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Prize Sharing Rules in Collective Contests: Towards Strategic Foundations

Dhritiman Gupta

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Indian Statistical Institute, Delhi
Economics and Planning Unit
7, S. J. S. Sansanwal Marg, New Delhi, 110016, India

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†Indian Statistical Institute, New Delhi, Email: gupta.dhritiman@gmail.com

1 Introduction

Collective contests are situations where agents organize into groups to compete over a given prize. Such situations are quite common: funds to be allocated among different departments of an organization, team sports, projects to be allocated among different divisions of a firm, regions within a country vying for shares in national grants, party members participating in pre-electoral campaigns, disputes between tribes over scarce resources.

Prizes in such contests may be purely private, e.g. money. Or the prizes may have some public characteristic like reputation or glory for the winning team. In this paper we focus on purely private prizes. For prizes with public characteristics the reader may refer to Baik (2008), Balart et al. (2016).

One essential feature of collective contests is that a groups' performance depends on the individual contribution of its members. Departments in universities usually receive funds depending on the publication record of the department, which in turn depends on the individual publication of its members. So the group needs to coordinate and establish some rules regarding its internal organization, in particular how to share the prize in case of success in a contest. In this study we focus on two such important sharing rules. One such prize sharing rule, which was proposed by Nitzan (1991) suggests the following way of sharing the prize within the group, if the group wins the collective contest:

$$(1-\alpha_i)\frac{x_{ki}}{X_i} + \alpha_i\frac{1}{n_i} \tag{1}$$

where x_{ki} is the effort put in by the k^{th} member of group i, X_i is the total effort of group i and n_i is the size of group i. α_i is weight put on egalitarian sharing of the prize within the group and $1-\alpha_i$ is the weight put on a sharing rule, which rewards higher efforts within the group, thereby inducing intra-group competition. Rule N introduces intra-group externalities by making each members reward depend on efforts of all other members of the group.

This prize sharing rule has been extensively studied in the literature on collective contests, see e.g. Flamand et al. (2015). The popularity of this rule lies in its intuitive appeal. It combines two extreme forms of internal organization, capturing the tension between intra-group competition and the tendency to free ride on efforts of other group members. Despite its popularity the rule is ad hoc. This paper tries to provide strategic foundations to these prize sharing rules, which we denote N throughout the paper.

In order to do that, we introduce another rule E, which represents cooperative behavior within a group. According to this rule, the net expected group payoff is divided equally among all group members, thereby aligning individual and group interests. In other words, using rule E helps to internalize all intra-group externalities. It is defined as follows:

$$\frac{1}{n_i}(P_i(X_i, X_j) - X_i) \tag{2}$$

where $P_i(X_i, X_j)$ is the probability with which group i wins the prize and X_i is aggregate effort of the group i.

We consider a situation in which a group has access to these two prize sharing rules E and N. We construct a two stage game where the groups choose between the rules simultaneously in the first stage. The rules having been chosen, the individual group members simultaneously put in efforts in the second stage. The question we ask is whether this game has any subgame perfect Nash equilibrium in which rule N is chosen by any group.

We find that both groups choosing E always constitutes a subgame perfect Nash equilibrium in pure strategies. However, we also uncover a class of games, that we call Coordination games, in which both groups choosing N is also a subgame perfect Nash equilibrium in pure strategies.

The reason why such Coordination games arise is that, when the weight on intra-group competition is high enough in both groups, a situation of strategic uncertainty is created between the groups. In these cases rule N is a powerful instrument to increase chances of winning the contest. If a particular group chooses N, it generates high efforts and wins the contest with a high probability. The other group should, in that case, choose N to increase its own efforts to counter the first group and keep its probability of winning from falling too much. The upward spiral in efforts comes at the cost of a vastly reduced net surplus [1], which harms both groups in terms of payoffs.

In fact, we go on to show that the Nash equilibrium in which N is chosen payoff dominates the one in which both groups choose E. So it is not survive the equilibrium selection criterion of payoff dominance, first suggested in Harsanyi et al. (1988).

However, when we consider criteria of equilibrium selection, which are based on the "riskiness" of the equilibrium point, the results change. First, we consider the selection criterion of risk dominance as suggested in Harsanyi (1995). We are able to provide necessary and sufficient conditions for equilibrium profile NN to risk dominate EE. We show the existence of such games by considering a special subclass of coordination games we call symmetric coordination games.

Finally, we consider a equilibrium selection criterion called the Security Principle. According to it the players choose the strategy that maximizes their minimum possible payoff, see e.g. Van Huyck et al. (1990). We show that equilibrium profile NN is always selected by this criterion.

Even though different equilibrium selection criterion make different prescriptions, the fact that equilibrium NN is selected by some of them helps us establish that there exists a strategic basis to the prize sharing rules N introduced by Nitzan (1991).

The paper is structured as follows. In Section 2 we discuss the relevant literature. In Section 3 we describe the model. In Section 4 we analyze the second stage of the game, where individuals make effort choices. In Section 5 we analyze the first stage of the game where the group leaders make their choice between E and N. In Section 6 we study the robustness of the equilibria to equilibrium refinement criteria of Payoff Dominance and Risk Dominance and the Security Principle. Section 7 contains a discussion of the results and things left out

[1] Surplus minus total efforts put in the contest

of the main body. Section 8 concludes. All proofs can be found in the Appendix 1.

2 Literature

The literature on prize sharing rules (rule N) in collective contests started with an influential paper by Nitzan (1991). Thereafter, these class of rules have been widely applied to the analysis of group competition. The popularity of this class of rules owes to the fact that it very nicely captures effects of intra-group competition on the welfare of the groups in the collective contest. For an extensive survey the reader can look at Flamand et al. (2015).

These rules have been used to study two very important features of collective contests *(a) Monopolization* and *(b) Group Size Paradox (GSP).*

In two group contests Davis and Reilly (1999) uses the term monopolization to refer to a situation where one group withdraws from the competition. Ueda (2002) extended the idea of monopolization to multi-group contests. In our analysis monopolization is possible but plays a supplementary role with regard to the main aim of the paper.

Group Size Paradox (GSP) is a situation where a smaller group outperforms a larger one in terms of payoffs. The notion dates back to the seminal work by Olson (1965), who focused on the detrimental effects of free riding within large groups. Our focus not being on GSP, the interested reader is referred to Flamand et al. (2015).

There is an extensive literature on strategic choice of sharing rules under different restrictions on publicness of the prize and the sharing rule itself. One part of the literature (Baik (1994), Lee (1995), Noh (1999), Ueda (2002)) focuses on the case where the prize can be shared at most proportionally to individual contributions. Another part of the literature weakens this assumption (Baik and Shogren (1995), Lee and Kang (1998), Baik and Lee (1997), Baik and Lee (2001), Lee and Kang (1998), Gürtler (2005)) and allows transfers from worse performing group members to better performing group members. A recent strand of literature, (Nitzan and Ueda (2014), Vázquez-Sedano (2014)) has studied cost sharing schemes with purely public prizes, where prize sharing is not possible.

There are a few other papers, which study the effect of publicness of the prize on group welfare. The purely public prizes case, where the prize sharing rules do not apply, has been analyzed by (Baik (1993), Baik (2008)). Esteban and Ray (2001) considers the case of a mixed private-pubic goods, with exogenous and fully egalitarian sharing rules, which was later endogenized in a private information framework in Nitzan and Ueda (2011). Balart et al. (2016) analyze the case of a mixed public-private prize with strategic choice of sharing rules in a complete information setting.

This paper differs in focus from all the strands of literature cited above, in that it attempts to provide non-cooperative foundations to these prize sharing rules N instead of studying its effects on group welfare. We assume the prize to be fully private and we also abstract from strategic choice of sharing rules. Instead we provide the groups a strategic choice between an exogenous and internally non-cooperative prize sharing rule N and an intra-group cooperative prize sharing rule E and ask whether a group chooses rule N in any subgame perfect Nash equilibrium of an appropriately defined two stage game.

There are two papers, which analyze the choice between E and N, when both options are available. Cheikbossian (2012) questions the validity of GSP, by giving individual members of the groups a choice between N with $\alpha_i = 1$, which captures maximal internal non-cooperation and cooperative rule E. He goes onto show that it is easier to sustain E as a subgame perfect Nash equilibrium within the larger group, where the punishment used for a group member deviating from E is that other group members deviate to N then on.

The focus of our paper is different. We focus on how the presence of different options creates strategic uncertainty between the groups and why that may lead to N being chosen by both groups in equilibrium. In our model, individuals cannot deviate from the sharing rule chosen by their leaders. Cheikbossian (2012), on the other hand, focuses on the question of the ease of maintaining cooperation within a group, given that non-cooperative options are present for each individual member.

To the best of our knowledge, the only other paper with the motivation to find a strategic foundation for rule N is Ursprung (2012). He considers two groups of the same size. He gives the groups a choice among E, and the two extreme points of rule N, i.e. $\alpha_i = 0$ and $\alpha_i = 1$. He goes onto show that in an evolutionary game, N with $\alpha_i = 0$ crowds out E in the long run. In our model, there is no choice between different points of rule N. Also groups can be of different sizes. Besides, our study does not take the evolutionary game route. Instead, we try to characterize which parts of rule N can arise in equilibrium of an appropriately constructed two stage game. As our paper differs on important features from Ursprung (2012), our analysis can be considered to be complementary to theirs.

3 Model

There are two groups A and B, of size n_i , $i = \{A, B\}$, where $n_i \in \{2, 3, ...\}$. We assume without loss of generality that group B is at least as large as A, i.e. $n_B \geqslant n_A$. We denote the total number of agents as N, so that $N = n_B + n_A$. All agents are assumed to be risk neutral.

Both groups compete for a purely private prize, the size of which we normalize to 1. The groups cannot write binding contracts among themselves regarding sharing the prize. Instead they indulge in a rent-seeking Tullock contest spending efforts trying to win the contest. The outcome of this contest depends on the aggregate effort spent by the two groups. Let x_{ki} denote the effort level of individual k belonging to group i, where effort costs are $C(x_{ki})$. In particular $C(x_{ki}) = x_{ki}$. The aggregate effort of group i is $X_i = \sum_{k=1}^{n_i} x_{ki}$. The aggregate effort of the groups in the contest is denoted X, i.e., $X = X_1 + X_2$.

Efforts do not add to productivity, and only determine the probability $P_i(X_i, X_j)$ that group i wins the contest. We assume that $P_i(X_i, X_j)$ takes the ratio form, i.e.

$$P_i(X_i, X_j) = \begin{cases} \frac{X_i}{X_i+X_j}, & \text{if } X_i > 0 \text{ or } X_j > 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{3}$$

Every group has a leader, who has the authority to enforce a sharing rule that specifies how the expected groups payoffs are to be shared within the group. Both leaders are benevolent, maximizing the expected group payoff while making their decisions.

The leaders can choose between two alternative sharing rules, either a *cooperative sharing rule* denoted E, or a *non-cooperative sharing rule* denoted N. We next turn to discussing these two rules.

■ **Cooperative Sharing Rule E:** The cooperative sharing rule E, introduced in (2), involves the group leader committing to share the net expected group payoff equally among all its members. Given $P_i(X_i, X_j)$ takes the ratio form in (3), that is equivalent to the leader committing to divide the surplus net of aggregate efforts, i.e., $1 - X$, equally among all members in case of success[2]. It is important to note that this rule implies that the net surplus is contractible, i.e., $1 - X$ is verifiable. The expected net utility of member k of group i is as follows:

$$EU_{ki}(\mathrm{E}) = \frac{1}{n_i}(P_i(X_i, X_j) - X_i) = P_i(X_i, X_j)\Big(\frac{1-X}{n_i}\Big). \tag{4}$$

Individual k in group i chooses effort x_{ki} to maximize equation (4).

As this scheme gives each member a fixed share in the net group payoff, each individual's interest gets aligned with group interest. That is why we call the rule *cooperative*. The equal sharing assumption is of course not necessary for perfect alignment of individual and group interests. Any asymmetric sharing scheme which gives all members a fixed positive share in the net group payoff will also work. We fix it at equal shares because it has natural appeal in a setting where all agents are symmetric. More importantly, the equal sharing assumption

[2] $P_i(X_i, X_j) - X_i = \frac{X_i}{X_i+X_j} - X_i = \frac{X_i}{X_i+X_j}(1 - X_i - X_j) = P_i(X_i, X_j)(1 - X)$

makes the leader a representative agent of his group, which makes concerns about his identity irrelevant.

■ **Non-cooperative Rule N:** The group leader can instead opt for the prize sharing rules introduced by Nitzan (1991). We denote this prize sharing rule by N. If group i leader chooses Rule N, then in case of success, the share of the k^{th} member of group i (s_{ki}) is as follows:

$$s_{ki}(x_{ki}, X_i; \alpha_i, n_i) = (1 - \alpha_i)\frac{x_{ki}}{X_i} + \frac{\alpha_i}{n_i}, \tag{5}$$

where $\alpha_i \in [0, 1]$. α_i is fixed for a group and cannot be manipulated by the leaders [3]. N is feasible as $\sum_{k \in n_i} s_{ki} = 1$. It should also be noted that in this case only the ratio of the individual to the total group effort needs to be verifiable.

Note that this rule is a weighted average of an egalitarian component $\frac{1}{n_i}$ and a competitive component $\frac{x_{ki}}{X_i}$. The egalitarian part tends to reduce group effort because individual members of a group free ride on effort provision, given that his share is independent of his efforts. The competitive component, on the other hand, tends to increase group efforts because individual members compete internally to get a larger share of the prize in case of success.

It should be noted that a change in group efforts has two countervailing effects. On the one hand, an increase in groups efforts increases the chances that the group wins the contest. On the other hand, higher group efforts also dissipates the prize leaving a lower ex-post surplus.

This is the trade off, which the literature on strategic choice of prize sharing rules focuses on, see e.g. Flamand et al. (2015). While abstracting from this trade-off in our paper by fixing the weights α_i, we focus on a qualitatively similar trade-off which is generated when the groups choose between E and N.

When group i leader chooses N, individual k in group i chooses efforts x_{ki} to maximize

[3] We endogenize the choice of α_i in Chapter 2

his expected utility, which is as follows:

$$EU_{ki}(N) = \begin{cases} s_{ki}(x_{ki}, X_i; \alpha_i, n_i) P_i(X_i, X_j) - x_{ki} & \text{if } \; X_i > 0, X_j \geqslant 0, \\ \frac{1}{2n_i} & \text{if } \; X_i = X_j = 0, \\ 0 & \text{if } \; X_i = 0, X_j > 0. \end{cases} \tag{6}$$

■ **Leader's Objective:** Recall that the leader of both groups are benevolent social planners. The strategy of the leader of group i is denoted $\sigma_i \in \{E, N\}$, $i \in \{A, B\}$. The leader chooses σ_i, i.e., either the cooperative rule E or non-cooperative rule N, to maximize the net group payoffs. The maximization problem of leader of group i is as follows:

$$\max_{\sigma_i \in \{E,N\}} P_i\Big(X_i(\sigma_i, \sigma_j), X_j(\sigma_i, \sigma_j)\Big)\Big(1 - X(\sigma_i, \sigma_j)\Big) \tag{7}$$

where $X(\sigma_i, \sigma_j) = X_i(\sigma_i, \sigma_j) + X_j(\sigma_i, \sigma_j)$.

The payoff representation in equation (7) is intuitive, and captures the trade-off inherent in the group leader's maximization problem. X measures the amount of prize dissipated in the competition between the two groups. Therefore, $1 - X$ is the surplus net of efforts,which remains for *ex post* consumption. The probability with which group i wins this net surplus is $P_i(X_i, X_j)$. If leader of group i wants to win the contest with a higher probability she has to take measures, which increase group efforts X_i. But when X_i goes up so does X, which reduces the size of the net surplus.

■ **Description of the Game:** Our game consists of two stages. In the first stage the two leaders simultaneously choose between E and N. Having observed the choice of the sharing rules, in stage 2 all agents simultaneously decide on their own effort levels.

We denote an equilibrium strategy profile of the game $\sigma^* = (\sigma_A^*, \sigma_B^*)$.

We solve for the Subgame Perfect Nash equilibrium (SPNE) of the game described above.

4 Choice of Individual Efforts

In this section we characterize the Nash equilibrium effort choices of individual members of the groups taking as given the sharing rules, which are chosen by the group leaders in the first stage.

Before stating the results we define the phenomenon of Monopolization of a group in the contest, which is well recognized in the collective contest literature, see e.g. Davis and Reilly (1999).

Definition 1 ***Monopolization***

A SPNE $\langle \sigma_A^*, \sigma_B^* \rangle$ is said to involve monopolization of group i, if group i does not put in any effort in the contest.

Convention: In what follows we denote generic efforts as X_A and X_B. But when we talk about equilibrium efforts, surpluses and probabilities of winning we use superscripts. We fix the first component of the superscripts to be the strategy chosen by group A and the second component to be the strategy chosen by group B in the first stage.

4.1 Equilibrium Net Surplus and Probabilities of Success

In the following proposition we report only the surplus net of effort S, which remains for consumption, i.e. $S = 1 - X$, and the probabilities with which each group wins the net surplus, P_i and P_j. Such a choice was made to keep the discussion in line with the basic trade-off in the model. In the Appendix 1 we provide all the details. Before proceeding we introduce the following notations:

For $i, j \in \{A, B\}$ and $j \neq i$ we define

$$\chi_i = n_i + n_i(n_j - 1)\alpha_j - n_j(n_i - 1)\alpha_i. \tag{8}$$

χ_i can be interpreted as a measure of the competitiveness of group i relative to group j. In fact, when both groups choose N, the probability that group i wins the contest P_i is directly proportional to χ_i. Note that χ_i is increasing in α_j and decreasing in α_i. When α_j is large relative to α_i, group j is relatively less competitive, which gives group i an advantage in the contest. On the other hand when α_i large relative to α_j, group j wins the contest more often.

In Proposition 1 we report the net surplus and probabilities of winning in an equilibrium of the second stage of our game. For features of the best response functions the readers are encouraged to go to Appendix 2. There we do a detailed analysis of individual and aggregate best response functions and analyze when aggregate efforts are strategic substitutes and when they are strategic complements. As mentioned, we relegate that analysis to Appendix 2 as it is subsidiary to the focus of the paper.

Proposition **1**

(A) If both groups choose E then in any Nash equilibrium of the effort subgame

(a) The net surplus in the contest is $S^{EE} = \frac{1}{2}$.

(b) The probabilities of winning are $(P_i^{EE}, P_j^{EE}) = (\frac{1}{2}, \frac{1}{2})$.

(B) If group i chooses E and group j chooses N, $i, j \in \{A, B\}$ and $j \neq i$, then in any Nash equilibrium of the effort subgame

(a) The net surplus in the contest is $S^{\sigma_A \sigma_B} = 1 - \frac{1+(1-\alpha_j)(n_j-1)}{n_j+1}$.

(b) The probabilities of winning are $(P_i^{\sigma_A \sigma_B}, P_j^{\sigma_A \sigma_B}) = (\frac{1+\alpha_j(n_j-1)}{(n_j+1)}, 1 - \frac{1+\alpha_j(n_j-1)}{(n_j+1)})$.

(C) If both groups choose N then

(1) If $\chi_i \leqslant 0$, $i, j \in \{A, B\}$ and $j \neq i$ [4], then group i is monopolized by group j. In the unique intra-group symmetric Nash Equilibrium of the effort subgame

[4] If $\chi_i \leqslant 0$ then $\chi_j > 0$ as $\chi_i + \chi_j = N$

(a) *The net surplus in the contest is* $S^{NN} = 1 - \frac{(1-\alpha_j)(n_j-1)}{n_j}$.

(b) *The probabilities of winning are* $(P_i^{NN}, P_j^{NN}) = (0, 1)$.

(2) *If* $\chi_i > 0$ *and* $\chi_j > 0$, $i, j \in \{A, B\}$ *and* $j \neq i$, *then neither group is monopolized and in any Nash equilibrium of the effort subgame*

(a) *The net surplus in the contest is* $S^{NN} = 1 - \frac{1+(1-\alpha_i)(n_i-1)+(1-\alpha_j)(n_j-1)}{N}$.

(b) *The probabilities of winning are* $(P_i^{NN}, P_j^{NN}) = (\frac{\chi_i}{N}, 1 - \frac{\chi_i}{N})$.

We next discuss the results summarized in Proposition 1.

■ **Both groups choose E:** When both groups choose E in the first stage, there exists a continuum of Nash equilibria in individual efforts in all of which $X^{EE} = \frac{1}{2}$ and so the net surplus is $S^{EE} = \frac{1}{2}$. Both groups win with equal probabilities $P_i^{EE} = P_j^{EE} = \frac{1}{2}$. Therefore, $X_i^{EE} = X_j^{EE} = \frac{1}{4}$, but the individual effort choices can be asymmetric. Given the fact that aggregate effort choices are all that matters, we find that the equilibrium levels of aggregate efforts are independent of group sizes. We will treat this case as our benchmark for comparison as it represents full cooperation within both the groups.

■ **Group i chooses E, group j chooses N:** Here we analyze the individual effort choices of group members when group i has chosen E and group j has chosen N in the first stage. For ease of exposition, let us assume that group $i = A$ and $j = B$. Just as in the benchmark case, the individual effort choices in the Nash equilibrium is not unique but the aggregate efforts X_A^{EN} and X_B^{EN} are. The Nash equilibrium levels of net surplus S^{EN} and the probability of group A winning, P_A^{EN} are stated in Proposition 1. In Figure 1, we make a comparison to the benchmark case.

The total effort X^{EN} monotonically decreases and net surplus S^{EN} monotonically increases in α_B, equaling the benchmark level of $\frac{1}{4}$, at $\alpha_B = \frac{1}{2}$. For $\alpha_B > \frac{1}{2}$ aggregate effort costs X^{EN} is lower compared to the benchmark case, and hence the net surplus, S^{EN} is higher.

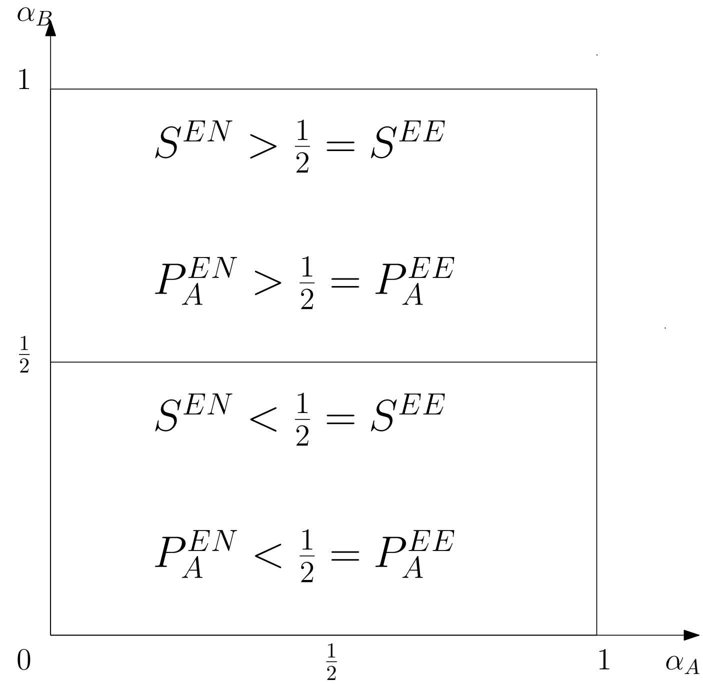

Figure 1: Comparison of EN to EE

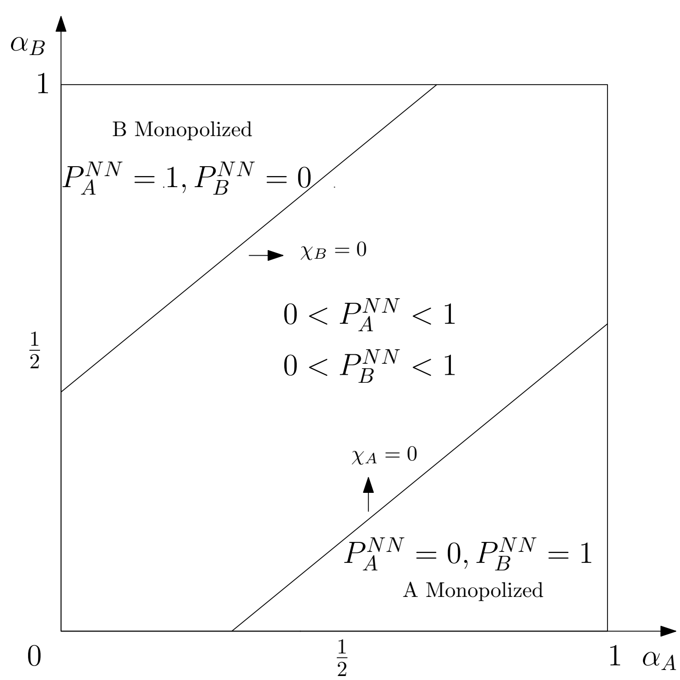

Figure 2: Probabilities of winning under NN

On the other hand, the probability that group A wins the contest, P_A^{EN}, monotonically increases in α_B, equaling the benchmark level at $\alpha_B = \frac{1}{2}$. As α_B rises, free riding increases within group B, thereby not only creating a larger net surplus but also reducing the probability that group B wins the contest.

■ **Both groups choose N:** When both groups choose N in the first stage, we may have Monopolization of one group by the other, in that the equilibrium effort level of the other group is zero,(see Figure 2). It is clear that the probability with which group i wins the contest is 0 when $\chi_i \leqslant 0$, which happens when α_i is large relative to α_j.

We now focus on the more interesting case, where neither group is Monopolized, which happens when $\chi_i > 0$. From Proposition 1,

The net surplus $S^{NN} > \frac{1}{2}$ if:

$$(n_i - 1)(1 - 2\alpha_i) + (n_j - 1)(1 - 2\alpha_j) < 0 \tag{9}$$

whereas the probability that group i wins $P_i^{NN} > \frac{1}{2}$ if:

$$\chi_i > \frac{N}{2} \tag{10}$$

The equations are represented in Figure 3. For relatively low levels of both α_A and α_B the effort expended in the contest is more than the benchmark level of $\frac{1}{2}$, which makes the net surplus less than $\frac{1}{2}$. The probability of group i winning is lower the closer we are to the line where it is monopolized.

The total effort X^{NN} is monotonically decreasing and the net surplus S^{NN} is monotonically increasing in both α_A and α_B. When α_A goes up free riding goes up within group A reducing the total effort put in the contest, thereby increasing the net surplus. Similarly for α_B.

The probability that group i wins, P_i^{NN}, goes up as α_j rises as free riding goes up within group j. But, P_i^{NN} falls with α_i, as now there is more free riding among its own members.

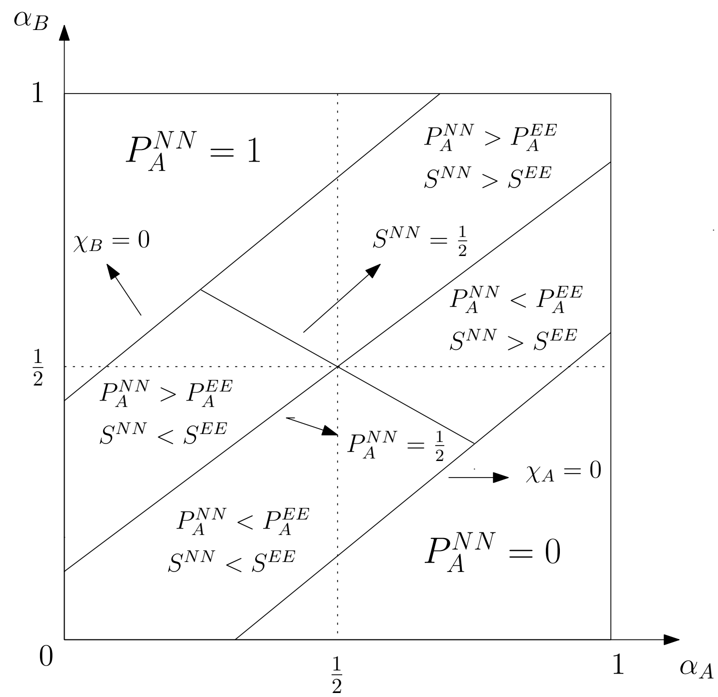

Figure 3: Comparison of NN to EE

4.2 Group Payoff Functions

In the previous subsection we analyzed properties of the equilibrium in the second stage of our game, specifically focusing on the associated net surplus and the probabilities of winning. In Appendix 2, we analyze how changing group sizes affects the net surplus and probabilities of winning. But given that we are primarily interested in group payoffs instead of its individual components, we next we analyze what happens to the group payoffs when the parameters in the model are changed.

As mentioned at the beginning under any strategy profile the payoff of group i can be expressed in the following form

$$\Pi_i = P_i S \tag{11}$$

where P_i is the probability with which group i wins the contest and S is the surplus net of efforts of the groups.

So, the growth rate of group payoffs with respect to a particular parameter, will just be the sum of the growth rate of the probability of winning and the growth rate of the net surplus with respect to that parameter. Suppose we change parameter K, then the following will be true

$$g_K^{\Pi_i} = g_K^{P_i} + g_K^S \tag{12}$$

where $g_K^Y = \frac{1}{Y}\frac{dY}{dK}$, for any variable Y.

In the previous subsection we analyzed $\frac{dP_i}{d\alpha_i}$ and $\frac{dS}{d\alpha_i}$. Here, we analyze the composition of the two effects when α_i is changed. Given, that there exists a trade-off between P_i and S, analyzing the composition of the two separate growth rates helps us pin down the growth rate of group payoffs. Obviously, the growth rate of group payoffs will be of the same sign as $\frac{d\Pi_i}{d\alpha_i}$.

Changing α_i

Here, we will change α_A and α_B and see how it affects group payoffs. The following Proposition contains the information.

Before stating the proposition we introduce the following notation:

$$\alpha_B^o = \frac{(n_B - n_A)(1 + \alpha_A(n_A - 1))}{2n_A(n_B - 1)} \tag{13}$$

α_B^o is the value of α_B, which maximizes the payoff of group B, Π_B^{NN}.

PROPOSITION **2**

(A) If group i chooses E and group j chooses N, $i, j \in \{A, B\}$ and $j \neq i$, then

- *(a) $\Pi_j^{\sigma_A\sigma_B}$ is strictly increasing (decreasing) in α_j iff $\alpha_j < (>)\frac{1}{2}$ and achieves global maximum at $\alpha_j = \frac{1}{2}$.*
- *(b) $\Pi_i^{\sigma_A\sigma_B}$ is strictly increasing in α_j.*

(B) If both groups choose N and neither group is monopolized, then

(a) Π_A^{NN} *is strictly decreasing in* α_A.

(b) Π_A^{NN} *is strictly increasing in* α_B.

(c) Π_B^{NN} *is strictly increasing in* α_A.

(d) Π_B^{NN} *is strictly increasing (decreasing) in* α_B *iff* $\alpha_B < (>)\alpha_B^o$ *and achieves global maximum at* $\alpha_B = \alpha_B^o$.

■ **Group A chooses E, Group B chooses N:**

▶ **Case 1:** $\alpha_B < \frac{1}{2}$.

In this case the payoffs of the groups depend only on α_B. When $\alpha_B < \frac{1}{2}$, we have $P_B^{EN} > S^{EN}$, so that the base probability of winning for group B is higher than the base net surplus.

It is also true that X_A and X_B are strategic substitutes in this case [5]. An increase in α_B reduces X_B^{EN} as free riding increases within group B. But, X_A^{EN} increases as the strategies are substitutes. This causes X_B^{EN} to fall farther. The net surplus S^{EN} rises as X_B^{EN} falls more than X_A^{EN} rises, thereby reducing aggregate efforts X^{EN}.

As X_A^{EN} increases so does the probability of winning for group A, P_A^{EN}. As the growth rates of both S^{EN} and P_A^{EN} are positive, Π_A^{EN} is increasing with α_B.

The payoff of group B, Π_B^{EN}, also rises in this case as the base probability of winning P_B^{EN} is quite high and S^{EN} is low to start with. So, the growth in S^{EN} dominates the deceleration in probability of success P_B^{EN}, causing group B payoffs to increase with α_B.

▶ **Case 2:** $\alpha_B > \frac{1}{2}$.

In this case, we have $P_B^{EN} < S^{EN}$, so that the base net surplus higher than the base probability of winning for group B.

As α_B rises, X_B^{EN} falls due to increased free riding in group B. But, X_A^{EN} also declines as X_A is a strategic complement to X_B. But, X_B^{EN} falls more than X_A^{EN}, so that P_A^{EN} is

[5] See Appendix 2

still increasing. Again, as the growth rates of both S^{EN} and P_A^{EN} are positive, Π_A^{EN} keeps on increasing with α_B

For group B, on the other hand, the deceleration in P_B^{EN} is now more than positive growth the in net surplus S^{EN}, by the base effect. So, the payoff of group B declines as α_B increases.

■ **Both groups choose N:** In this case it is easier to clarify part (b) and (c) of the proposition. As α_B rises, S^{NN} rises and so does P_A^{NN}. The growth rates of both are positive and so Π_A^{NN} also grows with α_B. Similarly, as α_A goes up, Π_B^{NN} is increasing.

To understand part (a) of the proposition, notice that as α_A goes up so does S^{NN}. Therefore, the growth rate of the net surplus, S^{NN}, is positive. But, the growth rate of P_A^{NN} is negative when α_A rises. Given that group A is the smaller group, when α_A increases, a small number of agents reduce their efforts, causing a minute growth of net surplus. However, decreased efforts contribute more to a reduction of the group's chances of victory. So, the growth rate in net surplus is always outdone by the slowdown in winning probabilities for group A. So, Π_A^{NN} is decreasing in α_A.

To understand part (d), notice that when α_B goes up, S^{NN} goes up but P_B^{NN} falls. When, $\alpha_B < \alpha_B^o$, the growth rate of net surplus dominates the deceleration in chances of winning for group B. This happens because, at such a low level of α_B the larger group B is also very competitive. It generates a lot of effort X_B^{NN}, causing a lot of the rent to be dissipated. This makes the base net surplus S^{NN} lower than the base P_B^{NN} here. When α_B rises, the growth rate in net surplus dominates the deceleration in probability of winning due to a lower base. So, the payoffs of group B is rising here.

When, $\alpha_B > \alpha_B^o$, the bases switch and therefore the deceleration in probabilities of winning dominates the growth in net surplus and the payoffs of group B start to fall.

		Group B	
		E	N
Group A	E	Π_A^{EE}, Π_B^{EE}	Π_A^{EN}, Π_B^{EN}
	N	Π_A^{NE}, Π_B^{NE}	Π_A^{NN}, Π_B^{NN}

Table 1: Game Γ

5 Choice of Sharing Rules by Group Leaders

In this section we consider the choice made by the group leaders in the first stage. Given the effort choices made by individual group members in the second stage, the group leaders play a normal form game in the first stage. A strategy profile is a Nash equilibrium of the normal form game, if both leaders choose strategies, which maximize (7), taking the other groups strategy choice as fixed.

Given any configuration of parameters $(\alpha_A, \alpha_B, n_A, n_B)$, we have a normal form game we denote $\Gamma(\alpha_A, \alpha_B, n_A, n_B)$. We denote the set of all such normal form games Γ. Games in Γ are bi-matrix games as represented in Table 1.

Proposition **3**

Consider any game $G \in \Gamma$. EE is a pure strategy Nash equilibrium of G.

This result is quite convenient and serves as a benchmark for us. The fact that E constitutes mutual best responses means that the only way we can generate N as a part of a Nash Equilibrium of any $G \in \Gamma$ is when both groups choose N, which takes the structure of a Coordination Game. To prove that EE is a Nash Equilibrium we have to show that

For $i, j \in \{A, B\}$ and $j \neq i$ and $\forall G \in \Gamma$

$$\Pi_i^{\sigma_A \sigma_B}(\sigma_i = N, \sigma_j = E) \leqslant \Pi_i^{EE}$$

Using Proposition 1 the inequality can be written as follows:

$$\left(\frac{1+(1-\alpha_i)(n_i-1)}{n_i+1}\right)\left(1-\frac{1+(1-\alpha_i)(n_i-1)}{n_i+1}\right)\leqslant\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\frac{1}{4} \tag{14}$$

where the first term in brackets is the probability that group i wins the contest P_i and the second term in brackets is the net surplus $1-X$. But (14) follows directly from part (A) of Proposition 2 and the fact that $\Pi_i^{EE}=\Pi_i^{EE}=\frac{1}{4}$.

When group j chooses E, group i can guarantee a payoff of $\frac{1}{4}$ by responding with E. At profile EE, the net surplus is $S^{EE}=\frac{1}{2}$ and each group wins it with $P_i^{EE}=P_j^{EE}=\frac{1}{2}$. On the other hand, if group i responds with N it can get a maximum of $\frac{1}{4}$ when $\alpha_i=\frac{1}{2}$. Otherwise, it gets a lower payoff. Therefore E is always a best response for group i when group j plays E. Look at Figure 4, where we plot Π_i^{EE} and $\Pi_i^{\sigma_A\sigma_B}(\sigma_i=N,\sigma_j=E)$.

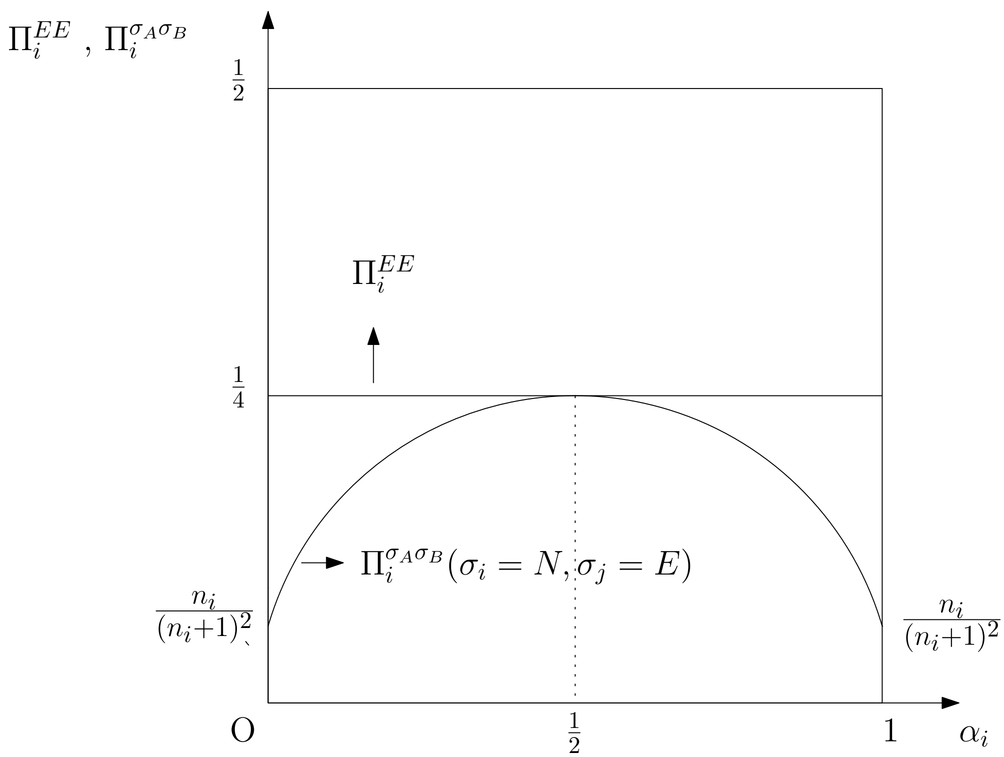

Figure 4: Payoff Comparison of EE and EN

▶ **Case 1:** $\alpha_i < \frac{1}{2}$

Consider $i = A$ and $j = B$. In this case, we know that $P_A^{NE} > S^{NE}$. We also know that $P_A^{NE} > P_A^{EE} = \frac{1}{2}$ and $S^{NE} < S^{EE} = \frac{1}{2}$, so that group A gets a larger share of a smaller net surplus. As, X_A and X_B are strategic substitutes in this case, as α_A increases, X_A falls and X_B increases. S^{NE} increases but P_A^{NE} falls. This means that the incremental net surplus, which is a public good created by a reduction in efforts by group A, is mostly captured by group B. Even, though the payoff of group A is increasing due to a lower base S^{NE}, choosing N cannot be an optimal response because group A could switch to E, where both groups contribute equally to the net surplus and take away an equal share of it.

▶ **Case 2:** $\alpha_i > \frac{1}{2}$

Consider $i = A$ and $j = B$. In this case, we know that $P_A^{NE} < S^{NE}$. It is also true that $P_A^{NE} < P_A^{EE} = \frac{1}{2}$ and $S^{NE} > S^{EE} = \frac{1}{2}$. Here, as α_A increases X_A falls but so does X_B as it is strategic complement to X_A. But X_A falls more and P_A^{NE} keeps on decreasing. So, again group A gets a smaller share of the public good it largely creates. It would be better for group A to switch to E, and get an equal share in a lower net surplus, which both groups have contributed to equally.

Given that EE is a Nash equilibrium of any $G \in \Gamma$, we need to check when games in Γ also have as Nash equilibrium the strategy profile NN.

Definition 2 ***Coordination game***

Consider any game $G \in \Gamma$. G will be called a Coordination game iff $\Pi_A^{EE} > \Pi_A^{NE}$, $\Pi_A^{NN} > \Pi_A^{EN}$, $\Pi_B^{EE} > \Pi_B^{EN}$ and $\Pi_B^{NN} > \Pi_B^{NE}$. The set of Coordination games is denoted Γ^C.

For $i = A, B$ and $j \neq i$, we introduce the following notations:

$$\overline{\alpha}_i = \frac{1 + \alpha_j(n_j - 1)}{n_j + 1}, \tag{15}$$

and

$$\underline{\alpha}_i = \frac{(1+\alpha_j(n_j-1))(n_i-n_j^2)}{n_j(n_j+1)(n_i-1)}, \tag{16}$$

where $\overline{\alpha}_i$ is the larger and $\underline{\alpha}_i$ is the smaller root of the following quadratic equation [6]

$$\Pi_i^{\sigma_A\sigma_B}(\sigma_i = E, \sigma_j = N) = \Pi_i^{NN}$$

We are now in a position to state and analyze the main result of the paper. Proposition 4 confirms the existence and helps us clearly identify the Coordination games we are looking for. This result helps us establish strategic foundations of the prize sharing rules N, which have been subjected to extensive analysis in the collective contests literature, see e.g. Flamand et al. (2015).

PROPOSITION 4

Consider any game $G \in \Gamma$

(A) EE and NN are pure strategy Nash equilibria of G iff $\alpha_A \in [0, \overline{\alpha}_A]$ and $\alpha_B \in [max\{0, \underline{\alpha}_B\}, \overline{\alpha}_B]$.

(B) Otherwise, G is dominance solvable and EE is its unique pure strategy Nash equilibrium.

This is the main result of this paper. We have been able to show, that there exist games $G \in \Gamma$ such that NN is a Nash equilibrium outcome, thereby providing strategic foundations to the prize sharing rules N.

G belongs to the set of Coordination games Γ^C when $\alpha_A \in [0, \overline{\alpha}_A)$ and $\alpha_B \in (\underline{\alpha}_B, \overline{\alpha}_B)$ if $\underline{\alpha}_B \geqslant 0$. On the other hand, when $\underline{\alpha}_B < 0$ then G belongs to the set of Coordination games Γ^C, if $\alpha_A \in [0, \overline{\alpha}_A)$ and $\alpha_B \in [0, \overline{\alpha}_B)$. Under the conditions specified above N is a strict best response to N for both the groups and hence satisfies the requirements for any $G \in \Gamma$ to be a Coordination game.

[6] Notice that $\underline{\alpha}_i = \frac{n_i - n_j^2}{n_j(n_i-1)}\overline{\alpha}_i$. So the roots are multiples of each other, i.e., $\underline{\alpha}_i = C\overline{\alpha}_i$, where $C < 1$.

If $\alpha_A = \overline{\alpha}_A$, then N is a weak best response to N for group A. The cooperative strategy E weakly dominates the non-cooperative strategy N for group A. This follows from Proposition 3. Similarly, E weakly dominates N for group B, when $\alpha_B = \overline{\alpha}_B$ or $\alpha_B = \underline{\alpha}_B$, in case $\underline{\alpha}_B$ is positive [7]. Even though we can see in Part (A) of Proposition 4 that both EE and NN are Nash equilibria in such cases [8], we ignore them while considering Coordination games Γ^C as they are defined to have N as a strict best response to N for both groups.

To check when N is a best response to N for group i we need to check the following inequality:

For $i, j \in \{A, B\}$ and $j \neq i$

$$P_i^{NN} S^{NN} \geqslant [P_i^{\sigma_A \sigma_B} S^{\sigma_A \sigma_B}](\sigma_i = E, \sigma_j = N) \tag{17}$$

It can be easily verified that $S^{NN} > S^{\sigma_A \sigma_B}(\sigma_i = E, \sigma_j = N)$ iff $\alpha_i > \overline{\alpha}_i$. Similarly, it can be verified that $P_i^{NN} > P_i^{\sigma_A \sigma_B}(\sigma_i = E, \sigma_j = N)$ iff $\alpha_i < \overline{\alpha}_i$. At, $\alpha_i = \overline{\alpha}_i$, the strategies N and E are equivalent for group i both in terms of net surplus and probabilities of winning the contest.

Let us first consider group A and refer to Figure 5. Let us start from $\alpha_A = \overline{\alpha}_A$, where $\Pi_A^{NN} = \Pi_A^{EN}$. Now from Proposition 2 we know that Π_A^{NN} is strictly decreasing in α_A. So, starting from, $\alpha_A = \overline{\alpha}_A$, if we reduce α_A, then Π_A^{NN} will strictly increase, while Π_A^{EN}, being independent of α_A, will remain unchanged. Given that the smaller root $\underline{\alpha}_A$ is negative, it follows that for all $\alpha_A \in [0, \overline{\alpha}_A)$, N is a strict best response to N for group A.

The story for group B is slightly different and can be seen in Figures 6 and 7. If $\alpha_B = \overline{\alpha}_B$, then $\Pi_B^{NN} = \Pi_B^{NE}$. From Proposition 2 we know that Π_B^{NN} is decreasing in α_B if $\alpha_B > \alpha_B^o$. So, starting from $\alpha_B = \alpha_B^1$, if we reduce α_B, Π_B^{NN} first increases upto α_B^o and then decreases. Π_B^{NE}, being independent of α_B is unchanged. Given that Π_B^{NN} decreases when we reduce α_B

[7] Of course in these cases equilibrium NN will be lost if we apply Iterated Elimination of Weakly Dominated Strategies (IEWDS)

[8] It can also be easily verified that $\overline{\alpha}_A$ and $\overline{\alpha}_B$ intersect at $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$. At $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$ all strategy profiles, i.e., EE, NN, EN and NE are Nash equilibria of Γ.

below α_B^o, gives rise to the possibility that the smaller root of $\Pi_B^{NN} = \Pi_B^{NE}$, which we denote $\underline{\alpha}_B$, is positive.

It can be easily verified that $\underline{\alpha}_B$ is negative when $n_B < n_A^2$. So in this case N is a strict best response to N for group B when $\alpha_B \in [0, \overline{\alpha}_B)$. See Figure 6.

In the other case, when $n_B \geqslant n_A^2$, the smaller root $\underline{\alpha}_B$ is non negative and N is a strict best response to for group B when $\alpha_B \in (\underline{\alpha}_B, \overline{\alpha}_B)$. This is captured in Figure 7.

In Figures 8 and 9 we represent the Coordination games for the two different cases in the $\alpha_A \alpha_B$ plane. The case, where $\underline{\alpha}_B$ is negative is captured in Figure 8. The case where $\underline{\alpha}_B$ is non-negative is captured in Figure 9. The Coordination games are marked in blue.

■ **Intuition:** To see why NN turns out to be a Nash equilibrium when $G \in \Gamma^C$, we have to understand how presence of the non-cooperative rule N creates a situation of strategic uncertainty for the groups. The main feature of this rule N is that it allows the groups a chance to enhance its probability of winning at the expense of the other group, when $G \in \Gamma^C$. Even, though the net surplus is lower a group wins with a higher chance by choosing N. If both groups believe that the other is going to choose N to increase its chances of winning the contest, both end up choosing N, not to give up a substantial winning advantage to the other group. Of course, coordinating on NN comes at the cost of a substantially reduced net surplus.

For example, consider the case where group B chooses N. If group A were to choose E, then it gives up the option of increasing its chances of winning the contest. If α_B is sufficiently low, then group B puts in a lot of effort and wins with a very high probability a net surplus, which is lower. But, group A has no way to counter group B. However, if group A were to respond with N, then it would be able to stop its probability of winning from falling too much.

Therefore, in the race to keep its probability of winning high, a group may choose N if it believes the other group will also do so. These kind of perverse incentives of groups results from the fact the net surplus behaves exactly like a public good between the groups, leading

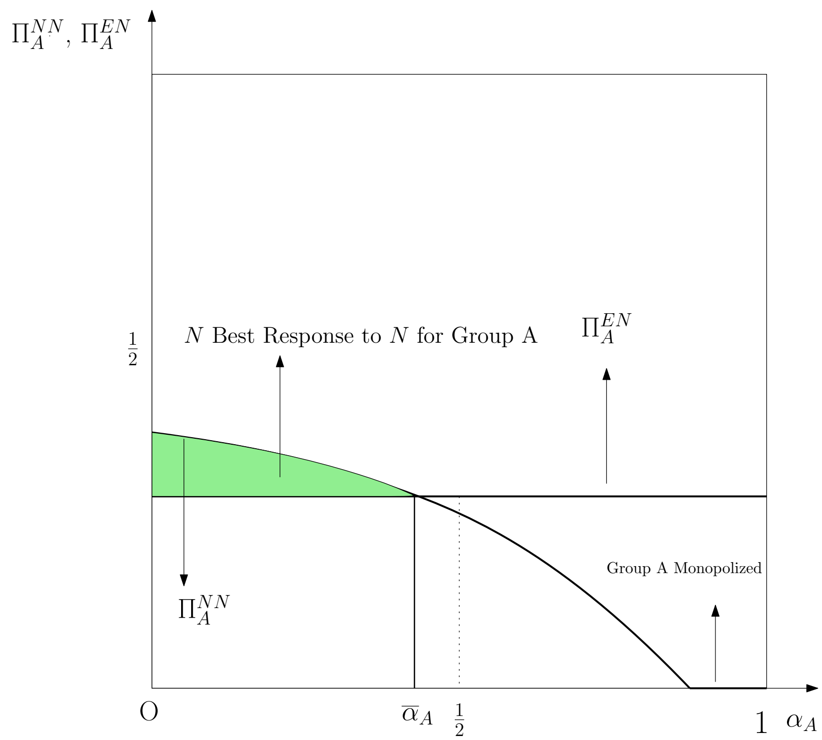

Figure 5: N best response to N for group A

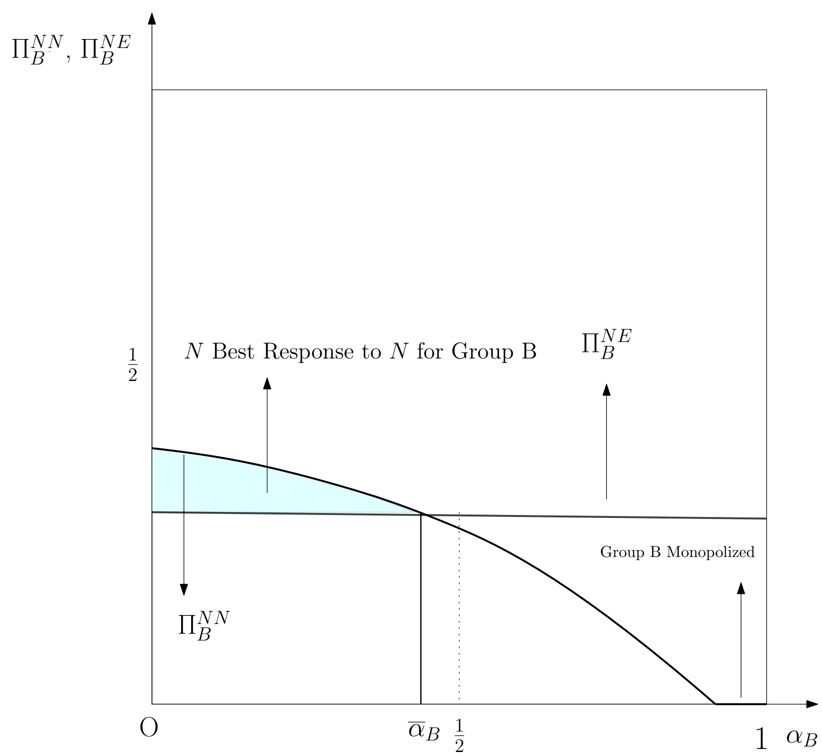

Figure 6: N best response to N for group B when $n_B < n_A^2$

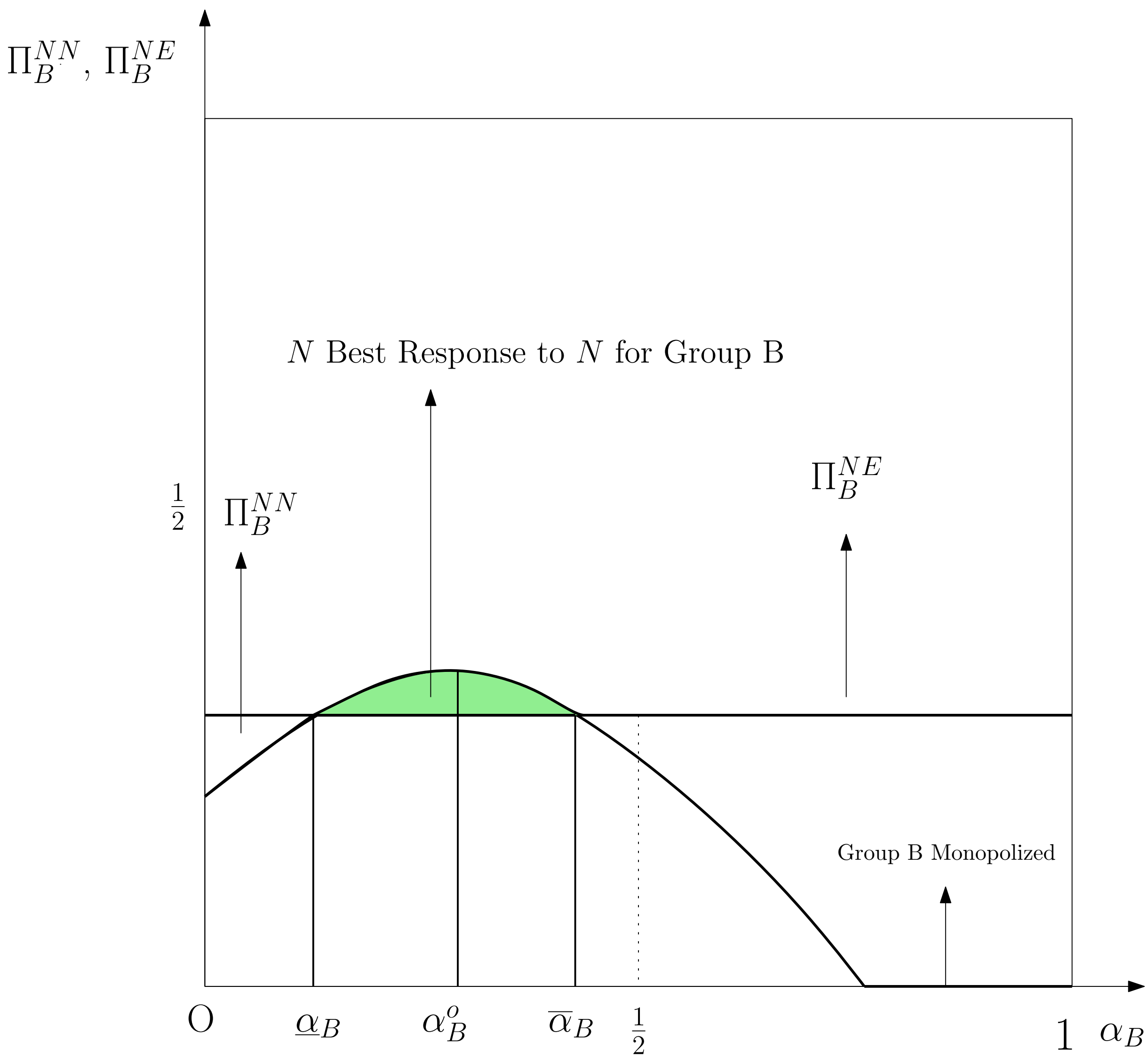

Figure 7: N best response to N for group B when $n_B \geqslant n_A^2$

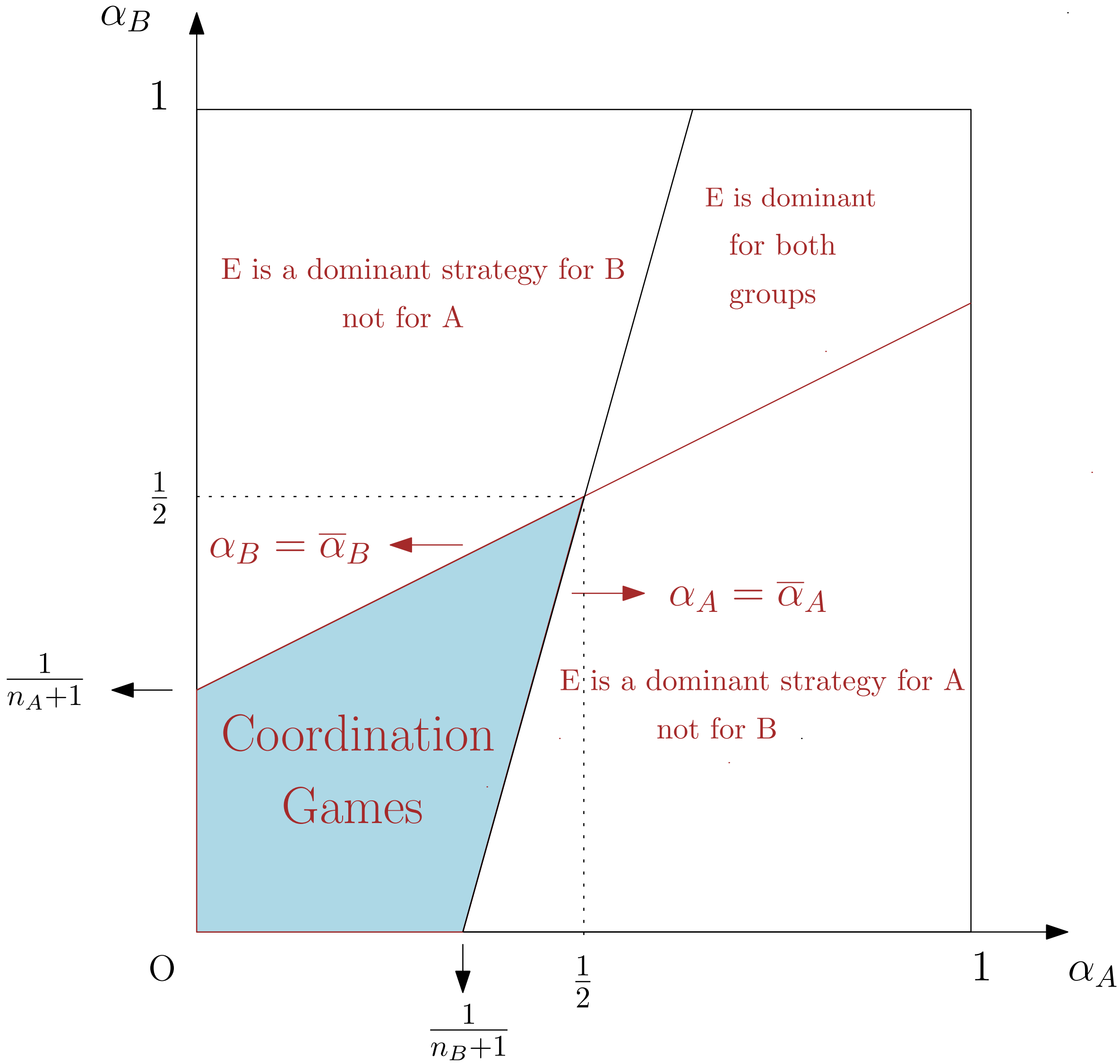

Figure 8: Coordination Games when $n_B < n_A^2$

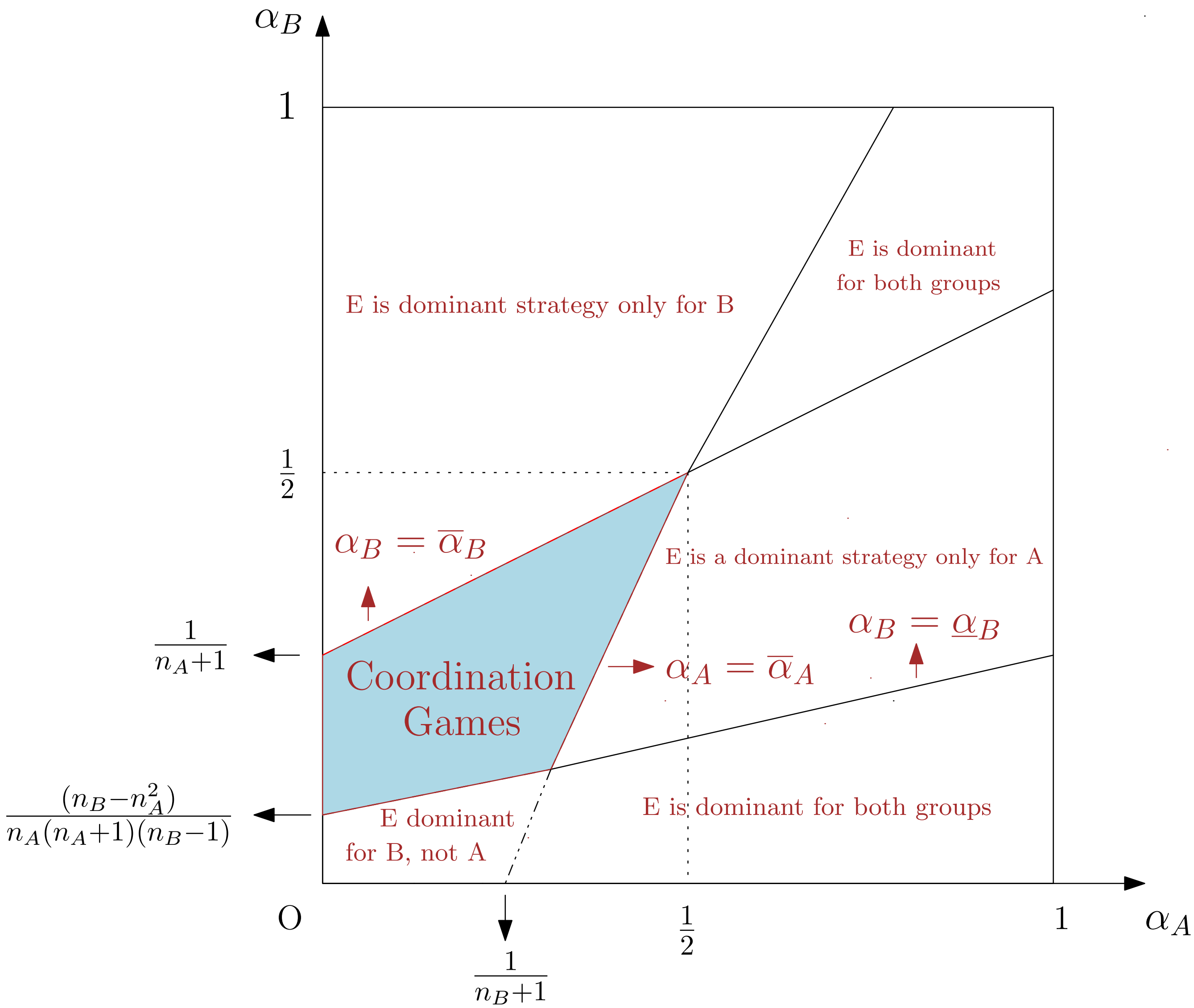

Figure 9: Coordination Games when $n_B \geqslant n_A^2$

to free riding on its maintenance by both groups. Instead, both groups have an incentive to increase their winning chances by putting in more effort. Therefore, if one group believes that the other is trying to enhance its chances of winning by choosing N, it should respond by doing the same to maintain parity. Given efforts eat into the prize, none of the groups ideally want to end up in this spiral of higher efforts. But, given the strategic uncertainty embodied in the normal form game $G \in \Gamma^C$, NN turns out to be an equilibrium outcome. This result essentially has the flavor of a failure to coordinate on the Pareto efficient outcome EE.

6 Equilibrium Selection

We have been able to generate NN as a subgame perfect Nash equilibrium of an appropriately constructed two stage game, thereby providing a strategic foundations to the non-cooperative prize sharing rule N, which has been so extensively analyzed in the collective contests literature. But, given that it is an equilibrium of a Coordination game, where EE is also a Nash equilibrium, the natural next step is to consider the question of equilibrium selection, i.e., which of the equilibria are the groups likely to coordinate on? To tackle this we introduce the three refinement criteria of the Nash equilibrium solution concept, namely payoff dominance, risk dominance and the security principle.

If a game has multiple Nash equilibria and there is one Nash equilibrium which is Pareto superior to all other Nash equilibria then it is called payoff dominant. The notion of payoff dominance is based on the idea of collective rationality, which leads to a coordination on the Pareto superior equilibrium. The readers may refer to Harsanyi et al. (1988) for the first discussions of this refinement concept. Readers are also referred to (Schelling, 1980), who argues that efficiency based considerations may make decision makers to focus on and select a payoff dominant equilibrium point if it is unique.

A Nash equilibrium is said to be risk dominant if the losses from deviation from it is the largest among all other Nash equilibria. In the presence of high degree of uncertainty about other player's actions, this criterion seems to be more natural as players have an incentive to coordinate on it to minimize losses. A risk dominant equilibrium is defined to be one which generates the highest product of losses for the players, when there is a deviation from it. Harsanyi (1995) first made a case for risk dominance as an equilibrium selection criterion.

Interestingly, there can be a tension between the criteria of risk dominance and payoff dominance in the sense that they may make conflicting prescriptions. A Nash equilibrium can be payoff dominant but not risk dominant and vice versa. This leads to the obvious concern about the relative appropriateness of the criteria? Researches have built evolutionary game theory models in an attempt to justify one or the other of the refinements, see e.g. Samuelson (1997). As it turns out, the tension between the two criterion is also a feature of our model under certain circumstances.

There also exists a substantial experimental literature, which studies how real subjects actually select between payoff dominance and risk dominance, when the two criteria make conflicting prescriptions. For a guide to that literature, the readers may look at Keser et al. (2000) and the references therein. The major takeaways from this literature is that the number of players, time horizon, pre-play communication and the structure of interactions matter. Interestingly, Keser et al. (2000) report an experiment where despite the two criteria making the same equilibrium prescription, subjects systematically deviate from playing it. Based on their conclusions, the authors claim that it is important to look for new criteria that may play an important role in equilibrium selection.

Therefore, we also consider the Security principle, see e.g. Van Huyck et al. (1990), as an additional selection criteria in our paper. The security principle suggests players to select a course of action that maximizes their minimum payoffs over all possible actions. The idea is based on the notion of maximin introduced by Von Neumann and Morgenstern (1944). This criterion, like risk dominance, is based on the "riskiness" of the equilibrium point. Therefore, it will be salient when there is sufficient uncertainty regarding the other player's actions.

We now take up the equilibrium selection criteria one at a time. We first formally define a criterion tailored to our game Γ^C. Then we state the corresponding result.

First, we take up the equilibrium selection criteria based on "riskiness"of the equilibrium point, i.e., risk dominance and the security principle. Then, we consider the selection criterion of payoff dominance.

Definition **3** ***Risk Dominance***

Consider any game $G \in \Gamma^C$. NN is said to risk dominate EE in G iff $(\Pi_A^{NN} - \Pi_A^{EN})(\Pi_B^{NN} - \Pi_B^{NE}) \geqslant (\Pi_A^{EE} - \Pi_A^{NE})(\Pi_B^{EE} - \Pi_B^{EN})$. If the inequality holds strictly NN is said to strictly risk dominate EE.

For ease of stating the result we start by introducing some notations.
For $i, j \in \{A, B\}$ and $j \neq i$ we define

$$\Delta_i = n_j(n_j+1)^2\Big[(\overline{\alpha}_i - \alpha_i)(\alpha_i - \underline{\alpha}_i)\Big], \tag{18}$$

where $\overline{\alpha}_i$ and $\underline{\alpha}_i$ are the roots of $\Pi_i^{\sigma_A\sigma_B}(\sigma_i = E, \sigma_j = N) = \Pi_i^{NN}$ as defined in (15) and (16). Δ_i is a measure of $\Pi_i^{NN} - \Pi_i^{\sigma_A\sigma_B}(\sigma_i = E, \sigma_j = N)$. As we consider only Coordination games, it is true that $\alpha_i \in (\underline{\alpha}_i, \overline{\alpha}_i)$ and therefore the right hand side of (18) is positive. We are now in a position to state a condition which is necessary and sufficient for equilibrium profile NN to risk dominate EE.

Proposition **5**

Consider any game $G \in \Gamma^C$. NN risk dominates EE in G iff $N^4(1-2\alpha_A)^2(1-2\alpha_B)^2 \leqslant 16\Delta_A\Delta_B$.

The Prosposition provides us a very easy to check condition for NN to risk dominate EE. It can be written out as follows:

$$N^4(1-2\alpha_A)^2(1-2\alpha_B)^2 \leqslant 16 n_A n_B (n_A+1)^2(n_B+1)^2(\overline{\alpha}_A-\alpha_A)(\alpha_A-\underline{\alpha}_A)(\overline{\alpha}_B-\alpha_B)(\alpha_B-\underline{\alpha}_B) \tag{19}$$

The left hand side of (19) is a measure of $(\Pi_A^{EE}-\Pi_A^{NE})(\Pi_B^{EE}-\Pi_B^{EN})$. It is close to zero if either α_A or α_B is close to $\frac{1}{2}$. But it is clear from Figures 8 and 9 that Γ is a Coordination game when α_A and α_B are relatively symmetric, i.e., not too far from each other. So if the left hand side has to be small when Γ is a Coordination game, we must have $\alpha_A \approx \alpha_B$ and close to $\frac{1}{2}$.

The the right hand side of (19) is a measure of $(\Pi_A^{NN}-\Pi_A^{EN})(\Pi_B^{NN}-\Pi_B^{NE})$. Its size depends on the product $\Delta_A\Delta_B$. The product will be close to zero if either α_A or α_B approaches any of its respective roots. But, it is clear from Figures 8 and 9 that when $\alpha_A \approx \alpha_B$ and close to $\frac{1}{2}$, both α_A and α_B are at some distance from its roots, which may make the product $\Delta_A\Delta_B$ large enough to dominate left hand side of (19), which is close to zero.

Therefore, Coordination games in which NN risk dominates EE, if they exist, are likely to be located around $\alpha_A = \alpha_B$. To show that the set of games in which equilibrium profile NN risk dominates EE is non-empty, we consider a subclass of Coordination games of Γ we call Symmetric Coordination games.

Definition 4 ***Symmetric Cordination games***

Consider any game $G \in \Gamma^C$. G is said to be a Symmetric Coordination game iff $n_A = n_B = n$ and $\alpha_A = \alpha_B = \alpha$. The set of all Symmetric Coordination games is denoted Γ^{SC}.

Corollary **1**

Consider any game $G \in \Gamma^{SC}$. NN risk dominates EE in G iff $\alpha \in [\frac{1}{4}-\frac{1}{4n}, \frac{1}{2})$ [9].

[9] These games are in fact Stag Hunt games, see e.g. Skyrms (2004)

This result can be obtained by replacing $n_A = n_B = n$ and $\alpha_A = \alpha_B = \alpha$ in (19) [10] [11]. This corollary of Proposition 5 establishes that when groups participating in the collective contest are symmetric in all respects, there is a robust strategic basis of N based on the equilibrium selection criterion of risk dominance. In order to understand why these games arise, first note that at $\alpha_A = \alpha_B = \frac{1}{2}$, both the right hand side and left had side of (19) are zero. As we approach $\alpha_A = \alpha_B = \frac{1}{2}$ from below, along $\alpha_A = \alpha_B$, the right hand side falls at faster rate than the left hand side and therefore has to dominate it along the path, given that both have to be zero at $\alpha_A = \alpha_B = \frac{1}{2}$.

If we introduce asymmetries between groups it is unlikely that NN will pass the test of risk dominance as it becomes harder to satisfy (19).

Next, we consider the equilibrium selection criterion called the Security Principle, see e.g. Van Huyck et al. (1990). A secure strategy for a player is one in which the smallest payoff is at least as large as the smallest payoff to any other feasible strategy. Security principle selects equilibrium points implemented by secure strategies. The Security Principle, as we will see, always selects NN unlike the criterion of payoff dominance, never selects it (Proposition 7).

Definition 5 ***Secure Strategy***

A strategy $\overline{\sigma}_i$ of group i is said to be secure iff $\overline{\sigma}_i = arg\Big(max_{\sigma_i \in \{E,N\}} min_{\sigma_j \in \{E,N\}} \Pi_i(\sigma_i, \sigma_j)\Big)$, $i, j \in \{A, B\}$ and $j \neq i$.

The strategy $\overline{\sigma}_i$ guarantees group i the best out of the worst of its outcomes.

Definition 6 ***Security Principle***

Consider any game $G \in \Gamma^C$. NN will be said to satisfy the Security Principle in G iff N is a secure strategy for both groups A and B.

[10] It is easiest to see if we use the form of Δ_i in (59).

[11] This case corresponds to Figure 8, with $n_A = n_B = n$.

Proposition **6**

Consider any game $G \in \Gamma^C$. NN satisfies the Security Principle in G.

Proof:

We do the proof assuming $i = A$.

We know that $\Pi_A^{NN} > \Pi_A^{EN}$ when Γ is a Coordination game. We also know from Proposition 7 that $\Pi_A^{EE} > \Pi_A^{NN}$ when Γ is a Coordination game. Therefore it follows that we must have $\Pi_A^{EE} > \Pi_A^{NN} > \Pi_A^{EN}$ when Γ is a Coordination game.

We can also see in proof of Proposition 3 that $\Pi_A^{EE} > \Pi_A^{NE}$ when $\alpha_A < \frac{1}{2}$. And, it can also be easily verified from Proposition 1 and 4 that $\Pi_A^{NE} > \Pi_A^{NN}$ when Γ is a Coordination game. This is true because both $P_A^{NE} > P_A^{NN}$ and $S^{NE} > S^{NN}$, i.e., not only is the net surplus higher in this case, but group A also wins the contest with a higher probability. Therefore, when Γ is a Coordination game, we have $\Pi_A^{EE} > \Pi_A^{NE} > \Pi_A^{NN} > \Pi_A^{EN}$.

As $\Pi_A^{NN} > \Pi_A^{EN}$, i.e., the minimum payoff from choosing N is strictly larger than the minimum payoff from choosing E for group A, N is a secure strategy for group A. The argument is similar for group B. ■

Finally, we consider the equilibrium selection criterion of payoff dominance.

Definition **7** ***Payoff Dominance***

Consider any game $G \in \Gamma^C$. EE is said to payoff dominate NN in G iff $\Pi_A^{EE} \geqslant \Pi_A^{NN}$ and $\Pi_B^{EE} \geqslant \Pi_B^{NN}$ with one inequality holding strictly. If both inequalities hold strictly we will say that EE strictly payoff dominates NN in G.

Proposition **7**

Consider any game $G \in \Gamma^C$. EE strictly payoff dominates NN in G.

To prove this result we need to show that, for $G \in \Gamma^C$ and $i = A, B$

$$P_i^{NN} S^{NN} > P_i^{EE} S^{EE} \tag{20}$$

We proceed by identifying games $G \in \Gamma$, such that strategy profile EE is Pareto superior to strategy profile NN, i.e., $P_i^{NN} S^{NN} \geqslant P_i^{EE} S^{EE}$, $i = A, B$. We denote such games Γ^{PS}. Then we go on to show that the set of Coordination games Γ^C is a proper subset of Γ^{PS} , i.e., $\Gamma^C \subset \Gamma^{PS}$.

The following equation represents the bigger root [12] of the quadratic equation of (20)

$$\alpha_j^+ = \frac{(n_j - n_i)(n_i - 1)\alpha_i + N\sqrt{((n_i - 1)\alpha_i)^2 + n_i} - 2n_i}{2n_i(n_j - 1)} \tag{21}$$

For instance, when $\alpha_B = \alpha_B^+$, we have $\Pi_A^{NN} = \Pi_A^{EE} = \frac{1}{4}$. If $\alpha_B < \alpha_B^+$, group B is more competitive and generates more effort, which leads to a lower S^{NN} and P_A^{NN} and hence a lower Π_A^{NN} compared to $\Pi_A^{EE} = \frac{1}{4}$. Similarly, when $\alpha_A < \alpha_A^+$, we have $\Pi_B^{NN} < \Pi_B^{EE}$. For the shapes of α_A^+ and α_B^+ look at Figure 10.

When both $\alpha_B \leqslant \alpha_B^+$ and $\alpha_A \leqslant \alpha_A^+$ with one inequality holding strictly, EE is Pareto superior to NN. This can be observed in Figure 10. It is clear from the diagram that EE Pareto superior to NN, when both α_A and α_B are substantially less than $\frac{1}{2}$.

To understand why this must be the case we refer to Figure 3. We start from $\alpha_A = \alpha_B = \frac{1}{2}$, where strategy profiles EE and NN are equivalent. Now, if either α_A or α_B falls then $S^{NN} < \frac{1}{2}$ and decreasing. For example, if α_A falls substantially but α_B falls infinitesimally, then P_A^{NN} rises and P_B^{NN} falls and we approach $P_A^{NN} = 1$. Here, group A captures almost the whole of the reduced net surplus, thereby getting a payoff $\Pi_A^{NN} > \Pi_A^{EE} = \frac{1}{4}$. For this case not to arise we need α_B to fall sufficiently as well.

It can also be observed in Figure 11 and 12, that $\overline{\alpha}_A$ supports α_A^+ from below and $\overline{\alpha}_B$ supports α_B^+ from above at $(\frac{1}{2}, \frac{1}{2})$ in the $\alpha_A\alpha_B$ plane. This fact helps us establish our result.

[12] We do not report the smaller root α_j^- as it is negative and can be ignored. See proof of Proposition 7 in Appendix 1

If $(\alpha_A, \alpha_B) < (\frac{1}{2}, \frac{1}{2})$, then $\alpha_B^+ > \overline{\alpha}_B$ and $\alpha_A^+ > \overline{\alpha}_A$. Therefore, the set of Coordination games Γ^C, is a proper subset of the games in which EE is Pareto superior to NN.

■ **Intuition:** To understand the result, it is best to begin by noticing that the issue is only relevant in Coordination games. Further note that the Coordination games are clustered around $P_A^{NN} = P_B^{NN} = \frac{1}{2}$ (see Figures 3 and 8). The difference in probabilities of winning between the groups cannot be too large if Γ has to be a Coordination game.

We know from Proposition 1 that $P_A^{EE} = P_B^{EE} = \frac{1}{2}$. Given that the disparity in probabilities of winning between the groups cannot be large, i.e., $P_A^{NN} \approx P_A^{NN}$, and $S^{NN} < \frac{1}{2} = S^{EE}$, when Γ is Coordination game, it will be the case that each group achieves a payoff strictly less than $\frac{1}{4}$, i.e., EE Payoff dominates NN. In Coordination games, both groups essentially cancel out the gain in winning probabilities each wishes to have by choosing N. But, as both groups efforts are higher under NN net surplus is lower compared to EE. The net effect is that both groups lose by choosing N.

In this section we introduced several equilibrium selection criterion to check whether equilibrium NN is prescribed by any of them. When we consider the criterion of risk dominance we are able to show that there exist Coordination games in which NN risk dominates EE. We provide a necessary and sufficient conditions for NN to risk dominate EE in Proposition 5 and then go onto show existence of such games using a symmetric subclass of Coordination games in Corollary 1. When we consider equilibrium selection criterion called the Security Principle, we are able to show that NN is always prescribed over EE. However, when we consider the principle of payoff dominance, equilibrium profile NN is never selected as is shown in Proposition 7. The results are therefore mixed. However, given that there exist equilibrium criteria which prescribe selection of equilibrium NN, allows us to claim that there exists a robust strategic basis of prize sharing rules N, first introduced in Nitzan (1991).

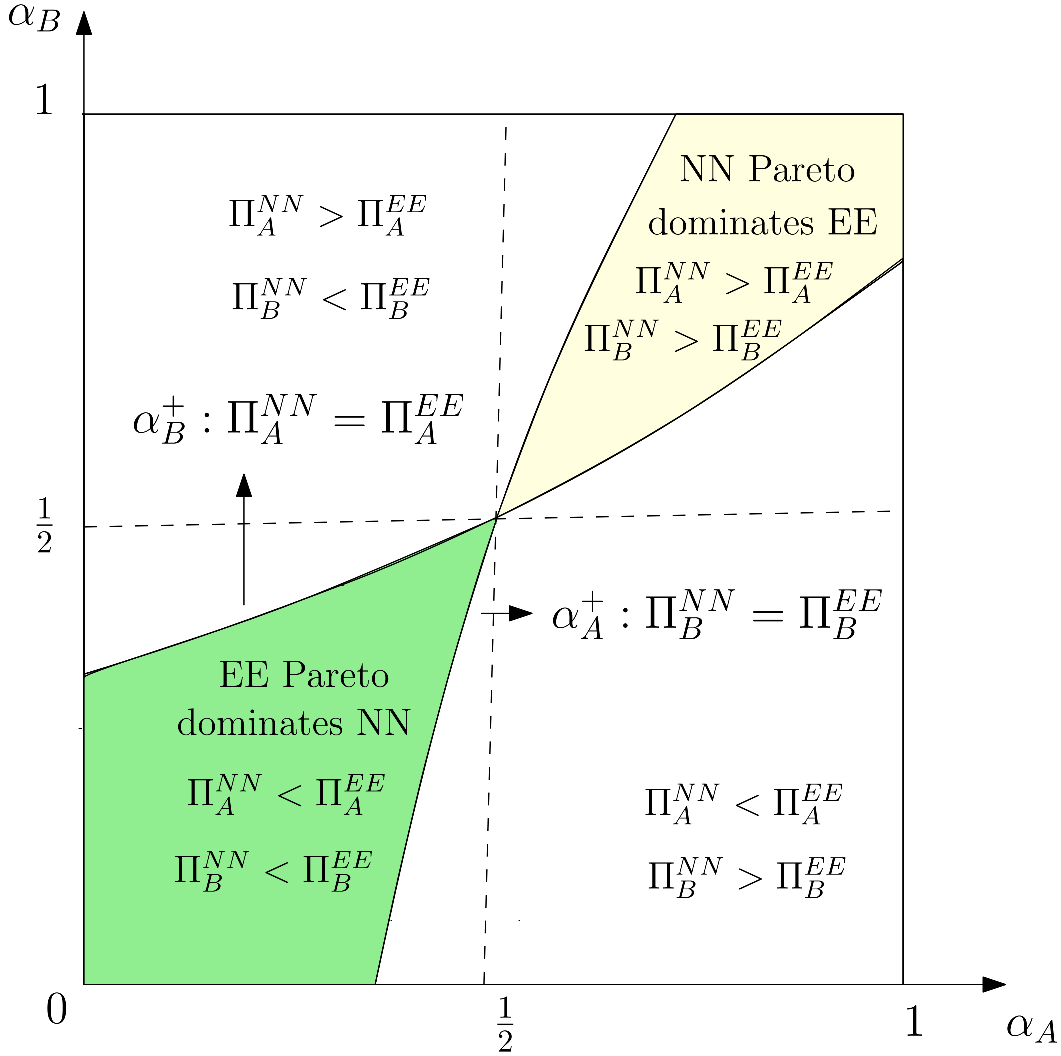

Figure 10: The locus of $\Pi_i^{NN} = \Pi_i^{EE}$

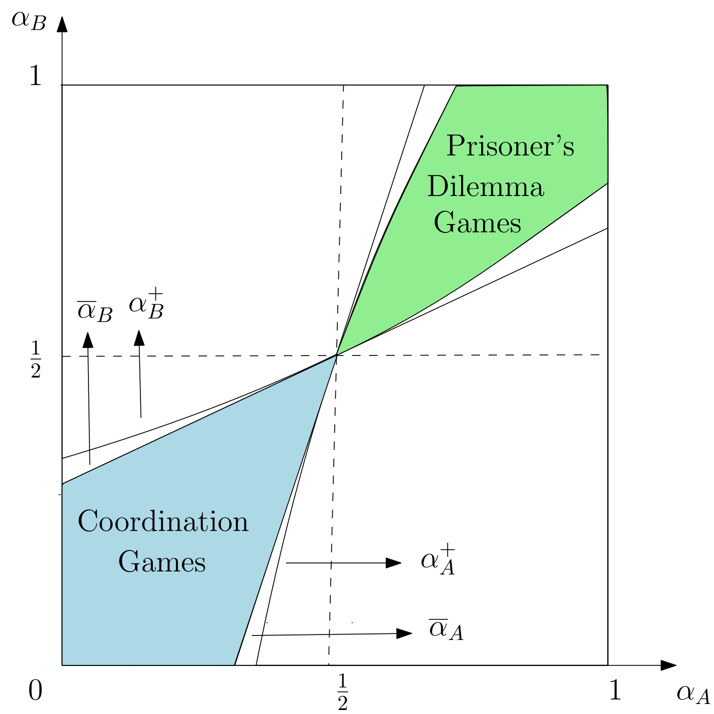

Figure 11: Payoff Dominance: When $n_B < n_A^2$

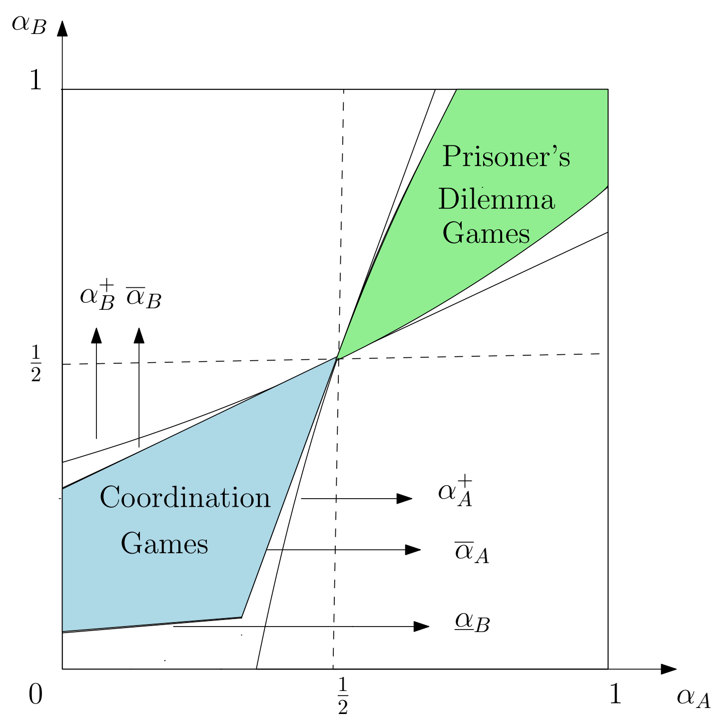

Figure 12: Payoff Dominance: When $n_B \geqslant n_A^2$

7 Discussion

In this section we discuss a few assumptions we made and some other properties, which we have skipped in the main body.

■ **Coordination Devices:** Given that in our model selecting equilibrium NN is essentially a failure to coordinate on a Pareto efficient equilibrium point EE, we discuss a few coordination devices, which may help the groups circumvent the problem.

(1) Timing of the Game: In our game we assume that in the first stage the group leaders move simultaneously to choose between E and N and having observed those choices the agents make their effort decisions simultaneously. But, it is clear that if one leader moves first, then the groups will coordinate on EE. Given the EE payoff dominates NN (Proposition 7), if one of the group leaders could choose the rule first, he would choose E and coordination failure on N will be avoided. But, the assumption of simultaneous choice of the rules is justified because in our framework of direct conflict and no communication between the groups, there is no reason to assume otherwise.

(2) Strategic Choice of Sharing Rules: In our game we have kept the α_i's fixed and provided the leader a choice between E and N. Another part of the literature considers the case, where the leaders do not have access to E. The only rule available is N but the leaders can choose $\alpha_i \in [0, 1]$. This part of the literature mostly focuses on the phenomenon of Group Size Paradox (GSP), whereby a larger group wins the contest less often due to free riding.

If we allow the leaders to choose $\alpha_i \in [0, 1]$ in our model, then all equilibria will be payoff equivalent to EE. Given that the group leaders have some adjustment room with N, they will adjust N in such a manner that both groups will get fully cooperative payoffs. In fact it can shown that EE, NE ,EN will all be equilibrium profiles, with the leader of group i, choosing $\alpha_i = \frac{1}{2}$ under N. Only NN will not be an equilibrium profile. So, allowing strategic choice of sharing rules essentially gives the leaders an extra degree of freedom and help them avoid coordination failures.

■ **Prisoner's Dilemma Games:** There also exists a class of Prisoner's Dilemma games in our model. We primarily focused on the case where $(\alpha_A, \alpha_B) < (\frac{1}{2}, \frac{1}{2})$, because the focus of the paper was on providing strategic foundations to N. But if $(\alpha_A, \alpha_B) > (\frac{1}{2}, \frac{1}{2})$, and $\alpha_A > \alpha_A^+$ and $\alpha_B > \alpha_B^+$ (α_A^+ and α_B^+ defined in (21)) , then Γ turns out to be Prisoner's Dilemma games. Both groups have a dominant strategy E, but the strategy profile NN payoff dominates EE. So the use of grim trigger strategies, would allow us to generate NN as a subgame perfect Nash equilibrium if the first stage game is infinitely repeated [13]. The Prisoners Dilemma games can be seen in Figures 11 and 12.

When, $(\alpha_A, \alpha_B) > (\frac{1}{2}, \frac{1}{2})$, rule N makes both groups less competitive in the contest for the prize. The benefit is that a lot of net surplus gets saved and both groups benefit. But of course, given that rule N is not competitive enough, both groups have unilateral incentives of deviating to E. If the groups could write enforceable agreements they would have chosen $X = 0$. In this case mutually beneficial agreements are ones where $(\alpha_A, \alpha_B) > (\frac{1}{2}, \frac{1}{2})$, $\alpha_A > \alpha_A^+$ and $\alpha_B > \alpha_B^+$. But in absence of the possibility of explicit agreements between groups, one way to sustain NN as an equilibrium outcome is to repeat our stage game infinitely and use reverting to the Nash equilibrium EE forever as a punishment strategy for deviation from strategy N by any group at any stage.

8 CONCLUSION

The explicit aim of the paper was to provide strategic foundations to the prize sharing rules introduced by Nitzan (1991), which has subsequently become the standard in the collective contests literature. To that end, we were able to uncover a class of Coordination games, where in fact the groups may end up coordinating on the Nitzan rule N, even though a cooperative option E is present. But, coordinating on this rule looks like a case of coordination failure,

[13] Ursprung (2012) recognizes that if $\alpha_A = 1$ and $\alpha_B = 1$ then Γ is a Prisoner's Dilemma game.

because the equilibrium with mutual cooperation EE payoff dominates the one in which both groups choose the prize sharing rules NN.

However, when we introduce equilibrium selection criterion of risk dominance and security principle, which are based on the "riskiness" of the equilibrium point, we find that NN does indeed survive both these criterion. We provide a necessary and sufficient condition for NN to risk dominate EE and show existence of such a class of coordination games. When we use the security principle, we find that the prescription is always to select NN. In light of these selection criterion, which prescribe selection of equilibrium profile NN, we claim that there exists a robust strategic basis to the prize sharing rules N.

We also uncover a class of Prisoner's Dilemma games where, the prize sharing rule N has a robust basis if the game is repeated infinitely and the leaders can use grim-trigger like punishment strategies.

Previously Ursprung (2012) showed in an evolutionary game theoretic model, that the extreme point $\alpha_i = 0$ of the prize sharing rule N crowds out E in the long run. We considered the whole class of rules in a 2 stage game and showed that there exist games, where the prize sharing rules may arise in equilibrium. Our analysis is complementary to theirs. It seems a worthwhile exercise to check, which parts of the rule N can actually crowd out E in the long run, given that we have been able to compute precise the conditions under which N is a Nash equilibrium in the static context.

Given, the complexity of the analysis we also did not consider what would happen if there are more than two groups. Another question which deserves attention is whether these prize sharing rules N will ever be chosen in equilibrium if efforts also had a productive component. All these issues and more, are beyond the aims and scope of the current analysis and warrant future research.

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Appendix 1

Proof of Proposition 1

First notice that both

Proposition 1 will be proved with the help of a few Lemmas, which we prove next.

Lemma **1**

If both group A and B choose Rule E, then in any Nash Equilibrium

- *Group effort levels are* $(X_i^{EE}, X_j^{EE}) = (\frac{1}{4}, \frac{1}{4})$.
- *The net surplus in the contest is* $S^{EE} = \frac{1}{2}$
- *The probabilities of winning are* $(P_i^{EE}, P_j^{EE}) = (\frac{1}{2}, \frac{1}{2})$
- *The payoffs of the groups are* $(\Pi_i^{EE}, \Pi_j^{EE}) = (\frac{1}{4}, \frac{1}{4})$.

Proof:

The payoff of member k of Group i is as follows:

$$\Pi_{ki}^{EE} = \frac{1}{n_i}(\frac{X_i}{X_i + X_j} - X_i) \tag{22}$$

The individual members of the groups choose efforts x_{ki} to maximize (22).

The following equation represents the F.O.C of any member k in group i:

$$\frac{X_j}{(X_i + X_j)^2} = 1 \tag{23}$$

Similarly, the following equation represents the F.O.C. of any member k in group j:

$$\frac{X_i}{(X_i + X_j)^2} = 1 \tag{24}$$

Adding (23) and (24) and we find that

$$X_i + X_j = \frac{1}{2} \tag{25}$$

Using (25) back in (23) and (24) we obtain that in any Nash equilibrium we must have:

$$(X_i, X_j) = (\frac{1}{4}, \frac{1}{4})$$

Again using (25) we get that the net surplus $S^{EE} = 1 - X_i - X_j = 1 - \frac{1}{2} = \frac{1}{2}$

The probabilities can be obtained by dividing the equilibrium efforts by (25) and we get $(P_i^{EE}, P_j^{EE}) = (\frac{1}{2}, \frac{1}{2})$

Using the equilibrium effort levels in (22) we obtain the payoffs of the groups in equilibrium are as follows:

$$(\Pi_i^{EE}, \Pi_j^{EE}) = (\frac{1}{4}, \frac{1}{4})$$

. ■

Lemma **2**

If Group i chooses E and j chooses N, then in the intra-group symmetric Nash Equilibrium

- *Group effort levels are* $(X_i, X_j) = (\frac{1+(1-\alpha_j)(n_j-1)}{(n_j+1)} - \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2}, \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2})$.
- *The net surplus in the contest is* $S^{\sigma_A\sigma_B} = 1 - \frac{1+(1-\alpha_j)(n_j-1)}{n_j+1}$.
- *The probabilities of winning are* $(P_i^{\sigma_A\sigma_B}, P_j^{\sigma_A\sigma_B}) = (\frac{1+\alpha_j(n_j-1)}{(n_j+1)}, 1 - \frac{1+\alpha_j(n_j-1)}{(n_j+1)})$.
- *The payoffs of the groups are:*
  $(\Pi_i^{\sigma_A\sigma_B}, \Pi_j^{\sigma_A\sigma_B}) = (\frac{(1+\alpha_j(n_j-1))^2}{(n_j+1)^2}, \frac{1+(1-\alpha_j)(n_j-1)}{(n_j+1)} - \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2})$

Proof:

The payoff of member k in group i is as follows:

$$\Pi_{ki} = \frac{X_i}{X_i + X_j} - X_i \tag{26}$$

The payoff of member k of Group j (which chooses N) is as follows:

$$\Pi_{kj} = \frac{X_j}{X_i + X_j}[(1-\alpha_j)\frac{x_{kj}}{X_j} + \frac{\alpha_j}{n_j}] - x_{kj} \tag{27}$$

The following equation represents the F.O.C. of member k of group i:

$$\frac{X_j}{(X_i + X_j)^2} = 1 \tag{28}$$

The following equation represents the F.O.C. of member k of group j:

$$\frac{X_i}{(X_i + X_j)^2}[(1-\alpha_j)\frac{x_{kj}}{X_j} + \frac{\alpha_j}{n_j}] + \frac{X_j}{X_i + X_j}[\frac{(1-\alpha_j)(X_j - x_{kj})}{X_j^2}] = 1 \tag{29}$$

Adding (29) over members in group j we reach the following condition:

$$\frac{X_i}{(X_i + X_j)^2} + \frac{(1-\alpha_j)(n_j - 1)}{X_i + X_j} = n_j \tag{30}$$

Adding (28) and (30) we find the total effort expended in the contest in equilibrium to be :

$$X_i + X_j = \frac{1 + (1-\alpha_j)(n_j - 1)}{n_j + 1} \tag{31}$$

The net surplus can obtained from (31) and is as follows

$$S^{\sigma_A \sigma_B} = 1 - X_i - X_j = 1 - \frac{1 + (1-\alpha_j)(n_j - 1)}{n_j + 1}$$

.

Using (31) in (28) we find that in equilibrium group j puts in

$$X_j = \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2} \tag{32}$$

Replacing X_j in (32) in (31) we solve for X_i in equilibrium to be

$$X_i = \frac{1+(1-\alpha_j)(n_j-1)}{(n_j+1)} - \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2} \tag{33}$$

To figure out the payoff of Group i we divide (33) by (31) we get the probability of group i winning the contest to be

$$P_i^{\sigma_A \sigma_B} = \frac{X_i}{X_i + X_j} = 1 - \frac{1+(1-\alpha_j)(n_j-1)}{n_j+1} \tag{34}$$

Subtracting X_i in (33) from (34) gives us group i's payoff in equilibrium to be

$$\Pi_i^{\sigma_A \sigma_B} = \frac{(1+\alpha_j(n_j-1))^2}{(n_j+1)^2}$$

Similarly dividing (32) by (31) we the probability that group j wins the contest and subtracting X_j from the result we get the payoff of group j.

■

Lemma **3**

If both groups choose N and $\alpha_i n_j(n_i-1) - \alpha_j n_i(n_j-1) \geqslant n_i$ then group i is monopolized by group j. In the unique intra-group symmetric Nash Equilibrium

- *Group efforts are $(X_i, X_j) = (0, \frac{(1-\alpha_j)(n_j-1)}{n_j})$.*
- *The net surplus in the contest is $S^{NN} = 1 - \frac{(1-\alpha_j)(n_j-1)}{n_j}$.*
- *The probabilities of winning are $(P_i^{NN}, P_j^{NN}) = (0, 1)$.*
- *The payoffs of the groups are $(\Pi_i^{iM}, \Pi_j^{iM}) = (0, \frac{1+\alpha_j(n_j-1)}{n_j})$.*

Proof:

If both groups choose rule N, then the payoff of member k in group i is as follows

$$\Pi_{ki} = \frac{X_i}{X_i + X_j}[(1-\alpha_i)\frac{x_{ki}}{X_i} + \frac{\alpha_i}{n_i}] - x_{ki} \tag{35}$$

The following is the F.O.C. for member k of group i

$$\frac{X_j}{(X_i + X_j)^2}[(1-\alpha_i)\frac{x_{ki}}{X_i} + \frac{\alpha_i}{n_i}] + \frac{X_i}{X_i + X_j}[\frac{(1-\alpha_i)(X_i - x_{ki})}{X_i^2}] \leqslant 1 \tag{36}$$

If both groups choose rule N, then the payoff of member k in group j is as follows

$$\Pi_{kj} = \frac{X_j}{X_i + X_j}[(1-\alpha_j)\frac{x_{kj}}{X_j} + \frac{\alpha_j}{n_j}] - x_{kj} \tag{37}$$

The F.O.C. for member k in group j is

$$\frac{X_i}{(X_i + X_j)^2}[(1-\alpha_j)\frac{x_{kj}}{X_j} + \frac{\alpha_j}{n_j}] + \frac{X_j}{X_i + X_j}[\frac{(1-\alpha_j)(X_j - x_{kj})}{X_j^2}] \leqslant 1 \tag{38}$$

For all members of group i to choose $x_{ki} = 0$, the F.O.C. of group i members in (36) satisfied at $x_{ki} = 0$, which boils down to the following condition after we sum the F.O.C s

$$\frac{1}{X_j} + \frac{\theta_i}{X_j} \leqslant n_i \tag{39}$$

And summing the F.O.C.s of group j members in (38) ,at $X_i = 0$ we get the following condition

$$n_j X_j = \theta_j \tag{40}$$

For i to be monopolized in a Nash equilibrium both (39) and (40) have to be satisfied. Replacing X_j from (40) into (39) and simplifying we find that group i is monopolized if

$$\alpha_i n_j(n_i-1)-\alpha_j n_i(n_j-1)\geqslant n_i$$

Using (40) we get $X_j=\frac{\theta_j}{n_j}=\frac{(1-\alpha_j)(n_j-1)}{n_j}$

Therefore, net surplus is $S^{NN}=1-X_j=1-\frac{(1-\alpha_j)(n_j-1)}{n_j}$. Group j wins the contest with probability 1. The payoff of group i is 0, because it is monopolized. The payoff of group j is the net surplus S^{NN}, which it wins with probability 1.

■

Lemma 4

If both groups choose N and none of the groups is monopolized then in the unique intra-group symmetric Nash Equilibrium

- *Group efforts are* $(X_i,X_j)=(n_j(X^{NN})^2-(1-\alpha_j)(n_j-1)X^{NN},n_i(X^{NN})^2-(1-\alpha_i)(n_i-1)X^{NN})$ *where* $X^{NN}=\frac{1+(1-\alpha_i)(n_i-1)+(1-\alpha_j)(n_j-1)}{N}$.
- *The net surplus in the contest is* $S^{NN}=1-\frac{1+(1-\alpha_i)(n_i-1)+(1-\alpha_j)(n_j-1)}{N}$.
- *The probabilities of winning are* $(P_i^{NN},P_j^{NN})=(\frac{\chi_i}{N},1-\frac{\chi_i}{N})$ *where* $\chi_i=n_i+n_i(n_j-1)\alpha_j-n_j(n_i-1)\alpha_i$.
- *The payoffs of the groups are:* $(\Pi_i^{NN},\Pi_j^{NN})=((\frac{\chi_i}{N})(1-\frac{1+(1-\alpha_i)(n_i-1)+(1-\alpha_j)(n_j-1)}{N}),(1-\frac{\chi_i}{N})(1-\frac{1+(1-\alpha_i)(n_i-1)+(1-\alpha_j)(n_j-1)}{N}))$.

Proof:

As none of the groups is monopolized the F.O.C. (36) and (38) hold with equality at some $x_{ki}>0$, $\forall k\in\{2,3..n_i\}$ and $x_{kj}>0$, $\forall k\in\{2,3..n_j\}$.

Using (36) which the F.O.C. for Group i members and summing it over all the members in i we get the following condition

$$\frac{X_j}{X^2}+\frac{\theta_i}{X}=n_i \tag{41}$$

Summing (38) over members of group j , we get the following condition

$$\frac{X_i}{X^2} + \frac{\theta_j}{X} = n_j \tag{42}$$

Adding (41) and (42) and simplifying we can solve for total effort X to be

$$X^{NN} = \frac{1 + \theta_i + \theta_j}{n_i + n_j} = \frac{1 + (1 - \alpha_i)(n_i - 1) + (1 - \alpha_j)(n_j - 1)}{N} \tag{43}$$

From (43) it follows that the net surplus is

$$S^{NN} = 1 - X^{NN} = 1 - \frac{1 + (1 - \alpha_i)(n_i - 1) + (1 - \alpha_j)(n_j - 1)}{N}$$

From (41) and (42) and using $\theta_r = (1 - \alpha_r)(n_r - 1)$, $r = i, j$ we can deduce that

$$X_i = n_j X^2 - (1 - \alpha_j)(n_j - 1)X$$

and

$$X_j = n_i X^2 - (1 - \alpha_i)(n_i - 1)X$$

From these equations it is clear that the probability that group i wins the contest is

$$P_i^{NN} = \frac{X_i}{X} = n_j X - (1 - \alpha_j)(n_j - 1) \tag{44}$$

Replacing X^{NN} from (43) in (44) and simplifying we get that $P_i^{NN} = \frac{\chi_i}{N}$ where $\chi_i = n_i + n_i(n_j - 1)\alpha_j - n_j(n_i - 1)\alpha_i$. Of course, the chances that group j wins the contest is just $P_j^{NN} = 1 - \frac{\chi_i}{N}$.

Note that $\Pi_i^{NN} = P_i^{NN} S^{NN}$. Replacing values of P_i^{NN} and S^{NN} we get our result. Similarly we can obtain the payoff of group j.

■

Proposition 1 is just sub-parts of Lemma 1, 2, 3, 4.

Proof of Proposition 2

■ Part A of the Proposition

Notice in Lemma 2 that both $\Pi_i^{\sigma_A\sigma_B}$ and $\Pi_j^{\sigma_A\sigma_B}$ are independent of α_i.

Again from Lemma 2

$$\Pi_i^{\sigma_A\sigma_B} = \frac{(1+\alpha_j(n_j-1))^2}{(n_j+1)^2}$$

This is clearly a strictly increasing function of α_j.

$$\Pi_j^{\sigma_A\sigma_B} = \frac{1+(1-\alpha_j)(n_j-1)}{(n_j+1)} - \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2} \tag{45}$$

Define $C = \frac{1+(1-\alpha_j)(n_j-1)}{(n_j+1)}$. It is easy to see that $\frac{dC}{d\alpha_j} < 0$.

Replacing value of C in (45) we simplify it to $\Pi_j^{\sigma_A\sigma_B} = C - C^2$

Differentiating with respect to α_j we get

$$\frac{d\Pi_j^{\sigma_A\sigma_B}}{d\alpha_j} = (1-2C)\frac{dC}{d\alpha_j}$$

As $\frac{dC}{d\alpha_j} < 0$, the sign of $\frac{d\Pi_j^{\sigma_A\sigma_B}}{d\alpha_j}$ depends on the sign of $1-2C$. If $1-2C < 0$ then $\frac{d\Pi_j^{\sigma_A\sigma_B}}{d\alpha_j} > 0$. But $1-2C < 0$ when $\alpha_j < \frac{1}{2}$. If $\alpha_j > \frac{1}{2}$, then $1-2C > 0$ and then we have $\frac{d\Pi_j^{\sigma_A\sigma_B}}{d\alpha_j} < 0$.

■ Part (B) of the Proposition

Using Lemma 4 we can write the payoff of group i as follows

$$\Pi_i^{NN} = \Big(\frac{n_i + n_i(n_j-1)\alpha_j - n_j(n_i-1)\alpha_i}{N}\Big)\Big(\frac{1+\alpha_i(n_i-1)+\alpha_j(n_j-1)}{N}\Big) \tag{46}$$

Notice that in both the terms within the brackets α_j enters with a positive sign. Therefore, it is the case that $\frac{d\Pi_i^{NN}}{d\alpha_j} > 0$. So we have $\frac{d\Pi_A^{NN}}{d\alpha_B} > 0$ and $\frac{d\Pi_B^{NN}}{d\alpha_A} > 0$.

Differentiating Π_i^{NN} in (46) with respect to α_i we get

$$\frac{d\Pi_i^{NN}}{d\alpha_i} = \frac{(n_i-1)[(n_i-n_j)(1+(n_j-1)\alpha_j)-2n_j(n_i-1)\alpha_i]}{N^2} \tag{47}$$

The sign of $\frac{d\Pi_i^{NN}}{d\alpha_i}$ is the same as the sign of $(n_i-n_j)(1+(n_j-1)\alpha_j)-2n_j(n_i-1)\alpha_i$, which is the second term in brackets in the numerator.

Consider $i = A$. The term then is $(n_A-n_B)(1+(n_B-1)\alpha_B)-2n_B(n_A-1)\alpha_A$. It is negative as we have assumed $n_B \geqslant n_A$. Therefore, $\frac{d\Pi_A^{NN}}{d\alpha_A} < 0$.

Consider $i = B$. The term $(n_B-n_A)(1+(n_A-1)\alpha_A)-2n_A(n_B-1)\alpha_B > 0$ when $\alpha_B < \frac{(n_B-n_A)(1+(n_A-1)\alpha_A)}{2n_A(n_B-1)} = \alpha_B^o$. Therefore, $\frac{d\Pi_B^{NN}}{d\alpha_B} > 0$ if $\alpha_B < \alpha_B^o$. And $\frac{d\Pi_B^{NN}}{d\alpha_B} < 0$ if $\alpha_B > \alpha_B^o$.

Proof of Proposition 3

Strategy profile EE will be a pure strategy Nash equilibrium of Γ if $\Pi_A^{EE} \geqslant \Pi_A^{NE}$ and $\Pi_B^{EE} \geqslant \Pi_B^{EN}$.

From Lemma 1 we know that $\Pi_A^{EE} = \frac{1}{4}$. And from Lemma 2 we know that

$$\Pi_A^{NE} = \frac{1+(1-\alpha_A)(n_A-1)}{(n_A+1)} - \frac{(1+(1-\alpha_A)(n_A-1))^2}{(n_A+1)^2}$$

E is a best response to E for group A if the following inequality is satisfied

$$\frac{1}{4} \geqslant \frac{1+(1-\alpha_A)(n_A-1)}{(n_A+1)} - \frac{(1+(1-\alpha_A)(n_A-1))^2}{(n_A+1)^2} \tag{48}$$

To see why (48) holds we define $x = \frac{1+(1-\alpha_A)(n_A-1)}{(n_A+1)}$. Then (48) can be written as

$$\frac{1}{4} \geqslant x - x^2$$

$$\Rightarrow (x-\frac{1}{2})^2 \geqslant 0$$

But this is true irrespective of the values of the parameters. Playing strategy E is a best response for group A to group B playing E.

When $\alpha_A = \frac{1}{2}$, then $x = \frac{1}{2}$ and we have

$$\Rightarrow (x - \frac{1}{2})^2 = 0$$

So in this case E is a weak best response to E for group A. In all other cases E is a strong best response for group A to E.

Similarly we can show that $\Pi_B^{EE} \geqslant \Pi_B^{EN}$ which means group B playing E is a best response to group A playing E.

Proof of Proposition 4

For strategy profile NN to be a Nash equilibrium we must have $\Pi_A^{NN} \geqslant \Pi_A^{EN}$ and $\Pi_B^{NN} \geqslant \Pi_B^{NE}$.

In general it must be true that for $i = A, B$

$$\Pi_i^{NN} \geqslant \Pi_i^{\sigma_A \sigma_B}(\sigma_i = E, \sigma_j = N) \tag{49}$$

Replacing the payoffs from Lemma 2 and 4 in (49) we get

$$\frac{\Big(n_i + n_i(n_j-1)\alpha_j - n_j(n_i-1)\alpha_i\Big)\Big(1 + (n_i-1)\alpha_i + (n_j-1)\alpha_j\Big)}{N^2} \geqslant \frac{\Big(1 + \alpha_j(n_j-1)\Big)^2}{(n_j+1)^2} \tag{50}$$

We solve (50) as a quadratic equation using the Sridharacharya formula and get the following two roots:

The smaller root is

$$\underline{\alpha}_i = \frac{(n_i - n_j^2)(1 + \alpha_j(n_j-1))}{n_j(n_j+1)(n_i-1)} \tag{51}$$

The larger root is

$$\overline{\alpha}_i = \frac{1+\alpha_j(n_j-1)}{(n_j+1)} \tag{52}$$

It can be easily shown using Proposition 2 that $\Pi_i^{NN} \geqslant \Pi_i^{\sigma_A\sigma_B}(\sigma_i = E, \sigma_j = N)$ iff $\alpha_i \in [\underline{\alpha}_i, \overline{\alpha}_i]$. [14] In other words, if $\alpha_i \in [\underline{\alpha}_i, \overline{\alpha}_i]$ for $i = A, B$, then NN is a Nash equilibrium profile.

Now consider $i = A$. Given the assumption that $n_B \geqslant n_A$ it is clear from (51) that $\underline{\alpha}_A < 0$. Therefore the lower root can be ignored and the relevant range is $\alpha_A \in [0, \overline{\alpha}_A]$.

Consider $i = B$. From equation (51) it is clear that $\underline{\alpha}_B < 0$ iff $n_B < n_A^2$. Otherwise it is positive. If $\underline{\alpha}_B < 0$, then the relevant range for NN to be a Nash equilibrium is $\alpha_B \in [0, \overline{\alpha}_B]$. If $\underline{\alpha}_B \geqslant 0$, then the relevant range is $\alpha_B \in [\underline{\alpha}_B, \overline{\alpha}_B]$. We can write this range in a concise manner as $\alpha_B \in [max\{0, \underline{\alpha}_B\}, \overline{\alpha}_B]$.

Therefore, NN is a Nash equilibrium profile of Γ iff $\alpha_A \in [0, \overline{\alpha}_A]$ and $\alpha_B \in [max\{0, \underline{\alpha}_B\}, \overline{\alpha}_B]$. If the condition is not satisfied then in light of Proposition 3 it follows that strategy E is a strictly dominant strategy for at least one of the groups in Γ. Given that there are only two groups, Γ will be dominance solvable with the unique Nash equilibrium strategy profile EE. See Figures 8 and 9.

Proof of Proposition 7

For EE to strictly payoff dominate NN we find when is it that $\Pi_A^{NN} < \Pi_A^{EE}$ and $\Pi_B^{NN} < \Pi_B^{EE}$.

In general for $i = A, B$ we must have

$$\Pi_i^{NN} < \Pi_i^{EE} \tag{53}$$

Using Lemma 1 and 4 in (53) we get the following inequality which needs to hold

[14] For instance, consider group A. Starting from $\overline{\alpha}_A$ where (50) holds with equality, if we decrease α_A slightly, the LHS of (50) increases by Proposition 2 but the RHS being independent of α_A is unaffected.

$$\frac{\Big(n_i + n_i(n_j-1)\alpha_j - n_j(n_i-1)\alpha_i\Big)\big(1+(n_i-1)\alpha_i+(n_j-1)\alpha_j\Big)}{N^2} < \frac{1}{4} \tag{54}$$

Solving (54) as a quadratic equation using the Sridharacharya formula we get the following two roots

The larger root is

$$\alpha_j^+ = \frac{(n_j-n_i)(n_i-1)\alpha_i + N\sqrt{((n_i-1)\alpha_i)^2+n_i} - 2n_i}{2n_i(n_j-1)} \tag{55}$$

The smaller root is

$$\alpha_j^- = \frac{(n_j-n_i)(n_i-1)\alpha_i - N\sqrt{((n_i-1)\alpha_i)^2+n_i} - 2n_i}{2n_i(n_j-1)} \tag{56}$$

Using Proposition 2 we can easily verify that EE will payoff dominate NN iff $\alpha_j \in (\alpha_j^-, \alpha_j^+)$, $j = A, B$.

We first consider group $i = A$. The roots of $\Pi_A^{NN} = \Pi_A^{EE}$ are α_B^+ and α_B^-. We now state a few important properties which these roots satisfy.

Property **1**

In the $\alpha_A\alpha_B$ plane α_B^+ lies completely above the α_A axis and α_B^- lies completely below the α_A axis and can therefore be ignored [15].

This can be verified by trying to solve either $\alpha_B^+ = 0$ or $\alpha_B^- = 0$, which gives us values of α_A at which these roots cut the α_A axis. Neither equation has a real solution as the discriminant for both these problems is $N\sqrt{1-n_B}$, which is a complex number. Therefore, there does not exist a real α_A such that $\alpha_B^+ = 0$ or $\alpha_B^- = 0$. Therefore, neither $\alpha_B^+ = 0$ nor $\alpha_B^- = 0$ cut the α_A axis.

Replacing, $\alpha_A = 0$ in α_B^+ we find that it cuts the α_B axis at $\frac{N\sqrt{n_A}-2n_A}{2n_A(n_B-1)} > 0$. This combined with the observation made above helps us conclude that α_B^+ lies completely above the α_A

[15] This means that the relevant zone for payoff dominance will be $\alpha_B \in [0, \alpha_B^+)$

axis

Replacing, $\alpha_A = 0$ in α_B^- we find that it cuts the α_B axis at $\frac{-N\sqrt{n_A}-2n_A}{2n_A(n_B-1)} < 0$. Therefore, α_B^- lies completely below the α_A axis and can be ignored.

PROPERTY **2**

α_B^+ is increasing and convex in the α_A [16].

To prove this we just look at the first and the second derivatives of α_B^+ with respect to α_A

$$\frac{d\alpha_B^+}{d\alpha_A} = \frac{(n_B - n_A)(n_A - 1) + \frac{N(n_A-1)^2\alpha_A}{\sqrt{n_A+(\alpha_A(n_A-1))^2}}}{2n_A(n_B - 1)} > 0$$

$$\frac{d^2\alpha_B^+}{d\alpha_A^2} = \frac{N(n_A - 1)^2}{2n_A(n_B - 1)}\Big(\frac{n_A}{(n_A + (\alpha_A(n_A - 1)^2)^{\frac{3}{2}}}\Big) > 0$$

PROPERTY **3**

α_B^+ passes through $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$. At $\alpha_A = \frac{1}{2}$ it is supported from below by the line $\overline{\alpha}_B$.

The first part is easily shown by replacing $\alpha_A = \frac{1}{2}$ in α_B^+. We get $\alpha_B^+ = \frac{1}{2}$

To prove the second part we note from (52) that the slope of $\overline{\alpha}_B$ is $\frac{d\overline{\alpha}_B}{d\alpha_A} = \frac{n_A-1}{n_A+1}$.

The slope of α_B^+ is

$$\frac{d\alpha_B^+}{d\alpha_A} = \frac{(n_B - n_A)(n_A - 1) + \frac{N(n_A-1)^2\alpha_A}{\sqrt{n_A+(\alpha_A(n_A-1))^2}}}{2n_A(n_B - 1)}$$

At $\alpha_A = \frac{1}{2}$, the slope is

$$\frac{d\alpha_B^+}{d\alpha_A} = \frac{(n_B - n_A)(n_A - 1) + \frac{N(n_A-1)^2}{n_A+1}}{2n_A(n_B - 1)} = \frac{n_A - 1}{n_A + 1}$$

[16] For clear visualization note that in the $\alpha_A\alpha_B$ plane α_B^+ plots as an increasing and convex function

Therefore, Slope of $\overline{\alpha}_B$ = Slope of α_B^+ at $\alpha_A = \frac{1}{2}$. Also at $\alpha_A = \frac{1}{2}$ we have $\overline{\alpha}_B = \frac{1}{2}$ and $\alpha_B^+ = \frac{1}{2}$. So, the curve α_B^+ and line $\overline{\alpha}_B$ have a common point and same slope at $\alpha_A = \frac{1}{2}$. Given that α_B^+ is convex and increasing and $\overline{\alpha}_B$ is increasing and linear in α_A, it follows that $\overline{\alpha}_B$ supports α_B^+ from below at $\alpha_A = \frac{1}{2}$.

Now we consider $i = B$ and state similar properties for α_A^+ and α_A^-

Property **4**

In the $\alpha_A\alpha_B$ plane α_A^+ lies completely to the right of the α_B axis and α_A^- lies completely to the left of α_A axis and can therefore be ignored.

We skip the proof as it follows exactly the same steps as Property 1.

Property **5**

α_A^+ in increasing (decreasing) in α_B if $\alpha_B > (<) \frac{n_B - n_A}{2\sqrt{n_A}(n_B-1)}$. α_A^+ is convex in α_B[17].

To prove this we just look at the first and the second derivatives of α_A^+ with respect to α_B

$$\frac{d\alpha_A^+}{d\alpha_B} = \frac{\frac{N(n_B-1)^2\alpha_B}{\sqrt{\alpha_B^2(n_B-1)^2+n_B}} - (n_B - n_A)(n_B - 1)}{2n_B(n_A - 1)}$$

Therefore, $\frac{d\alpha_A^+}{d\alpha_B} > 0$ iff

$$\frac{N(n_B - 1)^2\alpha_B}{\sqrt{\alpha_B^2(n_B - 1)^2 + n_B}} > (n_B - n_A)(n_B - 1)$$

Simplifying we get that this happens iff $\alpha_B > \frac{n_B - n_A}{2\sqrt{n_A}(n_B-1)}$

For convexity of α_A^+ we look at the second derivative, which is

$$\frac{d^2\alpha_A^+}{d\alpha_B^2} = \frac{N(n_B - 1)^2}{2n_B(n_A - 1)}\Big(\frac{n_B}{((n_B - 1)^2\alpha_B^2 + n_B)^{\frac{3}{2}}}\Big) > 0$$

[17] In the $\alpha_A\alpha_B$ plane it plots as a concave function when α_A^+ is increasing and convex function when α_A^+ is decreasing. This happens because the domain of the function α_A^+. i.e., $\alpha_B \in [0, 1]$ is the vertical axis

Property 6

α_A^+ passes through $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$. At $\alpha_B = \frac{1}{2}$ it is supported from below by the line $\overline{\alpha}_A$ [18].

We skip the proof as it follows exactly the same steps as Property 3.

Properties 1 to 6 are captured in Figure 10.

We now proceed to show that the set of games Γ with Nash equilibria EE ad NN, i.e., $\alpha_A \in [0, \overline{\alpha}_A]$ and $\alpha_B \in [max\{0, \underline{\alpha}_B\}, \overline{\alpha}_B]$, is a proper subset of the set of games where EE payoff dominates NN, i.e., $\alpha_A \in [0, \alpha_A^+)$ and $\alpha_B \in [0, \alpha_B^+)$.

This fact directly follows from Property 3 and 6. Given for $i = A, B$, $\overline{\alpha}_i$ supports α_i^+ from below it is true that $\overline{\alpha}_i < \alpha_i^+$ except at $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$ [19], where they are equal. But we can remove $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$ as at that point all strategy profiles are Nash equilibria. In the set of games we are interested in we have $\overline{\alpha}_A < \alpha_A^+$ and $\overline{\alpha}_B < \alpha_B^+$. A look at the parametric ranges in the previous paragraph immediately confirms that the games which have Nash equilibria EE and NN are a proper subset of the games in which EE strictly payoff dominates NN. Look at Figures 11 and 12.

Proof of Proposition 5

Let us first consider the terms $\Pi_A^{NN} - \Pi_A^{EN}$ and $\Pi_B^{NN} - \Pi_B^{NE}$. In general, for $i = A, B$ we are have to consider $\Pi_i^{NN} - \Pi_i^{\sigma_A \sigma_B}$, where group i is the one which chooses E when the two group choose different strategies.

From Lemma 2 and 4 we can write the difference as follows

$$\Pi_i^{NN} - \Pi_i^{\sigma_A \sigma_B} = \left(\frac{n_i + n_i(n_j-1)\alpha_j - n_j(n_i-1)\alpha_i}{N}\right)\left(\frac{1+\alpha_i(n_i-1)+\alpha_j(n_j-1)}{N}\right) - \frac{(1+\alpha_j(n_j-1))^2}{(n_j+1)^2} \tag{57}$$

Simplifying we get the following condition

[18] In the diagram in the $\alpha_A \alpha_B$ plane it seems that $\overline{\alpha}_A$ supports α_A^+ from above not below. But it has to be noted that that the domain $\alpha_B \in [0, 1]$ is the vertical axis and not the horizontal axis

[19] $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$ is the point at which the lines $\overline{\alpha}_i$ s support the curves

$$\begin{aligned}\Pi_i^{NN} - \Pi_i^{\sigma_A \sigma_B} &= \tfrac{(n_i-1)}{N^2(n_j+1)^2}\Bigg((n_j^2 - n_i)(1+\alpha_j(n_j-1))^2 + (n_j+1)^2(n_i-n_j)(1+\alpha_j(n_j-1))\alpha_i \\ &-n_j(n_j+1)^2(n_i-1)\alpha_i^2\Bigg)\end{aligned} \tag{58}$$

Let us define

$$\Delta_i = \frac{(n_j^2 - n_i)(1+\alpha_j(n_j-1))^2 + (n_j+1)^2(n_i-n_j)(1+\alpha_j(n_j-1))\alpha_i - n_j(n_j+1)^2(n_i-1)\alpha_i^2}{(n_i-1)} \tag{59}$$

Using the definition of $\overline{\alpha}_i$ and $\underline{\alpha}_i$ in (15) and (16) we can simplify and rewrite the above condition as follows

$$\Delta_i = n_j(n_j+1)^2\Big[(\overline{\alpha}_i - \alpha_i)(\alpha_i - \underline{\alpha}_i)\Big] \tag{60}$$

Using this definition of Δ_i in (59) we can write equation (58) as

$$\Pi_i^{NN} - \Pi_i^{\sigma_A \sigma_B} = \frac{(n_i-1)^2}{N^2(n_j+1)^2}\Delta_i \tag{61}$$

Now let us consider $\Pi_A^{EE} - \Pi_A^{NE}$ and $\Pi_B^{EE} - \Pi_B^{EN}$. In general for $i = A, B$ we are interested in $\Pi_i^{EE} - \Pi_i^{\sigma_A \sigma_B}$, where group i is the one which chooses N when the two groups choose different strategies.

From Lemma 1 and 2 we can write the difference as

$$\begin{aligned}\Pi_i^{EE} - \Pi_i^{\sigma_A \sigma_B} &= \tfrac{1}{4} - \tfrac{1+(1-\alpha_i)(n_i-1)}{(n_i+1)} - \tfrac{(1+(1-\alpha_i)(n_i-1))^2}{(n_i+1)^2} \\ &= (\tfrac{1}{2} - \tfrac{1+(1-\alpha_i)(n_i-1)}{(n_i+1)})^2\end{aligned} \tag{62}$$

This can be simplified and written as

$$\Pi_i^{EE} - \Pi_i^{\sigma_A \sigma_B} = \frac{(n_i - 1)^2}{4(n_i + 1)^2}(1 - 2\alpha_i)^2 \tag{63}$$

For NN to risk dominate EE we must have

$$(\Pi_A^{NN} - \Pi_A^{EN})(\Pi_B^{NN} - \Pi_B^{NE}) \geqslant (\Pi_A^{EE} - \Pi_A^{NE})(\Pi_B^{EE} - \Pi_B^{EN}) \tag{64}$$

Using equations (61) and (63) for groups $i = A, B$, we can immediately conclude that inequality (64) is satisfied iff

$$N^4(1 - 2\alpha_A)^2(1 - 2\alpha_B)^2 \leqslant 16\Delta_A\Delta_B$$

.

Appendix 2

Best Response Functions

Here we study the properties of the best response functions of the individual's in the two groups. To do that we start with a few notations.

We denote the best response function of the k^{th} member of group $i \in \{A, B\}$, when the group chooses $\sigma_i \in \{E, N\}$ as $R_{ik}^{\sigma_i}(X_j)$. For example, if group A chooses E, then the best response function of the k^{th} member of group A will be denoted $R_{Ak}^{E}(X_B)$, and if it chooses N, then $R_{Ak}^{N}(X_B)$.

When group i chooses E, the best response Function of member k , $R_{ik}^{E}(X_j)$ can be obtained by maximizing (4). It is implicitly characterized by the following first order condition:

$$\frac{X_j}{(X_i + X_j)^2} = 1 \tag{65}$$

Similarly, when group i chooses N, its best response function of member k, $R_{ik}^{N}(X_j)$ is obtained by maximizing (3). It is implicitly characterized by the following first order condition:

$$\frac{X_j}{(X_i + X_j)^2}[(1-\alpha_i)\frac{x_{ki}}{X_i} + \frac{\alpha_i}{n_i}] + \frac{X_i}{X_i + X_j}[\frac{(1-\alpha_i)(X_i - x_{ki})}{X_i^2}] = 1 \tag{66}$$

Because group members are symmetric in all respects, the best response functions are the same. We can therefore apply symmetry and obtain the best response function of a representative agent of the group, which we denote $R_i^{\sigma_i}(X_j)$. This is the same notation introduced above but without the subscript k.

In the Proposition that follows, we use the following notation:

For $i \in \{A, B\}$

$$\theta_i = (1-\alpha_i)(n_i - 1)$$

θ_i is a measure of competitiveness of group i weighted by group size. If α_i is low θ_i is high, so that more competitive groups will tend to have a higher θ_i. If such a group is also large, then the competitive nature of the group gets accentuated by its size. Therefore, larger groups with lower α_i's have higher θ_i's and are the most competitive ones.

Next, we state a general result about best response functions of the groups. We state the result without proof [20] but do a detailed diagrammatic analysis.

PROPOSITION **8**

For $i, j \in \{A, B\}$ and $j \neq i$

(A) If group i chooses E, then the slope of the best response function is as follows:

$$\frac{R_i^E(X_j)}{dX_j} = \frac{X_i - X_j}{2X_j}$$

Therefore, X_i is a strategic complement to X_j iff $X_i > X_j$.

(B) If group i chooses N, then the slope of the best response function is as follows:

$$\frac{R_i^N(X_j)}{dX_j} = \frac{(X_i - X_j) - \theta_i(X_i + X_j)}{2X_j + 2\theta_i(X_i + X_j)}$$

Therefore, X_i is a strategic complement to X_j iff $\frac{X_i}{X_j} > \frac{1+\theta_i}{1-\theta_i}$.

We next discuss the results summarized in Proposition 8.

■ **Both groups choose E:** The best Response Functions in this case are represented in Figure 13. Both $R_A^E(X_B)$ and $R_B^E(X_A)$ are strictly increasing when $X_A < \frac{1}{4}$ and $X_B < \frac{1}{4}$. Here, X_A and X_B are strategic complements.

The Best Response functions are well defined except at $(X_A, X_B) = (0, 0)$ and they intersect at $(X_A^{EE}, X_B^{EE}) = (\frac{1}{4}, \frac{1}{4})$, which is the unique Nash equilibrium in group efforts.

[20] Available on request

At the equilibrium point, X_A and X_B are strategically independent, i.e., neither strategic complements nor strategic substitutes.

It is also important to notice that the Best Response Functions are independent of the parameters in the model.

■ **Group i chooses E, Group j chooses N:** Here, we will analyze the Best response functions of group i, which chooses E and group j, which chooses N. For ease of exposition we will assume that $i = A$ and $j = B$. The Best Response Functions in this case are represented in Figure 14. The Best Response function for group A, $R_A^E(X_B)$, is the same as in the previous case.

The Best Response Function of group B, $R_B^N(X_A)$, is increasing when $\frac{X_B}{X_A} > \frac{1+\theta_B}{1-\theta_B}$. The term on the right hand side is positive only when $\alpha_B \in (\frac{n_B-2}{n_B-1}, 1]$. In all other cases, the condition is trivially satisfied.

To see this clearly, in Figure 14, we have plotted the Best Response Function of group B for $\alpha_B = 0, \frac{1}{2}, 1$. When we increase α_B, $R_B^N(X_A)$, shifts inwards, because free riding increases within group B, which causes X_B to fall for the same group size n_B.

The Best Response Functions have a unique intersection and it is always at a point, where $R_B^N(X_A)$ is strictly decreasing. So, X_B is a strategic substitute of X_A in the neighborhood of any Nash equilibrium in group efforts .

X_A, on the other hand, is a strategic substitute to X_B as, long as $X_B > \frac{1}{4}$. So, when $X_B > \frac{1}{4}$, X_A and X_B are strategic substitutes. The Nash equilibrium in group effort levels, is stable.

However, when $X_B < \frac{1}{4}$, X_A is a strategic complement to X_B, while X_B is a strategic substitute of X_A. The Nash equilibrium in group effort levels, is unstable.

■ **Both groups choose N:** The Best Response Functions in this case are represented in Figure 15. As in the previous case the Best Response Function of group B, $R_B^N(X_A)$, is strictly increasing when $\frac{X_B}{X_A} > \frac{1+\theta_B}{1-\theta_B}$. However, now the Best Response function of group A, $R_A^N(X_B)$, is also increasing when $\frac{X_A}{X_B} > \frac{1+\theta_A}{1-\theta_A}$. The functions intersect uniquely to yield the Nash equilibrium in group efforts.

The functions intersect at a point, where $R_B^N(X_A)$ is decreasing. Therefore, X_B is a strategic substitute for X_A in the neighborhood of any Nash equilibrium. If, additionally at the equilibrium we have that $\frac{X_A}{X_B} < \frac{1+\theta_A}{1-\theta_A}$, so that $R_A^N(X_B)$ is also decreasing, then X_A is also a strategic substitute for X_B and the Nash equilibrium is stable.

If, however, $\alpha_A \in (\frac{n_A-2}{n_A-1}, \frac{n_A(2-N)}{(n_B-n_A)(n_A-1)} + \frac{2n_A(n_B-1)}{(n_B-n_A)(n_A-1)}\alpha_B)$, the functions intersect at a point where $R_A^N(X_B)$ is increasing. Here, X_A is a strategic complement to X_B. In this case the Nash equilibrium is unstable.

For this case to arise, we need both α_A and α_B to be sufficiently high and close to 1. One example of such a case is where $\alpha_A = 1$ and $\alpha_B = 1$. This is shown in Figure 15. When α_i rises, $i \in \{A, B\}$, $R_i^N(X_j)$ shifts in as free riding increases within group i but $R_j^N(X_i)$ is unaffected.

One interesting phenomenon, which arises in this case, is Monopolization of a group from the contest. If $\frac{\theta_B}{n_B} \geqslant \frac{1+\theta_A}{n_A}$, then the Best Response Function of group A is contained within the Best Response Function of group B, so that they do not intersect at any point in the interior, where both $X_A > 0$ and $X_B > 0$. Then in the Nash equilibrium in efforts, group B puts in an aggregate effort of $X_B^{NN} = \frac{1+\theta_A}{n_A}$ and group A members best respond with zero effort, so that $X_A^{NN} = 0$. So, we say that group A has been monopolized by group B. This phenomenon is captured in Figure 16. In a similar manner, group B is monopolized by group A when, $\frac{\theta_A}{n_A} \geqslant \frac{1+\theta_B}{n_B}$.

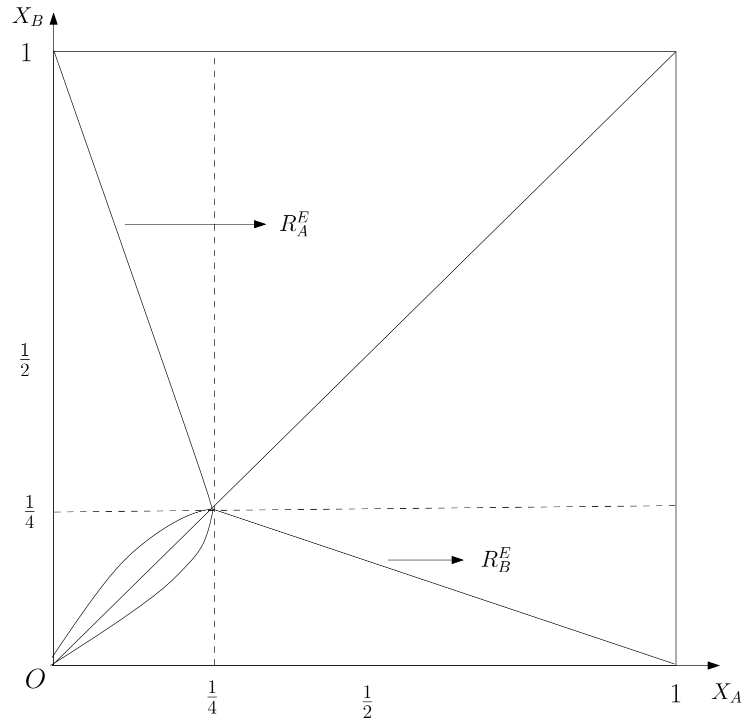

Figure 13: Best Responses with EE

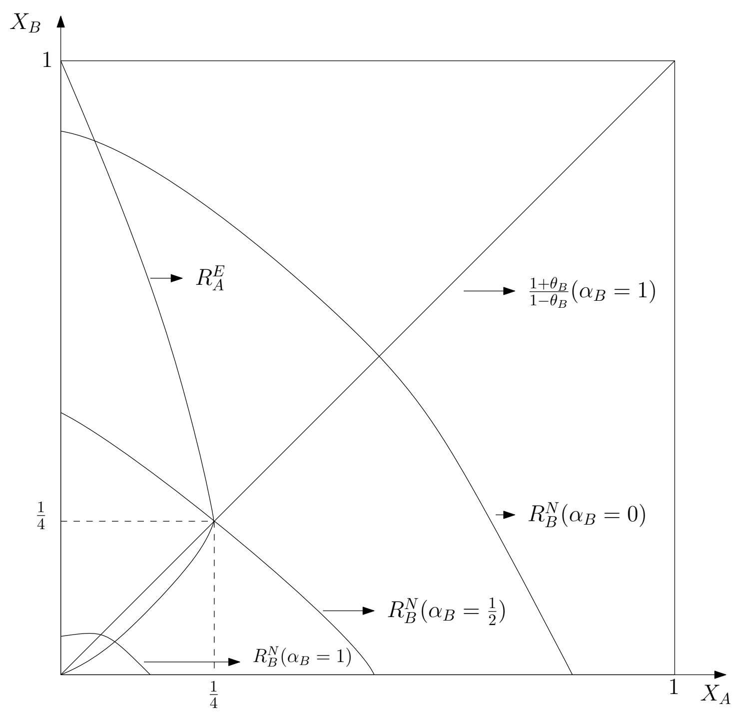

Figure 14: Best Responses with EN

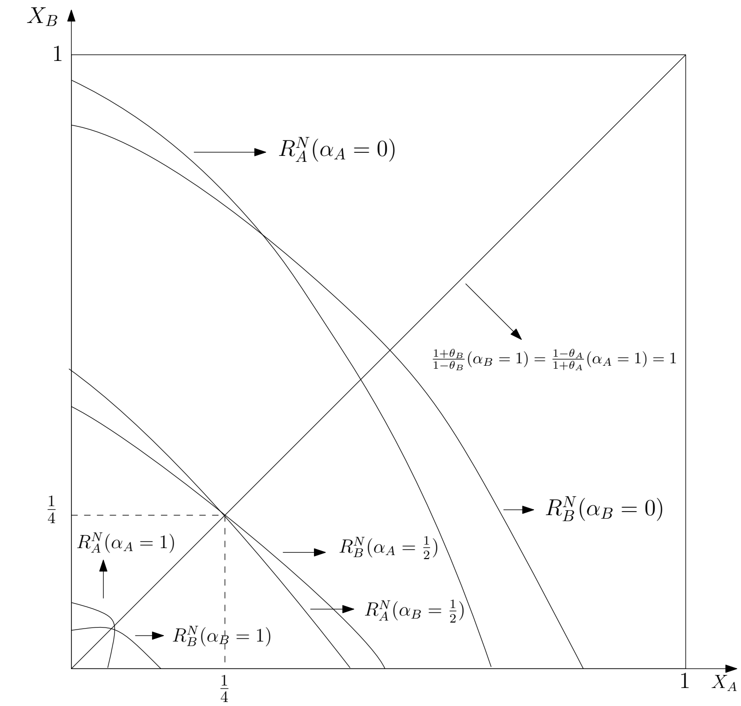

Figure 15: Best Responses with NN

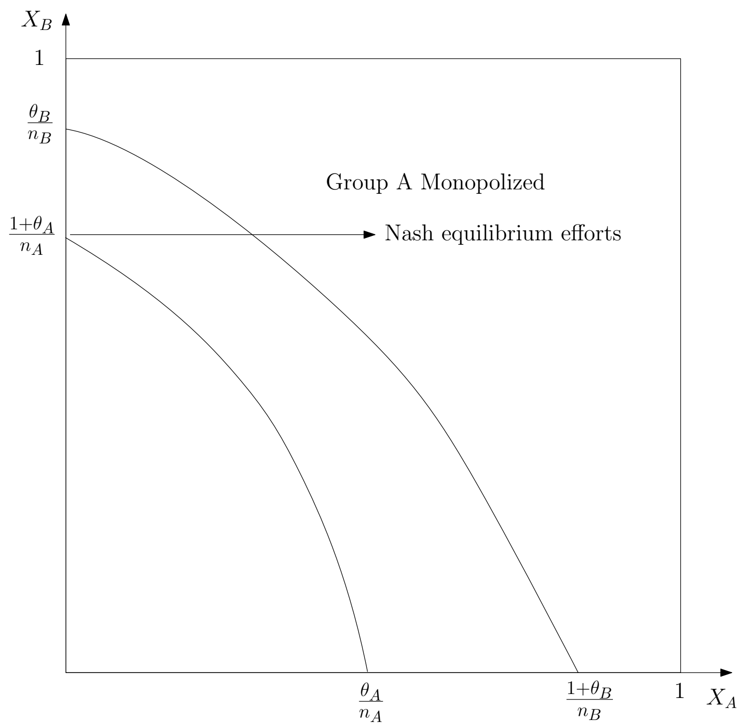

Figure 16: Group A Monopolized