Sequential Learning with Predecessor Choice

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Abstract

This work analyses a model of sequential learning with predecessor choice. There are two states of the world and two types of agents. The two types differ in their initial beliefs about the state of the world, and are identical otherwise. Agents can choose to observe one of their immediate predecessors, either of their own type or the opposite type. After choosing a predecessor, agents receive a private signal and learn the action of their chosen predecessor, as well as the action of their predecessor's predecessor, and so on. They then take their own action. We are primarily interested in the conditions under which agents prefer to observe their opposite type across generations, leading to assimilation between the two groups. We find that asymptotically agents will always prefer to observe their own type; however, early generations may choose to observe their opposite types. We derive conditions under which this occurs, and characterize decision rules specifying the action taken by these agents. Using simulations, we demonstrate that such kind of opposite-observing behaviour cannot continue forever – at some point agents switch to their own type – and discuss comparative statics for the timing of this switch.

1 Introduction

The way in which individuals are connected to each other plays a vital role in the information they receive and the opinions they form. Since agents make important decisions based on these opinions and information, the structure of ties between them becomes vital. For example, who people vote for, how much they invest in education, or whether they adopt a new technology are all influenced by the beliefs and actions of others around them. Learning from others is complicated because we can often observe only the final actions of others and not the information it was based on. In addition, one may face cost constraints and have to choose which agents to interact with. If agents hold different beliefs or have different preferences over outcomes, what their actions reveal about the information received by them will vary, and we may find it more useful to observe the actions of some kind of people over others. If agents form ties in order to gain information, what will the pattern of ties look like? Will two groups of people find it more useful to engage with each other, or will they keep to themselves?

Recent advancements in the technology of communication, especially the rise of social media, Jackson (2014) notes, "can lead to an increasingly dense network of interactions, but also could result in more segregated interactions, as it becomes easier to locate and stay in communication with others who have similar characteristics or interests." Thus, it is possible that agents over time have expanded the set of people they interact with, possibly making this set more diverse. At the same time, it has been noted that (especially in the virtual world), agents often end up in 'echo-chambers' or 'filter bubbles' where, either through preference or as a result of algorithmic recommender systems, they are much more likely to encounter opinions that are similar to their own, and receive information that confirms what they already believe (Halberstam & Knight, 2016; Sunstein 2001, 2009, 2017).

Homophily, the tendency of agents to be connected to people who are similar to them, has been frequently observed in social networks. The reasons for this observation may be related to preferences or biases in matching (Currarini, Jackson & Pin, 2009) but could also result from other factors. As in Kets & Sandroni (2016), agents could face lower strategic uncertainty when coordinating with people who are similar to them. Similarly, norms existing in a group may make it easier to select on an equilibrium in games with multiple equilibria.

When agents limit their interactions to specific groups, it becomes more likely that we will observe different outcomes over these groups. With the internet and social media rising in importance as news sources, and with the possibility that agents engage in 'motivated reasoning'- for example giving undue weight to their initial positions - homophily may lead not only to persistent disagreement among agents but also to increased polarization of opinions (Goel, Dandekar & Lee, 2013; Halberstam & Knight, 2016).
In this context, it becomes important to consider the reasons why agents may choose to interact with those similar to them, and how, in general, agents from different groups interact with each other in order to gain information, form opinions and take actions.

This works builds on the specific model of sequential learning in Bikhchandani, Hirshleifer & Welch (1992). We assume that agents, instead of being identical, belong to two groups who have different prior beliefs. These prior beliefs prescribe different optimal actions. We introduce an 'uninformative' signal on which agents act according to their prior beliefs, representing the possibility of not receiving any new information, and thus being an intuitive addition. As in Bikhchandani, Hirshleifer & Welch (1992), agents of both types sequentially choose their actions based on their private information and the actions of agents before them. However, we admit endogeneity of information acquisition as well as costly information in a simplistic way, by assuming that (i) agents may choose which type of predecessor they wish to form a link with, and (ii) they can only observe *one* of their immediate predecessors, who then also reveals information about previous generations.

Consider a motivating example. Suppose people belong to one of two different castes and that one person from each caste is nominated to be a leader. The rest of the agents must choose which candidate to support. All agents would like to support the better candidate, regardless of his caste. However, to begin with, agents expect the person from their own caste to be better, and would thus support him in the absence of any other information. Any new information they receive is rationally incorporated in their decision-making. Suppose agents sequentially choose which candidate to support. An agent may approach individuals who came before her to find out their decision, but suppose she only has the time to approach one person. This person also passes on information about the actions of previous generations. In addition, the agent receives a private signal which may favour one of the candidates, but could also offer no new information. Which type of person should an individual approach for information? Will an agent find it more useful to observe the action of someone from her own caste or the other caste? Can we have a situation where these two groups *always* prefer to observe each other?

Formally, our model considers two states of the world and two types of agents, who have different prior beliefs but are identical otherwise. The type of each agent is common knowledge. Agents have to choose one of two actions, each being optimal in one given state. There are countably infinite generations of agents who arrive sequentially, with one agent of each type in a given period. Agents can choose to observe *one* of their immediate predecessors, either of their own type or the opposite type. After choosing a predecessor, agents observe the action of their chosen predecessor. They also observe the action of their predecessor's chosen predecessor, and so on, up to the first generation. In addition, agents receive a private signal, which may suggest one of the two states as being more likely, but could also be uninformative. Agents then choose their own actions. Depending on the true state, they receive their payoffs.

The questions we ask are the following: how do agents form links in this setting? Would an agent rather have her immediate predecessor be similar to her, or dissimilar? What will this choice depend on, and how will it vary across generations? Can it be possible that agents always wish to observe dissimilar individuals? Under what conditions do agents form cross-group links for longer periods of time?

The answers to these questions are important for a number of reasons. One, how people form links with each other is interesting in itself from a network formation perspective. Two, it is possible that the sequence of these (individual) predecessor choices is not optimal from the point of view of information aggregation. This may invite the possibility of a social planner choosing to mandate that people of these different types interact with each other in a certain way. Finally, the predecessors that agents choose will determine their likelihood of taking different actions, and thus determine the kind of learning outcomes we are likely to see.

Our results are as follows. We find that, with two signals, each suggesting one of the states as being more likely, agents are indifferent between observing predecessors of their own type and the opposite type. When the uninformative signal is introduced,

this indifference no longer necessarily holds. We find that asymptotically, agents always prefer to observe their same type, but early generations may prefer to observe the opposite type. We derive conditions under which this occurs. The preference for opposite type is driven by the possibility of receiving an inconsistent signal. When an agent receives her inconsistent signal, she prefers to observe the opposite type. This is because, if they take the agent's default action, their action will reveal more precise information about their signal. This is the case the agent is most concerned with, that she will be unable to take her default action because the information received from her predecessor was is 'muddy'. (The very fact that Type X is more likely than type Y to take action A, makes the observation of action A by type X less revealing of the signal received by him.) We next characterise decision rules specifying the action taken by agents for case where opposite types are observed initially. Interestingly, unless the agent is on his prior, the last two actions he observes are sufficient to predict his optimal action at a given signal. Under parameter restrictions on the prior beliefs, we find that predecessor choice depends on the conditional probabilities of private signals and varies for each generation. Finally, using simulations of the model we confirm that some generation eventually switches in the case where we start with opposite-observing behaviour, and discuss comparative statics for the timing of this switch.
This work attempts to demonstrate, in the context of social learning, that even unbiased agents may choose to interact with others similar to them, in this case because doing so provides an informational advantage.

2 Literature Review

An extensive literature, starting with Bikhchandani, Hirshleifer & Welch (1992) and Banerjee (1992), covers models of observational learning where agents move sequentially. In such models, agents use private signals and the actions of others to learn about some underlying state of the world, in order to choose their own action. Learning generally occurs through the Bayesian updation of agents' beliefs about the state of the world. There are several interesting questions that have been asked in this setting. For example, with enough number of people arriving down the line, will agents eventually figure out the true state of the world, i.e. will 'asymptotic learning' occur? A related question concerns convergence. Do agents eventually act in the same way or end up with the same beliefs?

In Bikhchandani, Hirshleifer & Welch (1992) and Banerjee (1992), with discrete signals and agents observing actions of all agents before them, asymptotic learning fails. Bikhchandani, Hirshleifer & Welch (1992) define the concept of an information cascade, a situation where agents rationally choose to ignore their own private signal and follow the action taken by their predecessors. With homogeneous agents, one agent ignoring her signal would trigger everybody after her to do the same, and further signal information would not be incorporated in decision-making. Bikhchandani, Hirshleifer & Welch (1992) show that the probability of being in a cascade approaches 1 as the number of individuals becomes very large. Banerjee (1992), independently studying a similar model, finds that agents may exhibit 'herd behaviour' and (rationally) take the same action as people before them, thereby inflicting a negative externality on later individuals who don't learn anything from their actions.

Generalising the above models, Smith and Sørensen (2000) show that in sequential learning models with discrete signals, learning will always fail, because there will exist action histories that no signal is strong enough to overcome. However, when signals are continuous and private beliefs are "unbounded"[1], they show that asymptotic learning goes through, with belief convergence implying action convergence when agents are identical.

The possibility of information cascades arises in our model for similar reasons as in Bikhchandani, Hirshleifer & Welch (1992). This may be understood in the context of Smith and Sørensen (2000) by noting that, since we have assumed discrete signals, private beliefs will be bounded. Thus, asymptotic learning will fail. We note the following. One, there may be situations where discrete signals are a more realistic assumption than continuous signals. For example, bond ratings or the educational institution attended by a prospective job candidate are discrete signals. Two, as noted by Acemoglu and Ozdaglar (2011) "useful models of learning should not always predict consensus, and certainly not the weeding out of incorrect beliefs. Instead, these should merely be possible outcomes among others".

Instead of assuming that agents observe all previous actions, more recent papers have applied social learning models to networks, assuming more realistically that agents only observe the actions of some subset of their previous generations (their 'neighbourhood'). Acemoglu, Dahleh, Lobel, and Ozdaglar (2011), for example, characterize the conditions required for asymptotic learning on social networks under Bayesian learning. They find that in addition to unbounded private beliefs, the network structure must have the property of 'expanding observations', meaning roughly that there should be a minimum amount of new information arriving in the network. There has also been a development of literature on "non-Bayesian" social learning. Learning in these papers is generally communication-based, with individuals repeatedly exchanging their beliefs or opinions with each other and updating them in the process. The most commonly used rule is based on linear updation which, following DeGroot (1974), assumes that an agent's final belief in a given period is a weighted average of her neighbours' beliefs in the previous period and her own belief in the previous period updated after observing a private signal in this period. While these updation methods are most often employed in situations where Bayesian calculations would be too complex (e.g. when neighbourhoods are generated randomly), they are also consistent with the findings of experimental evidence. Anderson and Holt (1997), for example, find a considerable fraction of subjects deviating from Bayesian preferences, especially when rules of thumb (such as counting signals) are available.

Papers such as Bikhchandani, Hirshleifer & Welch (1992) which assume identical

[1] Private beliefs are 'unbounded' if they have full support in [0,1]. This corresponds to signals having no 'maximum informativeness'.

agents attempt to explain conformity or other such properties within groups, rather than within society at large. The heterogeneity among agents has often been considered explicitly, for example in Smith and Sørensen (2000), who work with multiple (private) types of rational agents having different preferences, as well as "crazy" individuals who have a strict preference for one action. They find that, among other possibilities, these groups may end up in different type-specific cascades. Confounded learning may also arise, where agents take different actions with the same (positive) probability in all states.

Heterogeneity among individuals may be modelled in different ways. Sethi and Yildiz (2016), for example, in the context of agents' choice to observe others over repeated interactions, consider heterogeneity by assuming that agents have differing prior beliefs, which are unobserved, and which they explain as arising due to some subjective 'perspectives' of these individuals. Banerjee and Somanathan (2001), on the other hand, assume *observable* heterogeneous priors, with the goal of understanding information aggregation when agents share opinions strategically.
Sethi and Yildiz (2016) discuss the trade-off between observing agents who are better informed (receive more precise signals) versus agents who are better understood (perspectives are better known) and find that a wide variety of observation patterns may arise, including the possibility of the population being partitioned into different groups, with individuals observing only those belonging to their own group. In a recent paper, Sethi & Yildiz (2019) extend the framework used in Sethi & Yildiz (2016) to address questions about homophily in their model. Agents face the same trade-off as before, however, agents here belong to one of two groups, with prior beliefs being correlated within a group, and no information being available, to begin with, on the prior beliefs of agents outside one's group. it is interesting to consider, in particular, the results of Sethi and Yildiz (2019) for the case where priors are perfectly correlated within a group and uncertainty about the other group is low. This is the closest case to our model. They find that cultures merge in the long run– agents learn all perspectives, and choose the best-informed individual as their target regardless of group membership. In contrast, our model predicts segregation in the long run, with agents preferring to observe their own type. Of course, these results are not directly comparable - in the current work, for example, no agent is a priori better informed than any other agent, and prior beliefs of the other group are perfectly known. Under the case where uncertainty about the other group is high, Sethi & Yildiz (2019) find that agents exhibit in-breeding homophily. For moderately high and low levels of correlation, heterogeneity in behaviour is observed.
Banerjee and Somanathan (2001), in the context of information aggregation under strategic opinion sharing, find that communication between those with extreme and opposed views is only possible if there is no possibility of false reporting, or if the group is relatively homogeneous. Their results suggest that there may be advantages in communication if the agents interacting are not too dissimilar.

Heterogeneity may alternatively be modelled as a differences in preferences. Lobel and Sadler (2016) consider a sequential learning model where, even though all agents match actions to states in an identical way, the weights assigned to the errors in the two states are different across individuals. They introduce homophily by assuming that the structure of links in the network is correlated with individual preferences.

The point of interest in studying heterogeneous agents is to understand the differences in the amount or quality of information derived from similar versus dissimilar people, as well the implication for information aggregation. The extent of heterogeneity will be an important factor in this context. Bikchandani, Hershliefer and Welch (1998) note that even if individual preferences are not completely opposing, the presence of uncertainty about predecessors could slow the rate of learning. In a slightly contrasting result, Golub and Jackson (2012) find that with communication based-learning and average-based updating, the speed of learning is slower when there is homophily in the social network.

Lobel and Sadler (2016) note that while the actions of dissimilar others are less informative, this leads individuals to pay more attention to their own signal, which may be desirable for information aggregation. They find that in sparse networks, heterogeneity introduces additional noise and may act as a barrier to information aggregation. The presence of homophily is then desirable because it 'rescues' the improvement principle[2]. By contrast, in dense networks, even though hetereogeneous preferences lead to less informative actions, the independence among these (and the fact that there are many of them) may be useful for learning.

We consider heterogeneous agents in this work by assuming that the two types of agents differ in their prior beliefs but have identical preferences. As discussed by Banerjee and Somanathan (2001), heterogenous prior beliefs with identical preferences can often be alternatively considered as identical priors with heterogeneous preferences. This holds true in our case. We assume that the types of individuals are observable. When types are unobservable, the problem of interest concerns the way in which individuals derive information from others, being unsure about their type. By assuming observed types, we instead ask the following question: when individuals are completely aware of which agents are similar to them and which are dissimilar, who would they choose to get information from?

While the literature on social learning has largely considered observation links to be either given or randomly generated, recent work has looked at endogenous observation of predecessors (Ali, 2018; Kultti and Miettinen, 2006,2007; Song, 2016) by introducing a cost of information. The focus of these papers, however, has been on the impact on learning, in particular the conditions under which complete learning can be attained. The choices of agents themselves, especially with regard to types, has not received attention. Agents in Kultti and Miettinen (2006, 2007) and Song (2016) are identical; while agents in Ali (2018) are heterogeneous with respect to the cost of acquiring information, interaction between similar versus dissimilar agents has not been discussed (and may not be interesting in this case). The focus of Lobel & Sadler (2016) too, while they specifically consider agents with differing preferences, is on assessing the impact of homophily on

[2]The improvement principle provides insight into asymptotic learning and is based on the idea that an agent can always do better than his neighbours. By imitating a neighbour, the agent guarantees himself their expected payoff, which can be improved upon using his private information. This principle may break down in the presence of heterogeneous preferences (Golub and Sadler, 2017).

learning, which they begin by assuming. An important distinction of this work lies in the fact that we are interested here in *how* agents will form links in the context of social learning, when there is some notion of cost of information involved, as well as heterogeneity among agents. Since agents don't have any inherent behavioural preference for similar or opposite types, they will choose to form links with others based on the informativeness of their actions. Understanding this process and its implications is the purpose of this work.

3 The Model

There are two states of the world, denoted by $\theta \in \{0, 1\}$ and two types of agents, denoted by $t \in \{X, Y\}$. The state θ is unknown, and types X and Y differ in their prior beliefs about the true state of the world. The prior belief of type t is denoted by q_t and represents the probability assigned to the state $\theta = 1$ by t. We assume that $q_Y = 1 - q_X$[3] with $q_X > \frac{1}{2}$.

Countably infinite generations (N) of agents arrive sequentially, one of each type in a given period. Agents are indexed by nt, where n stands for generation and t for type.

An agent nt has to take one of two actions, $a \in \{A, R\}$, where A corresponds to 'accept' and R to 'reject'. Let a^{nt} denote the action taken by agent nt. Based on this action and the underlying state, agent nt receive the payoff $U(a^{nt} \,|\, \theta)$. We assume that each action is optimal in exactly *one* state, with A being the preferred action when $\theta = 1$ and R being preferred otherwise:

$$\begin{aligned} &U(R \,|\, \theta = 0) = U(R \,|\, \theta = 1) = 0 \\ &U(A \,|\, \theta = 0) = -1 \\ &U(A \,|\, \theta = 1) = 1 \end{aligned}$$

Note that $U(A \,|\, \theta)$ and $U(R \,|\, \theta)$ are independent of the identity of the agent taking these actions, and that the penalty of taking the wrong action is identical across states.

Agents have two sources of information- (i) they receive a private signal, (ii) they observe the actions of some subset of the agents that came before them.

Private signals: Private signals are denoted by $s \in S$ with $S = \{s_L, s_N, s_H\}$. An agent may receive one of three possible signals - a low signal (s_L) suggests that the "low" state i.e. $\theta = 0$ is more likely to be the true state, a "no information" signal (s_N), referred to as the uninformative signal, suggests that both states are equally likely, and a high signal (s_H), suggests that $\theta = 1$ is more likely. We therefore call s_L (s_H) the "correct" signal in state $\theta = 0$ ($\theta = 1$) and the "incorrect" signal in state $\theta = 1$ ($\theta = 0$). Signals have the following conditional probability distribution :

[3]While this is a strong assumption, it is useful in simplifying the analysis. It may be noted that the results of Theorem 1 will be robust to qY not being exactly equal to $1 - q_X$ as long as q_Y satisfies the 'mirror image' constraint of Assumption 1 i.e. $q_Y > \max\left\{\frac{p_2(1-p_2)}{p_1(1-p_1)+p_2(1-p_2)}, \frac{p_1(1-p_1)^2}{p_1(1-p_1)^2+p_2(1-p_2)^2}\right\}$.

	$\theta = 0$	$\theta = 1$
s_L	p_1	p_2
s_N	p_3	p_3
s_H	p_2	p_1

with $p_1 + p_2 + p_3 = 1$. For a given θ, p_1 denotes the probability of receiving the correct signal, p_2 denotes the probability of receiving the incorrect signal and p_3 denotes the probability of receiving the uninformative signal[4].

We assume that $p_1 > p_2$ so that signals are informative by the monotone likelihood ratio property. Additionally, we have the following definitions:

Definition 1 (Consistent State)**.** *An agent's consistent state is that state on which the agent places a higher initial belief.*

Definition 2 (Consistent and Inconsistent Signals)**.** *The correct signal in the agent's consistent state is defined as her consistent signal. The incorrect signal for this state is called her inconsistent signal.*

An agent's consistent signal 'agrees with' her prior. It is clear that the signals s_L and s_H will constitute consistent and inconsistent signals for all agents.

Definition 3 (Default and Contrary Actions)**.** *The optimal action for an agent based (only) on her prior beliefs is called her default action. The remaining action in $\{A, R\}$ is called her contrary action.*

We denote the default action of type t by $a^{\text{def}}(t)$.

We thus have the following:

Type	Consistent State	Consistent Signal	Default Action
X	$\theta = 1$	s_H	A
Y	$\theta = 0$	s_L	R

Social Information: An agent can observe the actions of a subset of the agents who came before her, with *some* control over this subset. In particular, the agent nt can form a link with (i.e. choose to observe) *one* person of the previous generation. We denote the "predecessor choice" of agent nt by $\mu^{nt} \in \{(n-1)X, (n-1)Y\}$.
When she chooses μ^{nt}, nt is able to view not only this agent's action, but also the action of the agent with whom μ^{nt} formed a link, and so on, all the way back to the first generation. Thus, if $3X$ chooses $2X$ who chose $1Y$, then the fourth generation on observing $3X$ will be able to see all three actions a^{1Y}, a^{2X} and a^{3X}. If an agent is indifferent between

[4]One may alternatively say that there are only two signals s_L and s_H and think of p_3 as the probability of not receiving a signal.

observing either of her predecessors, we assume that she observe her own type[5].

The sequence of events is as follows. The first generation, having no history to observe (and thus no predecessor choice to make), take actions based on their signals. Subsequent generations on arrival must first choose their predecessors. Note that choosing a predecessor essentially means choosing a particular *kind* of history to observe. There are two branches starting from the first generation where each subsequent agent is connected to their chosen predecessor. The problem of choosing a predecessor is then the problem of deciding which of these branches to attach oneself to.
Upon choosing a predecessor, agents receive their signal and can view the actions taken by the agents in their chosen branch. Based on this information, they choose which action to take. Then, given their action and the true state θ, they receive their payoff.

The sequence of actions of previous generations which an agent might observe, along with the identities of the people taking those actions, constitute a 'history'. Denote a history upto generation n by

$$h_n = \left\{a^{it_i}\right\}_{i=1}^{n}$$

with

$$t_n \in \{X, Y\} \quad \text{and} \quad t_{i-1} = \mu^{it_i} \text{ for } i = 2, ..., n-1$$

where $t_i \in \{X, Y\}$ represents the type of the i^{th} generation agent in the given history.
For example, $h_3 = R^{1Y}, A^{2X}, R^{3X}$ denotes one possible history upto the third generation. This history gives us the following information: one, it tells us that $3X$ chose to observe $2X$ and $2X$ chose to observe $1Y$ and two, it gives us the actions taken by these agents. We denote the set of all possible histories upto the n^{th} generation by H_n.

We will particularly be interested in histories where agents choose to observe their opposite types e.g. $2X$ chooses $1Y$, $3Y$ chooses $2X$, $4X$ chooses $3Y$ and so on.
Let $H_n^O \subset H_n$ (where O stands for 'opposite') denote the set of all possible histories upto generation n where each agent it for $i = 2, ..., n$ chooses a predecessor of the opposite type. We will refer to such a history as a "history of opposites".
Denote by $H_n^O(X)$ the set of those histories in H_n^O where the last i.e. n^{th} agent is of type X. Similarly define $H_n^O(Y)$. Clearly, $H_n^O = H_n^O(X) \cup H_n^O(Y)$ and $H_n^O(X) \cap H_n^O(Y) = \phi$.
Note that, given n, the identity of every agent in *any* history belonging to the sets $H_n^O(X)$ and $H_n^O(Y)$ is fixed. For example, if $n = 4$, then the agents in any history $h_4 \in H_4^O(X)$ are necessarily $1Y, 2X, 3Y, 4X$.

Suppose that agent $(n+1)t$ observes some signal s and some history h_n. Her posterior belief (on the state $\theta = 1$) at the information set (s, h_n) is then given by $r^{(n+1)t}(s, h_n)$

[5]We assume that the tie-breaking rule does not randomize between predecessors in order to avoid complicated histories where both agents of a generation have observed the same predecessor. Although we assume here than an indifferent agent observes her own type, we can replace this by a rule choosing the opposite type without affecting the result in Theorem 1 (a weak inequality will replace the strong inequality in equation 4.7).

and is calculated using Bayes' rule:

$$r^{(n+1)X}(s, h_n) = \frac{q_X\,\mathrm{P}(s \,|\, \theta = 1)\,\mathrm{P}(h_n \,|\, \theta = 1)}{q_X\,\mathrm{P}(s \,|\, \theta = 1)\,\mathrm{P}(h_n \,|\, \theta = 1) + (1 - q_X)\,\mathrm{P}(s \,|\, \theta = 0)\,\mathrm{P}(h_n \,|\, \theta = 0)}$$

This can be rewritten in the following way:

$$\frac{r^{(n+1)X}(s, h_n)}{1 - r^{(n+1)X}(s, h_n)} = \frac{q_X\ \mathrm{P}(s \,|\, \theta = 1)\ \mathrm{P}(h_n \,|\, \theta = 1)}{(1 - q_X)\ \mathrm{P}(s \,|\, \theta = 0)\ \mathrm{P}(h_n \,|\, \theta = 0)}$$

Agent $(n+1)t$ is said to be "on her prior" after seeing (s, h_n) if her posterior belief $r^{(n+1)t}(s, h_n)$ is exactly equal to her prior belief q_t. It is clear that if one of $(n+1)X$ and $(n+1)Y$ is on prior at (s, h_n), then so is the other one.

Note that the predecessor choice of an agent is not based on any private information. In addition, as discussed above, the tie-breaking rule is deterministic. Using common knowledge of rationality, it is thus easy to see that an agent nt can figure out the predecessor choices of all generations before her. Thus, when agent $(n+1)X$ is choosing whether to observe nX or nY, the predecessor choices of generations upto n are known to her. Given these predecessor choices and the state θ, let the probability of $(n+1)X$ being on his prior when he observes nt be

$$\mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, \theta\,,\, nt\Big) = \sum_{s \,\in\, S}\ \sum_{h_n \,\in\, H} 1_{r^{(n+1)X}(s,h_n) = q_X}\,.\,\mathrm{P}(s \,|\, \theta)\,.\,\mathrm{P}(h_n \,|\, \theta)$$

where $1_{r^{(n+1)X}(s,h_n)=q_X}$ is an indicator function taking value 1 when $(n+1)X$'s posterior belief at (s, h_n) is equal to q_X, and 0 otherwise, and some set H is the set of all the possible histories that $(n+1)X$ can see when he chooses to observe agent nt.
If $(n+1)X$ is on his prior at (s, h_n), then

$$\begin{aligned} \mathrm{P}(s \,|\, \theta = 0)\,.\,\mathrm{P}(h_n \,|\, \theta = 0) \ &= \ \mathrm{P}(s \,|\, \theta = 1)\,.\,\mathrm{P}(h_n \,|\, \theta = 1) \\ \Rightarrow \quad \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, \theta = 0\,,\, nt\Big) \ &= \ \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, \theta = 1\,,\, nt\Big) \end{aligned}$$

Thus, given any state θ, we may denote by $\mathrm{P}(r^{(n+1)X} = q_X \,|\, nt\,)$ the probability of $(n+1)X$ being on his prior when he observes nt.

Similarly, for $s_1 \in S$,

$$\mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, \theta\,,\, s_1,\, nt\Big) = \mathrm{P}(s_1 \,|\, \theta)\,.\, \sum_{h_n \,\in\, H} 1_{r^{(n+1)X}(s_1,h_n) = q_X}\,.\,\mathrm{P}(h_n \,|\, \theta)$$

Let $a^{(n+1)t}(s, h_n)$ denote the action that agent $(n+1)t$ would take on the information set (s, h_n). We call this her 'optimal' action given s and h_n. We assume that agent $(n+1)t$ chooses between the actions A and R in the following way:

$$a^{(n+1)t}(s, h_n) = \begin{cases} R & \text{if } \ r^{(n+1)t}(s, h_n) < \frac{1}{2} \\ a^{\mathrm{def}}(t) & \text{if } \ r^{(n+1)t}(s, h_n) = \frac{1}{2} \\ A & \text{if } \ r^{(n+1)t}(s, h_n) > \frac{1}{2} \end{cases}$$

We denote by $S(a^{(n+1)t}\,|\,h_n)$ the set of signals for which, given the history h_n, agent $(n+1)t$ takes action $a^{(n+1)t}$. We call this the implied signal set of agent $(n+1)t$ given this history and her action. Given h_n we call $S(R^{(n+1)t}\,|\,h_n)$ agent $(n+1)t$'s 'rejection set', and $S(A^{(n+1)t}\,|\,h_n)$ her 'acceptance set'. These sets are mutually exclusive and exhaustive subsets of the signal set S. Thus we say that $(n+1)t$ creates a 'partition' of her signal set given h_n. Note that writing the set $S(A^{(n+1)t}|h_n)$ is sufficient for describing this partition.

For any given history, we can derive the implied signal set of each agent in that history. Thus, the history can alternatively be written as a sequence of subsets of the signal set S. Consider an example. Suppose $2X$ observes $1X$ and we have the history $R^{1X}R^{2X}$ with $S(R^{1X}) = \{s_L\}$ and $S(R^{2X}\,|\,R^{1X}) = \{s_L, s_N\}$. Then we say that $R^{1X}R^{2X} \Rightarrow \{s_L\}\times\{s_L, s_N\}$. Similarly, $R^{1X}A^{2X} \Rightarrow \{s_L\}\times\{s_H\}$. Since s_L and s_H cancel each other out, $R^{1X}A^{2X}$ provides no net information and we may say in addition that $R^{1X}A^{2X} \Rightarrow \phi$.

Let $\mathrm{P}(a^{(n+1)X}\,|\,\theta,\, nt\,)$ denote the probability that agent $(n+1)X$ observing agent nt takes action a, where $a \in \{A, R\}$, given state θ. $\mathrm{P}(a^{(n+1)X}\,|\,\theta\,,\, nt)$ is then the probability of being at those information sets (out of whichever information sets are feasible when observing nt) where $(n+1)X$ takes action a.
Note that when agent $(n+1)t$ is deciding between observing nX and nY, the predecessor choices upto generation n are already fixed. Thus, $\mathrm{P}(a^{(n+1)X}\,|\,\theta\,,\, nt)$ is defined for a *given* sequence of predecessor choices upto generation n.
In cases where the predecessor choice of agent $(n+1)X$ is already given or is not the point of discussion, we use the notation $\mathrm{P}(a^{(n+1)X}\,|\,\theta)$.

Let $EU^{(n+1)X}(nt)$ denote the expected payoff of agent $(n+1)X$ when observing agent nt, where $t \in \{X, Y\}$. $EU^{(n+1)Y}(nt)$ is similarly defined. Since their choice must be made *before* their own signal is realised or they are able to see the history of actions in their chosen branch, it is this payoff that the $(n+1)^{th}$ generation compares for $t = X$ and $t = Y$ when choosing who to observe. For example, for $t = Y$, we have

$$\begin{aligned}
EU^{(n+1)X}(nY) &= q_X\,.\,\mathrm{P}(A^{(n+1)X}\,|\,\theta=1, nY\,)\,.\,U(A\,|\,\theta=1)\\
&\quad +\ (1-q_X)\,.\,\mathrm{P}(A^{(n+1)X}\,|\,\theta=0, nY\,)\,.\,U(A\,|\,\theta=0)\\
&\quad +\ q_X\,.\,\mathrm{P}(R^{(n+1)X}\,|\,\theta=1, nY\,)\,.\,U(R\,|\,\theta=1)\\
&\quad +\ (1-q_X)\,.\,\mathrm{P}(R^{(n+1)X}\,|\,\theta=0, nY\,)\,.\,U(R\,|\,\theta=0)\\
&= q_X\,.\,\mathrm{P}^{(n+1)X}(A\,|\,\theta=1, nY\,)\ -\ (1-q_X)\,.\,\mathrm{P}^{(n+1)X}(A\,|\,\theta=0, nY\,)
\end{aligned}$$

Let $\mathrm{P}\Big(a_1^{nt_1}\, a_2^{(n+1)t_2}\,\Big|\,\theta\Big)$, where $a_1, a_2 \in \{A, R\}$ and $t_1, t_2 \in \{X, Y\}$, refer to the joint probability that agent nt_1 takes action a_1 and agent $(n+1)t_2$, *who chooses to observe* nt_1, takes the action a_2. Thus,

$$\mathrm{P}\Big(a_1^{nt_1}\, a_2^{(n+1)t_2}\,\Big|\,\theta\Big) = \mathrm{P}(a_1^{nt_1}\,|\,\theta\,)\,.\,\mathrm{P}(a_2^{(n+1)t_2}\,|\,\theta,\, nt_1,\, a_1^{nt_1}\,)$$

Notice that there is a considerable amount of symmetry in this model. For example, s_L and s_H together cancel each other out– they are mirror images in this sense. Similarly, for any history of opposites in the set $H_n^O(X)$ there will be a corresponding mirror image history in the set $H_n^O(Y)$. The priors of the two types are also mirror images around $\frac{1}{2}$. We introduce the following notation in order to be able to exploit this symmetry.

Loosely speaking, a hat over a notation signifies that we are referring to its mirror image. To see this more precisely in the case of private signals, consider the involution defined by $\hat{s}$:

Let $\hat{s} = f(s) : S \to S$ be such that

$$f(s_L) = s_H, \qquad f(s_N) = s_N, \qquad f(s_H) = s_L$$

For a given θ, we use $\hat{\theta}$ to refer to the remaining state in $\{0, 1\}$. Similarly, we have, for actions, $\hat{A} = R$ and $\hat{R} = A$, and for types, $\hat{X} = Y$ and $\hat{Y} = X$.

For a history h_n, the mirror image history $\hat{h}_n$ contains, for each action a^{it} in h_n $(i = 1, ..., n)$, the action $\hat{a}^{i\hat{t}}$ in its place, i.e. both the actions and identities of agents are reversed. For example, the mirror image of the history $h_3 = A^{1X}R^{2Y}R^{3Y}$ is $\hat{h}_3 = R^{1Y}A^{2X}A^{3X}$.

Implied signal sets will also have mirror images. For $S_1 \subset S$, we have $\hat{S}_1 := \{\hat{s} \mid s \in S_1\}$. For example, if $S_1 = \{s_L, s_N\}$, then $\hat{S}_1 = \{s_N, s_H\}$. It is then easy to see that, similar to the way in which s_L and s_H cancel each other out, the sets $\{s_L, s_N\}$ and $\{s_N, s_H\}$ will also cancel each other out.

4 Analysis

The first generation takes their action based on their private signal. At the uninformative signal, it is clear that $1X$ and $1Y$ will both take their default actions. It follows that they take their default action at their consistent signal as well.

If these agents were to take their default action at their inconsistent signal, it would mean that signals don't really matter, not just for them but also for subsequent generations, who will be in a similar position as the first generation. Moreover, predecessor choice will not be of any importance. Thus, in order to ensure that signals matter and predecessor choice is relevant, we have the following assumption:

$$q_X < \frac{p_1}{p_1 + p_2} \tag{4.1}$$

(4.1) implies that $1X$ rejects on s_L and by symmetry, that $1Y$ accepts on s_H. Thus, the first generation take their contrary action at their inconsistent signal and $1X$ and $1Y$ partition their signal sets in the following way:

$$S(A^{1X}) = \{s_N, s_H\}, \quad S(A^{1Y}) = \{s_H\}$$

We also make the following assumption:

Assumption 1. $q_X < \min\left\{ \frac{p_1(1-p_1)}{p_1(1-p_1) + p_2(1-p_2)}, \frac{p_2(1-p_2)^2}{p_1(1-p_1)^2 + p_2(1-p_2)^2} \right\}$

If we think of $\{s_L, s_N\}$ as a 'weak low signal' and of $\{s_N, s_H\}$ as a 'weak high signal', then Assumption 1 describes conditions defining the strength of a 'weak' signal relative to the standard signal.
To be precise, Assumption 1 implies the following actions for any agent $(n+1)t$,

$$\text{if } s^{(n+1)t} = s_L \text{ and } h_n \Rightarrow \{s_N, s_H\}, \qquad \text{then R}$$
$$\text{if } s^{(n+1)t} = s_L \text{ and } h_n \Rightarrow \{s_N, s_H\} \times \{s_N, s_H\}, \text{ then A}$$

The first statement implies that, given q_X and $\frac{p_1}{p_2}$, the probability of getting the uninformative signal p_3 is high enough that type X rejects at $\{s_L\} \times \{s_N, s_H\}$ even though he would have accepted at $\{s_L\} \times \{s_H\}$. Similarly, the second statement implies that given q_X and $\frac{p_1}{p_2}$, p_3 is low enough that type Y accepts at $\{s_L\} \times \{s_N, s_H\} \times \{s_N, s_H\}$ just like he would have at $\{s_L\} \times \{s_H\} \times \{s_H\}$.
Thus, Assumption 1 states that, for either type of agent, (a) one 'weak' signal is not strong enough to overcome a 'standard' signal in the opposite direction, and (b) two 'weak' signals are strong enough for this task.

Note that since $q_X > \frac{1}{2}$, if $q_X < \frac{p_2(1-p_2)^2}{p_1(1-p_1)^2 + p_2(1-p_2)^2}$, we must have

$$p_2\,(1-p_2)^2 \;>\; p_1\,(1-p_1)^2$$
$$\Rightarrow \quad \frac{1-p_2}{(1-p_1)\;+\;(1-p_2)} > \frac{p_1\,(1-p_1)}{p_1\,(1-p_1)\;+\;p_2\,(1-p_2)}$$

Thus Assumption 1 implies the following:

$$q_X < \frac{p_1\,(1-p_1)}{p_1\,(1-p_1)\;+\;p_2\,(1-p_2)} \tag{4.2}$$

$$q_X < \frac{1-p_2}{(1-p_1)\;+\;(1-p_2)} \tag{4.3}$$

$$q_X < \frac{p_2(1-p_2)^2}{p_1(1-p_1)^2 + p_2(1-p_2)^2} \tag{4.4}$$

Note also that (4.1) automatically holds when Assumption 1 is satified.

4.1 Results

We begin by noting that, asymptotically, agents in this model always prefer to observe their own type. Let us see why. Consider some generation n. If n is sufficiently large, we know that given the structure of the model, the probability of being in an informational cascade will be very close to one (Bikchandani, Hershleifer & Welsch, 1992; Lee 1993). Now suppose that this generation n has chosen to observe their own type. With probability close to one, agents nX and nY are in an information cascade, which means that their actions convey no information to the next generation. Thus, the predecessor choice problem for generation $(n+1)$ is identical to that of generation n, and so $(n+1)$ will also prefer to connect to the branch where generation $(n-1)$ is of the same type. This means that $(n+1)$ would choose to observe the *same* type in its previous generation. Now suppose that generation n chose to observe the opposite type. By a similar argument as above, the next generation i.e. $(n+1)$ would also like to attach itself to the branch where the type of generation $(n-1)$ is opposite to her own type. However, this means that in the immediately previous generation, $(n+1)$ should choose to observe the *same* type. Thus, generation $(n+1)$ always chooses to observe it's own type. Using the first part of the argument again, it is easy to see that all later generations will also prefer to observe their same type.

The predecessor choices of agents in general, however, are not as predictable. We find that agents will often prefer to form a link with their opposite type, and this preference is not necessarily short-lived across generations. In order to arrive at the predecessor choice of any agent $(n+1)t$, one may, given the sequence of predecessor choices up to generation n, calculate all information sets agent $(n+1)t$ may be on, find her posterior belief and optimal action for each of these, and thus calculate and compare her expected payoff from observing a predecessor of type X with one of type Y. However, it would be useful to find some simple rules describing how agent $(n+1)t$ will behave at different information sets, bypassing the need for repeated calculations of posterior beliefs. The following lemma helps us do so.

Lemma 1 (Decision rules with three signals). *For $n \geq 2$, suppose that all generations up to n have observed their opposite types. Let agent $(n+1)t'$, where $t' \in \{X, Y\}$, receive the private signal $s \in S$ and observe some history $h_n \in H_n^O$. If Assumption 1 is satisfied, then the following hold:*

1. *(a) $r^{(n+1)X}(s, h_n) \notin \left[\frac{1}{2}, q_X\right)$, $\forall s \in S$ and $h_n \in H_n^O$*
   (b) $r^{(n+1)Y}(s, h_n) \notin \left(q_Y, \frac{1}{2}\right]$, $\forall s \in S$ and $h_n \in H_n^O$
2. *$(n+1)t'$ decides between A and R using the following "decision rules":*
   - *(i) On receiving her* consistent *signal, if $(n+1)t'$ is on her prior, she takes her default action. If she is not on her prior, she takes her contrary action if the last two actions in h_n are her contrary actions, and otherwise takes her default action.*
   - *(ii) On receiving the* uninformative *signal, if $(n+1)t'$ is on her prior, she takes her default action. If she is not on her prior, $(n+1)t'$ follows (i.e. takes the same action as) her predecessor.*
   - *(iii) On receiving her* inconsistent *signal, $(n+1)t'$ follows the latest agent of the opposite type in h_n.*
3. *(a) If $S(A^{(n+1)}X \mid h_n) = \phi$, then*

$$\frac{P(h_n \mid \theta = 1)}{P(h_n \mid \theta = 0)} \in \left\{ \frac{p_2\,(1-p_1)}{p_1\,(1-p_2)}, \frac{(1-p_1)^2}{(1-p_2)^2} \right\}$$

(b) If $S(A^{(n+1)Y} \,|\, h_n) = S$*, then*

$$\frac{P(h_n \,|\, \theta = 1)}{P(h_n \,|\, \theta = 0)} \in \left\{ \frac{p_1\,(1-p_2)}{p_2\,(1-p_1)}, \frac{(1-p_2)^2}{(1-p_1)^2} \right\}$$

Proof. Appendix A. ■

Given a history of opposites, Lemma 1 is helpful in two ways. One, as discussed in Corollary 1 below, it tells us that under the given assumptions, agents of type X and Y generally agree in their actions when receiving the same information, disagreeing only when this information is effectively null. Two, it provides some useful rules describing the actions of a given generation in terms of actions taken by previous generations.
Let us note a few collorаries of Lemma 1. Since these follow almost immediately from Lemma 1, their proofs are omitted.

Corollary 1. *Let* $h_n \in H_n^O$*.*

(a) If the agent $(n+1)t$*, for* $t \in \{X, Y\}$*, is on her prior at some signal* s_1*, then for any* $s < s_1$*, she will take action* R *at* (s, h_n) *and for any* $s > s_1$*, she will take action* A *at* (s, h_n)*. At* (s_1, h_n)*, she takes her default action.*

(b) Consider any $s \in S$*.*
If $r^{(n+1)X}(s, h_n) \neq q_X$*, then* $a^{(n+1)X}(s, h_n) = a^{(n+1)Y}(s, h_n)$ *i.e. both* $(n+1)X$ *and* $(n+1)Y$ *take the same action at* (s, h_n)*.*
If $r^{(n+1)X}(s, h_n) = q_X$*, both agents take their default actions.*

Corollary 1 allows us, for any given history, to fully determine the acceptance and rejection sets of an agent who is on prior at some signal in S, and notes that the two types of agents, if they observe the same history of opposites, take differing actions only when they are on prior. This implies the following: For $t \in \{X, Y\}$, and $h_n \in H_n^O(t)$,

$$\begin{aligned} \mathrm{P}\Big(A^{(n+1)X} \,\Big|\, \theta, nt\,\Big) &= \mathrm{P}\Big(A^{(n+1)Y} \,\Big|\, \theta, nt\,\Big) + \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, \theta, nt\Big) \\ \mathrm{P}\Big(R^{(n+1)X} \,\Big|\, \theta, nt\,\Big) &= \mathrm{P}\Big(R^{(n+1)Y} \,\Big|\, \theta, nt\,\Big) - \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, \theta, nt\Big) \end{aligned} \tag{4.5}$$

Corollary 2. *Let* $t \in \{X, Y\}$*. If* $h_n \in H_n^O(\hat{t})$ *and* $S(A^{(n+1)t} \,|\, h_n) = S(A^{1t})$*, then* $r^{(n+1)t}(s_N, h_n) = q_t$*.*

This result states that if, given a history of opposites, an agent partitions her signal set in the same way that the first generation of her type partitioned their own signal set, then this agent will be on prior at s_N (given this history) just like the first generation was (at the null history).

Corollary 3. *Let* $t \in \{X, Y\}$ *and* $h_n \in H_n^O(t)$*.*

1. *If a^{nt} is type t's default action, then the $(n+1)^{th}$ generation cannot be on prior at t's consistent signal given h_n.*

2. *If a^{nt} is type t's contrary action, then the $(n+1)^{th}$ generation cannot be on prior at t's inconsistent signal or the uninformative signal.*

Corollary 3 describes situations where an agent can not be on her prior, and implies the following: For $h_n \in H_n^O(Y)$,

$$\begin{aligned} \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, s_L,\, nY,\, R^{nY}\Big) = 0 \\ \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, s_N,\, nY,\, A^{nY}\Big) = 0 \\ \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, s_H,\, nY,\, A^{nY}\Big) = 0 \end{aligned} \tag{4.6}$$

Corollary 4. *Let $t \in \{X, Y\}$. If $h_n \in H_n^O(\hat{t})$ and the last action in h_n i.e. $a^{n\hat{t}}$ is type t's default action, then $(n+1)t$ on seeing h_n takes his default action for all signals in S. Therefore,*

$$\mathrm{P}(A^{nY}\, R^{(n+1)X} \,|\, \theta\,) = 0$$
$$\mathrm{P}(R^{nX}\, A^{(n+1)Y} \,|\, \theta\,) = 0$$

Corollary 4 implies that when an agent $(n+1)t$ observes a history of opposites where the last agent is of his opposite type $(\hat{t})$, and this agent has already "switched over" i.e. taken his own contrary (and type t's default) action, then $(n+1)t$ will never "switch over" himself.

Corollary 5. *Let $t \in \{X, Y\}$ and $h_n \in H_n^O$. If $(n+1)t$ receives her inconsistent signal and the last action in h_n is her contrary action, then $(n+1)t$ will take her contrary action.*

Corollary 5 notes that for an agent receiving her inconsistent signal, an (immediate) predecessor taking the agent's contrary action is sufficient to make her do so as well, given a history of opposites.

The following theorem describes the predecessor choice of the second generation given Assumption 1 and describes the condition required for a history of opposites to arise up to some generation $(n+1)$.

Theorem 1 (Predecessor choice with three signals)**.** *Suppose Assumption 1 holds. The second generation then strictly prefers to observe their opposite type. Any subsequent generation $(n+1)$ (for $n \geq 2$) will also observe opposite type if all previous generations have observed their opposite type and*

$$P\Big(A^{(n+1)X} \,\Big|\, \theta = 1,\, nY\Big) > P\Big(R^{(n+1)X} \,\Big|\, \theta = 0,\, nY\Big) + P\Big(r^{(n+1)X} = q_X \,\Big|\, nY\Big) \tag{4.7}$$

with the above probabilities given by Propositions 1 and 2.

Proof. For second generation:
The second generation agents when choosing between $1X$ and $1Y$ consider all possible

	Information Set (I)	$\mathrm{P}(I \mid \theta = 0)$	$\mathrm{P}(I \mid \theta = 1)$	a^{2X}	a^{2Y}
s_L, R^{1X}	$\{s_L\} \times \{s_L\}$	p_1^2	p_2^2	R	R
s_L, A^{1X}	$\{s_L\} \times \{s_N, s_H\}$	$p_1\,(1-p_1)$	$p_2\,(1-p_2)$	R	R
s_N, R^{1X}	$\{s_N\} \times \{s_L\}$	$p_1\,p_3$	$p_2\,p_3$	R	R
s_N, A^{1X}	$\{s_N\} \times \{s_N, s_H\}$	$p_3\,(1-p_1)$	$p_3\,(1-p_2)$	A	A
s_H, R^{1X}	$\{s_H\} \times \{s_L\}$	$p_1\,p_2$	$p_1\,p_2$	A	R
s_H, A^{1X}	$\{s_H\} \times \{s_N, s_H\}$	$p_2(1-p_1)$	$p_1\,(1-p_2)$	A	A

Table 1: Observing $1X$

information sets they may encounter to calculate their optimal action for each case, as done below.

Observing 1X: Recall that $R^{1X} \Rightarrow \{s_L\}$ and $A^{1X} \Rightarrow \{s_N, s_H\}$.

In the given table, a^{2X} refers to $a^{2X}(s, h_1)$ where h_1 refers to a given history upto (involving) 1st generation X. For example, if the second generation sees the private signal s_L and observes action R taken by $1X$, her posterior belief $r^{2X}(s_L, R^{1X})$ is given by

$$\frac{r^{2X}(s_L, R^{1X})}{1 - r^{2X}(s_L, R^{1X})} = \frac{q_X}{1-q_X}\,\frac{\mathrm{P}(s_L|\theta=1)}{\mathrm{P}(s_L|\theta=0)}\,\frac{\mathrm{P}(R^{1X}|\theta=1)}{\mathrm{P}(R^{1X}|\theta=0)}$$
$$= \frac{q_X}{1-q_X}\,\frac{p_2^2}{p_1^2}$$

We then compare $r^{2X}(s_L, R^{1X})$ to $\frac{1}{2}$ to determine $a^{2X}(s_L, R^{1X})$.

$$r^{2X}(s_L, R^{1X}) \lesseqgtr \frac{1}{2}$$
$$\frac{q_X}{1-q_X} \lesseqgtr \frac{p_1^2}{p_2^2} > \frac{p_1}{p_2}$$

Using the assumption in (4.1), we have $r^{2X}(s_L, R^{1X}) < \frac{1}{2}$ and thus $a^{2X}(s_L, R^{1X}) = R$. The calculation for $2Y$ is done in a similar way. In this manner, one can calculate $a^{2X}(s, h_1)$ and $a^{2X}(s, h_1)$ for all possible combinations of s and h_1, as done in the last two columns.

This exercise is repeated for the case where the second generation observes $1Y$.

Observing 1Y: Recall that $R^{1Y} \Rightarrow \{s_L, s_N\}$ and $A^{1Y} \Rightarrow \{s_H\}$.

Note that in certain cases, we need the help of assumptions in order to determine what a^{2X} or a^{2Y} will be. For example, we need Assumption 1 to get $r^{2Y}(s_H, R^{1Y}) > \frac{1}{2}$.

The 2nd generation then calculates their expected payoff from observing $1X$ and $1Y$:

$$EU^{2X}(1X) = q_X \,.\, \mathrm{P}(A^{2X} \mid \theta = 1,\, 1X) \;-\; (1-q_X) \,.\, \mathrm{P}(A^{2X} \mid \theta = 0,\, 1X)$$

$$
\begin{aligned}
&= q_X \,.\, [\,\mathrm{P}(A^{2X} \,|\, \theta = 1,\, 1X) \;+\; \mathrm{P}^{\,2X}(A \,|\, \theta = 0,\, 1X)\,] \\
&\quad -\; \mathrm{P}(A^{2X} \,|\, \theta = 0,\, 1X) \\
&= q_X \Big\{ [\,p_3\,(1-p_2) \;+\; p_1\,p_2 \;+\; p_1\,(1-p_2)\,] \\
&\qquad +\; [\,p_3\,(1-p_1) \;+\; p_1\,p_2 \;+\; p_2\,(1-p_1)\,] \Big\} \\
&\quad -\; [\,p_3\,(1-p_1) \;+\; p_1\,p_2 \;+\; p_2\,(1-p_1)\,]
\end{aligned}
$$

Similarly, one may write $EU^{2X}(1Y)$ to get

$$
EU^{2X}(1X) - EU^{2X}(1Y) = \; (2q_X - 1) \Big\{ p_3^2 - p_1\,p_2 \Big\}
$$

From Assumption 1 we have $p_2\,(1-p_2)^2 > p_1\,(1-p_1)^2$. Since $p_1 > p_2$, this is equivalent to $p_3^2 \,<\, p_1\,p_2$. Since $q_X > \frac{1}{2}$ we have $EU^{2X}(1X) < EU^{2X}(1Y)$ and thus $2X$ chooses $1Y$. A similar argument shows that $2Y$ will choose $1X$.

For subsequent generations:
Suppose some generation $(n+1)$, for $n > 2$, is trying to decide which predecessor they should observe, given that up to generation n, everybody has chosen opposite types. Consider the problem from the perspective of agent $(n+1)X$ (the problem for $(n+1)Y$ will be symmetric). If $(n+1)X$ chooses nX, then

$$
\begin{aligned}
EU^{(n+1)X}(nX) &= q_X \,.\, \mathrm{P}(A^{\,(n+1)X} \,|\, \theta = 1,\, nX) \;-\; (1-q_X)\,.\, \mathrm{P}(A^{\,(n+1)X} \,|\, \theta = 0,\, nX) \\
&= q_X \,.\, \Big[\,\mathrm{P}(A^{\,(n+1)X} \,|\, \theta = 1,\, nX) \;+\; \mathrm{P}(A^{\,(n+1)X} \,|\, \theta = 0,\, nX)\Big] \\
&\quad -\; \mathrm{P}(A^{\,(n+1)X} \,|\, \theta = 0,\, nX)
\end{aligned}
$$

Similarly deriving $EU^{(n+1)X}(nY)$, we get

$$
\begin{aligned}
&EU^{(n+1)X}(nX) - EU^{(n+1)X}(nY) \\
&\quad = q_X \bigg[\mathrm{P}\Big(A^{\,(n+1)X} \,\Big|\, \theta = 1,\, nX\Big) - \mathrm{P}\Big(A^{\,(n+1)X} \,\Big|\, \theta = 1\;nY\Big) \\
&\qquad\quad +\; \mathrm{P}\Big(A^{\,(n+1)X} \,\Big|\, \theta = 0,\, nX\Big) - \mathrm{P}\Big(A^{\,(n+1)X} \,\Big|\, \theta = 0,\, nY\Big) \bigg] \\
&\quad - \bigg[\mathrm{P}\Big(A^{\,(n+1)X} \,\Big|\, \theta = 0,\, nX\Big) - \mathrm{P}\Big(A^{\,(n+1)X} \,\Big|\, \theta = 0,\, nY\Big) \bigg] \\
&\quad = (2q_X - 1) \bigg[\mathrm{P}\Big(A^{\,(n+1)X} \,\Big|\, \theta = 1,\, nX\Big) - \mathrm{P}\Big(A^{\,(n+1)X} \,\Big|\, \theta = 1,\, nY\Big) \bigg] \\
&\quad = -\,(2q_X - 1) \bigg[\mathrm{P}\Big(A^{\,(n+1)X} \,\Big|\, \theta = 1,\, nY\Big) - \mathrm{P}\Big(R^{\,(n+1)X} \,\Big|\, \theta = 0,\, nY\Big) \\
&\qquad\quad - \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, \theta = 0,\, nY\Big) \bigg] \qquad \text{(using (4.5))}
\end{aligned}
$$

Since $q_X > \frac{1}{2}$, we have

$$
EU^{(n+1)X}(nY) \;\lesseqgtr\; EU^{(n+1)X}(nX)
$$

$$
\begin{aligned}
\Leftrightarrow \quad & \mathrm{P}\Big(A^{(n+1)X} \,\Big|\, \theta=1,\, nY\Big) \lesseqgtr \mathrm{P}\Big(A^{(n+1)X} \,\Big|\, \theta=1,\, nX\Big) \\
\Leftrightarrow \quad & \mathrm{P}\Big(A^{(n+1)X} \,\Big|\, \theta=1,\, nY\Big) \lesseqgtr \mathrm{P}\Big(R^{(n+1)X} \,\Big|\, \theta=0,\, nY\Big) \\
& \qquad\qquad\qquad\qquad + \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, nY\Big)
\end{aligned} \tag{4.8}
$$

We notice that $(n+1)X$ chooses that predecessor with whom he is more likely to accept (his default action).

■

It is interesting to think about why early generations prefer to observe opposite types. Consider the second generation. If $2X$ observes the action A taken by $1X$, she knows that $1X$ either received s_N or s_H (in terms of the terminology introduced, she knows that $1X$ received a 'weak high' signal). On the other hand, if she observes the action A taken by $1Y$, it is clear that $1Y$ received the high signal s_H. Thus, an action when taken by a type for which it is a contrary action, reveals more precise information about their signal. On receiving an inconsistent signal, an agent would therefore prefer to observe an agent of the opposite type. This turn out to be the driving factor in the unconditional preference for opposite type for early generations. While agents prefer to observe their own type conditional on receiving the uninformative signal, Assumption 1 implies that the probability of receiving the uninformative signal is not too high. Conditional on receiving the consistent signal, while the second generation is indifferent, later generations seem to prefer their own type. However, since an agent's own prior belief favours taking his default action, the preference here for his own type is weaker than the preference for opposite type in the first case.

With Theorem 1, we have now written the entire problem of $(n+1)X$'s predecessor choice in terms of $(n+1)X$ hypothetically observing nY. We now work with the problem in this form and will therefore, in what follows, mostly omit the notation describing who $(n+1)X$ is observing.

Proposition 1 (Recurrence of joint probabilities)**.** *Suppose all agents up to generation n observe opposite types. The joint action probabilities for nY and $(n+1)X$ are then given by*

$$
\begin{bmatrix} \mathrm{P}\Big(A^{nY}A^{(n+1)X} \,\Big|\, \theta\Big) \\ \mathrm{P}\Big(R^{nY}A^{(n+1)X} \,\Big|\, \theta\Big) \\ \mathrm{P}\Big(R^{nY}R^{(n+1)X} \,\Big|\, \theta\Big) \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 1 \\ 0 & \mathrm{P}\big(s_H \,|\, \theta\big) & 0 \\ 1 & 1-\mathrm{P}\big(s_H \,|\, \theta\big) & 0 \end{bmatrix}
\begin{bmatrix} \mathrm{P}\Big(A^{(n-1)Y}A^{nX} \,\Big|\, \hat{\theta}\Big) \\ \mathrm{P}\Big(R^{(n-1)Y}A^{nX} \,\Big|\, \hat{\theta}\Big) \\ \mathrm{P}\Big(R^{(n-1)Y}R^{nX} \,\Big|\, \hat{\theta}\Big) \end{bmatrix}
$$
$$
+ \begin{bmatrix} 0 \\ \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, s_N,\, nY\Big) + \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, s_H,\, nY\Big) \\ -\mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, s_N,\, nY\Big) - \mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, s_H,\, nY\Big) \end{bmatrix}
$$

for $\theta \in \{0,1\}$. The joint action probabilities for nX and $(n+1)Y$ may be written similarly.

Proof. Suppose that all agents up to the n^{th} generation has chosen a predecessor of the opposite type and now it is the $(n+1)^{th}$ generation's turn to choose. Consider the problem from the point of view of $(n+1)X$. We know that

$$\begin{aligned} \mathrm{P}(A^{(n+1)X}\,|\,\theta,\, nY) &= \mathrm{P}(R^{nY}\,A^{(n+1)X}\,|\,\theta) + \mathrm{P}(A^{nY}\,A^{(n+1)X}\,|\,\theta) \\ \mathrm{P}(R^{(n+1)X}\,|\,\theta,\, nY) &= \mathrm{P}(R^{nY}\,R^{(n+1)X}\,|\,\theta) \end{aligned} \tag{4.9}$$

since $\mathrm{P}(A^{nY}\,R^{(n+1)X}) = 0$ from Corollary 4 of Lemma 1. From Lemma 1, we also know that $(n+1)X$ will follow the decision rules, and thus,

$$\begin{aligned} &\mathrm{P}\Big(R^{nY}\,R^{(n+1)X}\,\Big|\,\theta\Big) \\ &\quad= \mathrm{P}(s_L\,|\,\theta)\,.\,\mathrm{P}(R^{nY}\,|\,\theta)\,.\,1 \\ &\quad\quad+\mathrm{P}(s_N\,|\,\theta)\,.\,\mathrm{P}(R^{nY}\,|\,\theta) - \mathrm{P}\Big(r^{(n+1)X} = q_X\,\Big|\,\theta,\, s_N,\, nY\Big) \\ &\quad\quad+\mathrm{P}(s_H\,|\,\theta)\,.\,\mathrm{P}\Big(R^{(n-1)X}\,R^{nY}\,\Big|\,\theta\Big) - \mathrm{P}\Big(r^{(n+1)X} = q_X\,\Big|\,\theta, s_H,\, nY\Big) \qquad \text{(using 4.6)} \\ &\quad= \mathrm{P}\Big(R^{(n-1)X}R^{nY}\,\Big|\,\theta\Big) + \mathrm{P}(A^{(n-1)X}R^{nY}\,|\,\theta)\,.\,\Big[1-\mathrm{P}(s_H\,|\,\theta)\Big] \\ &\quad\quad-\mathrm{P}\Big(r^{(n+1)X} = q_X\,\Big|\,\theta,\, s_N,\, nY\,\Big) - \mathrm{P}\Big(r^{(n+1)X} = q_X\,\Big|\,\theta,\, s_H,\, nY\,\Big) \end{aligned}$$

Similarly, we derive the following:

$$\begin{aligned} \mathrm{P}\Big(R^{nY}\,A^{(n+1)X}\,\Big|\,\theta\Big) &= \mathrm{P}(A^{(n-1)X}R^{nY}\,|\,\theta)\,.\,\mathrm{P}(s_H\,|\,\theta\,) + \mathrm{P}\Big(r^{(n+1)X} = q_X\,\Big|\,\theta,\, s_N\,\Big) \\ &\quad+\mathrm{P}\Big(r^{(n+1)X} = q_X\,\Big|\,\theta,\, s_H\,\Big) \\ \mathrm{P}\Big(A^{nY}\,R^{(n+1)X}\,\Big|\,\theta\Big) &= 0 \\ \mathrm{P}\Big(A^{nY}\,A^{(n+1)X}\,\Big|\,\theta\Big) &= \mathrm{P}(A^{(n-1)X}A^{nY}\,|\,\theta) \end{aligned}$$

Using symmetry, we also have, for $a_1, a_2 \in \{R, A\}$,

$$\mathrm{P}\Big({a_1}^{(n-1)X}\,{a_2}^{nY}\,\Big|\,\theta\Big) = \mathrm{P}\Big({\hat{a}_1}^{(n-1)Y}\,{\hat{a}_2}^{nX}\,\Big|\,\hat{\theta}\Big)$$

Combining all of the above, we get the result in Proposition 1. ■

The following proposition gives us a similar recurrence rule for the probabilities of generation $(n+1)$ being on the prior with a given signal, using these probabilities of previous generations.

Proposition 2 (Recurrence of 'on prior' probabilities)**.** *Suppose $h_n \in H_n^O(Y)$. Then,*

$$\begin{bmatrix} \mathrm{P}(r^{(n+1)X} = q_X\,|\,s_L,\, nY) \\ \mathrm{P}(r^{(n+1)X} = q_X\,|\,s_N,\, nY) \\ \mathrm{P}(r^{(n+1)X} = q_X\,|\,s_H,\, nY) \end{bmatrix} = \begin{bmatrix} 0 & \frac{p_1 p_2}{p_3} & p_1 p_2 \\ 0 & 0 & p_3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{P}(r^{nX} = q_X\,|\,s_L,\,(n-1)Y) \\ \mathrm{P}(r^{nX} = q_X\,|\,s_N,\,(n-1)Y) \\ \mathrm{P}(r^{nX} = q_X\,|\,s_H,\,(n-1)Y) \end{bmatrix} + \begin{bmatrix} \mathrm{P}(r^{(n-2)X} = q_X\,|\,s_N,(n-3)Y) \\ 0 \\ 0 \end{bmatrix}$$

Proof. Appendix A. ■

Propositions (1) and (2) are central to the analysis because they provide us with useful recurrence relations - if we know the vectors of joint action probabilities and 'on prior' probabilities for the second generation, we can write these vectors for any subsequent generation, given that everybody before this generation observed their opposite type.

Using (4.8), since we know that nX chose to observe $(n-1)Y$, we have

$$\begin{bmatrix}1 & 1 & -1\end{bmatrix}\begin{bmatrix}\mathrm{P}\Big(A^{(n-1)Y}A^{nX}\,\Big|\,\theta=1\Big)\\ \mathrm{P}\Big(R^{(n-1)Y}A^{nX}\,\Big|\,\theta=1\Big)\\ \mathrm{P}\Big(R^{(n-1)Y}R^{nX}\,\Big|\,\theta=0\Big)\end{bmatrix} > \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}\mathrm{P}\Big(r^{nX}=q_X\,\Big|\,s_L,\,(n-1)Y\Big)\\ \mathrm{P}\Big(r^{nX}=q_X\,\Big|\,s_N,\,(n-1)Y\Big)\\ \mathrm{P}\Big(r^{nX}=q_X\,\Big|\,s_H,\,(n-1)Y\Big)\end{bmatrix}$$

The following inequality then describes the predecessor choice of $(n+1)X$:

$$EU^{(n+1)X}(nY) \lesseqgtr EU^{(n+1)X}(nX)$$

$$\begin{bmatrix}1 & 1 & -1\end{bmatrix}\begin{bmatrix}\mathrm{P}\Big(A^{nY}A^{(n+1)X}\,\Big|\,\theta=1\Big)\\ \mathrm{P}\Big(R^{nY}A^{(n+1)X}\,\Big|\,\theta=1\Big)\\ \mathrm{P}\Big(R^{nY}R^{(n+1)X}\,\Big|\,\theta=0\Big)\end{bmatrix} \lesseqgtr \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}\mathrm{P}\Big(r^{(n+1)X}(s_L,h_n)=q_X\Big)\\ \mathrm{P}\Big(r^{(n+1)X}(s_N,h_n)=q_X\Big)\\ \mathrm{P}\Big(r^{(n+1)X}(s_H,h_n)=q_X\Big)\end{bmatrix} \tag{4.10}$$

Now since the second generation has observed opposite types, we can use the above for $n=2$ and given values of p_1 and p_2 to check whether the third generation will observe their opposite type. If the third generation does do so, then the decision rules hold and so we can use the above inequality, now for $n=3$, to check whether the fourth generation will observe their opposite type. In this manner, we may figure out, given p_1 and p_2, which generation first chooses to observe their own type ('switches'). Let us denote this generation by n^{switch}.

4.2 Simulation

Given the following vectors of joint action probabilities for $1Y$ and $2X$, and 'on prior' probabilities for the second generation (calculated by hand),

$$\begin{bmatrix}\mathrm{P}\Big(A^{1Y}A^{2X}\,\Big|\,\theta=0\Big)\\ \mathrm{P}\Big(R^{1Y}A^{2X}\,\Big|\,\theta=0\Big)\\ \mathrm{P}\Big(R^{1Y}R^{2X}\,\Big|\,\theta=0\Big)\end{bmatrix} = \begin{bmatrix}p_2\\ p_2(1-p_2)\\ (1-p_2)^2\end{bmatrix} \quad \begin{bmatrix}\mathrm{P}\Big(A^{1Y}A^{2X}\,\Big|\,\theta=1\Big)\\ \mathrm{P}\Big(R^{1Y}A^{2X}\,\Big|\,\theta=1\Big)\\ \mathrm{P}\Big(R^{1Y}R^{2X}\,\Big|\,\theta=1\Big)\end{bmatrix} = \begin{bmatrix}p_1\\ p_1(1-p_1)\\ (1-p_1)^2\end{bmatrix}$$

$$\begin{bmatrix}\mathrm{P}(r^{2X}=q_X\,|\,s_L,\,1Y)\\ \mathrm{P}(r^{2X}=q_X\,|\,s_N,\,1Y)\\ \mathrm{P}(r^{2X}=q_X\,|\,s_H,\,1Y)\end{bmatrix} = \begin{bmatrix}p_1\,p_2\\ 0\\ 0\end{bmatrix}$$

we use Theorem 1 to simulate the model for given values of p_1, p_2 and p_3 with the help of Propositions 1 and 2.

Theorem 1 notes that when Assumption 1 is satisfied, the second generation starts by observing their opposite type and any subsequent generation $(n+1)$, for $n \geq 2$, continues to observe opposite types as long as (4.7) holds. When (4.7) holds for some generation $(n+1)$, we have a history of opposites upto $(n+1)$ and thus this same condition will tell us who the next generation will choose to observe.

When (4.7) is not satisfied for some subsequent generation, we know that this generation will be the first to observe same type, thus, this generation becomes n^{switch}. We represent our data pictorially in figures 1 to 6. We make the following observations from these:

1. Keeping p_3 fixed, n^{switch} weakly increases as $\frac{p_1}{p_2}$ falls.
2. Keeping $\frac{p_1}{p_2}$ fixed, n^{switch} weakly increases as p_3 falls.
3. If some even generation $2m$ (for $m \geq 1$) has chosen to observe opposite type, the odd generation $2m+1$ will also find it optimal to observe opposite type. i.e. n^{switch} is always even.

The first two results seem fairly intuitive. For example, if, keeping the likelihood of the signals s_L and s_H fixed, we increase the value of p_3, it means that, for a given generation, the probability of not receiving any information increases. Thus, with higher chance, an agent will have to follow her predecessor. Conditional on getting a neutral signal, an agent would rather observe her own type, since in this case there is no conflict in simply following her predecessor, even if their exact signal is not inferable. Thus, the incentive to observe one's own type goes up. Even if initial generations find the actions of opposite types more informative, later generations, wishing to avoid the possibility of taking their contrary action based on following an opposite type who received no information, will switch faster to their own type.

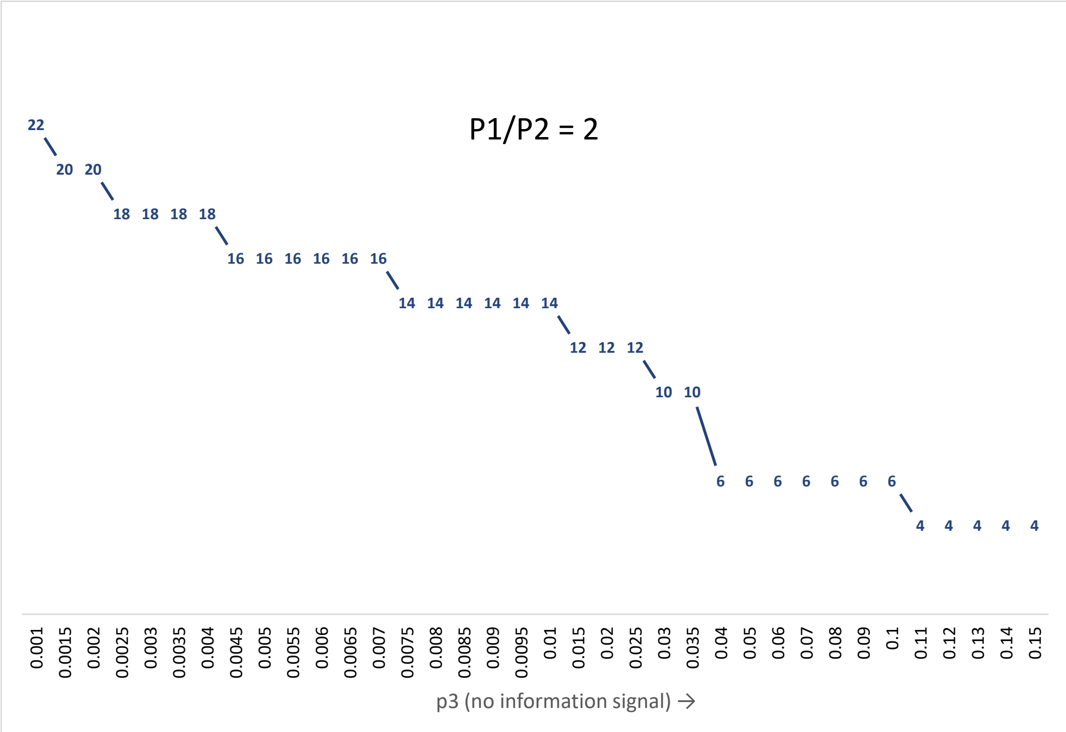

Figure 1: Values of n^{switch} when $\frac{p_1}{p_2} = 2$

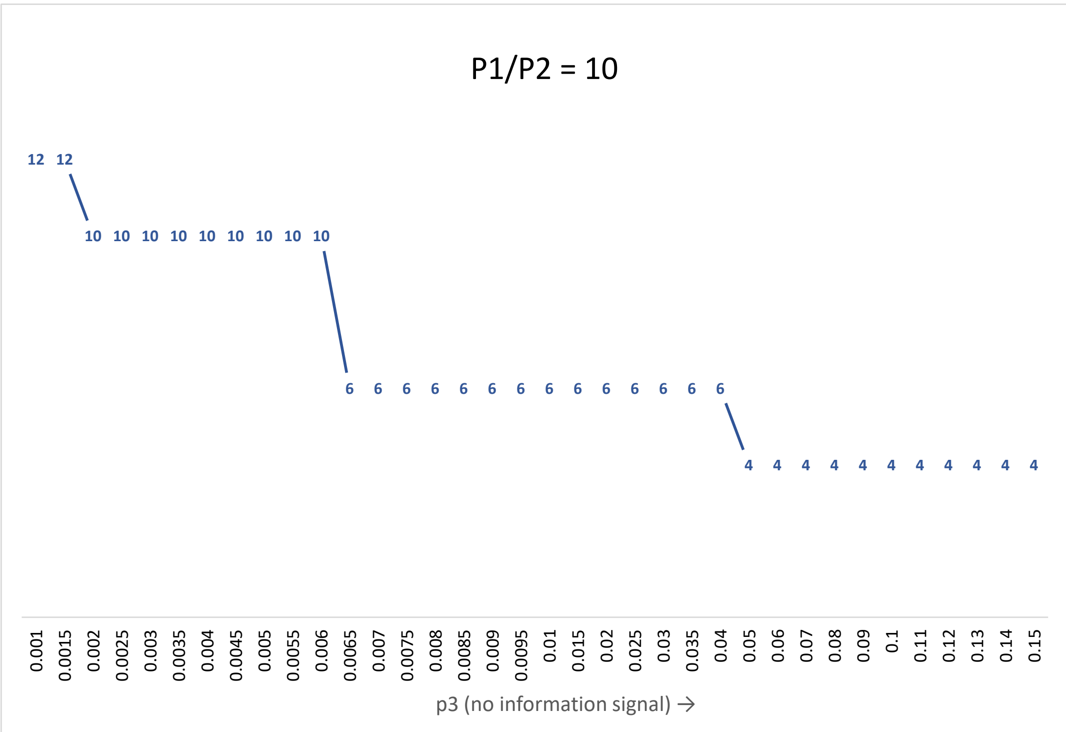

Figure 2: Values of n^{switch} when $\frac{p_1}{p_2} = 10$

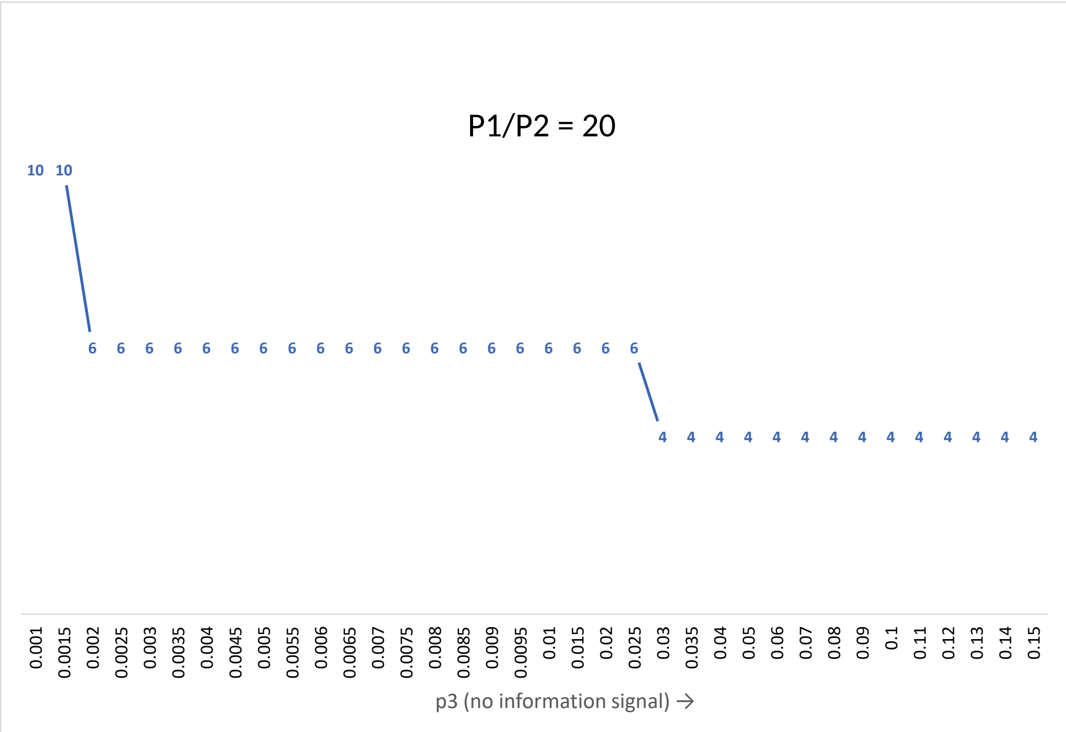

Figure 3: Values of n^{switch} when $\frac{p_1}{p_2} = 20$

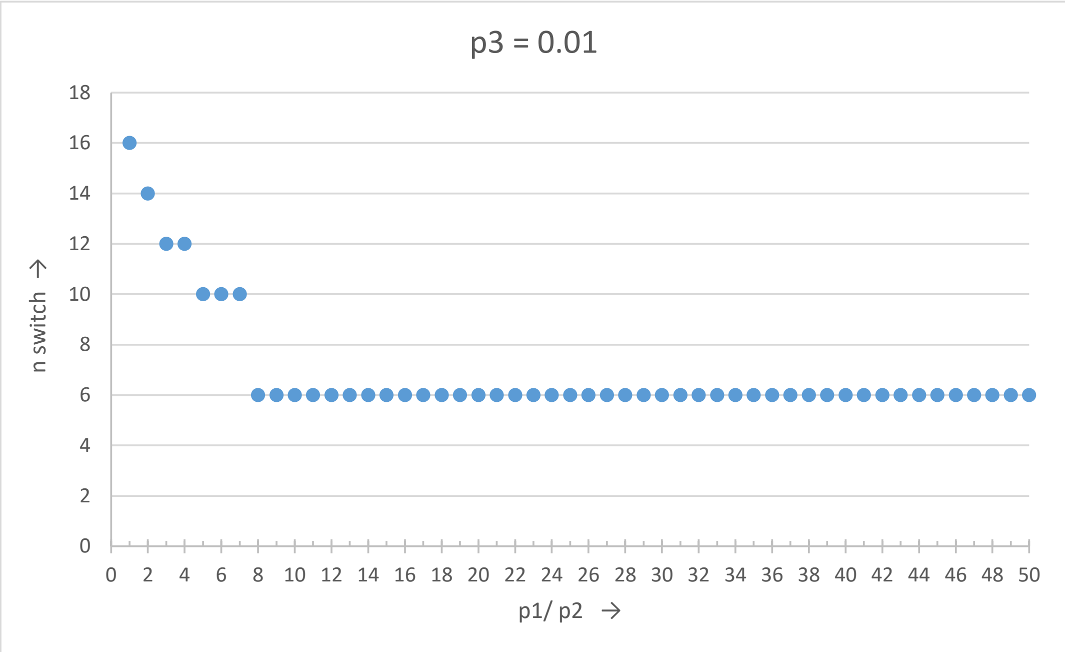

Figure 4: Values of n^{switch} when $p_3 = 0.01$

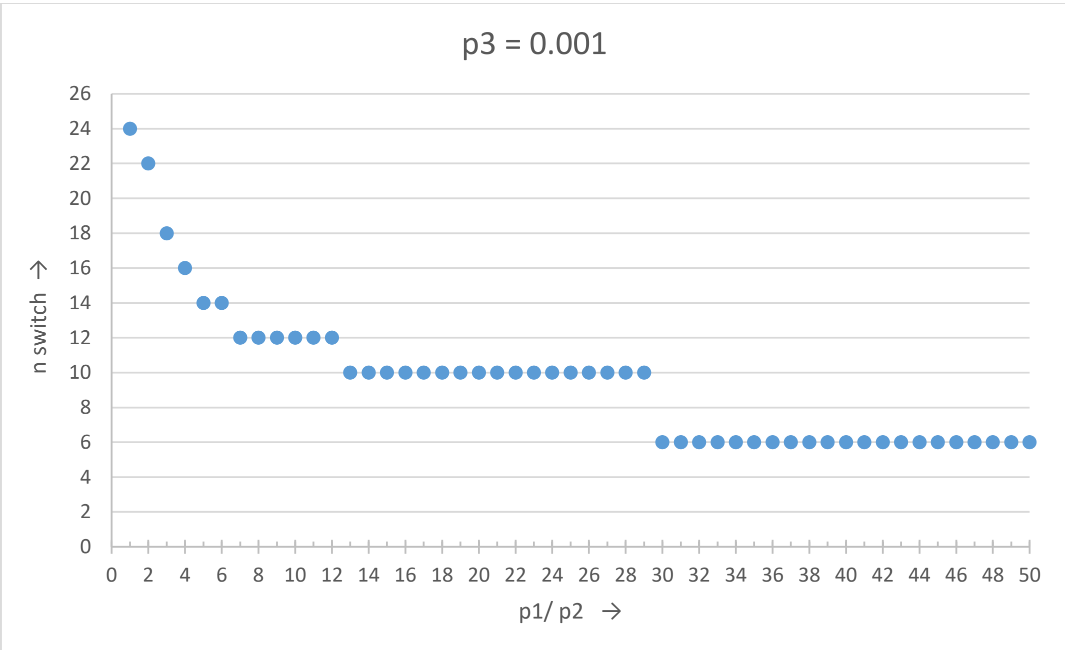

Figure 5: Values of n^{switch} when $p_3 = 0.001$

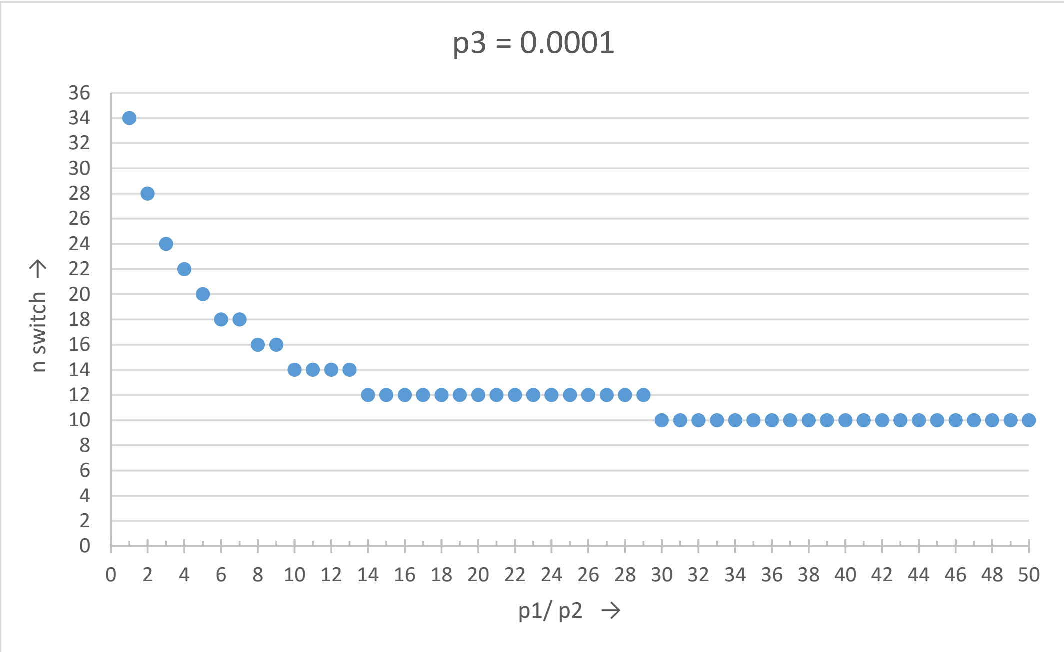

Figure 6: Values of n^{switch} when $p_3 = 0.0001$

4.3 Two signal case

In this subsection, we assume that $p_3 = 0$, and thus there are only two signals s_L and s_H. We retain our previous notations, keeping in mind that, now, $S = \{s_L, s_H\}$ and we have $p_1 + p_2 = 1$.

As before, our starting assumption is that first generation agents do not take their default action on all signals, which in this case implies that

$$q_X < p_1 \tag{4.11}$$

Similar to the previous subsection, we characterise decision rules for both types, as given in Lemma 1. There are two major differences. One, the rules in this case apply to *any* kind of history, and not just a history of opposites. Two, these rules do not require any additional parametric condition on the prior beliefs.

Lemma 2 (Decision rules for two signal case)**.** *The decision making rules of any generation (n+1) are as follows:*
For $n + 1 = 2$*:*

(a) On receiving his consistent signal, the agent takes his default action.
(b) On receiving his inconsistent signal, he follows his predecessor.

For $n + 1 > 2$*: Suppose the agent observes* (s, h_n)*.*

(a) If the agent is on his prior at (s, h_n)*, then he takes his default action.*

(b) If the agent is not on his prior,

i. On observing s_L he rejects unless the last two actions in h_n have been acceptances.

ii. On receiving s_H he accepts unless the last two actions in h_n have been rejections.

Proof. Proof omitted (available on request).

■

The expected payoff of agent $(n+1)X$ when he chooses to observe nt, for $t \in \{X, Y\}$, is given by:

$$EU^{(n+1)X}(nt) = q_X \cdot \Big[\mathrm{P}(A^{(n+1)X} \,|\, \theta = 1,\, nt) + \mathrm{P}(A^{(n+1)X} \,|\, \theta = 0,\, nt) \Big] - \mathrm{P}(A^{(n+1)X} \,|\, \theta = 0,\, nt)$$

Therefore,

$$\begin{aligned} &EU^{(n+1)X}(nX) - EU^{(n+1)X}(nY) \\ &= q_X \Bigg[\mathrm{P}\Big(A^{(n+1)X} \,\Big|\, \theta = 1,\, nX\Big) - \mathrm{P}\Big(A^{(n+1)X} \,\Big|\, \theta = 1,\, nY\Big) \\ &\quad + \mathrm{P}\Big(A^{(n+1)X} \,\Big|\, \theta = 0,\, nX\Big) - \mathrm{P}\Big(A^{(n+1)X} \,\Big|\, \theta = 0,\, nY\Big) \Bigg] \\ &- \Bigg[\mathrm{P}\Big(A^{(n+1)X} \,\Big|\, \theta = 0,\, nX\Big) - \mathrm{P}\Big(A^{(n+1)X} \,\Big|\, \theta = 0,\, nY\Big) \Bigg] \end{aligned}$$

The expression for $EU^{(n+1)Y}(nX) - EU^{(n+1)Y}(nY)$ will be similar. It then follows that

$$\begin{aligned} &\mathrm{P}(A^{(n+1)t} \,|\, \theta,\, nX) = \mathrm{P}(A^{(n+1)t} \,|\, \theta,\, nY) \quad \text{for } \theta \in \{0, 1\} \\ &\Rightarrow\; EU^{(n+1)t}(nX) = EU^{(n+1)t}(nY) \end{aligned} \tag{4.12}$$

Theorem 2 (Predecessor choice with two signals)**.** *When $p_3 = 0$ and the tie-breaking rule randomizes between the two types with equal probability, individual $(n+1)t$ for any $n \geq 1$ and $t \in \{X, Y\}$ is indifferent between observing her own type and the opposite type.*

Proof. Proof omitted (available on request). ■

It may be verified that Theorem 2 will hold for any tie-breaking rule that chooses the same type with probability p_s and the opposite type with probability $1 - p_s$, for $p_s \in [0, 1]$. It should be noted that while agents are able to figure out the predecessor choices of all generations before them in cases of strict preference, this is not generally true for cases of indifference. When the tie-breaking rule is not deterministic, subsequent agents will not know the actual realization of the predecessor choice for a generation that randomised between the two types.

It is interesting to note that in the two extreme cases where $p_s = 0$ and $p_s = 1$ (among others), we will see realizations of predecessor choices that are vastly different. Even though all agents will continue to remain indifferent between the two predecessors, in the first case we see mixed histories where there is interaction between the two groups, in the second case we will have segregation, with agents always observing only their own type.

5 Conclusion

This work studies a model of sequential learning with predecessor choice. We consider two types of agents, each having different prior beliefs, and analyse which type of their immediate predecessor agents prefer to form a link with. With two signals, we show that agents will be indifferent between observing predecessors of their own type and the opposite type. The tie-breaking rule, however, plays an important role in determining whether the two types interact with each other. If agents always break ties in favour of their own type, there will be segregation between the two groups, even though all generations will remain indifferent between types.

Introducing an additional uninformative signal results in the possibility of strict preference for the two types of predecessors. Under some parametric conditions on the prior beliefs, it is optimal for the second generation to observe their opposite type. We derive conditions under which subsequent generations continue to observe their opposite type and characterize decision rules which specify the action taken by these agents in this case. One of the important questions we ask in this work is whether it is possible for the two types of agents to continue observing each other even as the number of generations grows large. Results suggest that this is not possible. In the long run, we find that agents will prefer to observe their own type, regardless of the history of predecessor choices before them. Thus, even in the absence of any behavioural preference to interact with similar others, agents end up forming links *within* their own group in the long run. This suggests a bleak picture for societal cohesion in the context of this model. It is striking that this segregation occurs even if the prior beliefs q_X and q_Y are arbitrarily close to $\frac{1}{2}$, i.e. there are only extremely small differences between the two groups.
The conditions under which interaction between groups occurs more frequently is another point of interest of this work. For the case where initial generations prefer to observe their opposite type, we find that a lower probability of not receiving any private information, or a lower signal precision (as measured by the likelihood of receiving a correct versus an incorrect signal) are able to sustain opposite observing behaviour for longer. If one believes that there is some intrinsic value to different groups interacting with each other, outside of any gains from learning or welfare considerations, it becomes interesting to look for ways in which the two types can be influenced to have more cross-ties.

Future work on this topic would include an analysis of varying interaction between groups on learning, especially in the short run. We note here that opposite observing histories can sometimes prevent (delay) an information cascade from being transmitted to agents that follow, something which would be sure to happen if agents were all of the same type. In particular, if an agent takes the same action on all signals but is on her prior at one of these signals, then, the following agent, being her opposite type, takes a different action at this signal. Thus, this agent will *not* ignore his private information, even though the previous agent has done so. It would be interesting to extend the analysis to more than two types, and to consider a case where information is received only about the immediately previous generation. One may also consider a situation where links are not necessarily formed sequentially- this would capture a network in a more general sense.

A Appendix: Proofs

Proof of Lemma 1. We prove this lemma by induction.

For $n = 2$ *:*

If the second generation observes their opposite types, then the third generation on observing $2X$ sees the history (a^{1Y}, a^{2X}) and on observing $2Y$ see the history (a^{1Y}, a^{2X}).

Note that

$$S(A^{2X} \mid R^{1Y}) = \{s_H\}, \quad S(A^{2X} \mid A^{1Y}) = S$$
$$\text{and} \quad S(A^{2Y} \mid R^{1X}) = S, \qquad S(A^{2Y} \mid A^{1X}) = \{s_N, s_H\}$$

Let us consider the case when 3X observes 2X. Other cases will follow a similar logic. We prove the three parts of the statement one by one.

	Information Set (I)	$\text{P}(I \mid \theta = 0)$	$\text{P}(I \mid \theta = 1)$	a^{3X}	a^{3Y}
s_L, R^{1Y}, R^{2X}	$\{s_L\} \times \{s_L, s_N\} \times \{s_L, s_N\}$	$p_1\,(1-p_2)^2$	$p_2\,(1-p_1)^2$	R	R
s_L, R^{1Y}, A^{2X}	$\{s_L\} \times \{s_L, s_N\} \times \{s_H\}$	$p_1\,p_2\,(1-p_2)$	$p_1\,p_2\,(1-p_1)$	R	R
s_L, A^{1Y}, A^{2X}	$\{s_L\} \times \{s_H\} \times S$	$p_1\,p_2$	$p_1\,p_2$	A	R
s_N, R^{1Y}, R^{2X}	$\{s_N\} \times \{s_L, s_N\} \times \{s_L, s_N\}$	$p_3\,(1-p_2)^2$	$p_3\,(1-p_1)^2$	R	R
s_N, R^{1Y}, A^{2X}	$\{s_N\} \times \{s_L, s_N\} \times \{s_H\}$	$p_2\,p_3\,(1-p_2)$	$p_1\,p_3\,(1-p_1)$	A	A
s_N, A^{1Y}, A^{2X}	$\{s_N\} \times \{s_H\} \times S$	$p_2\,p_3$	$p_1\,p_3$	A	A
s_H, R^{1Y}, R^{2X}	$\{s_H\} \times \{s_L, s_N\} \times \{s_L, s_N\}$	$p_2\,(1-p_2)^2$	$p_1\,(1-p_1)^2$	R	R
s_H, R^{1Y}, A^{2X}	$\{s_H\} \times \{s_L, s_N\} \times \{s_H\}$	$p_2^2\,(1-p_2)$	$p_1^2\,(1-p_1)$	A	A
s_H, A^{1Y}, A^{2X}	$\{s_H\} \times \{s_H\} \times S$	p_2^2	p_1^2	A	A

Table 2: Observing $2X$

1. For the first part of the statement, we show that $r^{3X}(s, h_2) \notin \left[\frac{1}{2}, q_X\right)$ for any $s \in S$ and $h_2 \in H_2^O(X)$.

   Consider the history $R^{1Y}R^{2X}$.

$$\frac{r^{3X}(s_H, R^{1Y}R^{2X})}{1 - r^{3X}(s_H, R^{1Y}R^{2X})} = \frac{q_X}{1-q_X}\,\frac{p_1\,(1-p_1)^2}{p_2\,(1-p_2)^2}$$

   Using (4.4),

$$r^{3X}(s_H, R^{1Y}R^{2X}) < \frac{1}{2}$$
$$\Rightarrow \quad r^{3X}(s, R^{1Y}R^{2X}) < \frac{1}{2} \qquad \forall\, s \in S$$

$$\notin \left[\frac{1}{2}, q_X\right)$$

and $3X$ rejects at all signals given the history $R^{1Y}R^{2X}$.

One may similarly consider the other histories.

2. Let us now prove the second part of the statement i.e. the decision rules.

   At his consistent signal (s_H): When both $1Y$ and $2X$ reject, then $3X$ is not on prior at s_H $\left(r^{3X}(s_H, R^{1Y}R^{2X}) < \frac{1}{2}\right)$ and should therefore reject, which he does. For all other histories, he should accept, which he obeys as well.

   At his inconsistent signal (s_L): From table (2), it is clear that a^{3X} matches a^{1Y} for the first three rows, where $s^{3X} = s_L$. Thus $3X$ is following the latest opposite type.

   At the uninformative signal (s_N): From table (2), $3X$ is not at prior in any of the rows where he receives s_N, and he is clearly following $2X$ in all three cases.

3. The third part of the statement is concerned with $3X$ in instances where he rejects on all signals. This happens only with the history $R^{1Y}R^{2X}$ where,

$$\frac{r^{3X}(s_N, R^{1Y}R^{2X})}{1 - r^{3X}(s_N, R^{1Y}R^{2X})} = \frac{q_X}{1 - q_X} \frac{p_3\,(1-p_1)^2}{p_3\,(1-p_2)^2}$$

$$\Rightarrow \qquad \frac{\mathrm{P}(R^{1Y}R^{2X} \,|\, \theta = 1)}{\mathrm{P}(R^{1Y}R^{2X} \,|\, \theta = 0)} = \frac{(1-p_1)^2}{(1-p_2)^2}$$

$$\in \left\{ \frac{p_2\,(1-p_1)}{p_1\,(1-p_2)}, \frac{(1-p_1)^2}{(1-p_2)^2} \right\}$$

   as required.

Thus, all three parts of the statement are true in this case.

Now assume that the statement is true for some n.

We then have the following assumptions:

A1. (a) $r^{(n+1)X}(s, h_n) \notin \left[\frac{1}{2}, q_X\right)$, $\forall s \in S$ and $h_n \in H_n^O$

(b) $r^{(n+1)Y}(s, h_n) \notin (\,q_Y, \frac{1}{2}\,]$, $\forall s \in S$ and $h_n \in H_n^O$

A2. Given some $h_n \in H_n^O$, $(n+1)X$ and $(n+1)Y$, based on their signal s, behave in the following way:

(i) If $h_n \in H_n^O(X)$,

(n+1)X	(n+1)Y
$s_L \rightarrow$ follow $(n-1)Y$	$s_L \rightarrow$ R unless last two A's and not on prior
$s_N \rightarrow$ follow nX	$s_N \rightarrow$ follow nX unless on prior
$s_H \rightarrow$ A unless last two R's and not on prior	$s_H \rightarrow$ follow nX

(ii) If $h_n \in H_n^O(Y)$,

(n+1)X	(n+1)Y
$s_L \to$ follow nY	$s_L \to$ R unless last two A's and not on prior
$s_N \to$ follow nY unless on prior	$s_N \to$ follow nY
$s_H \to$ A unless last two R's and not on prior	$s_H \to$ follow $(n-1)X$

A3. (a) If $(n+1)X$ rejects on all signals given $h_n \in H_n^O$, then

$$\frac{\mathrm{P}(h_n \,|\, \theta = 1)}{\mathrm{P}(h_n \,|\, \theta = 0)} \in \left\{ \frac{p_2\,(1-p_1)}{p_1\,(1-p_2)}, \frac{(1-p_1)^2}{(1-p_2)^2} \right\}$$

(b) If $(n+1)Y$ accepts on all signals given $h_n \in H_n^O$, then

$$\frac{\mathrm{P}(h_n \,|\, \theta = 1)}{\mathrm{P}(h_n \,|\, \theta = 0)} \in \left\{ \frac{p_1\,(1-p_2)}{p_2\,(1-p_1)}, \frac{(1-p_2)^2}{(1-p_1)^2} \right\}$$

We now prove the induction statement for $(n+1)$ i.e. for the generation $(n+2)$ observing $(n+1)$.
We look at the problem from the point of view of $(n+2)X$. The proof will follow for $(n+2)Y$ along similar lines by symmetry.

Note that we can write $(n+2)X$'s posterior in the following way:

$$\frac{r^{(n+2)X}(s, h_{n+1})}{1 - r^{(n+2)X}(s, h_{n+1})} = \frac{r^{(n+1)X}(s, h_n)}{1 - r^{(n+1)X}(s, h_n)} \frac{\mathrm{P}(a^{(n+1)t} \,|\, \theta = 1, h_n)}{\mathrm{P}(a^{(n+1)t} \,|\, \theta = 0, h_n)}$$

Let t represent the type of the last i.e. $(n+1)^{th}$ agent in h_{n+1} and let $a^{(n+1)t}$ represent the action she takes on seeing her signal $s^{(n+1)t}$ and some history $h_n \in H_n^O(\hat{t}\,)$. Thus, $h_{n+1} = (h_n, a^{(n+1)t})$.

Consider any $s \in S$ and $h_{n+1} \in H_{n+1}^O$. Write h_{n+1} as $(h_n, a^{(n+1)t})$. There are four possible ways in which $(n+1)t$ can partition her signal set given h_n:

1. $S(A^{(n+1)t} \,|\, h_n) = S$
2. $S(A^{(n+1)t} \,|\, h_n) = \{s_N, s_H\}$
3. $S(A^{(n+1)t} \,|\, h_n) = \{s_H\}$
4. $S(A^{(n+1)t} \,|\, h_n) = \phi$

We proceed with the proof using the following steps:

(i.) Start with a particular partition.

(ii.) For this partition, consider $t = X$ and $t = Y$ in turn.

(iii.) For a given t, consider $a^{(n+1)t} = R$ and $a^{(n+1)t} = A$ (wherever possible).

(iv.) For each action, consider $s^{(n+2)X}$ being s_L, s_N and s_H by turn.

(v.) Prove the first two parts of the induction statement.

Calculate $(n+2)X$'s posterior belief $r^{(n+2)X}(s, h_{n+1})$. Check that this doesn't lie between half and q_X (including half). Based on how the belief compares to $\frac{1}{2}$, determine whether $(n+2)X$ will accept or reject.

Compare this action to the one suggested by the decision rules to determine whether the rules are followed or not.

(vi.) In case $(n+2)X$ rejects at all signals given h_{n+1}, prove the third part of the statement.

Let us consider the possible partitions one by one.

1. $S(A^{(n+1)t} \,|\, h_n) = S$.

   In this case, $A^{(n+1)t}$ provides no new information and thus,

$$\begin{aligned} r^{(n+2)X}(s, h_{n+1}) &= r^{(n+1)X}(s, h_n) \\ &\notin \Big[\frac{1}{2}, q_X\Big) \end{aligned}$$

   Since $(n+1)t$ accepts at all signals given h_n, this implies that $(n+1)X$ definitely accepts at all signals given h_n and thus, $(n+2)X$ will also accept at all signals given h_{n+1} with both $t = X$ and $t = Y$. In order to show that $(n+2)X$ obeys the decision rules, we thus have to show that the rules suggest acceptance at all signals.

   (a) t=X
   For s_L:
   Since $(n+1)X$ on receiving s_L must have followed nY, his acceptance means that nY must have accepted in h_n. According to the decision rules, at s_L, $(n+2)X$ should follow the latest opposite type in h_{n+1} i.e. nY and thus accept.

   For s_N:
   Since $(n+2)X$ is not at prior at s_N, the decision rules suggest that he should follow his predecessor i.e. $(n+1)X$ and thus accept.

   For s_H:
   The rule again suggests acceptance in this case, since at least one of the last two actions in h_{n+1}, namely, $a^{(n+1)X}$, is an acceptance.

(b) t=Y
From Corollary (4), it is clear that in this case, when $(n+1)Y$ has accepted, the decision rules state that $(n+2)X$ should also accept at all signals. This is exactly what $(n+2)X$ does, hence he follows the decision rules.

2. $S(A^{(n+1)t} \mid h_n) = \{s_N, s_H\}$

(a) $t = X$
From Corollary (2) it is clear that $r^{(n+1)X}(s_N, h_n) = q_X$.

If $\mathbf{a^{(n+1)X} = R}$,
For s_L:

$$\frac{r^{(n+2)X}(s_L, h_{n+1})}{1 - r^{(n+2)X}(s_L, h_{n+1})} = \frac{r^{(n+1)X}(s_L, h_n)}{1 - r^{(n+1)X}(s_L, h_n)} \frac{p_2}{p_1}$$

Since $p_2 < p_1$,

$$\begin{aligned} r^{(n+2)X}(s_L, h_{n+1}) &< r^{(n+1)X}(s_L, h_n) \\ &< \frac{1}{2} \\ &\notin \Big[\frac{1}{2}, q_X\Big) \end{aligned}$$

and $(n+2)X$ will reject at s_L. This is in accordance with the decision rules, since he is following the latest opposite type in h_{n+1} i.e. nY. We know nY must have rejected in the given history because at (s_L, h_n), $(n+1)X$'s rejection is coming from following nY, so nY must have rejected.

For s_N:
Since $r^{(n+1)X}(s_N, h_n) = q_X$, this means that s_N and h_n cancel each other out. Now, $(n+1)X's$ action reveals one low signal and so $(n+2)X$ is not at prior at (s_N, h_{n+1}) and should take action R. This is in accordance with the decision rules, which suggest following the previous person i.e. $(n+1)X$ at s_N, who rejected.

For s_H:

$$\begin{aligned} \frac{r^{(n+2)X}(s_H, h_{n+1})}{1 - r^{(n+2)X}(s_H, h_{n+1})} &= \frac{r^{(n+1)X}(s_H, h_n)}{1 - r^{(n+1)X}(s_H, h_n)} \frac{p_2}{p_1} \\ &= \frac{r^{(n+1)X}(s_N, h_n)}{1 - r^{(n+1)X}(s_N, h_n)} \end{aligned}$$

i.e. $r^{(n+2)X}(s_H, h_{n+1}) = r^{(n+1)X}(s_N, h_n) = q_X$.

Thus $(n+2)X$ accepts at s_H. This is in accordance with the decision rules which suggest taking A when at prior at s_H.

Now, **if $\mathbf{a^{(n+1)X} = A}$**, then
For s_L:

$$\begin{aligned}\frac{r^{(n+2)X}(s_L, h_{n+1})}{1 - r^{(n+2)X}(s_L, h_{n+1})} &= \frac{r^{(n+1)X}(s_L, h_n)}{1 - r^{(n+1)X}(s_L, h_n)} \frac{1 - p_2}{1 - p_1} \\ &= \frac{r^{(n+1)X}(s_N, h_n)}{1 - r^{(n+1)X}(s_N, h_n)} \frac{p_2}{p_1} \frac{1 - p_2}{1 - p_1} \\ &= \frac{q_X}{1 - q_X} \frac{p_2(1 - p_2)}{p_1(1 - p_1)}\end{aligned}$$

Using (4.2),

$$\begin{aligned}r^{(n+2)X}(s_L, h_{n+1}) &< \frac{1}{2} \\ &\notin \Big[\frac{1}{2}, q_X\Big)\end{aligned}$$

and $(n + 2)X$ will reject at s_L. This agrees with his decision rules, since he should follow nY at s_L who we know must have rejected in h_n since $(n + 1)X$ rejected at s_L.

For s_N:
At (s_N, h_{n+1}), $(n + 2)X's$ posterior is given by:

$$\begin{aligned}\frac{r^{(n+2)X}(s_N, h_{n+1})}{1 - r^{(n+2)X}(s_N, h_{n+1})} &= \frac{r^{(n+1)X}(s_N, h_n)}{1 - r^{(n+1)X}(s_N, h_n)} \frac{\mathrm{P}(A^{(n+1)X}|\theta = 1, h_n)}{\mathrm{P}(A^{(n+1)X}|\theta = 0, h_n)} \\ &= \frac{q_X}{1 - q_X} \frac{1 - p_2}{1 - p_1}\end{aligned}$$

Since $p_1 > p_2$, $r^{(n+2)X}(s_N, h_{n+1}) > q_X$ and thus $(n+2)X$ accepts at (s_N, h_{n+1}). According to the decision rules, $(n + 2)X$ at s_N should follow his predecessor i.e. $(n + 1)X$ when he is not on prior. Thus $(n + 2)X$ should accept. Thus he is following the rules here.

For s_H:

$$\begin{aligned}r^{(n+2)X}(s_H, h_{n+1}) &> r^{(n+2)X}(s_N, h_{n+1}) \\ &> q_X \\ &\notin \Big[\frac{1}{2}, q_X\Big)\end{aligned}$$

and $(n + 2)X$ accepts at s_H. This matches the action suggested by the decision rules, since at least one of the last two actions in h_{n+1} is not a rejection $(a^{(n+1)X} = A)$.

(b) $t = Y$

The proof for this part follows a similar logic as in part (a).

3. $S(A^{(n+1)t} \,|\, h_n) = \{s_H\}$

   Notice that there is a similarity between this case and the previous one, where the rejection set was $\{s_L\}$. Any given history here can be considered a mirror image of a history there. For example, if, here in h_{n+1}, we have $(n+1)X$ rejecting with $S(R^{(n+1)X} \,|\, h_n) = \{s_L, s_N\}$ then in $\hat{h}_{n+1}$, we have $(n+1)Y$ accepting with $S(A^{(n+1)Y} \,|\, \hat{h}_n) = \{s_N, s_H\}$. This is nothing but a sub-case under the previous heading, for which we have already shown that the induction statement holds.

   The proof for this case will therefore follow along parallel lines to the proof in the previous case.

4. $S(A^{(n+1)t} \,|\, h_n) = \phi$

   This case is symmetrically similar to the first case in which $(n+1)t$ accepted on all signals and will have a similar proof (which we omit).

We have thus proved, for every possible case, that if the statement holds for some n, it also holds for $(n+1)$. Since we have also shown that the statement is true for $n = 2$, it follows by induction that the statement is true for all $n \geq 2$. ■

Proof of Proposition 2. The proposition contains 3 parts:

1. $$\mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, s_L, nY\Big)$$
$$= \frac{p_1\, p_2}{p_3}\, \mathrm{P}\Big(r^{nX} = q_X \,\Big|\, s_N, (n-1)Y\Big)$$
$$+ \;\; p_1\, p_2 \left[\mathrm{P}\Big(r^{nX} = q_X \,\Big|\, s_H, (n-1)Y\Big) + \mathrm{P}\Big(r^{(n-2)X} = q_X \,\Big|\, s_N, (n-3)Y\Big) \right]$$

2. $$\mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, s_N, nY\Big) = \; p_3\, \mathrm{P}\Big(r^{nX} = q_X \,\Big|\, s_H, (n-1)Y\Big)$$

3. $$\mathrm{P}\Big(r^{(n+1)X} = q_X \,\Big|\, s_H, nY\Big) = \; \mathrm{P}\Big(r^{nX} = q_X \,\Big|\, s_L, (n-1)Y\Big)$$

Since $h_n \in H_n^O$ we will use the fact that everybody upto generation n is following the decision rules (see Lemma 1).

For sake of convenience, we denote the probabilities involved in the above lemma with shorthands, using the following rule: given $h_i \in H_i^O(\hat{t})$,

$$\mathrm{P}\Big(*_s^{(i+1)t}\Big) = \mathrm{P}\Big(r^{(i+1)t} = q_t \,\Big|\, s,\, i\hat{t}\Big)$$

for $s \in S$ and $1 \leq i \leq n$.

We also use the notation "$a^{it} \Rightarrow S_1$" to mean that $S(a^{it} \,|\, h_{i-1}) = S_1$, for $S_1 \subset S$.

Given h_{n-1}, nY can partition her signal set in the following ways:

(a) $S(A^{nY} \,|\, h_n) = S$

(b) $S(A^{nY} \,|\, h_n) = \{s_N, s_H\}$

(c) $S(A^{nY} \,|\, h_n) = \{s_H\}$

(d) $S(A^{nY} \,|\, h_n) = \phi$

We are interested in the ways in which $(n+1)X$ observing nY may be on his prior. Consider the above partitions one by one.

(a) $S(A^{nY} \,|\, h_n) = S$

Here, $a^{nY} = A$ so nY is clearly not on her prior at any signal given h_{n-1} (and therefore nX). Since $r^{(n+1)X}(s, h_n) = r^{nX}(s, h_{n-1})$ here, $(n+1)X$ will not be on prior at any signal in this case.

(b) $S(A^{nY} \,|\, h_n) = \{s_N, s_H\}$

In this case, nY can only be at her prior at s_L, if at all.

(i) Suppose nY is on prior at s_L.

This means that $h_{n-1} \Rightarrow \{s_H\}$. If nY rejects, then $h_n \Rightarrow \phi$ and thus $(n+1)X$ will be on prior at s_N (and therefore not at s_L or s_H), with

$$\begin{aligned} \mathrm{P}\Big((n+1)X \text{ on prior at } s_N \,\Big|\, \theta\Big) &= \mathrm{P}(s_N \,|\, \theta)\,.\,\mathrm{P}(nY \text{ on prior at } s_L \,|\, \theta\,) \\ &= p_3\,.\,\mathrm{P}(nX \text{ on prior at } s_H \,|\, \theta\,) \end{aligned}$$

If nY accepts, then $h_n \Rightarrow \{s_H\} \times \{s_N, s_H\}$ and thus $(n+1)X$ will not be on prior at any signal.

(ii) Suppose nY is not on prior at s_L.

If nY rejects, then since nY is not on prior at s_L, $(n+1)X$ will not be on prior at s_N (see above). Using (4.6), he cannot be on prior at s_L either. For s_H, it is easy to see that $r^{(n+1)X}(s_H, h_n) = r^{nX}(s_N, h_{n-1})$ in this case. Since nY accepts at s_N, $r^{nX}(s_N, h_{n-1}) \neq q_X$ and thus $(n+1)X$ cannot be on prior at s_H.

If nY accepts, then $(n+1)X$ will accept on all signals. Thus he may only be on prior at s_L. Since $A^{nY} \Rightarrow \{s_N, s_H\}$, we have, for this case,

$$\begin{aligned} &\mathrm{P}\Big((n+1)X \text{ on prior at } s_L \,\Big|\, \theta\Big) \\ &\qquad = \mathrm{P}(s_L \,|\, \theta)\,.\,\mathrm{P}\Big(\, h_{n-1} \Rightarrow \{s_H\} \times \{s_L, s_N\} \,|\, \theta\Big)\,.\,\mathrm{P}(s_N \vee s_H \,|\, \theta) \end{aligned}$$

(c) $S(A^{nY} \,|\, h_n) = \{s_H\}$

Using Corollary 2, nY is on prior at s_N and thus $h_{n-1} \Rightarrow \phi$.

If nY rejects, then $h_n \Rightarrow \{s_L, s_N\}$ which cannot lead to prior for any signal that $(n+1)X$ may get.

If nY accepts, then $h_n \Rightarrow \{s_H\}$ and thus $(n+1)X$ will be on prior at s_L (and not at other signals), with

$$\begin{aligned} &\mathrm{P}\Big((n+1)X \text{ on prior at } s_L \,\Big|\, \theta\Big) \\ &\qquad = \mathrm{P}(s_L \,|\, \theta) \,.\, \frac{\mathrm{P}(nY \text{ on prior at } s_N \,|\, \theta\,)}{\mathrm{P}(s_N \,|\, \theta)} \,.\, \mathrm{P}(s^{nY} = s_H \,|\, \theta) \\ &\qquad = \frac{p_1\, p_2}{p_3} \,.\, \mathrm{P}(nX \text{ on prior at } s_N \,|\, \theta\,) \end{aligned}$$

(d) $S(A^{nY} \,|\, h_n) = \phi$
In this case, nY may be on prior only at s_H, if at all.
Since R^{nY} contains no new information, $(n+1)X$ will be in the same position as nX. Thus, if the n^{th} generation is on prior at s_H, then (and only then), $(n+1)X$ will be on prior at s_H, with

$$\begin{aligned} \mathrm{P}\Big((n+1)X \text{ on prior at } s_H \,\Big|\, \theta\Big) &= \mathrm{P}(s_H \,|\, \theta) \,.\, \frac{\mathrm{P}(nY \text{ on prior at } s_H \,|\, \theta\,)}{\mathrm{P}(s_H \,|\, \theta)} \\ &= \mathrm{P}(nY \text{ on prior at } s_H \,|\, \theta\,) \\ &= \mathrm{P}(nX \text{ on prior at } s_L \,|\, \theta\,) \end{aligned}$$

Clearly, $(n+1)X$ cannot be on prior at any other signal.

The proof of part (2) and (3) stands complete. For each of the signals s_N and s_H, there is only one partition under which $(n+1)X$ can be on prior at this signal and thus it is easy to see that

$$\begin{aligned} &\mathrm{P}\Big(*_{s_N}^{(n+1)X}\Big) = p_3 \,.\, \mathrm{P}\Big(*_{s_H}^{nX}\Big) \\ \text{and}\quad &\mathrm{P}\Big(*_{s_H}^{(n+1)X}\Big) = \mathrm{P}\Big(*_{s_L}^{nX}\Big) \end{aligned}$$

Proof of part (1)
Combining at all cases where $(n+1)X$ can be at prior at s_L, we have the following:

$$\begin{aligned} \mathrm{P}\Big(*_{s_L}^{(n+1)X}\Big) &= \frac{p_1\, p_2}{p_3}\, \mathrm{P}\Big(*_{s_N}^{nX}\Big) \\ &\quad + \mathrm{P}(s_L \,|\, \theta) \,.\, \mathrm{P}(s_N \vee s_H \,|\, \theta) \,.\, \mathrm{P}\Big(\, h_{n-1} \Rightarrow \{s_H\} \times \{s_L, s_N\} \,|\, \theta\,\Big) \end{aligned} \tag{A.1}$$

If $h_{n-1} \Rightarrow \{s_H\} \times \{s_L, s_N\}$, then nY at (s_N, h_{n-1}) should accept. Since she clearly won't be on prior at s_N, she must be following $(n-1)X$. Thus, $a^{(n-1)X} = A$.
Moreover, $S(A^{(n-1)X} \,|\, h_{n-2}) = \{s_H\}$ which we prove by contradiction below:

Suppose $S(A^{(n-1)X} \,|\, h_{n-2}) = S$. Since $(n-1)X$ follows $(n-2)Y$ at s_L, $a^{(n-2)Y} = A$. Now, nX at s_L would also follow $(n-2)Y$ and accept. However, $h_{n-1} \times s_L \Rightarrow \{s_L, s_N\}$ and so nX should have rejected, which is a contradiction.

Suppose $S(A^{(n-1)X} \,|\, h_{n-2}) = \{s_N, s_H\}$. From Corollary 2, $(n-1)X$ must be on prior at s_N which implies that $h_{n-2} \Rightarrow \phi$. Then $h_{n-1} \Rightarrow \{s_N, s_H\} \neq \{s_H\} \times \{s_L, s_N\}$. Hence we have a contradiction.

Thus, $S(A^{(n-1)X} \,|\, h_{n-2}) = \{s_H\}$ and so

$$\mathrm{P}\Big(h_{n-1} \Rightarrow \{s_H\} \times \{s_L, s_N\} \,\Big|\, \theta\Big) = \mathrm{P}(s_H \,|\, \theta) \,.\, \mathrm{P}\Big(h_{n-2} \Rightarrow \{s_L, s_N\} \,\Big|\, \theta\Big)$$

Using a similar logic, one may check that if $h_{n-2} \Rightarrow \{s_L, s_N\}$ then $a^{(n-2)Y} = R$ and $R^{(n-2)Y} \Rightarrow \{s_L\}$ or $R^{(n-2)Y} \Rightarrow \{s_L, s_N\}$. Now,

$$\begin{aligned}
&\mathrm{P}\Big(\, h_{n-2} \Rightarrow \{s_L, s_N\} \,\Big|\, \theta\,\Big) \\
&\quad = \mathrm{P}\Big(h_{n-2} \Rightarrow \{s_L, s_N\} \wedge R^{(n-2)Y} \Rightarrow \{s_L\} \,\Big|\, \theta\,\Big) \\
&\qquad + \mathrm{P}\Big(\, h_{n-2} \Rightarrow \{s_L, s_N\} \wedge R^{(n-2)Y} \Rightarrow \{s_L, s_N\} \,\Big|\, \theta\,\Big) \\
&\quad = \mathrm{P}\Big(h_{n-3} \Rightarrow \{s_H\} \times \{s_L, s_N\} \,|\, \theta\Big) \,.\, \mathrm{P}(s_L \,|\, \theta) \\
&\qquad + \mathrm{P}\Big(h_{n-3} \Rightarrow \phi \wedge s^{(n-2)Y} \in \{s_L, s_N\} \,|\, \theta\Big) \qquad \text{(using Corollary 2)} \\
&\quad = \mathrm{P}\Big(h_{n-3} \Rightarrow \{s_H\} \times \{s_L, s_N\} \,|\, \theta\Big) \,.\, \mathrm{P}(s_L \,|\, \theta) \;+\; \frac{\mathrm{P}\Big(*_{s_N}^{(n-2)Y}\Big)}{\mathrm{P}(s_N \,|\, \theta)} \,.\, \Big[\mathrm{P}(s_L \,|\, \theta) + \mathrm{P}(s_N \,|\, \theta)\Big]
\end{aligned}$$

Using (A.1) to similarly write $\mathrm{P}\Big(*_{s_L}^{(n-1)X}\Big)$, we thus have

$$\mathrm{P}\Big(*_{s_L}^{(n+1)X}\Big) = \frac{p_1\, p_2}{p_3}\, \mathrm{P}\Big(*_{s_N}^{nX}\Big) \;+\; p_1\, p_2 \left[\mathrm{P}\Big(*_{s_H}^{nX}\Big) + \mathrm{P}\Big(*_{s_N}^{(n-2)X}\Big)\right]$$

■

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