the recursive relation is designed to achieve this objective.

Finally, we show that the set of games $\{D_T\}$ for all $T \subset N$, $T \neq \emptyset$ form a basis of $\mathcal{G}(N)$. Let $d = 2^n - 1$. Since the class of unanimity games $\{u_S | S \subseteq N, S \neq \emptyset\}$ makes a basis for the vector space $\mathcal{G}(N)$, therefore the dimension of $\mathcal{G}(N)$ is d. Let S_1, S_2, ..., S_d be a fixed sequence containing all non empty subsets of N such that $n = s_1 \leq s_2 \leq ... \leq s_d = 1$. Let $A = [a_{ij}]$ be the $d \times d$ matrix defined by $a_{ij} = D_{S_i}(S_j), i, j = 1, 2, 3, ..., d$. Then $a_{ii} = D_{S_i}(S_i) = 1$. For $i > j$, $s_i \geq s_j$. Then either $s_i = s_j$ or $s_i > s_j$. Suppose that $s_i = s_j$. Since $S_i \neq S_j$ therefore $S_i \not\subseteq S_j$. Let $s_i > s_j$. Then $S_i \not\subseteq S_j$. It follows that $a_{ij} = 0$ for $i > j$. Thus $A = [a_{ij}]$ is an upper triangular matrix with diagonal entries 1 meaning $det(A) = 1$. Therefore, the set $\{D_{S_i} : i = 1, 2, ..., d\}$ is comprised of d independent vectors in $\mathcal{G}(N)$. Since any linearly independent set containing d vectors form a basis of $\mathcal{G}(N)$ therefore $\{D_{S_i} | i = 1, 2, 3, ..., d\}$ forms a basis of $\mathcal{G}(N)$. □

Proof of Proposition 8

By *Eff*, we have $\sum_{i \in N} \Phi_i(D_T) = D_T(N) = f(n, t)$.

By Proposition 7, $i \in N \setminus T$ is a k^1-null player in the game D_T. Since Φ satisfies kNP_1 therefore $\Phi_i(D_T) = 0$ for $i \in N \setminus T$. Further, any two players $i, j \in T$ are symmetric which implies $\Phi_i(D_T) = \Phi_j(D_T)$. Thus we have,

$$\Phi_i(D_T) = \begin{cases} \dfrac{f(n,t)}{t}, & \text{if } i \in T \\ 0, & \text{otherwise} \end{cases} \tag{7.3}$$

This completes the proof. □

Proof of Proposition 9

By *Eff*, we have $\sum_{i \in N} \Phi_i(W_T) = W_T(N) = \dfrac{\alpha(n-t,k)\alpha(t,k)}{\alpha(n,k)}$.

For $i \in N \setminus T$, if $S \subset N \setminus i$ then $T \subset S \implies T \subset S \setminus i$. If $T \not\subset S$, $i \notin T$ then $T \not\subset S \setminus i$. Therefore $W_T(S) = W_T(S \setminus i) = 0$ for $T \not\subset S, i \notin T$. For $T \subset S, i \notin T$, by Remark [7], we have $W_T(S) = \dfrac{\alpha(n-s+1,k)\alpha(s-1,k)}{\alpha(n-s,k)\alpha(s,k)} W_T(S \setminus i)$. Therefore $i \in N \setminus T$ is a k-null player of type II in the k-unanimity game W_T. Since Φ satisfies kNP_2 therefore $\Phi_i(W_T) = 0$ for $i \in N \setminus T$. Any two players $i, j \in T$ are symmetric therefore $\Phi_i(W_T) = \Phi_j(W_T)$ for all $i, j \in T$. Thus

$$\Phi_i(W_T) = \begin{cases} \dfrac{\alpha(n-t,k)\alpha(t,k)}{t\alpha(n,k)}, & \text{if } i \in T \\ 0, & \text{otherwise} \end{cases}$$

□