Stable and efficient task assignment to pairs

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Abstract

We study a model in which agents are matched in pairs in order to undertake a task and have preferences over both the partner and the task they are assigned to. Preferences over partner-task pairs are non separable, but correlated in the following sense. Every agent has a set of tasks (possibly empty) that he likes to perform with a potential partner. This set is agent-specific and the set of tasks that agent i would like to perform with partner j may be different from the set that he likes to perform with agent k. Preferences are symmetric in the sense that the set of tasks that agent i likes to perform with agent j coincides with the set of tasks that agent j would like to perform with agent i. Individual preferences are such that all partner-task pairs belong to three indifference classes. The topmost indifference class consists of the pairs in which an agent is matched with a partner and a task they like to perform with each other. The second class contains all the pairs in which the agent is matched with a partner with whom she has a set of common good tasks, but the task assigned to the pair does not belong to this set. Finally, the bottom class contains all pairs in which the agent is matched with someone with whom she has no common good tasks. We propose an algorithm that identifies an assignment in the weak core and is Pareto efficient assignment. We show that the algorithm is strategy-proof. We conjecture that it is also pairwise strategy proof, meaning that pairs of agents do not have incentive to jointly misreport the set of common good tasks. We conjecture that the algorithm is group strategy-proof.

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Keywords: Matching; Core; Pareto-efficiency; Strategy-proofness

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1 Introduction

In many situations agents are matched in teams in order to perform a task. Agents have preferences over the task that they are asked to perform as well as over the partners that they are assigned to work with. Consequently, forming stable teams is important - it ensures that agents do not have opportunities to abandon their assignments and do better for themselves.

A centralized authority matches agents in pairs and assigns them a task. We are interested in mechanisms that satisfy stability, efficiency and provide incentives to agents to truthfully reveal their preferences. This problem shares some features with two-sided matching models like the roommate problem since agents have preferences over their potential partners. It also has common features with one-sided matching models like the house allocation model, the object assignment model because a task has to be assigned to each pair of agents. In this sense our model is a hybrid of the two classical models.

We consider a model where agents are matched in pairs to perform a task. For each pair of agents there is a set of tasks, possibly empty that the agents in the pair like to perform together. Preferences over tasks are dichotomous but not separable because the tasks that agent i would like to do with agent j can be different from the tasks that agent i would like to do with another partner $k \neq j$. The preferences over tasks are *pairwise symmetric* among agents because the set of tasks that agent i would like to perform with j coincides with the set of tasks that j would like to do with i. So for any pair of agents, there exists a set of common good tasks $T(i, j)$ (possibly empty).

Individuals have preferences over partner, task tuples. Individual preferences are such that all partner, task tuples can be placed in three indifference classes. The first indifference class consists of tuples where the agent is paired with a partner with whom she has a non empty set of common good tasks and is assigned a task from their set of common good tasks. The second class contains the tuples in which the agent is matched with a partner with whom she has a non empty set of common good tasks, but the task they are assigned does not belong to this set. Finally, the third class contains the tuples in which the agent is matched with someone with whom she has an empty set of common good tasks.

Our main results are as follows. We propose an algorithm, the object constrained maximum matching algorithm (OCMMA) that generates a weak core assignment. The OCMMA assignment is Pareto efficient. We also investigate the incentive properties of the OCMMA. Pairwise symmetry imposes a restriction on the preference profiles. We elaborate on this point in Section 5. We show that the OCMMA is strategy-proof. We also show that it satisfies pairwise strategy-proofness i.e. pairs of agents do not have incentive to jointly misreport the set of common good tasks. Finally we show that the OCMMA is is group strategy-proof.

Our paper considers a variant the model proposed in Nicoló et al. (2018). Both papers consider a model where agents have to be matched in pairs and each pair must be assigned an object. The agents have preferences over partner, project tuples. However the preference domain in Nicoló et al. (2018) is separable over partners and projects: the marginal component preferences over partners and projects are independent. The component preferences are assumed to be dichotomous. Thus the component preferences over projects do not depend on the partner assigned. The set of possible partners is partitioned into friends (good partners) and outsiders (bad partners). The set of projects is partitioned into good andbad projects. Therefore every partner, project pair can be placed in one of four indifference classes, depending on whether the partner and the project are good or bad. Friendship is mutual and transitive and so the set of agents can be partitioned into friendship components. Finally, preferences of friends satisfy *homophily*: for any pair of friends i, j, the good sets of projects for i and j satisfy the set inclusion property. The paper proposes the minimim demand priority algorithm (MDPA) to identify assignments in the weak core. The MDPA satisfies a restricted version of Pareto efficiency and is strategy proof.

However, in this paper we consider a model where the component preferences over projects are dependent on the partner assigned. Agents are friends only if they have a non-empty set of common good tasks. The preference domain and the proposed mechanism to find assignments in the weak core are different in the two models. Moreover in the current model, the algorithm generates Pareto efficient assignments and is group strategy proof.

The remainder of the paper is organized as follows. In Section 2 we present the model. Section 3 describes the OCMMA algorithm. Sections 4 and 5 describe the normative properties of the OCMMA. Section 6 concludes.

2 The Model

There is a finite set of agents $N = \{1, 2, \ldots, i, j, \ldots, n\}$ where n is even. The set of objects is denoted by A.

An assignment σ is a collection of triples (i, j, a) with the interpretation that the agent pair (i, j) is assigned object a. In order to ensure feasibility, we require each agent to be paired with one other agent and one object. In addition, each object is assigned exactly to one pair or left unassigned. We require all agents to be assigned a partner and an object. Finally let u^σ denote the set of unassigned objects in the assignment σ i.e. it is the set of objects a such that there does not exist agents i and j with $(i, j, a) \in \sigma$.

The set of all possible triples is denoted by T, where $T = \{(i, j, a) : i, j \in N, a \in A, i \neq j\}$. Let T_i denote the set of triples to which agent i belongs i.e. $T_i = \{i\} \times N \setminus \{i\} \times A$.

A partial assignment $\alpha \subset T$ such that

1. $\forall\, t, s \in \alpha,\, t \cap s = \emptyset$.
2. $|\alpha| \leq \frac{|N|}{2}$.

For a partial assignment α, let $N(\alpha)$ denote the set of agents in α and $A(\alpha)$ denote the set of objects in α.

Let Σ denote the set of partial assignments. A partial assignment α is an (complete) assignment if $|\alpha| = \frac{|N|}{2}$. Σ_c denotes the set of complete assignments. So $\Sigma_c \subset \Sigma$.

2.1 Preferences

Each agent has a preference ordering over possible partner, project pairs i.e. each agent has preferences over the set of triples that she belongs to.

Consider agent $i \in N$. Agent i has a set of tasks (possibly empty) that she likes to perform with a potential partner j, denoted by $S(i,j)$ where $S(i,j) \in 2^A$. Note that this set is agent-specific i.e. the set of tasks that agent i would like to perform with partner j may be different from the set that he likes to perform with agent k.

We assume that for any pair of agents $i, j \in N$, we have $S(i,j) = S(j,i)$. We refer to this assumption as pairwise alignment in the preferences of any pair of agents.

Each agent i has a preference ordering[1] $\succ_i$ over elements in T_i. The triples in T_i can be classified into three indifference classes as described below. The first indifference class is

$$H_1(\succ_i) = \{(i,j,a) \in T_i : a \in S(i,j)\}.$$

The second indifference class for agent i is

$$H_2(\succ_i) = \{(i,j,a) \in T_i \setminus H_1(\succ_i) : \exists b \in A \text{ such that } (i,j,b) \in H_1(\succ_i)\}.$$

Finally the third indifference class is $H_3(\succ_i) = T_i \setminus [\cup_{k=1}^{2} H_k(\succ_i)]$.

REMARK: For any pair of agents $i, j \in N$, we assume $S(i,j) = S(j,i)$. This implies

$$[(i,j,a) \in H_1(\succ_i) \iff (i,j,a) \in H_1(\succ_j)].$$

This means that $H_1(\succ_i) \cap T_j = H_1(\succ_j) \cap T_i$.

Let $\succ$ denote a preference profile where $\succ = (\succ_1, \ldots, \succ_n)$. For any preference profile $\succ$, we define the sets $H_k(\succ) = \cup_{i \in N} H_k(\succ_i)$ for $k \in \{1, 2, 3\}$.

2.2 Blocking and Stability

Let σ be an assignment. A coalition of agents $S \in 2^N$ is admissible if $|S|$ is even. The set of projects available to the coalition S is $A(S)$ is

$$A(S) = \{a \in A : (i,j,a) \in \sigma \text{ and either } i \in S \text{ or } j \in S\} \cup u^{\sigma}.$$

[1] An ordering is a binary relation which is complete, reflexive and transitive.

Note that for any agent i in S, the object assigned to i in σ is included in $X(S)$. Thus $|X(S)| \geq \frac{|S|}{2}$. Let $\sigma'(S)$ be a partial assignment such that $N(\sigma') = S$ and $A(\sigma') \subseteq A(S)$.

Definition 1 *An admissible coalition S strongly blocks assignment σ at preference profile $\succ$ if there exists a partial assignment $\sigma'(S)$ such that every agent in S is better off in $\sigma'(S)$.*

Definition 2 *An assignment is in the weak core if it is not strongly blocked by any admissible coalition.*

Definition 3 *An admissible coalition S weakly blocks assignment σ at preference profile $\succ$ if there exists a partial assignment σ' such that every agent in S is weakly better off and some agent in S is strictly better off.*

Definition 4 *An assignment is in the strong core if it is not weakly blocked by any admissible coalition.*

Another possibility is to consider a smaller set of available objects for coalition S,

$$A^0(S) = \{a \in A : (i, j, a) \in \sigma \text{ and } i, j \in S\} \cup u^{\sigma}$$

with $A^0(S)| \geq \frac{|S|}{2}$.

The set $A^0(S)$ includes an assigned object a in σ only if both agents who have been assigned a in σ are present in S. Since $A^0(S) \subseteq A(S)$, the previous notion of blocking using $A(S)$ allows for more deviations. Thus if an assignment cannot be blocked by an admissible coalition S using the objects in $A(S)$, it cannot be blocked by S using the objects in $A^0(S)$.

We can show by a simple example that the strong core is empty.

3 Object Constrained Maximum Matching Algorithm

We describe an algorithm to generate an assignment which we refer to as Object Constrained Maximum Matching Algorithm (OCMMA). We provide the formal description of the algorithm below and discuss it's properties in the next section.

Let $\succ^{\Sigma}$ be a complete order on the set Σ (set of partial assignments) which satisfies the following for all $\sigma, \tau \in \Sigma$,

$$N(\sigma) \supset N(\tau) \implies \sigma \succ^{\Sigma} \tau$$

Fix an arbitrary profile $\succ$. This profile induces the sets $\{H_1(\succ), H_2(\succ), H_3(\succ)\}$. We shall make assignments in the sets $H_1(\succ)$, $H_2(\succ)$, $H_3(\succ)$ in sequence. These will be labelled Steps 1 to 3 respectively. At the start of the generic step q where $q \in \{1, 2, 3\}$, the algorithm is provided two inputs: (i) the set of available projects and (ii) the set of unassigned agents in N.

Step 1: Here the set of available objects is A and the set of unassigned agents is N. An assignment is (fully) contained in $H_1(\succ)$ if all agents in the assignment are in the first indifference class according to $\succ$.[2] The set of all assignments with this property is $F^1(\succ)$ where,

$$F^1(\succ) = \{\sigma \in \Sigma : \sigma \subseteq H_1(\succ)\}.$$

Choose the assignment which is maximal in the set $F(\succ)$ according to the order $\succ^\Sigma$. Denote the maximal assignment by σ^1_{max}.

The set of agents who have been assigned a partner and an object in Step 1 is $N(\sigma^1_{max})$. Similarly the set of objects assigned is $A(\sigma^1_{max})$. After the completion of Step 1, the set of unassigned agents is $N \setminus N(\sigma^1_{max})$ and the set of available objects is $A \setminus A(\sigma^1_{max})$. Proceed to Step 2.

Step 2: Step 2 repeats Step 1 with the sets $A \setminus A(\sigma^1_{max})$ and $N \setminus N(\sigma^1_{max})$ but with an important difference.

An assignment is (fully) contained in $H_2(\succ)$ if all agents in the assignment are in the second indifference class according to $\succ$. The set of all assignments with this property is $F^2(\succ)$ where,

$$F^2(\succ) = \{\sigma \in \Sigma : \forall\ (i, j, a) \in \sigma, i, j \in N \setminus N(\sigma^1_{max}); a \in A \setminus A(\sigma^1_{max}) \text{ and } (i, j, a) \in H_2(\succ)\}.$$

Choose the assignment which is maximal in $F^2(\succ)$ according to $\succ^\Sigma$. Denote the maximal assignment by σ^2_{max}.

The sets of the assigned agents and objects in Step 2 are $N(\sigma^2_{max})$ and $A(\sigma^2_{max}$ respectively. Thus the set of unassigned agents after the completion of Step 2 is $N \setminus N(\sigma^1_{max} \cup \sigma^2_{max})$. The set of available objects is $A \setminus A(\sigma^1_{max} \cup \sigma^2_{max})$. Proceed to Step 3.

Step 3: Here the set of available objects is $A \setminus [\cup^2_{k=1} A_k]$. The set of unassigned agents is $N \setminus [\cup^2_{k=1} N_k]$. An assignment is (fully) contained in $H_3(\succ)$ if all agents in the assignment are in the third indifference class according to $\succ$. The set of all assignments with this property is $F^3(\succ)$ where,

$$F^3(\succ) = \{\sigma \in \Sigma : \forall\ (i, j, a) \in \sigma, i, j \in N \setminus N(\sigma^1_{max} \cup \sigma^2_{max}); a \in A \setminus A(\sigma^1_{max} \cup \sigma^2_{max}) \text{ and } (i, j, a) \in H_3(\succ)\}.$$

Choose the assignment which is maximal in $F^3(\succ)$ according to the order $\succ^\Sigma$. Denote the maximal assignment by σ^3_{max}.

The assignment generated by the OCMMA algorithm is $\sigma^1_{max} \cup \sigma^2_{max} \cup \sigma^3$. This completes the description of the algorithm.

[2] Note that for an agent i in triple t of the assignment, this means $t \in H_1(\succ)$.

4 Properties of the OCMMA

In this section, we show that the OCMMA algorithm satisfies several important properties.

4.1 Pareto Efficency

Theorem 1 *The OCMMA algorithm generates a Pareto efficient assignment at every* $\succ$.

Proof: Let $\succ$ be an arbitrary preference profile and σ be the assignment generated by the algorithm at $\succ$. We assume for contradiction that σ is not Pareto efficient. This implies that there exists $\tau \in \Sigma_c$ such that τ Pareto dominates σ.

Since σ is generated by the algorithm, we know that it can be decomposed into three partial assignments where each partial assignment corresponds to a step in the algorithm. The assignment σ can be decomposed into three partial assignments $(\sigma^1_{max}, \sigma^2_{max}, \sigma^3)$.

The set $N(\sigma^1_{max})$ consists of the agents who are matched in Step 1 of the OCMMA. The set A_1 consists of objects allocated in Step 1 of the OCMMA.

Consider the assignment τ and the partial assignment $\tau \cap H_1(\succ)$. The set $N(\tau \cap H_1(\succ))$ consists of agents who are in the first indifference class in the assignment τ.

Claim 1: $N_1(\tau \cap H_1(\succ)) = N(\sigma^1_{max})$.

Proof: Since τ Pareto dominates σ, we have $N(\sigma^1_{max}) \subseteq N(\tau \cap H_1(\succ))$. $F^1(\succ)$ is the set of partial assignments in Step 1 of the algorithm. Since $F^1(\succ) = \{\alpha \in \Sigma : \alpha \subseteq H_1(\succ)\}$, we have $\tau \cap H_1(\succ) \in F(\succ)$.

The partial assignment σ^1_{max} is chosen in Step 1 of the algorithm. So $\sigma^1_{max} \succ^{\Sigma} \tau \cap H_1(\succ)$. This implies $N(\sigma^1_{max}) \not\subset N(\tau \cap H_1(\succ))$. Thus $N(\sigma^1_{max}) = N(\tau \cap H_1(\succ))$.

We have shown that the assignment τ has the same set of agents in the first indifference class as the assignment σ.

The set $N(\sigma^2_{max})$ consists of agents in the second indifference class in the assignment σ. Similarly $N(\tau \cap H_2(\succ))$ is the set of agents in the second indifference class in the assignment τ.

Claim 2: $N(\tau \cap H_2(\succ)) = N(\sigma^2_{max})$.

Proof: We know $N_2 \subseteq N_2(\tau)$. This follows from the assumption that τ Pareto dominates σ. In Step 2 of the algorithm, σ^2_{max} is chosen as the maximal element from the set $F^2(\succ)$ according to the order $\succ^{\Sigma}$. There are two possibilities with respect to the partial assignment $\tau \cap H_2(\succ)$: either it belongs to the set $F^2(\succ)$ or it does not. We will consider these two cases separately.

(i) $\tau \cap H_2(\succ) \in F^2(\succ)$.

We have $\sigma^2_{max} \succ^{\Sigma} \tau \cap H_2(\succ)$. This implies $N(\sigma^2_{max}) \not\subset N(\tau \cap H_2(\succ))$. So $N(\sigma^2_{max}) = N(\tau \cap H_2(\succ))$.

(ii) $\tau \cap H_2(\succ) \notin F^2(\succ)$.

We will show that there exists a partial assignment $\alpha \in F^2(\succ)$ such that for any $(i, j, a) \in \tau \cap H_2(\succ)$, there exists $(i, j, x) \in \alpha$ with $x \in A \setminus A_1$. The partial assignment α contains the same set of agents as $\tau \cap H_2(\succ)$ i.e. $N(\alpha) = N(\tau \cap H_2(\succ))$. In addition, α contains the same set of pairs as the assignment $\tau \cap H_2(\succ)$, where each pair is assigned an object from the set $A \setminus A(\sigma^1_{max})$. Note that the two assignments are welfare equivalent for the agents in $\tau \cap H_2(\succ)$ as the set of pairs remains unchanged in the two assignments.

We know that $N(\tau \cap H_2(\succ)) \in N \setminus N(\tau \cap H_1(\succ))$. By Claim 1, we have $N(\tau \cap H_1(\succ)) = N(\sigma^1_{max})$. So $N(\tau \cap H_2(\succ)) \in N \setminus N(\sigma^1_{max})$. We construct the partial assignment α as follows. For any $(i, j, a) \in \tau \cap H_2(\succ)$,

1. If (i, j, a) such that $a \in A \setminus A(\sigma^1_{max})$, then $(i, j, a) \in \alpha$.
2. If (i, j, a) such that $a \notin A \setminus A(\sigma^1_{max})$, then $(i, j, x) \in \alpha$ for some $x \in A \setminus A(\sigma^1_{max})$.

In Step 2 of the algorithm, the set of agents is $N \setminus N(\sigma^1_{max})$ and the set of objects is $A \setminus A(\sigma^2_{max})$. Recall the set $F^2(\succ)$

$$F^2(\succ) = \{\beta \in \Sigma : \forall\,(i, j, a) \in \beta \,\text{such that}\, (i, j, a) \in H_2(\succ); i, j \in N \setminus N(\sigma^1_{max}) \,\text{and}\, a \in A \setminus A(\sigma^1_{max})\}.$$

By construction, $\alpha \in F^2(\succ)$. We have $\sigma^2_{max} \succ^{\Sigma} \alpha$. This implies $N(\sigma^2_{max}) \not\subset N(\alpha)$. Since $N(\alpha) = N(\tau \cap H_2(\succ))$, we have $N(\sigma^2_{max} \not\subset N(\tau \cap H_2(\succ))$. So $N(\sigma^2_{max}) = N(\tau \cap H_2(\succ))$.[3]

By Claims 1 and 2, we know $N_1(\tau) = N_1$ and $N_2(\tau) = N_2$. Thus τ does not Pareto dominate σ.

■

4.2 Stability

We show below the OCMMA generates an assignment in the weak core.

Lemma 1 *Consider the assignment σ generated by the OCMMA algorithm at the preference profile $\succ$. There does not exist $i, j \in N$ and $a \in A$ such that $(i, j, a) \in H_1(\succ)$ and i, j are in σ^3_{max}.*

[3]This follows from that fact that $N(\sigma^2_{max} \subseteq N(\tau \cap H_2(\succ))$.

Proof: Let $\succ$ be an arbitrary preference profile and σ be the assignment generated by the OCMMA algorithm at $\succ$. Note that σ can be decomposed into three partial assignments $(\sigma^1_{max}, \sigma^2_{max}, \sigma^3)$. We assume for contradiction that there exist $i, j \in N$ and $a \in A$ such that $(i, j, a) \in H_1(\succ)$ and i, j are in σ^3. There exists $b \in A \setminus A(\sigma^1_{max} \cup \sigma^2_{max})$ such that $(i, j, b) \in H_2(\succ)$. Since $A \setminus A(\sigma^1_{max} \cup \sigma^2_{max}) \subset A \setminus A(\sigma^1_{max})$, $b \in A \setminus A(\sigma^1_{max})$. Also $i, j \in N \setminus N(\sigma^1_{max})$ as i, j are in σ^3. Thus in Step 2 of the OCMMA, we have

$$\sigma^2_{max} \cup (i, j, b) \in F^2(\succ). \tag{1}$$

Also $\sigma^2_{max} \subset \sigma^2_{max} \cup (i, j, b)$. This contradicts the fact that σ^2_{max} is the maximal assignment in $F^2(\succ)$ according to $\succ^\Sigma$. ■

LEMMA **2** *Consider the assignment σ generated by the OCMMA algorithm at the preference profile $\succ$. There does not exist $i, j \in N \setminus N(\sigma^1_{max})$ and $a \in A \setminus A(\sigma^1_{max})$ such that $(i, j, a) \in H_1(\succ)$.*

Proof: Let $\succ$ be an arbitrary preference profile and σ be the assignment generated by the OCMMA algorithm at $\succ$. Note that σ can be decomposed into three partial assignments $(\sigma^1_{max}, \sigma^2_{max}, \sigma^3)$. We assume for contradiction that there exist $i, j \in N \setminus N(\sigma^1_{max})$ and $a \in A \setminus A(\sigma^1_{max})$ such that $(i, j, a) \in H_1(\succ)$. So the partial assignment $\sigma^1_{max} \cup (i, j, a) \in F^1(\succ)$. Also $\sigma^1_{max} \subset \sigma^1_{max} \cup (i, j, a)$. This contradicts the fact that σ^1_{max} is maximal in $F^1(\succ)$ according to $\succ^\Sigma$. ■

LEMMA **3** *Consider the assignment σ generated by the OCMMA algorithm at the preference profile $\succ$. There does not exist $i, j \in N \setminus N(\sigma^1_{max} \cup \sigma^2_{max})$ and $a \in A \setminus A(\sigma^1_{max} \cup \sigma^2_{max})$ such that $(i, j, a) \in H_2(\succ)$.*

Proof: Let $\succ$ be an arbitrary preference profile and σ be the assignment generated by the OCMMA algorithm at $\succ$. Note that σ can be decomposed into three partial assignments $(\sigma^1_{max}, \sigma^2_{max}, \sigma^3)$. We assume for contradiction that there exist $i, j \in N \setminus N(\sigma^1_{max} \cup \sigma^2_{max})$ and $a \in A \setminus A(\sigma^1_{max} \cup \sigma^2_{max})$ such that $(i, j, a) \in H_2(\succ)$. So the partial assignment $\sigma^2_{max} \cup (i, j, a) \in F^2(\succ)$. Also $\sigma^2_{max} \subset \sigma^2_{max} \cup (i, j, a)$. This contradicts the fact that σ^2_{max} is maximal in $F^2(\succ)$ according to $\succ^\Sigma$. ■

THEOREM **2** *The OCMMA algorithm generates an assignment in the weak core at every preference profile $\succ$.*

Proof: Let $\succ$ be an arbitrary preference profile and σ be the assignment generated by the OCMMA algorithm at $\succ$. We will show that σ cannot be blocked by an admissible coalition.

Suppose σ is strongly blocked by an admissible coalition i.e. there exists $S \subseteq N$ and a partial assignment $\sigma'(S)$ such that every agent in S is better off in $\sigma'(S)$.

We consider the following exhaustive possibilities: (I) $|S| = 2$ and (II) $S > 2$.

(I) $S = 2$. Let $S = \{i, j\}$ and $(i, k, a), (j, l, b) \in \sigma$. The set of projects available to S is $A(S) = \{a, b\} \cup u^\sigma$. Suppose S blocks σ via the partial assignment $\sigma'(S) = (i, j, c)$ where $c \in A(S)$. Since both agents i and j strictly improve by blocking, they are not in $H_1(\succ)$ in the assignment σ.

We consider the following exhaustive possibilities.

(i) Both agents i and j are in σ^3_{max}. Since agents i and j strictly improve by blocking, there exists $x \in A$ such that $(i, j, x) \in H_1(\succ)$. We know this is not possible by Lemma 1.

(ii) Agent i is in σ^3_{max} and j is not. In fact, j is in σ^2_{max}. Thus $i, j \in N \setminus N(\sigma^1_{max})$. Since both agents i and j strictly improve by blocking, it must be the case that $(i, j, c) \in H_1(\succ)$.[4] Note that $c \in A \setminus A(\sigma^1_{max})$. We have $i, j \in N \setminus N(\sigma^1_{max})$, $c \in A \setminus A(\sigma^1_{max})$ such that $(i, j, c) \in H_1(\succ)$. This is not possible by Lemma 2.

(iii) Agent j is in σ^3_{max} and i is not. This case is symmetric to Case (ii).

(iv) Agent i and j are not in σ^3_{max}. This implies that agents i, j are in σ^2_{max} and $i, j \in N \setminus N(\sigma^1_{max})$. Since agents i, j improve by blocking, $(i, j, c) \in H_1(\succ)$. Also $c \in A \setminus A \setminus A(\sigma^1_{max})$. This is not possible by Lemma 2.

(II) $|S| > 2$.[5]

The set of available projects to S is $A(S) = \{a \in A : (i, j, a) \in \sigma$ and either $i \in S$ or $j \in S\} \cup u^\sigma$. Suppose S blocks σ by the partial assignment $\sigma'(S)$. Consider agent $i \in S$ and let $(i, j, a) \in \sigma'(S)$. Since i strictly improves in $\sigma'(S)$, we have $i \notin N(\sigma^1_{max})$.

There are two subcases to consider.

(i) Agent i is in σ^2_{max}.

By assumption, $(i, j, a) \in \sigma'(S)$ where $j \in S$ and $a \in A(S)$. Since agent i strictly improves, we have $(i, j, a) \in H_1(\succ_i)$. Pairwise alignment implies $(i, j, a) \in H_1(\succ_j)$.

Suppose $a \in u^\sigma$. Then $i, j \in N \setminus N(\sigma^1_{max})$ and $a \in A \setminus A(\sigma^1_{max})$ such that $(i, j, a) \in H_1(\succ)$. We know this is not possible by Lemma 2.

So $a \in A(S) \setminus u^\sigma$. There exists $k \in S$ such that $(k, l, a) \in \sigma$. Since k also stricly improves by blocking, we know that the triple (k, l, a) does not belong to σ^1_{max}. So $a \in A \setminus A(\sigma^1_{max})$. Then there exist $i, j \in N \setminus N(\sigma^1_{max})$ and $a \in A \setminus A(\sigma^1_{max})$ such that $(i, j, a) \in H_1(\succ)$. This is not possible by Lemma 2.

[4] Agent j is in the second indifference class in σ and a strict improvement implies j moves to the first indifference class. Since $(i, j, c) \in H_1(\succ_j)$ implies $(i, j, c) \in H_1(\succ_j)$, we know agent i improves to the first indifference class.

[5] Note that $|S|$ is even.

(ii) Agent i is in σ^3_{max}.

By assumption, $(i, j, a) \in \sigma'(S)$ where $j \in S$ and $a \in A(S)$. There are two possibilities: either i improves to the first or the second indifference class by blocking.

Suppose agent i strictly improves to the first indifference class i.e. $(i, j, a) \in H_1(\succ_i)$. Pairwise alignment implies $(i, j, a) \in H_1(\succ_j)$. Let $a \in u^\sigma$. Then $i, j \in N \setminus N(\sigma^1_{max})$ and $a \in A \setminus A(\sigma^1_{max})$ such that $(i, j, a) \in H_1(\succ)$. We know this is not possible by Lemma 2. So $a \in A(S) \setminus u^\sigma$. There exists $k \in S$ such that $(k, l, a) \in \sigma$. Since k also stricly improves by blocking, we know that the triple (k, l, a) does not belong to σ^1_{max}. So $a \in A \setminus A(\sigma^1_{max})$. Then there exist $i, j \in N \setminus N(\sigma^1_{max})$ and $a \in A \setminus A(\sigma^1_{max})$ such that $(i, j, a) \in H_1(\succ)$. This is not possible by Lemma 2.

The only remaining possibility is that agent i improves to the second indifference class i.e. $(i, j, a) \in H_1(\succ_i)$. By pairwise alignment, we have $(i, j, a) \in H_1(\succ_j)$. Let $a \in u^\sigma$. Then $i, j \in N \setminus N(\sigma^1_{max} \cup \sigma^2_{max})$ and $a \in A \setminus A(\sigma^1_{max} \cup \sigma^2_{max})$ such that $(i, j, a) \in H_2(\succ)$. We know this is not possible by Lemma 3. So $a \in A(S) \setminus u^\sigma$. There exists $k \in S$ such that $(k, l, a) \in \sigma$. Since k also stricly improves by blocking, we know that the triple (k, l, a) does not belong to σ^1_{max}. So $a \in A \setminus A(\sigma^1_{max})$. We know $(i, j, a) \in H_2(\succ)$. Thus there exist $b \in A$ such that $(i, j, b) \in H_1(\succ)$. So we have $i, j \in N$ and $b \in A$ such that $(i, j, b) \in H_1(\succ)$ and agents i, j are in σ^3_{max}. This is not possible by Lemma 1.

■

5 Strategic Properties of the OCMMA

In this section we investigate the strategic properties of the OCMMA. We assume that the preferences of an agent are private information and can be misreported by an agent if she believes this could be advantageous. However the assumption of symmetry introduces some complications in the standard model as it imposes a restriction on preference *profiles*. Therefore individual announcements of preferences may lead to profile announcements that are inconsistent with symmetry. Below, we propose a variant of the OCMMA that satisfactorily deals with this issue. We show that it is strategy-proof. We also show that it is group strategy-proof.

We now describe a general mechanism in this setting. Each agent i announces a preference ordering $\succ_i$ that contains three pieces of information: $H_1(\succ_i)$, $H_2(\succ_i)$ and $H_3(\succ_i)$. Since $H_1(\succ_i)$ provides a complete description of $\succ_i$, it is sufficient for agent i to announce her first indifference class $H_1(\succ_i)$.

Consider the set of reports by the agents, $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$. The mechanism calculates the set $H_1(\tilde{\succ})$ as follows: for any pair of agents i, j, the mechanism considers triples containing i and j which are present in the intersection of the reported (first) indifference classes of i and j i.e. $H_1(\tilde{\succ}_i) \cap H_1(\tilde{\succ}_j)$. So

$$H_1(\tilde{\succ}) = \{(i,j,a) \in T : (i,j,a) \in H_1(\tilde{\succ}_i) \cap H_1(\tilde{\succ}_j)\}.$$

Note that considering intersection of the (first) indifference classes for every pair of agents ensures that the symmetry assumption is satisfied.

Let Γ_i denote the set of all possible announcements of agent i. Recall that Σ_c is the set of all complete (feasible) assignments.

An assignment rule is a map σ, $\sigma : \times_{i\in N}\Gamma_i \to \Sigma_c$.

5.1 Strategy Proofness

In this section, we show the OCMMA is strategy proof.

Definition **5** *An assignment rule σ is strategy-proof if there does not exist $\succ, \succ_i' \in \Gamma_i$ and $\succ_{-i} \in \times_{j\neq i}\Gamma_i$ such that $\sigma(\succ_i', \succ_{-i}) \succ_i \sigma(\succ_i, \succ_{-i})$.*

The notion of strategy-proofness is standard: an agent cannot strictly improve by misreporting her preferences for any possible announcements of the preferences of other agents. Our first result in this section is the following.

Theorem **3** *The OCMMA algorithm is strategy-proof.*

Proof: Consider an arbitrary agent i. Let $\succ_i$ be a preference ordering for agent i and $\succ$ be a preference profile. We will show that there does not exist $\tilde{\succ}_i$ such that $\sigma(\tilde{\succ}_i, \succ_{-i}) \succ_i \sigma(\succ)$. We therefore only need to show that i cannot benefit by misreporting $H_1(\tilde{\succ}_i)$. In view of the assumptions made, this is equivalent to checking that i cannot benefit by changing her first indifference class.

Let $(\sigma^1_{max}, \sigma^2_{max}, \sigma^3)$ be the assignment generated by OCMMA at the preference profile $\succ$.

Agent i can misreport in only one of two ways: (i) by announcing $H_1(\tilde{\succ}_i)$ such that $H_1(\tilde{\succ}_i) \subset H_1(\succ_i)$ i.e. by contracting the first indifference class and (ii) by announcing $H_1(\tilde{\succ}_i)$ such that $H_1(\succ_i) \subset H_1(\tilde{\succ}_i)$ i.e. by expanding the first indifference class.

Case (i): $H_1(\tilde{\succ}_i) \subset H_1(\succ_i)$.

Suppose there exists a misreport by agent i say, $\tilde{\succ}_i$ such that i strictly improves by reporting $\tilde{\succ}_i$.

Let $(\tilde{\sigma}^1_{max}, \tilde{\sigma}^2_{max}, \tilde{\sigma}^3)$ be the assignment generated by the mechanism at the preference profile $(\tilde{\succ}_i, \succ_{-i})$. Let $\tilde{N}_1$ and $\tilde{A}_1$ denote the set of agents and the set of objects allocated in Step 1 of the mechanism at the preference profile $(\tilde{\succ}_i, \succ_{-i})$.

For the preference profile $(\tilde{\succ}_i, \succ_{-i})$, the mechanism calculates the set $H_1(\tilde{\succ}_i, \succ_{-i})$ as follows: (i) For agents $k, l \in N\setminus\{i\}$, if $(k,l,a) \in H_k(\succ_k) \cap H_l(\succ_l)$ then $(k,l,a) \in H_1(\tilde{\succ}_i, \succ_{-i})$

and (ii) for agents $i \in N$ and $l \in N \setminus \{i\}$, if $(i,l,a) \in H_1(\tilde{\succ}_i) \cap H_1(\succ_l)$ then $(i,l,a) \in H_1(\tilde{\succ}_i, \succ_{-i})$.

This implies

$$H_1(\tilde{\succ}_i, \succ_{-i}) \subset H_1(\succ). \tag{2}$$

By Equation 2, $F^1(\tilde{\succ}_i, \succ_{-i}) \subset F^1(\succ)$.

We have assumed that agent i strictly improves by misreporting. So agent i does not belong to any triple in the partial assignment σ^1_{max} and $i \notin N_1$. There are two possibilities: (1) agent i improves to the first indifference class by misreporting and (2) agent i improves to the second indifference class by misreporting.

We will show that Cases 1 and 2 are not possible.

Case 1: Agent i cannot improve to the first indifference class by misreporting.

Claim 1: $\tilde{\sigma}^1_{max} = \sigma^1_{max}$.

Since i does not belong to any triple in σ^1_{max}, we have $\sigma^1_{max} \in H_1(\tilde{\succ}_i, \succ_{-i})$.[6]

Since $F^1(\tilde{\succ}_i, \succ_{-i}) \subset F^1(\succ)$ and $\sigma^1_{max} \in F^1(\tilde{\succ}_i, \succ_{-i})$, σ^1_{max} is the maximal assignment in $F^1(\tilde{\succ}_i, \succ_{-i})$ according to $\succ^{\Sigma}$.[7] Thus the mechanism chooses the partial assignment σ^1_{max} in Step 1 at the preference profile $(\tilde{\succ}_i, \succ_{-i})$. So $\tilde{\sigma}^1_{max} = \sigma^1_{max}$. Also $\tilde{N}_1 = N_1$ and $\tilde{A}_1 = A_1$. This implies agent i is not allocated in Step 1 of the mechanism at the preference profile $(\tilde{\succ}_i, \succ_{-i})$ i.e. $i \notin \tilde{N}_1$.

We have shown that agent i is not allocated in the first step of the mechanism when she misreports to $\tilde{\succ}_i$. Then agent i is allocated in Steps 2 or 3 of the mechanism at the preference profile $(\tilde{\succ}_i, \succ_{-i})$. The following claim shows that it is not possible that agent i improves to the first indifference class when she is allocated in Steps 2 or 3 of the mechanism at the preference profile $(\tilde{\succ}_i, \succ_{-i})$.

Claim 2: There does not exist a triple containing i, say (i,j,a) such that $(i,j,a) \in \tilde{\sigma}^2_{max} \cup \tilde{\sigma}^3$ and $(i,j,a) \in H_1(\succ_i)$.

We will prove the claim by contradiction. Suppose there exists a triple, $(i,j,a) \in \tilde{\sigma}^2_{max} \cup \tilde{\sigma}^3$ such that $(i,j,a) \in H_1(\succ)$. Then $i,j \in N \setminus N_1$ and $a \in A \setminus A_1$. Since $\tilde{N}_1 = N_1$ and $\tilde{A}_1 = A_1$, we have $i,j \in N \setminus N_1$ and $a \in A \setminus A_1$.

Consider the assignment σ generated by the mechanism at preference profile $\succ$. There exist $i,j \in N \setminus N_1$ and $a \in A \setminus A_1$ such that $(i,j,a) \in H_1(\succ)$. We know this is not possible by Lemma 2.

Case 2: Agent i cannot improve to the second indifference class by misreporting.

[6] All agents who belong to some triple in σ^1_{max} do not misreport.

[7] Suppose not. Then σ^1_{max} is not the maximal assignment in $F^1(\succ)$ according to $\succ^{\Sigma}$.

Since $\tilde{\sigma}^1_{max} = \sigma^1_{max}$ and i does not belong to any triple in σ^1_{max}, we know agent i is not allocated in the first step of the mechanism at the preference profile $(\tilde{\succ}_i, \succ_{-i})$. Thus agent i is allocated in the second or third step of the mechanism at $(\tilde{\succ}_i, \succ_{-i})$.

Claim 3: $\tilde{\sigma}^2_{max} = \sigma^2_{max}$.

Proof: We know $\tilde{\sigma}^1_{max} = \sigma^1_{max}$, $\tilde{N}_1 = N_1$ and $\tilde{A}_1 = A_1$. Also $H_2(\tilde{\succ}_i, \succ_{-i}) \subseteq H_2(\succ)$.
Then

$$F^2(\tilde{\succ}_i, \succ_{-i}) \subseteq F^2(\succ).$$

We have shown in Case 1 that agent i cannot improve to the first indifference class by misreporting. This implies that agent i does not belong to any triple in σ^2_{max}.[8] Thus any triple $(k, l, a) \in \sigma^2_{max}$ has the property that $k, l \in N \setminus \{i\}$ and $(k, l, a) \in H_2(\tilde{\succ}_i, \succ_{-i})$.

So $\sigma^2_{max} \in F^2(\tilde{\succ}_i, \succ_{-i})$. Since $F^2(\tilde{\succ}_i, \succ_{-i}) \subseteq F^2(\succ)$, we have σ^2_{max} is the maximal assignment in $F^2(\tilde{\succ}_i, \succ_{-i})$ according to $\succ^{\Sigma}$. Thus $\tilde{\sigma}^2_{max} = \sigma^2_{max}$. This completes the proof of Claim 3.

Claim 3 and the fact that it is not possible for agent i to improve to the first indifference class by misreporting (Case 1) imply that agent i does not belong to any triple in σ^2_{max}. Thus i does not belong to any triple in $\tilde{\sigma}^2_{max}$. This means that agent i is not allocated in Step 2 of the mechanism at $(\tilde{\succ}_i, \succ_{-i})$.

The only remaining possibility is agent i is allocated in Step 3 of the mechanism at $(\tilde{\succ}_i, \succ_{-i})$. Let $(i, j, a) \in \tilde{\sigma}^3$. Then $i, j \in N \setminus [\tilde{N}_1 \cup \tilde{N}_2]$ and $a \in A \setminus [\tilde{A}_1 \cup \tilde{A}_2]$. By Claims 1 and 3, we have $i, j \in N \setminus [N_1 \cup N_2]$ and $a \in A \setminus [A_1 \cup A_2]$.

We have assumed that agent i improves to the second indifference class by misreporting i.e. $(i, j, a) \in H_2(\succ_i)$. Thus there exists a triple (i, j, a) such that $i, j \in N \setminus [N_1 \cup N_2]$, $a \in A \setminus [A_1 \cup A_2]$ and $(i, j, a) \in H_2(\succ)$. This is not possible by Lemma 3.

Cases 1 and 2 show that it is not possible for agent i to improve to the first or the second indifference class by the misreport $\tilde{\succ}_i$.

Case (ii): $H_1(\succ_i) \subset H_1(\tilde{\succ}_i)$.
We will show that the misreport by agent i where she expands her first indifference class, leads to no change in her assignment.

Consider the preference profile $(\tilde{\succ}_i, \succ_{-i})$. The mechanism considers the following sets for pairs of agents: $H_1(\tilde{\succ}_i) \cap H_1(\succ_j) = H_1(\succ_i) \cap H_1(\succ)$ for agents i, j (where $j \neq i$) and $H_1(\succ_k) \cap H_1(\succ_l)$ for agents $k, l \in N \setminus \{i\}$. So

$$H_1(\tilde{\succ}_i, \succ_{-i}) = H_1(\succ). \tag{3}$$

[8]Suppose i belongs to some triple in σ^2_{max}. Then agent i has to move to the first indifference class to be able to strictly improve by misreporting.

We consider the three steps of the OCMMA at $(\tilde{\succ}_i, \succ_{-i})$. In the first step, we have $F^1(\tilde{\succ}_i, \succ_{-i}) = F^1(\succ)$ (by Equation 3). Thus σ^1_{max} is the assignment chosen in this step at the preference profile $(\tilde{\succ}_i, \succ_{-i})$. Since σ^1_{max} is chosen in Step 1, we have $F^2(\tilde{\succ}_i, \succ_{-i}) = F^2(\succ)$ in the second step. Thus the assignment chosen in the second step is σ^2_{max}. In the third step, the assignment chosen is σ^3.

We have shown that there is no change in the assignment generated by the OCMMA when agent i misreports. So agent i does not improve by the misreport.

■

5.2 Group strategy proofness

In this section, we show the OCMMA satisfies group strategy proofness.

Let $\succ$ denote a preference profile where $\succ = (\succ_1, \ldots, \succ_n)$. Consider a coalition of agents $C \subseteq N$. Let $\succ_C$ denote the collection of preferences of agents in C i.e. $\succ_C = \{\succ_i\}_{i \in C}$. Then the preference profile $\succ$ can be written as $(\succ_C, \succ_{N \setminus C})$.

Definition **6** *An assignment rule σ is manipulable at $\succ$ by a coalition $C \subseteq N$ if there exists $\succ'_C \in \times_{i \in C} \Gamma_i$ such that $\sigma(\succ'_C, \succ_{N \setminus C}) \succ_i \sigma(\succ)$ for all $i \in C$. An assignment rule σ is group strategy-proof if it is not manipulable by any coalition $C \subseteq N$.*

Since the OCMMA assignment is Pareto efficient, the grand coalition N cannot obtain an assignment where all agents are strictly better off by misreporting. Thus it is sufficient to consider coalitions C where $C \subset N$.

Theorem **4** *The OCMMA algorithm is group strategy-proof.*

Proof: Consider an arbitrary coalition $C \subseteq N$. Let $\succ$ be a preference profile. We assume for contradiction that there exists $\tilde{\succ}_C \in \times_{i \in C} \Gamma_i$ such that $\sigma(\tilde{\succ}_C, \succ) \succ_i \sigma(\succ)$ for all $i \in C$.

Let $\sigma = (\sigma^1_{max}, \sigma^2_{max}, \sigma^3_{max})$ be the assignment generated by the OCMMA at the preference profile $\succ$. Let N_1 be the set of agents who are allocated in Step 1 of the mechanism at $\succ$ i.e. N_1 is the set of agents in σ^1_{max}. Let A_1 denote the set of objects allocated in Step 1 of the mechanism at $\succ$, so A_1 is the set of objects which have been allocated in the partial allocation σ^1_{max}. Similarly, N_2 and A_2 denote the set of agents and objects which have been allocated in Step 2 of the OCMMA at profile $\succ$ (belong to the partial allocation σ^2_{max}).

Let $\tilde{\sigma} = (\tilde{\sigma}^{max}_1, \tilde{\sigma}^2_{max}, \tilde{\sigma}^3_{max})$ be the assignment generated by the OCMMA at the preference profile $\tilde{\succ}_C, \succ_{N \setminus C}$. Let $\tilde{N}_1$ be the set of agents in $\tilde{\sigma}^1_{max}$. Let $\tilde{N}_1$ and $\tilde{A}_1$ denote the set of agents and the set of objects allocated in Step 1 of the mechanism at $(\tilde{\succ}_C, \succ_{N \setminus C})$.

The first observation is that $C \cap N_1 = \emptyset$. This is because all agents in C must strictly improve when they misreport, and the agents in N_1 are in the first indifference class.

LEMMA 4 *For any pair of agents $i, j \in C$ such that $(i,j,a) \in \tilde{\sigma}$, we have $(i,j,a) \in H_1(\succ)$.*

Proof: Since agents i, j strictly improve by misreporting, we know $S(i,j)$ is non empty and there exists $x \in A$ such that $(i,j,x) \in H_1(\succ)$. By Lemma 1, we know that both agents i and j cannot belong to σ^3_{max}. So there is at least one agent from $\{i,j\}$ who belongs to the second indifference class in σ. Suppose i is the agent who belongs to the second indifference class in σ. This implies that agent i is in the first indifference class in $\tilde{\sigma}$ i.e. $(i,j,a) \in H_1(\succ_i)$. By pairwaise alignment, we have $(i,j,a) \in H_1(\succ_j)$. Thus $(i,j,a) \in H_1(\succ)$. ■

We will show that the assignment generated in the first step of the OCMMA at the preference profiles $\succ$ and $(\tilde{\succ}_C, \succ_{N\setminus C})$ is the same.

Claim 1: $\sigma^1_{max} = \tilde{\sigma}^1_{max}$.

Proof: We will prove the claim by contradiction i.e. we assume $\sigma^1_{max} \neq \tilde{\sigma}^1_{max}$.

We know $C \cap N_1 = \emptyset$. This means that the agents in N_1 do not misreport and $(\tilde{\succ}_C, \succ_{N\setminus C}) = (\tilde{\succ}_C, \succ_{N_1}, \succ_{N\setminus C\cup N_1})$. This implies $\sigma^1_{max} \in F^1(\tilde{\succ}_C, \succ_{N\setminus C})$.[9]

Since $\tilde{\sigma}^1_{max}$ is the partial allocation chosen in the first step of the OCMMA at profile $(\tilde{\succ}_C, \succ_{N\setminus C})$, we have $\tilde{\sigma}^1_{max} \in F^1(\tilde{\succ}_C, \succ_{N\setminus C})$ and $\tilde{\sigma}^1_{max} \succ^\Sigma \sigma^1_{max}$ (Statement 1).

Similarly, the partial allocation σ^1_{max} belongs to $F^1(\succ)$ and is the maximal element in $F^1(\succ)$ according to $\succ^\Sigma$. Thus we can conclude $\tilde{\sigma}^1_{max} \notin F^1(\succ)$. To see this, suppose not i.e. $\tilde{\sigma}^1_{max} \in F^1(\succ)$. Then $\sigma^1_{max} \succ^\Sigma \tilde{\sigma}^1_{max}$ and this contradicts Statement 1 above.

We have $\sigma^1_{max} \in F^1(\succ)$ and $\tilde{\sigma}^1_{max} \notin F^1(\succ)$. Recall that $F^1(\succ) = \{\sigma_\in \Sigma : \sigma \subseteq H_1(\succ)\}$. So there exists a triple $(i,j,a) \in \tilde{\sigma}^1_{max}$ such that $(i,j,a) \notin H_1(\succ)$ and $(i,j,a) \in H_1(\tilde{\succ}_C, \succ_{N\setminus C})$. Then it must be that case that $i, j \in C$. This follows from $(\tilde{\succ}_C, \succ_{N\setminus C}) = (\tilde{\succ}_C, \succ_{N_1}, \succ_{N\setminus C\cup N_1})$ and $H_1(\tilde{\succ}_C, \succ_{N_1}, \succ_{N\setminus C\cup N_1}) = [\cup_{i\in N_1} H_1(\succ_i)] \cup [\cup_{i\in C} H_1(\tilde{\succ}_i)] \cup [\cup_{i\in N\setminus C\cup N_1} H_1(\succ_i)]$.

The above arguments show that there exists $i, j \in C$ such that $(i,j,a) \in \tilde{\sigma}$ and $(i,j,a) \notin H_1(\succ)$. This contradicts Lemma 4. ■

Claim 2: For all $(i,j,a) \in \tilde{\sigma}$, [if $(i,j,a) \in H_1(\succ)$ then $(i,j,a,) \in \tilde{\sigma}^1_{max}$].

Proof: We assume for contradiction that Claim 2 is false i.e. there exists $(i,j,a) \in \tilde{\sigma}$ such that $(i,j,a) \in H_1(\succ)$ and $(i,j,a) \notin \tilde{\sigma}^1_{max}$. By Claim 1, we have $\tilde{\sigma}^1_{max} = \sigma^1_{max}$. So $(i,j,a) \notin \sigma^1_{max}$ and $(i,j,a) \in H_1(\succ)$. This contradicts Lemma 2. ■

Claim 2 establishes that there does not exist a pair of agents i, j in C such that $(i,j,a) \in \tilde{\sigma}$ who are not matched in Step 1 of the OCMMA at the profile $(\tilde{\succ}_C, \succ_{N\setminus C})$ and $(i,j,a) \in H_1(\succ)$. This means a pair $(i,j,a) \in \tilde{\sigma}^2_{max} \cup \sigma^3_{max}$ can never lead to an improvement to the first indifference class.

[9]This is because $F^1(\tilde{\succ}_C, \succ_{N_1}, \succ_{N\setminus C\cup N_1}) = \{\sigma \in \Sigma : \sigma \subseteq H_1(\tilde{\succ}_C, \succ_{N_1}, \succ_{N\setminus C\cup N_1})\}$ and $H_1(\tilde{\succ}_C, \succ_{N_1}, \succ_{N\setminus C\cup N_1}) = [\cup_{i\in N_1} H_1(\succ_i)] \cup [\cup_{i\in C} H_1(\tilde{\succ}_i)] \cup [\cup_{i\in N\setminus C\cup N_1} H_1(\succ_i)]$. Note that $\sigma^1_{max} \in \cup_{i\in N_1} H_1(\succ_i)$.

Claim 3: $\sigma^2_{max} = \tilde{\sigma}^2_{max}$.

Proof: We will prove the claim by contradiction, so we assume $\sigma^2_{max} \neq \tilde{\sigma}^2_{max}$.

By Claim 1, we have $\sigma^1_{max} \neq \tilde{\sigma}^1_{max}$, $N_1 = \tilde{N}_1$ and $A_1 = \tilde{A}_1$. By Claims 1 and 2, we can conclude that no agent in C can improve to the first indifference class by misreporting.

Consider set of the agents (N_2) who are matched in Step 2 of the OCMMA at profile $\succ$. It must be the case that $N_2 \cap C = \emptyset$. This follows from the fact that no agent in C can improve to the first indifference class by misreporting and an agent in N_2 can only improve if she moves to the first indifference class. Thus $(\tilde{\succ}_C, \succ_{N\setminus C}) = (\tilde{\succ}_C, \succ_{N_1}, \succ_{N_2}, \succ_{N\setminus C\cup N_1\cup N_2})$.

We will now show that $\sigma^2_{max} \in F^2(\tilde{\succ}_C, \succ_{N\setminus C})$. Consider a triple $(i,j,a) \in \sigma^2_{max}$. Since $N_1 = \tilde{N}_1$ and $A_1 = \tilde{A}_1$, we have $i,j \in N \setminus \tilde{N}_1$ and $a \in A \setminus \tilde{A}_1$. Also $(i,j,a) \in H_2(\tilde{\succ}_C, \succ_{N\setminus C})$.[10] Thus from the definition of set $F^2(\tilde{\succ}_C, \succ_{N\setminus C})$, we get $(i,j,a) \in F^2(\tilde{\succ}_C, \succ_{N\setminus C})$.

Since $\tilde{\sigma}^2_{max}$ is chosen in Step of the OCMMA at profile $(\tilde{\succ}_C, \succ_{N\setminus C})$, we have $\tilde{\sigma}^2_{max} \succ^{\Sigma} \sigma^2_{max}$ (Statement 2).

We know $\sigma^2_{max} \in F^2(\succ)$. Statement 2 and the assumption that $\sigma^2_{max} \neq \tilde{\sigma}^2_{max}$ imply $\tilde{\sigma}^2_{max} \notin F^2(\succ)$. Thus there exists $(i,j,a) \in \tilde{\sigma}^2_{max}$ such that $(i,j,a) \notin H_2(\succ)$ and $(i,j,a) \in H_2(\tilde{\succ}_C, \succ_{N\setminus C})$. Thus we can conclude that $i,j \in C$.

We have $(i,j,a) \in \tilde{\sigma}$ such that $(i,j,a) \notin \tilde{\sigma}^1_{max}$ (this is because $(i,j,a) \in \tilde{\sigma}^2_{max}$). By Claim 2, we have $(i,j,a) \notin H_1(\succ)$.

The above arguments establish that there exist agents $i,j \in C$ such that $(i,j,a) \in \tilde{\sigma}$ and $(i,j,a) \notin H_1(\succ)$. This contradicts Lemma 4. ■

By Claims 1 and 3, we have $\sigma^3_{max} = \tilde{\sigma}^3_{max}$. Thus in each step, the allocation generated by the OCMMA in at the preference profiles $\succ$ and $(\tilde{\succ}_C, \succ_{N\setminus C})$. Thus the agents in C cannot strictly improve by misreporting.

This completes the proof of the theorem.

■

6 Conclusion

In this paper, we have investigated a class of matching models where agents have to be matched in pairs with a project. We provide a domain restriction on partner, project pairs that guarantees the existence of an assignment in the weak core and is Pareto efficient. We provide an algorithm, the OCMMA that generates an assignment in the weak core at every preference profile. It satisfies Pareto efficiency. In addition it is strategy-proof. It also satisfies group strategy-proofness.

[10] This is because
$H_2(\tilde{\succ}_C, \succ_{N\setminus C}) = [\cup_{i\in N_2} H_2(\succ_i)] \cup [\cup_{i\in N_1} H_2(\succ_i)] \cup [\cup_{i\in C} H_2(\tilde{\succ}_i)] \cup [\cup_{i\in N\setminus N_1\cup N_2\cup C} H_2(\succ_i)]$.

In future, we hope to extend our work to teams of general size. Our current results on pairs do not extend in a straightforward manner to the more general case.

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