Discussion Papers in Economics

Prize sharing rules in collective contests: When does group size matter?

Dhritiman Gupta

July, 2020

Discussion Paper 20-04

Indian Statistical Institute, Delhi
Economics and Planning Unit
7, S. J. S. Sansanwal Marg, New Delhi, 110016, India

Prize Sharing Rules in Collective Contests: When does group size matter? *

Dhritiman Gupta †

July 16, 2020

*I am grateful to my supervisor Dr. Prabal Roy Chowdhury for useful comments.
†Dhritiman Gupta, Indian Statistical Institute, New Delhi, Email: gupta.dhritiman@gmail.com

Prize Sharing Rules in Collective Contests: When Does Group Size Matter?

Abstract

In this paper we deal with situations of collective contests between two groups over a private prize. A well known way to divide the prize within the winning group is the prize sharing rule introduced by Nitzan (1991). Since its introduction it has become a standard in the collective contests literature. We generalize this rule by introducing a restriction we call norms of competitiveness of a group. We fully characterize how group sizes interact with such norms. What we show is that the smaller group is generally aggressive, but the larger group needs to have really egalitarian norms to behave aggressively in the contest. We also take up the question of how group welfare relates to group sizes under the stated norms. We provide a complete set of conditions under which the larger group fares worse in the contest, a phenomenon called Group Size Paradox (GSP) in the literature.

JEL Classification: D23, D71, D72, H41, C72

Keywords: Rent Seeking, Collective Action, Prize Sharing Rules

1 Introduction

Collective contests are situations where agents organize into groups to compete over a given prize. Such situations are quite common: funds to be allocated among different departments of an organization, team sports, projects to be allocated among different divisions of a firm, regions within a country vying for shares in national grants, party members participating in pre-electoral campaigns,disputes between tribes over scarce resources.

Prizes in such contests may be purely private, e.g. money. Or the prizes may have some public characteristic like reputation or glory for the winning team. In this paper we focus on purely private prizes. For prizes with public characteristics the reader may refer to Baik (2008), Balart et al. (2016).

One essential feature of collective contests is that a groups' performance depends on the individual contribution of its members. Departments in universities usually receive funds depending on the publication record of the department, which in turn depends on the individual publication of its members. So the group needs to coordinate and establish some rules regarding its internal organization, in particular how to share the prize in case of success. In this study we focus the prize sharing rule proposed by Nitzan (1991). The rule suggests the following way of sharing the prize within the group, if the group wins the collective contest:

$$(1-\alpha_i)\frac{x_{ki}}{X_i} + \alpha_i\frac{1}{n_i} \tag{1}$$

where x_{ki} is the effort put in by the k^{th} member of group i, X_i is the total effort of group i and n_i is the size of group i. α_i is weight put on egalitarian sharing of the prize within the group and $1-\alpha_i$ is the weight put on a sharing rule, which rewards higher efforts within the group, thereby inducing intra-group competition, i.e. an outlay-based incentive scheme. An increase in the weight on the egalitarian component increases free riding incentives in the group members. Whereas, an increased weight on the outlay-based component incentivizes efforts by making each members reward depend on efforts of all other members of the group.

This prize sharing rule has been extensively studied in the literature on collective contests, see e.g. Flamand et al. (2015). The popularity of this rule lies in its intuitive appeal. It combines two extreme forms of internal organization, capturing the tension between intra-group competition and the tendency to free ride on the efforts of other group members. In the situation of a collective contest, a larger weight on the outlay-based scheme helps a group generate higher efforts, thereby increasing their chances of winning the contest. But, higher efforts also eat into the surplus the groups are competing for, thereby making internal competition costly. A larger weight on the egalitarian component increases internal free riding making a group less competitive in the contest but leaves a larger surplus to be consumed in case of success. This is the trade-off, which the group leaders face when choosing its organizational form i.e., the weight he wants to put on the respective components of the prize sharing rule.

The literature on strategic choice of sharing rules see e.g. Flamand et al. (2015), allows the leader exactly this choice. A group leader can optimally choose the weight α_i for his own group. But there are two separate strands in this literature, which differ on the restrictions which are placed on that choice.

In one strand, the choice of shares α_i is restricted to the interval $[0, 1]$, so that the leader can choose to reward individual efforts at most proportionally. This situation is referred to as the case of "bounded meritocracy" in Balart et al. (2016). In an alternate strand, the leader is allowed to reward efforts more than proportionally by fining members, who put in lower effort and transferring that amount to the hard working ones. In such a case the interval over which α_i is chosen is $(-\infty, 1]$ [1]. This case is called "unbounded meritocracy" in Balart et al. (2016). The literature finds that when the leaders choose the rules simultaneously, at least one of the groups chooses not to all the weight on the outlay-based component of the

[1]Readers can look at Hillman and Riley (1989) for a paper where such transfers between individuals is possible.

prize sharing rule in equilibrium i.e., the leader of at least one group chooses not to make the group maximally competitive in the contest. This is irrespective of whether the rule is "boundedly meritocratic" or "unboundedly meritocratic".

We generalize the above literature by fixing the choice of α_i to the interval $[\underline{\alpha}_i, 1]$, where $\underline{\alpha}_i \in (-\infty, 1]$ is a parameter in the model. It can be interpreted as a social norm of competitiveness within the groups. This social norm, just like group sizes, is taken as an exogenous property of the groups and denotes the maximum possible competitiveness of a group. So, we can have smaller groups with very competitive norms i.e., "small aggressive groups" or large groups with egalitarian norms i.e., "large docile groups" etc. One can imagine such group specific social norms to have developed through intra-group interactions in times of peace but which acts as constraints on the group leader in times of conflict. We assume that when competing with the other group, a leader has to respect this group specific norm while choosing how to share the prize in case of success in the contest. In our paper we make necessary adjustments and call group i "boundedly meritocratic" if $\underline{\alpha}_i \geqslant 0$. Otherwise, group i is called "unboundedly meritocratic".

The above modeling innovation allows us to unify the different strands of the literature, so that both strands emerge as special cases in our model [2]. Moreover, we are able to identify situations in which both groups choose to make their groups maximally competitive in equilibrium of the contest game between the groups, i.e., both groups put maximal weight on the outlay-based incentive scheme by choosing $\alpha_i = \underline{\alpha}_i$. We call a group "**hawkish**" when it chooses to put all the weight on the outlay-based scheme. Otherwise, we call a group "**dovish**".

We assume throughout that group B is at least as large as group A. We find that the smaller group A generally chooses to be hawkish. It counters the disadvantage of having smaller numbers in the contest by putting all the weight on the outlay-based component

[2] Both "bounded meritocracy" and "unbounded meritocracy" are special cases in our model

of the rule, thereby generating maximum possible efforts by its group members. In other words, the smaller group focuses exclusively on winning the contest. The larger group B, on the other hand, is usually not hawkish. In a sense, the onus of maintenance of a larger net surplus falls on the larger group, when $\underline{\alpha}_B$ is low enough. If it chooses to be hawkish, then it would win the contest more often, but most of the prize would have dissipated due to large efforts by its large numbers. It is only when $\underline{\alpha}_B$ is really high i.e., group B is sufficiently "boundedly meritocratic", that it too shifts to being hawkish in order to increase its chances of winning the contest. When $\underline{\alpha}_B$ is high, free riding becomes the overriding force in group B and larger size actually becomes a handicap. The best a larger group can do to counter the disadvantage, is take a hawkish stance. In Proposition 2 and Corollary 1, we precisely identify the conditions under which both groups choose to be hawkish in equilibrium. This is an important observation as taking a hawkish stance, which increases a group's chance of success in the contest, seems to be a natural path for a group leader to take in a collective contest.

Next, we focus on the welfare of the groups in the contest, specifically focusing on the following question: When does the larger group fare worse in the contest in terms of chances of success? The fact that larger groups may fare worse in competition with smaller ones was first identified by (Olson, 1965) and it was named The Group Size Paradox (GSP). We find that if smaller group A is "unboundedly meritocratic" then GSP cannot be avoided. This result is independent of the nature of meritocracy in the larger group B. Therefore, a necessary condition for Group B to fare better in the contest is for smaller group to be "boundedly meritocratic", i.e, the smaller group should not be in a position to undo the disadvantage of smaller numbers by being "hawkish". The situation where the larger group fares better is called Group Size Advantage (GSA) in our paper.

A sufficient condition for group B to fare better in the contest is for group A to be "boundedly meritocratic" and group B to be "unboundedly meritocratic". In this case group

A cannot undo the disadvantage of smaller numbers by using the prize sharing rule, while the rule imposes no constraint on the leader of the larger group B.

The most interesting case arises when both groups are "boundedly meritocratic". Whether group B fares better or not entirely depends on the asymmetry between the norms of comeptitiveness across groups. If the norms are too asymmetric i.e., $\underline{\alpha}_A$ very high and $\underline{\alpha}_B$ very low, or vice versa, then whichever group is less comeptitive does worse due to excessive free riding. In cases of extreme asymmetry, egalitarian groups may end up getting monopolized (Ueda (2002)).

If the norms of competitiveness are symmetric across groups, i.e.,$\underline{\alpha}_A$ and $\underline{\alpha}_B$ are very close to each other, then whether GSP arises or not depends on whether both groups egalitarian or both are competitive. If both groups are egalitarian i.e., ($\underline{\alpha}_A > \frac{1}{2}$ and $\underline{\alpha}_B > \frac{1}{2}$), then GSP occurs because free riding is the dominant force for both groups in this case and it affects the larger group more adversely. In fact, this case corresponds precisely to the type of groups (Olson, 1965) studied in *The Logic of Collective Action*. We call this class of groups **Olson's Groups.**

On the other hand, if both groups are competitive i.e., ($\underline{\alpha}_A < \frac{1}{2}$ and $\underline{\alpha}_B < \frac{1}{2}$), then intra-group competition is the dominant force for both groups. In such a case, having a larger group size is an advantage and we have GSA. This class of groups are a mirror image of the type of groups (Olson, 1965) studied [3]. We call this class of groups **Neo-Olson Groups**.

The paper is structured as follows. In Section 2 we discuss the relevant literature. In Section 3 we describe the model. In Section 4 we analyze the second stage of the game, where individuals make effort choices. In Section 5 we analyze the first stage of the game where the group leaders make their choice of the sharing rule. In Section 6 we discuss when the phenomenon of Group Size Paradox arises and when it does not. Section 7 concludes. All proofs can be found in the Appendix in Section 9.

[3](Olson, 1965), however, did not study a collective contest but focused on collective action problems within a single group and related it to its size. But, his insight generalizes to a situation of collective contests.

2 Literature

The literature on the prize sharing rules in collective contests owes its genesis to the influential paper by Nitzan (1991). Following its introduction the rule has become the gold standard in the field due to the simple manner it combines two extreme forms of internal organization of groups i.e. one form, which encourages intra-group competition and another which promotes egalitarinism thereby reducing internal competition. To be clear, the prize sharing rule was first analyzed in Sen (1966). But their analysis focused on the optimality of the rule in a labour cooperative (a single group of workers). Throughout this paper we focus on collective contests, where two groups compete for a rent and the influence that has on how the groups internally organize themselves.

The literature on strategic choice of sharing rules focuses on the endogenous choice of internal organization of groups i.e. the group leaders have an option to optimally choose the weight he wants to place on the outlay based incentive scheme, which encourages higher group efforts by promoting internal competition. Two strands have emerged in the literature, which differ on the restriction placed on the leaders choice parameter. In the first strand ((Baik, 1994), (Lee, 1995), (Noh, 1999), Ueda (2002)), the leaders of the groups are allowed to choose α_i on the interval $[0, 1]$. So the outlay-based incentive can be at most proportional to efforts, i.e. the leaders cannot fine members who slack. The second strand (Baik and Shogren (1995), Baik and Lee (1997), Baik and Lee (2001), Lee and Kang (1998), Gürtler (2005)), makes the choice unrestricted , so that $\alpha_i \in (-\infty, 1]$.

In both cases the larger group chooses a less outlay-based incentive scheme than the smaller group i.e. the larger group takes a dovish stance. The reason is that there exists a trade-off between the chances of winning the contest and the size of the surplus net of efforts, which remains for *ex post* consumption. If the larger group implements maximum competition within its group, then given the advantage of size it wins the collective contest more often but the surplus that is left over is too small. As it turns out, the larger group optimally chooses a dovish stance to preserve a larger portion of the surplus.

We extend the above literature by proposing the restriction on the leaders choice of α_i to be over the $[\underline{\alpha}_i, 1]$, where $\underline{\alpha}_i \in (-\infty, 1]$ is a parameter in the model. Both strands emerge as special cases in our model. Our analysis generalizes the literature cited above and in the process allows us to analyze the conditions under which both groups choose to be hawkish , focusing just on winning the contest by putting maximal weight on the outlay-based scheme.

Additionally, we discuss conditions under which the larger group loses the contest more often, so that Group Size Paradox (GSP) applies. Even though it is not central to the main question addressed in this paper, we still report the results given that this has been a primary focus of the literature on collective contests. For example, look at (Nitzan and Ueda, 2011), (Balart et al., 2016) and (Esteban and Ray, 2001).

3 Model

There are two groups A and B, of size n_i, $i = \{A, B\}$, where $n_i \in \{2, 3, ...\}$. We assume without loss of generality that group B is at least as large as A, i.e. $n_B \geqslant n_A$. We denote the total number of agents as N, so that $N = n_B + n_A$. All agents are assumed to be risk neutral.

Both groups compete for a purely private prize, the size of which we normalize to 1. The groups cannot write binding contracts among themselves regarding sharing the prize. Instead they indulge in a rent-seeking Tullock contest spending effort trying to win the contest. The outcome of this contest depends on the aggregate effort spent by the two groups. Let x_{ki} denote the effort level of individual k belonging to group i, where effort costs are $C(x_{ki})$. For simplicity we take $C(x_{ki}) = x_{ki}$. The aggregate effort of group i is $X_i = \sum_{k=1}^{n_i} x_{ki}$.

The efforts do not add to productivity, and only determine the probability $P_i(X_i, X_j)$

that group i wins the contest. We assume that $P_i(X_i, X_j)$ takes the ratio form, i.e.

$$P_i(X_i, X_j) = \begin{cases} \frac{X_i}{X_i + X_j}, & \text{if } X_i > 0 \text{ or } X_j > 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{2}$$

Every group has a leader, who has the authority to enforce a sharing rule that specifies how the groups payoffs are to be shared within the group in case the groups wins the contest. Both leaders are benevolent, maximizing the expected group payoff while making their decisions.

We assume that the group leader has access to the prize sharing rules introduced by Nitzan (1991), which is described follows:

$$s_{ki}(x_{ki}, X_i; \alpha_i, n_i) = (1 - \alpha_i)\frac{x_{ki}}{X_i} + \frac{\alpha_i}{n_i}. \tag{3}$$

We also assume that, a group leader can choose the level of α_i for his group. Given the choice of α_i, the share of the prize the k^{th} member of group i gets is s_{ki}. It should be noted that this prize sharing rule is feasible as $\sum_{k \in n_i} s_{ki} = 1$.

The rule is a weighted average of an egalitarian component $\frac{1}{n_i}$ and a competitive component $\frac{x_{ki}}{X_i}$. The egalitarian component is an incentive scheme, which makes individual rewards independent of efforts. Therefore, a positive weight on it causes individual members of a group to free ride in effort provision. This reduces aggregate group efforts, leading to lower prize dissipation. The result is that a larger *ex post* surplus can be enjoyed by the group in case of success at the cost of lower chances of winning the contest itself.

The competitive component, on the other hand, is an outlay based incentive scheme, which rewards more those individuals, who have put in higher efforts within the group. The resultant competition within the group raises individual efforts, which in turn increases aggregate group effort. As a consequence, the chances of success in the contest increases for the group but now most of the prize gets dissipated in costly effort provision, which reduces the *ex post* surplus to be enjoyed in case of success.

In line with the literature on strategic choice of prize sharing rules, e.g. Flamand et al. (2015), our paper explicitly focuses on how this trade-off influences the choice of α_i by the group leader.

We assume that when choosing the weights to put on the different components of the prize sharing rule, a leader is subject to group specific norms of competitiveness. In particular, the leader of group i, $i \in \{A, B\}$, is assumed to choose $\alpha_i \in [\underline{\alpha}_i, 1]$, where $\underline{\alpha}_i \in (-\infty, 1]$ [4]. In other words, the "lower bound" $\underline{\alpha}_i$ corresponds to the maximum amount of competition that a group leader can generate within his group, i.e., the maximum weight he can place on the outlay-based incentive component. This limit on the competitiveness, which is a feature specific to a group, may be imagined to have developed out of long term interactions among group members. To be clear, the restriction implies that the leader can lower competition within the group with respect to the group norm, by choosing $\alpha_i > \underline{\alpha}_i$. He, however, cannot increase internal competition beyond a certain limit given by $\underline{\alpha}_i$. In this paper we do not go into the sources of such group specific norms and take them as fixed.

It should also be made clear at this point that these restrictions generate an interplay of the two main forces in our model. If $\underline{\alpha}_i$ is high enough then free riding is a dominant force within group i and a larger group size is then a disadvantage as far as chances of winning the contest is concerned. On the other hand if $\underline{\alpha}_i$ is low enough then the force of competition is dominates and a larger group size would be an advantage . How these different intra-group forces play out , where two groups of different sizes and different social norms are matched in a collective contest, is the meat of the paper.

After group i leader chooses α_i, individual k in group i chooses efforts x_{ki} to maximize his expected utility, which is as follows:

[4] In the existing literature the cases considered are $\underline{\alpha}_i = 0$ and $\underline{\alpha}_i = -\infty$

$$EU_{ki}(\mathrm{N}) = \begin{cases} s_{ki}(x_{ki}, X_i; \alpha_i, n_i) P_i(X_i, X_j) - x_{ki} & \text{if } X_i > 0, X_j \geqslant 0, \\ \frac{1}{2n_i} & \text{if } X_i = X_j = 0, \\ 0 & \text{if } X_i = 0, X_j > 0. \end{cases} \tag{4}$$

It should be noted that in this case only the ratio of the individual to the total group effort needs to be verifiable.

■ **Leader's Objective:** Recall that the leaders of both groups are benevolent social planners who choose $\alpha_i \in [\underline{\alpha}_i, 1]$, where $\underline{\alpha}_i \in (-\infty, 1]$, to maximize net group payoffs.

The maximization problem of leader of group i can be written as follows:

$$\max_{\alpha_i \in [\underline{\alpha}_i, 1]} P_i(X_i, X_j) - X_i \tag{5}$$

Given that $P_i(X_i, X_j)$ takes the ratio form it is straight forward to check that leader i's maximization problem can be re-written as follows [5]:

$$\max_{\alpha_i \in [\underline{\alpha}_i, 1]} P_i(X_i, X_j)(1 - X) \tag{6}$$

where $X = X_i + X_j$.

The payoff representation in (6) is intuitive, and captures the trade-off inherent in the group leader's maximization problem. X measures the amount of prize dissipated in the competition between the two groups. Therefore $1 - X$ is the surplus net of efforts, which remains for *ex post* consumption in case of success. The probability with which group i wins this net surplus is $P_i(X_i, X_j)$. If leader of group i wants to win the contest with a higher probability he has to take measures, which increase group efforts X_i. But when X_i goes up

[5] $P_i(X_i, X_j) - X_i = \frac{X_i}{X_i + X_j} - X_i = \frac{X_i}{X_i + X_j}(1 - X_i - X_j) = P_i(X_i, X_j)(1 - X)$

so does X, which reduces the size of the net surplus.

■ **Description of the Game:** Our game consists of two stages. In the first stage the leaders simultaneously choose their respective sharing rule $\alpha_i \in [\underline{\alpha}_i, 1]$, $i = A, B$. Having observed the choice of the sharing rules, in stage 2 all agents simultaneously decide on their own effort levels.

We denote the equilibrium of the game $\sigma^* = (\sigma_A^*, \sigma_B^*)$.

We solve for the Subgame Perfect Nash equilibrium (SPNE) of the game described above.

4 Choice of Individual Efforts

In this section we characterize the Nash equilibrium effort choices of individual members of the groups taking as given the sharing rules α_A and α_B, which are chosen by the group leaders in the first stage.

Before stating the results we need to state a few definitions, which we will use throughout the paper.

First, we define the phenomenon of Monopolization of a group in the contest, which is well recognized in the collective contest literature, see e.g. Davis and Reilly (1999), Ueda (2002).

Definition 1 ***Monopolization***

A SPNE $\langle \alpha_A^*, \alpha_B^* \rangle$ is said to involve monopolization of group i, if in equilibrium group i does not put in any effort in the contest.

Equilibrium Net Surplus and Probabilities of Success

In the following proposition we report the surplus net of effort, which remains for consumption, i.e. $1 - X$, which we denote S. We also report the probabilities with which each group wins the net surplus, P_i and P_j. Such a choice was made to keep the discussion in line with the basic trade-off in the model. In the Appendix we provide the relevant details. Before proceeding we introduce the following notations:

Henceforth, we denote the surplus net of efforts as S, so that $S = 1 - X$.[6]

For $i, j \in \{A, B\}$ and $i \neq j$ we define

$$\chi_i = n_i + n_i(n_j - 1)\alpha_j - n_j(n_i - 1)\alpha_i. \tag{7}$$

χ_i can be interpreted as a measure of the competitiveness of group i relative to group j. Note that χ_i is increasing in α_j and decreasing in α_i. When α_j is large relative to α_i, group j is relatively less competitive, which gives group i an advantage in the contest. On the other hand when α_i is large relative to α_j, group j wins the contest more often. In fact, as we see in the following Proposition, the probability with which group i wins the contest is directly proportional to χ_i.

Proposition **1**

Consider $i, j \in \{A, B\}$ and $j \neq i$.

(A) If $\chi_i \leqslant 0$ [7] then group i is monopolized by group j. In the unique intra-group symmetric Nash equilibrium in the effort subgame

- *(a) The net surplus in the contest is $S^{iM} = \frac{1+\alpha_j(n_j-1)}{n_j}$ [8].*
- *(b) The probabilities of winning are $(P_i^{iM}, P_j^{iM}) = (0, 1)$.*

(B) If $\chi_i > 0$ and $\chi_j > 0$ then neither group is monopolized. In the unique intra-group symmetric Nash equilibrium in the effort subgame

[6] This stands for the effective prize over which the contest takes place. See (6).

[7] When $\chi_i \leqslant 0$ then $\chi_j > 0$ as $\chi_i + \chi_j = N$

[8] The first component in the superscript is the group which is monopolized and the second is the the word monopolized

(a) *The net surplus in the contest is* $S^{NM} = \frac{1+\alpha_j(n_j-1)+\alpha_i(n_i-1)}{N}$ [9].

(b) *The probabilities of winning are* $(P_i^{NM}, P_j^{NM}) = (\frac{\chi_i}{N}, 1 - \frac{\chi_i}{N})$.

We next discuss the results summarized in Proposition 1

■ **Group i is Monopolized:** When $\chi_i \leqslant 0$ group i retires from the contest. This is exactly the same monopolization condition found by Ueda (2002). Furthermore, $\chi_i \leqslant 0$ when we have a low α_j and a high α_i. Therefore, group j members are extremely active due to individual incentives to exert effort, whereas free riding is such a dominant force in group i that individual efforts fall to zero. The effort group j exerts in this case is $X_j^{iM} = \frac{(n_j-1)(1-\alpha_j)}{n_j}$, which leaves a net surplus of $S^{iM} = \frac{1+\alpha_j(n_j-1)}{n_j}$. S^{iM} increases in α_j because the effort necessary to monopolize group i decreases with α_j, which leaves more surplus more consumption of group j.

■ **Neither group is Monopolized:** This case arises when $\chi_i > 0$ and $\chi_j > 0$, which immediately implies α_i and α_j cannot be too asymmetric. Notice that the probability that group i wins is directly proportional to χ_i. For χ_i to be high we need a α_i to be low relative to α_j, i.e., members of group i are relatively more active than members of group j.

It can be seen that the net surplus S^{NM} is increasing in both α_i and α_j. This follows from the fact that an increase in α_i or α_j exacerbates free riding within the groups, causing aggregate efforts in the contest to fall.

Proposition 1 helps us set up the optimization problems that the leaders face in the first stage. We now move to the first stage and characterize the Nash equilibrium.

5 Choice of Sharing Rules by Group Leaders

In this section we analyze the Nash equilibrium choice of the group leaders in the first stage. This leads us to the main result of this paper.

[9] The first component in the superscript stands for neither and the second is the the word monopolized

First, we define the stances taken by the group leaders in equilibrium. Group i is called *hawkish* if in equilibrium its leader chooses to implement maximal competition by putting all the weight on the outlay-based component of the prize sharing rule, i.e., $\alpha_i = \underline{\alpha}_i$. A group i is called *dovish* if in equilibrium its leader puts some weight on the egalitarian component of the prize sharing rule, thereby not implementing maximum group efforts, i.e. $\alpha_i > \underline{\alpha}_i$.

It should be made clear that in our paper the terms hawkish and dovish are not meant in the usual sense of extremes on a uni-dimensional scale. Hawkish and dovish behavior are with respect to group specific norms of competitiveness. A "hawk" focuses entirely on winning the contest by choosing $\alpha_i = \underline{\alpha}_i$. A "dove", on the other hand, does not entirely focus on winning the contest. It puts some attention on maintaining a larger net surplus by choosing $\alpha_i > \underline{\alpha}_i$.

Definition **2**

We call group i hawkish iff its leader chooses $\alpha_i = \underline{\alpha}_i$ in equilibrium. Otherwise, we call group i dovish.

5.1 Leader's Optimization Problem

In view of Proposition 1, we can set up the optimization problem of the group leaders noted in (6). We look at how the leader of group i optimally chooses α_i, given a fixed α_j.

If leader of group i wants to monopolize group j then he has to choose α_i such that $\chi_j \leqslant 0$. This observation follows from part (A) in Proposition 1. In that case we can write down his optimization problem as follows:

$$\max_{\alpha_i \in [\underline{\alpha}_i, 1]} \frac{1 + \alpha_i(n_i - 1)}{n_i} \quad \text{s.t.} \quad \chi_j \leqslant 0 \tag{8}$$

The solution to this problem is simple. As both the objective function and χ_j are increasing in α_i the leader will just set α_i such that $\chi_j = 0$ for given α_j. We now define a cutoff α_i^{jM} and call it the Monopolization cutoff. α_i^{jM} solves $\chi_j = 0$ at $\alpha_j = \underline{\alpha}_j$.

Definition 3 ***Monopolization Cutoff** (α_i^{jM})*

For $i, j \in \{A, B\}$ and $j \neq i$, the Monopolization Cutoff α_i^{jM} is defined as follows:

$$\alpha_i^{jM} = -\frac{1}{n_i - 1} + \frac{(n_j - 1)n_i}{(n_i - 1)n_j}\underline{\alpha}_j$$

.

The cutoff α_i^{jM} is such that if group j chooses $\alpha_j = \underline{\alpha}_j$, then the best choice of group i if it wants to monopolize group j is α_i^{jM}.

Now we consider the case where neither group is monopolized, i.e., $\chi_i > 0$ and $\chi_j > 0$. In that case using part (B) of Proposition 1 and (6) we can write the optimization problem of the leader of group i as follows:

$$\max_{\alpha_i \in [\underline{\alpha}_i, 1]} \left(\frac{\chi_i}{N}\right)\left(\frac{1 + \alpha_j(n_j - 1) + \alpha_i(n_i - 1)}{N}\right) \quad \text{s.t.} \quad \chi_i > 0 \quad \text{and} \quad \chi_j > 0 \tag{9}$$

The solution to problem (9) is non- trivial as χ_i is decreasing in α_i but the second term in brackets, which is the net surplus S^{NM}, is increasing in α_i. So to solve it we set up the Kuhn Tucker problem. The Lagrangian of group i given $i, j \in \{A, B\}$ and $j \neq i$, can be written as follows:

$$L_i = \left(\frac{\chi_i}{N}\right)\left(\frac{1 + \alpha_j(n_j - 1) + \alpha_i(n_i - 1)}{N}\right) + \lambda_i\left(\alpha_i - \underline{\alpha}_i\right) \tag{10}$$

Notice that we ignore the constraints $\underline{\alpha}_i \leqslant 1$ and $\chi_i > 0$ and $\chi_j > 0$ while setting up the Lagrangian. We check later that they are satisfied. Maximizing the function in (10) leads to a few cutoffs we need to define. These cutoffs help us delineate the parametric space by which group's constraint binds and which group's does not in equilibrium.

DEFINITION 4 ***Group i-Binding Cutoff*** (α_j^{iB})

For $i, j \in \{A, B\}$ and $j \neq i$, Group i-the Binding Cutoff α_j^{iB} [10] *is defined as follows:*

$$\alpha_j^{iB} = \frac{n_j - n_i}{2n_i(n_j - 1)}(1 + \underline{\alpha}_i(n_i - 1))$$

.

The Group i-Binding Cutoff α_j^{iB} arises from the Kuhn-Tucker conditions associated with L_i and L_j in (10). It arises when we assume that $\alpha_j > \underline{\alpha}_j$ and $\alpha_i = \underline{\alpha}_i$, so that $\lambda_j = 0$ and $\lambda_i \geqslant 0$. The cutoff helps us identify the parametric region where groups i's constraint will bind but group j's will not in equilibrium[11].

DEFINITION 5 ***Non-Binding Cutoffs***(α_i^{NN})

For $i, j \in \{A, B\}$ and $j \neq i$ the Non-Binding cutoffs are defined as follows:

$$\alpha_i^{NN} = \frac{n_i - n_j}{N(n_i - 1)}$$

.

The Non-Binding cutoffs, α_i^{NN} and α_j^{NN}, are obtained from the Kuhn-Tucker conditions associated with L_i and L_j in (10). The cutoff arises when we assume that group i chooses $\alpha_i > \underline{\alpha}_i$ and group j chooses $\alpha_j > \underline{\alpha}_j$, so that $\lambda_j = 0$ and $\lambda_i = 0$ This cutoff helps us identify the parametric zone where neither groups constraints bind in equilibrium[12].

[10]The first component of the superscript represents the group whose constraint binds and the second denotes the word binds

[11]Derived in Lemma 6 and 7

[12]Derived in Lemma 5

Proposition **2**

$\forall\ i, j \in \{A, B\}$ and $j \neq i$

(a) Group i is monopolized in a Nash equilibrium iff $\underline{\alpha}_i \in [\frac{1}{n_i-1}, 1]$ and $\underline{\alpha}_j \in (-\infty, \alpha_j^{iM}]$. In this case any combination of prize sharing rules (α_i^*, α_j^*), such that $\alpha_i^* \geqslant \underline{\alpha}_i$ and $\alpha_j^* = -\frac{1}{n_j-1} + \frac{(n_i-1)n_j}{(n_j-1)n_i}\alpha_i^*$ is a Nash equilibrium.

(b) In the unique Nash equilibrium group i is hawkish and group j is dovish iff $\underline{\alpha}_i \in [\alpha_i^{NN}, \frac{1}{n_i-1})$ and $\underline{\alpha}_j \in (-\infty, \alpha_j^{iB})$. The equilibrium prize sharing rules are (α_i^*, α_j^*)=$(\underline{\alpha}_i, \alpha_j^{iB})$.

(c) In the unique Nash equilibrium both groups are dovish iff $\underline{\alpha}_i \in (-\infty, \alpha_i^{NN})$ and $\underline{\alpha}_j \in (-\infty, \alpha_j^{NN})$. The equilibrium prize sharing rules are (α_i^*, α_j^*)=$(\alpha_i^{NN}, \alpha_j^{NN})$.

(d) In all other cases in the unique Nash equilibrium both groups are hawkish. The equilibrium prize sharing rules are (α_i^*, α_j^*)=$(\underline{\alpha}_i, \underline{\alpha}_j)$.

Next we discuss the results summarized in Proposition 2

■ **Group i Monopolized:** It is clear from the bounds stated in part (a) of the result that for group i to be monopolized in equilibrium, $\underline{\alpha}_i$ has to be sufficiently high and $\underline{\alpha}_j$ sufficiently low (see Figure 1). Furthermore, α_j^{iB} and α_j^{iM} intersect at $\underline{\alpha}_i = \frac{1}{n_i-1}$, so that for all $\underline{\alpha}_i < \frac{1}{n_i-1}$ we have $\alpha_j^{iM} < \alpha_j^{iB}$. Here group j has the option to monopolize group i by choosing $\alpha_j = \alpha_j^{iM}$. But group j chooses not to do that because by choosing $\alpha_j = \alpha_j^{iB}$, which is higher, it can maintain more of the net surplus and give up only a tiny chance of winning upto group A. The choice $\alpha_j > \underline{\alpha}_j$, implies group j chooses more free riding within its group, which allows group A to survive in the contest. Of course, the benefits of a larger net surplus dominates the cost of decreased chances of winning for group j in this case.

In case $\alpha_j^{iM} > \alpha_j^{iB}$, it is again optimal for group j to choose the higher of the two in equilibrium, in order to save net surplus. But at $\alpha_j = \alpha_j^{iM}$, group i is monopolized. Given that group i will be monopolized at $\alpha_j = \alpha_j^{iM}$, any $\alpha_i \geqslant \underline{\alpha}_j$ is best response for group i, as at all such choices it gets zero payoff. For group j on the other hand, the best response is to choose a α_j, which is consistent with α_j^{iM}, given whatever choice group i makes.

■ **Group A is hawkish, Group B is dovish:** From part (b) of the proposition it is clear that this case arises when both $\underline{\alpha}_A$ and $\underline{\alpha}_B$ are low, so that both groups are potentially very competitive. Look at Figure 1 and 2. Because both groups are sufficiently competitive, having a larger size is an advantage in the contest. But again, because both groups are competitive, it is more difficult for group A to compete against the larger group B. So the optimal choice of group A is to be maximally competitive by choosing a hawkish stance. In other words, group A focuses entirely on its chances of winning instead of saving net surplus.

The larger group B , on the other hand, chooses to save some surplus by choosing $\alpha_B = \alpha_B^{iB} > \underline{\alpha}_B$. It has the competitive advantage of a larger size. But the larger size also means a lot of surplus will be dissipated if it focuses primarily on winning the contest by choosing a hawkish stance. So, group B leader compromises on its chances of winning by choosing to be dovish in order to save some net surplus.

Similarly, we can analyze the case, where group B is hawkish and group A is dovish. This case arises when group B is "boundedly meritocratic" but group A is "unboundedly meritocratic". This being the case, free riding is the dominant force within group B, which makes its larger size a disadvantage. On the other hand, the smaller hand has very competitive norms. Given the larger group is not much of a competition for it, group A shifts focus to saving some net surplus by taking a dovish stance.

■ **Both groups are dovish:** As can be seen in part (c) of the result, this case arises when both $\underline{\alpha}_A$ and $\underline{\alpha}_B$ are extremely low, so that both groups have extremely competitive norms. If either group focuses entirely on chances of winning by taking a hawkish stance, then a lot of surplus will be lost in costly efforts. Hence, both groups compromise on chances of winning by shifting some attention to saving net surplus.

■ **Both groups are hawkish:** This is the main result of the paper and is succinctly summarized in Corollary 1. Look at Figure 2.

In this case both groups choose to be hawkish, i.e. both focus on winning the contest instead of trying to save net surplus. This case arises when group B is "boundedly meritocratic". The smaller group A could be "boundedly meritocratic" or "unboundedly meritocratic".

This case arises when social norms are such that a larger group size is a disadvantage for group B, as free riding is the dominant force within it. Group B tries to counter that disadvantage by choosing the lowest possible α_i and making its group maximally competitive in the contest.

For the smaller group on the other hand, the numbers are still a disadvantage. So, irrespective of the degree of meritocracy in its norms it tries to counter the disadvantage of smaller numbers by choosing hawkish stance.

This situation arises, when social norms of both groups are such that group sizes are a disadvantage. Hence both groups exclusively try to maximize their winning chances by choosing $\alpha_i = \underline{\alpha}_i$.

This is main observation of this paper. We have clearly identified the circumstances under which both groups will be hawkish, which seems to be a natural stance to take in a situation of pure conflict. This has not been identified in the literature til now. It is succinctly summarized in the following corollary of Proposition 2.

Corollary **1**

In the unique Nash equilibrium both group A and group B are hawkish iff $\underline{\alpha}_B \geqslant max\{\alpha_B^{AM}, \alpha_B^{AB}\}$ *and* $\underline{\alpha}_A \geqslant max\{\alpha_A^{BM}, \alpha_A^{BB}\}$

Corollary 1 provides a lower bounds on egalitarianism, which ensure that both groups will choose to be hawkish in equilibrium. As mentioned before, it is an important observation because in the context of group conflicts, the natural path for a group leader to follow would be to try and maximize chances of winning by generating maximal efforts. In other words, it precisely captures the circumstances under which social norms have a bite for both groups. The result can be seen clearly in Figures 1 and 2.

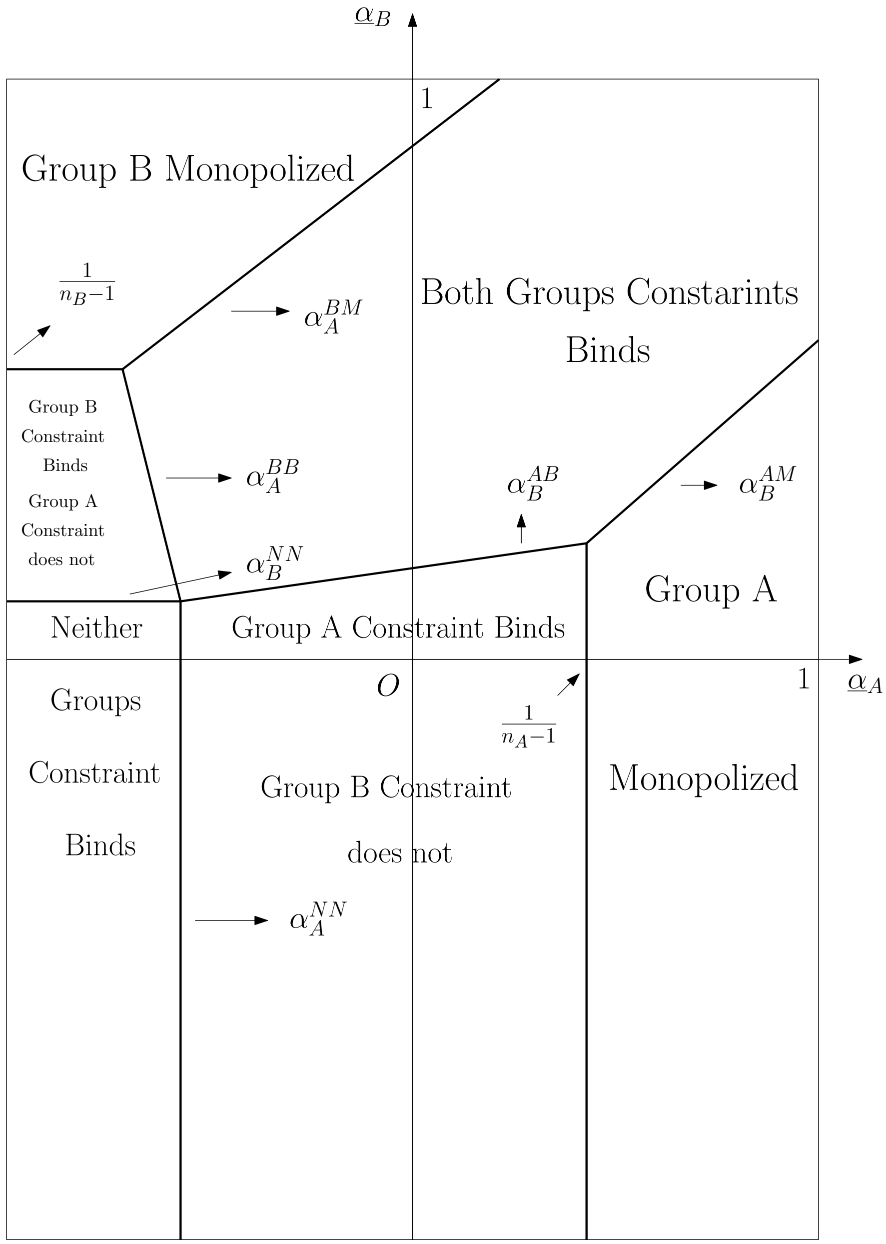

Figure 1: Leader's Choice in Nash Equilibrium

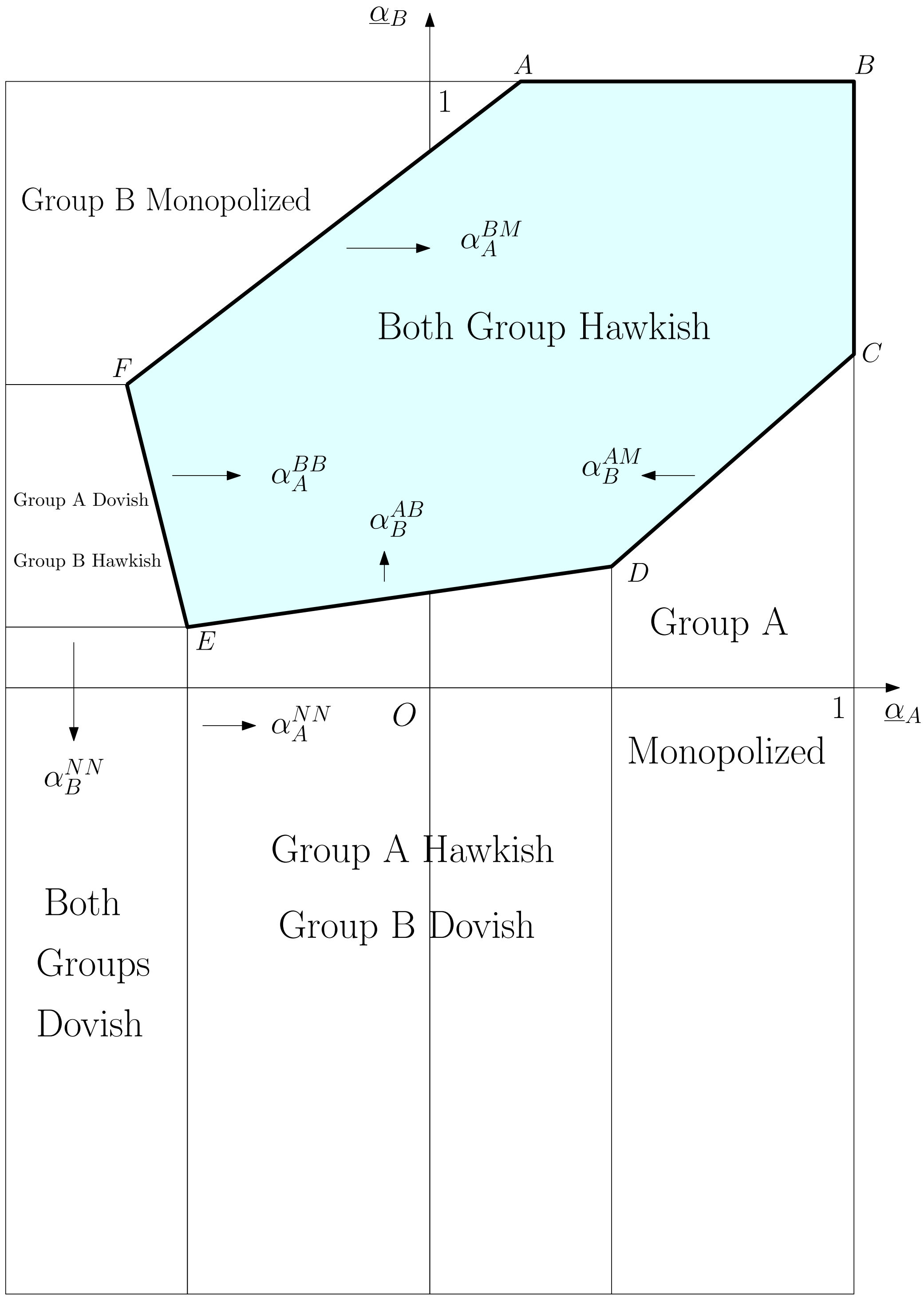

Figure 2: Leader's Choice in Nash Equilibrium

Before we state the intuition behind Proposition 2, let us take a closer look at Figure 2. In Figure 2 let us consider the polygon $ABCDEF$. This is the polygon of Nash equilibrium choices made by the leaders. If $(\underline{\alpha}_A, \underline{\alpha}_B)$ lies inside or on the boundary of the polygon then the Nash equilibrium is $(\alpha_A^*, \alpha_B^*) = (\underline{\alpha}_A, \underline{\alpha}_B)$. If $(\underline{\alpha}_A, \underline{\alpha}_B)$ lies outside the polygon then the Nash equilibrium is the nearest point on the boundary closest to it.

Intuition: This result points to the fact that for the larger group B to entirely focus on winning the contest by taking a hawkish stance, it needs to have sufficiently egalitarian norms, which makes free riding the dominant force within it. In that case, having larger numbers is a disadvantage, which can only be countered by taking a hawkish stance. If it had competitive norms, larger numbers would be an advantage in terms of winning the contest but would dissipate a lot of the surplus if it tried to generate maximal efforts. So, in such a case, the larger group leader takes a dovish stance, which reduce its efforts and chances of winning below maximum but retains a larger amount of surplus, which can be had in case of success. For the smaller group, on the other hand, numbers are a disadvantage. So it generally takes a hawkish stance to counter that disadvantage by taking a hawkish stance.

We conclude this section by summarizing the main takeaways. Firstly, we find that the smaller group generally takes a "hawkish" stance in the contest. The larger group, however, chooses a "hawkish" stance only in cases where it has sufficiently egalitarian norms, i.e., the incentive to free ride is so high within the group that larger numbers are actually a disadvantage. When it has sufficiently competitive internal norms, the larger group chooses a "dovish" stance to reduce its efforts and save surplus, which can be consumed *ex post* in case of success. But, the main observation is made in Corollary 1, which precisely identifies conditions under which both groups take a "hawkish" stance. Even though adoption of a "hawkish" stance by all participating groups seems to be the most natural thing to do in a purely competitive situation like ours, the conditions required for it to happen had not been identified in the previous literature.

6 Equilibrium Characterization

In this section we characterize the subgame perfect Nash equilibrium (SPNE) of the whole game. In Propositions 1 and 2, we characterized the Nash equilibrium of stage two and one of the game respectively. Now, we use the two propositions to characterize the (SPNE) of the game. We denote χ_i at $(\underline{\alpha}_A, \underline{\alpha}_A)$ as $\underline{\chi}_i$.

Proposition **3**

(A) If group i is monopolized, then in the SPNE

(a) The net surplus in the contest is $S^{iM} = \frac{\alpha_i(n_i-1)}{n_i}$ [13].

(b) The probabilities of winning are $(P_i^{iM}, P_j^{iM}) = (0, 1)$.

(B) If neither group is monopolized, then in the SPNE

(1) If both groups are dovish then

(a) The net surplus in the contest is $S^{NN} = \frac{1}{N}$.

(b) The probabilities of winning are $(P_i^{NN}, P_j^{NN}) = (\frac{n_j}{N}, \frac{n_i}{N})$.

(2) If group i is hawkish but group j is dovish then

(a) The net surplus in the contest is $S^{iB} = \frac{1+\underline{\alpha}_i(n_i-1)}{2n_i}$.

(b) The probabilities of winning are $(P_i^{iB}, P_j^{iB}) = (\frac{1-\underline{\alpha}_i(n_i-1)}{2}, \frac{1+\underline{\alpha}_i(n_i-1)}{2})$.

(3) If both groups are hawkish then

(a) The net surplus in the contest is $S^B = \frac{1+(n_A-1)\underline{\alpha}_A+(n_B-1)\underline{\alpha}_B}{N}$.

(b) The probabilities of winning are $(P_i^B, P_j^B) = (\frac{\underline{\chi}_i}{N}, \frac{\underline{\chi}_j}{N})$.

[13] The first component in the superscript is the group which is monopolized and the second is the the word monopolized

We next discuss the results summarized in Proposition 3.

■ **Group i is Monopolized:** This case arises when $\chi_i \leqslant 0$ as can be seen from Proposition 1. Group j's best response to any α_i is to choose α_j which solves $\chi_=0$. The effort is $X_j^{iM} = 1 - \frac{\alpha_i(n_i-1)}{n_i}$, which leaves a net surplus $S^{iM} = \frac{\alpha_i(n_i-1)}{n_i}$. The effort required to monopolize group i is decreasing in α_i as it easier for group j to crowd out group i, when free riding has increased within it. Therefore, the net surplus is increasing in α_i.

Given that α_i is high enough in this case, means that free riding is the dominant force in group i in this case. If now group i gets larger still, it becomes easier to monopolize group i as free riding will increase. Therefore, net surplus is rising in n_i as well.

Next, we focus on cases, where neither group is monopolized.

■ **Both groups are dovish:** Both groups are dovish means that in equilibrium $\alpha_i > \underline{\alpha}_i$ and $\alpha_i > \underline{\alpha}_i$. When both groups take a dovish stance, the total effort in equilibrium is $X^{NN} = 1 - \frac{1}{N}$, which leaves a net surplus $S^{NN} = \frac{1}{N}$. Because neither constraint binds, the probabilities of winning and net surplus are independent of $\underline{\alpha}_i$ and only depends on group sizes. In this case only groups sizes matter, i.e. social norms have no bite.

Given that both groups get to choose the globally best rules in this case, the only difference which applies between groups is one due to sizes. Increasing the size of group i decreases the effort of group i due to increased free riding. Efforts are strategic substitutes here and so the effort of group j goes up. But aggregate effort increases, thereby lowering net surplus S^{NN}. However, as the effort of group i falls, the probability of group i winning the contest goes down.

■ **Group i is hawkish, Group j is dovish:** This case arises when in equilibrium $\alpha_i = \underline{\alpha}_i$ and $\alpha_i > \underline{\alpha}_i$. In this case the aggregate effort in the Nash equilibrium is $X^{iB} = \frac{1}{2} + \frac{(n_i-1)(1-\underline{\alpha}_i)}{2n_i}$, which leaves a net surplus $S^{iB} = \frac{1}{2} - \frac{(n_i-1)(1-\underline{\alpha}_i)}{2n_i}$. When $\underline{\alpha}_i$ rises, the effort of group i decreases due to increased free riding. The effort of group j rises as efforts are strategic substitutes. Aggregate efforts decline and so the net surplus rises as $\underline{\alpha}_i$ rises. As effort of group i decreases, the probability that group i wins goes down with $\underline{\alpha}_i$.

When n_i increases, aggregate effort increases, thereby reducing the net surplus. When $\underline{\alpha}_i < 0$, effort of group i rises with n_i increasing its chances of winning. $\underline{\alpha}_i = 0$ denotes the cutoff above which the force of free riding dominates the force of competition in group i. Therefore, in terms of payoffs, larger numbers are a disadvantage for group i when $\underline{\alpha}_i > 0$ and is an advantage otherwise.

■ **Both groups are hawkish:** This case arises when in equilibrium $\alpha_i = \underline{\alpha}_i$ and $\alpha_i = \underline{\alpha}_i$. The aggregate effort level X^B is declining in $\underline{\alpha}_A$ and $\underline{\alpha}_B$ due to increased free riding. Therefore, the net surplus S^B increases in $\underline{\alpha}_A$ and $\underline{\alpha}_B$. As $\underline{\alpha}_i$ rises free riding in group i rises and so effort of group i falls. Unless both $\underline{\alpha}_A$ and $\underline{\alpha}_B$ are close to 1, efforts are strategic substitutes, so that when X_i^B rises, X_j^B falls. However, irrespective of whether X_j is a strategic complement or substitute to X_i, it can be easily verified that the aggregate efforts decline with $\underline{\alpha}_i$. Furthermore, the probability of group i winning decreases in $\underline{\alpha}_i$ and increases in $\underline{\alpha}_j$.

It should be noted that the efforts are higher when both groups are "doves" than when both groups are "hawks". This happens due to the way we have defined hawkish and dovish behavior in this paper. A group chooses a hawkish stance in equilibrium when it has egalitarian norms and a dovish stance when it has competitive norms. If a group is egalitarian then free riding is the dominant force within it. On the other hand, if a group has competitive norms then the dominant force is that of internal competition. Even though the groups choose dovish stances under competitive norms, the reduction in efforts is not to the extent that it falls below the efforts chosen by hawkish groups, which have egalitarian norms.

7 When does GSP occur?

In this section we turn to the question of welfare of the groups in the collective contest. We focus on the phenomenon of Group Size Paradox (GSP), which denotes situations in which the bigger group fares worse than the smaller group in the contest. In particular we link the incidence of GSP to whether the groups are "boundedly meritocratic" or "unboundedly meritocratic". Even though GSP has been a primary focus of the literature on collective contests, e.g. (Nitzan and Ueda, 2011), (Balart et al., 2016), there is no paper we know of which analyzes how group specific social norms affect the welfare of the groups.

Definition **6**

The group size paradox (GSP) occurs in equilibrium if the bigger group wins the contest with a lower probability i.e. $P_B < P_A$. If the bigger group has at least as much chance to win the contest as the smaller group i.e., $P_B \geqslant P_A$, then we say group size advantage (GSA) occurs in equilibrium.

There is no loss in defining GSP in terms of probabilities of success. We could have alternatively defined it in terms of group efforts or payoffs, as all of them are equivalent in this framework.

Next we define a cutoff, which we will need in the next proposition.

Definition **7** ***GSP Cutoff (α_B^{GSP})***

The GSP cutoff α_B^{GSP} is defined as follows:

$$\alpha_B^{GSP} = \frac{n_B - n_A}{2n_A(n_B - 1)} + \frac{(n_A - 1)n_B}{(n_B - 1)n_A}\underline{\alpha}_A$$

This cutoff is obtained by checking when $P_B^{BB} > P_A^{BB}$ i.e. when it the case that group B wins the contest with a higher probability, where both groups are hawkish (Proposition 3).

Proposition 4

GSP occurs iff $\underline{\alpha}_A < 0$ *or* $\underline{\alpha}_B > \alpha_B^{GSP}$.

We next discuss the result summarized in Proposition 4 by breaking it up into three different cases.

■ **Smaller group is "unboundedly meritocratic"** ($\underline{\alpha}_A < 0$):

In this case the smaller group can choose to put a larger than proportional weight on the competitive component of the rule. Allowing the smaller group this freedom allows it to counter the disadvantage of having smaller numbers in the collective contest. This is irrespective of whether the larger group is "boundedly meritocratic" or "unboundedly meritocratic".

If $\underline{\alpha}_B > 0$ then group B is "boundedly meritocratic". Being larger and "boundedly meritocratic" is doubly disadvantageous for group B. Essentially, group B contains a large number of free riders. Moreover, it does not have enough freedom to counter the force of free riding by choosing a rule, which rewards efforts more than proportionally. Therefore, the larger group always fares worse in this case.

If, on the other hand, group B is also "unboundedly meritocratic", so that $\underline{\alpha}_B < 0$, it faces the trade off between winning the contest and saving net surplus because it is larger. Group A being smaller does not face this trade off. It is optimal for group B to try and save net surplus by taking a dovish stance. In the process, group B ends up doing worse than group A, as the dovish stance increases free riding in it.

Therefore, $\underline{\alpha}_A = 0$ captures the cutoff level of competitiveness, such that below it group A is competitive enough to outdo the bigger group. In other, words group A being "boundedly meritocratic" is a sufficient condition for GSP to occur.

■ **Smaller group is "boundedly meritocratic" ($\underline{\alpha}_A > 0$) and larger group is "unboundedly meritocratic"($\underline{\alpha}_B < 0$):**

In this case the larger group has the advantage of rewarding efforts in its group more than proportionally, thereby being in a position to generate substantial efforts from its larger numbers. So it is in an advantageous position vis a vis the smaller group both with respect to size and potential level of competitiveness and hence efforts. Therefore, in equilibrium it fares better than the smaller group. We call this situation Group Size Advantage (GSA). Even though group B is dovish, the fact that group A is "boundedly meritocratic", allows it to fare better than group A in equilibrium.

■ **Both groups are "boundedly meritocratic" ($\underline{\alpha}_A \geqslant 0$ and $\underline{\alpha}_B \geqslant 0$) :**

This case, where both groups are "boundedly meritocratic" turns out to be the most interesting one. What turns out to be important is the degree of asymmetry of the norms of competitiveness across the groups. If the asymmetry is substantial, then the group with more egalitarian norms does worse unequivocally.

If the norms of competitiveness are relatively symmetric across groups, i.e. $\underline{\alpha}_A$ and $\underline{\alpha}_B$ are close to each other [14], then what determines the occurrence of GSP is whether both groups have egalitarian norms or both groups have competitive norms. Given that norms of competitiveness are symmetric across groups, what creates the difference between the groups is their relative sizes. But, the difference in sizes operate differently depending on whether both groups have competitive norms or both have egalitarian norms. Look at Figure 3.

If both groups are egalitarian i.e., $\underline{\alpha}_B > \frac{1}{2}$ and $\underline{\alpha}_A > \frac{1}{2}$, then the dominant force is one of free riding in both groups. Therefore, having a larger group is a disadvantage in this case. So, group B does worse than group A and GSP operates. Incidentally, this case perfectly characterizes the type of groups Olson (1965) talked about in *The Logic of Collective Action*

[14] In Figure 3 the idea of relative symmetry is captured by drawing the 45° line and looking at clusters of $\underline{\alpha}_A$ and $\underline{\alpha}_B$ around it

. Olson (1965) [15] studied the case where the norms of competitiveness were symmetric across groups. Specifically, he focused on the case of full egalitarianism, i.e., $\underline{\alpha}_A = 1$ and $\underline{\alpha}_B = 1$, making the force of free riding maximal within both groups. With that situation in mind, he reached the conclusion that larger numbers are not ideal for successful collective action. We show that the force of free riding dominates as long as $\underline{\alpha}_B > \frac{1}{2}$ and $\underline{\alpha}_A > \frac{1}{2}$, thereby providing a precise characterization of the types of groups, which were the focus of Olson (1965). We call this collection of groups **Olson's Groups**.

On the other hand, if both groups are sufficiently competitive i.e., $0 \leqslant \underline{\alpha}_B < \frac{1}{2}$ and $0 \leqslant \underline{\alpha}_A < \frac{1}{2}$, then the competitive force dominates. In such a case having larger numbers is an advantage and group B fares better, so that GSA operates. Olson (1965) spoke at length about how "selective incentives" could be used to outdo the force of free riding, making collective action possible in larger groups. This case provides a perfect characterization of such a situation. The norms being symmetric across groups, only group sizes matter. Here the "selective incetives", allows the larger group to overcome the force of free riding and fare better than the smaller group. We call the collection of groups with equally competitive norms i.e., $0 \leqslant \underline{\alpha}_B < \frac{1}{2}$ and $0 \leqslant \underline{\alpha}_A < \frac{1}{2}$, the **Neo-Olson Groups**. Look at Figure 3.

8 Conclusion

In this paper we generalized the prize sharing rule proposed by Nitzan (1991) in the context of collective contests. We propose a way to model group specific norms of competitiveness and then analyze how such internal norms affect a group's chances in external conflict. The modeling innovation allowed us to characterize situations in which both groups would choose focus entirely on winning an external conflict i.e. both group take the hawkish stance. This feature despite being the most natural thing to expect in a situation of conflict, had been

[15] To be preciseOlson (1965) studied the issue of free riding in collective action with only one group. But his conclusions generalize to the collective contest scenario.

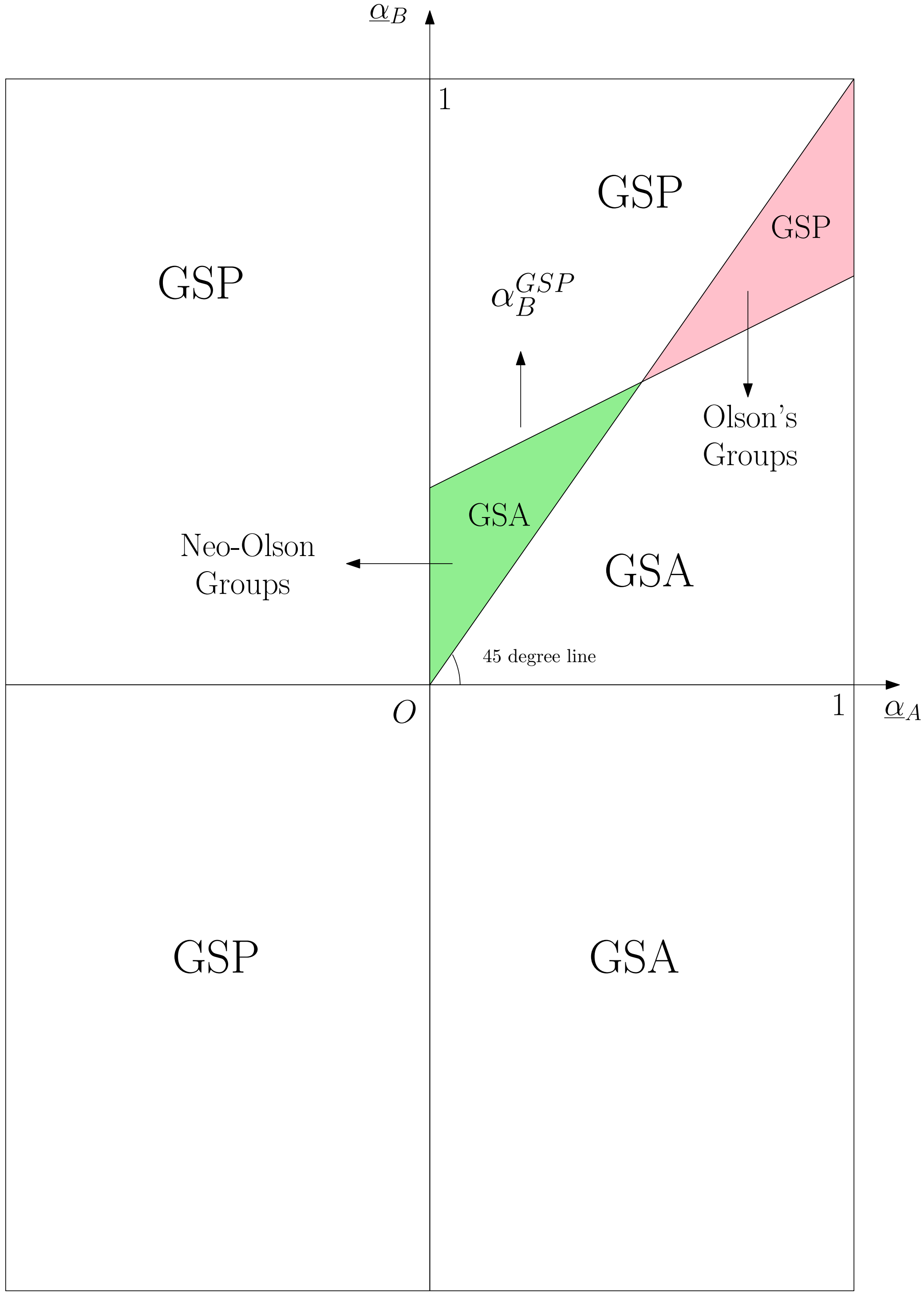

Figure 3: When does GSP Occur?

overlooked in literature till now.

We find that the smaller group A generally chooses to be hawkish. For group B to also behave in a hawkish manner, it has to be the case that it is sufficiently "boundedly meritocratic" i.e., $\underline{\alpha}_B \geqslant 0$ and high enough. This allows us to identify types of group conflicts, where both groups take the extremest stance possible in order to maximize the likelihood of success in the contest.

We also provide the conditions under, which GSP occurs. We find that group A being "unboundedly meritocratic" is a sufficient condition for GSP to occur. If group A is "boundedly meritocratic" and group B is "unboundedly meritocratic" then larger group size is an advantage for group B and it fares better than the smaller group. If both groups are "boundedly meritocratic", then whether GSA applies or GSP depends critically on whether the norms are symmetric across groups or not. If both group's norms are symmetric and competitive, then having a larger group is an advantage and GSA applies. If both group's norms are sufficiently egalitarian then free riding is the dominant force in both groups. In that case, being larger in size is a disadvantage and therefore GSP applies.

Even though the modeling innovation of imposing restrictions on the prize sharing rule allows us to clarify when group sizes matter and when social norms matter, what remains to be understood is where such social norms themselves come from. Given that these restrictions are interpreted as norms of competitiveness in surplus division within a group, modeling how such norms arise as a function of economic conditions a group faces in times of peace or how such norms relate to group sizes, are interesting questions that are left for future research.

References

Baik, K. H. (1993). Effort levels in contests: The public-good prize case. *Economics Letters*, 41(4):363–367.

Baik, K. H. (1994). Winner-help-loser group formation in rent-seeking contests. *Economics & Politics*, 6(2):147–162.

Baik, K. H. (2008). Contests with group-specific public-good prizes. *Social Choice and Welfare*, 30(1):103–117.

Baik, K. H. and Lee, S. (1997). Collective rent seeking with endogenous group sizes. *European Journal of Political Economy*, 13(1):121–130.

Baik, K. H. and Lee, S. (2001). Strategic groups and rent dissipation. *Economic Inquiry*, 39(4):672–684.

Baik, K. H. and Shogren, J. F. (1995). Competitive-share group formation in rent-seeking contests. *Public choice*, 83(1-2):113–126.

Balart, P., Flamand, S., and Troumpounis, O. (2016). Strategic choice of sharing rules in collective contests. *Social Choice and Welfare*, 46(2):239–262.

Davis, D. D. and Reilly, R. J. (1999). Rent-seeking with non-identical sharing rules: An equilibrium rescued. *Public Choice*, 100(1-2):31–38.

Esteban, J. and Ray, D. (2001). Collective action and the group size paradox. *American political science review*, 95(3):663–672.

Flamand, S., Troumpounis, O., et al. (2015). Prize-sharing rules in collective rent seeking. *Companion to political economy of rent seeking, London: Edward Elgar*, pages 92–112.

Gürtler, O. (2005). Collective contests with externalities: corrigendum. *European Journal of Political Economy*, 21(2):533–536.

Hillman, A. L. and Riley, J. G. (1989). Politically contestable rents and transfers. *Economics & Politics*, 1(1):17–39.

Katz, E., Nitzan, S., and Rosenberg, J. (1990). Rent-seeking for pure public goods. *Public Choice*, 65(1):49–60.

Lee, S. (1995). Endogenous sharing rules in collective-group rent-seeking. *Public Choice*, 85(1-2):31–44.

Lee, S. and Kang, J. H. (1998). Collective contests with externalities. *European Journal of Political Economy*, 14(4):727–738.

Nitzan, S. (1991). Collective rent dissipation. *The Economic Journal*, 101(409):1522–1534.

Nitzan, S. and Ueda, K. (2011). Prize sharing in collective contests. *European Economic Review*, 55(5):678–687.

Nitzan, S. and Ueda, K. (2014a). Cost sharing in collective contests.

Nitzan, S. and Ueda, K. (2014b). Intra-group heterogeneity in collective contests. *Social choice and Welfare*, 43(1):219–238.

Noh, S. J. (1999). A general equilibrium model of two group conflict with endogenous intra-group sharing rules. *Public Choice*, 98(3-4):251–267.

Olson, M. (1965). The logic of collective action, cambridge, mass. *Harvard Univ. Pr.*

Sen, A. K. (1966). Labour allocation in a cooperative enterprise. *The Review of Economic Studies*, 33(4):361–371.

Ueda, K. (2002). Oligopolization in collective rent-seeking. *Social Choice and Welfare*, 19(3):613–626.

9 Appendix

To prove Proposition 3 we have to first set up the individual effort choice problem of group members in stage two of the game. Then we propose and prove a set of Lemmas which help us prove the result.

9.1 Individual Effort Choice Problem

Taking as given (α_A, α_B) chosen by the group leaders in stage one of the game, the payoff of the k^{th} member in Group A is given as follows:

$$\pi^{kA}(X_A, X_B) = \frac{X_A}{X_A + X_B}[(1-\alpha_A)\frac{x_{kA}}{X_A} + \frac{\alpha_A}{n_A}] - x_{kA} \tag{11}$$

Similarly the payoff of the k^{th} member of Group B is as follows:

$$\pi^{kB}(X_A, X_B) = \frac{X_B}{X_A + X_B}[(1-\alpha_B)\frac{x_{kB}}{X_B} + \frac{\alpha_B}{n_B}] - x_{kB} \tag{12}$$

Both (11) and (12) are continuous except at $(X_A, X_B) = (0, 0)$. The functions are concave in x_{ki}. for $i = A, B$.

We can compute the Nash Equilibrium in individual efforts by examining the First Order Conditions of (11) and (12).

We ignore the constraint $0 \leqslant x_{ki} \leqslant 1$ while solving the problem and check later that they are indeed satisfied. We characterize within group symmetric Nash Equilibrium in our analysis.

Before proceeding we define the sets $N_i = \{1, 2...n_i\}$ for $i = A, B$.

First, we examine the First Order Conditions of the individual effort choice problem for members of both the groups. The F.O.C of (11) w.r.t. x_{kA} , $\forall k \in N_A$ is as follows:

$$\frac{X_B}{(X_A+X_B)^2}[(1-\alpha_A)\frac{x_{kA}}{X_A}+\frac{\alpha_A}{n_A}]+\frac{X_A}{X_A+X_B}[(1-\alpha_A)\frac{X_A-x_{kA}}{X_A^2}] \leq 1 \tag{13}$$

Similarly, the F.O.C of (12) w.r.t x_{kB} , $\forall k \in N_B$ is as follows:

$$\frac{X_A}{(X_A+X_B)^2}[(1-\alpha_B)\frac{x_{kB}}{X_B}+\frac{\alpha_B}{n_B}]+\frac{X_B}{X_A+X_B}[(1-\alpha_B)\frac{X_B-x_{kB}}{X_B^2}] \leq 1 \tag{14}$$

If (13) holds strictly then $x_{kA}=0$, $\forall k \in N_A$. Similarly in (14). Both inequalities cannot hold strictly at $(x_{kA}, x_{kB}) = (0,0)$, because it does not constitute a Nash Equilibrium. Given the Tullock Contest Sucess Function at $(x_{kA}, x_{kB}) = (0,0)$, a member in one of the groups will deviate because then his group will win the contest for sure and he will get a share of the incremental group payoff. It can also be easily verified that the Second Order Conditions hold.

Therefore, there are 3 mutually exclusive cases to take care of.

▶ **CASE 1**:

Inequality (13) holds weakly at $x_{kA} = 0$, Inequality (14) holds with equality at some $x_{kB} > 0$.

Lemma **1**

If $\alpha_A n_B(n_A-1) - \alpha_B n_A(n_B-1) \geqslant n_A$ then Group A is Monopolized by Group B.In the symmetric within group Nash Equilibrium , $x_{kA}^{AM} = 0$, $\forall k \in N_A$ and $x_{kB}^{AM} = \frac{(n_B-1)(1-\alpha_B)}{n_B^2}$, $\forall k \in N_B$. The aggregate effort of group B is $X_B^{AM} = \frac{(n_B-1)(1-\alpha_B)}{n_B^2}$.

Proof: If $x_{kA}^A = 0$, $\forall k \in N_A$, then $X_A^A = n_A x_{kA}^A = 0$. But notice that at $X_A^A = 0$ the L.H.S of (13) is not well defined. So we will consider the limit of of L.H.S. of (13) as $X_A^A \to 0$.

Define $x^A_{kA} = \epsilon > 0$, $\forall k \in N_A$. Then $X^A_A = n_A x^A_{kA} = n_A \epsilon$. As n_A is finite $X^A_A \rightarrow 0$ as $\epsilon \rightarrow 0$.

We need L.H.S. of (13) to be well defined and (13) to be satisfied as a weak inequality at $x^A_{kA} = \epsilon$ and $X^A_A = n_A \epsilon$ as $\epsilon \rightarrow 0$.

We replace $x^A_{kA} = \epsilon$ and $X^A_A = n_A \epsilon$ in (13) and sum it over all $k \in N_A$ to arrive at the following condition:

$$\lim_{\epsilon \rightarrow 0} \frac{X^A_B}{(n_A \epsilon + X^A_B)^2} + \frac{(n_A - 1)(1 - \alpha_A)}{(n_A \epsilon + X^A_B)} \leqslant n_A \tag{15}$$

As this limit is well-defined we need the following condition to be satisfied if Group A is to be Monopolized.

$$n_A X^A_B \geqslant 1 + (n_A - 1)(1 - \alpha_A) \tag{16}$$

At $X^A_A = 0$, the L.H.S. of (14) is well defined. We sum (14) over all $k \in N_B$, to arrive at the following condition:

$$n_B X^A_B = (n_B - 1)(1 - \alpha_B) \tag{17}$$

For $x^{AM}_{kA} = 0$ and $x^{AM}_{kB} = \frac{(n_B - 1)(1 - \alpha_B)}{n_B^2}$ to be mutual best responses, both (16) and (17) need to be satisfied. Replacing X^A_B from (17) in (16) we arrive at the following condition:

$$\alpha_A n_B (n_A - 1) - \alpha_B n_A (n_B - 1) \geqslant n_A \tag{18}$$

Equation (42) needs to be satisfied if group A is to be monopolized.

$\blacksquare$

▶ **CASE 2**:

Inequality (14) holds weakly at $x_{kB} = 0$, Inequality 13 holds with equality at some $x_{kA} > 0$.

Lemma **2**

If $\alpha_B n_A(n_B - 1) - \alpha_A n_B(n_A - 1) \geqslant n_B$ then Group B is Monopolized by Group A. In the symmetric within group Nash Equilibrium , $x^B_{kB} = 0$, $\forall k \in N_B$ and $x^B_{kA} = \frac{(n_A-1)(1-\alpha_A)}{n_A^2}$,$\forall k \in N_A$. The aggregate effort of group A is $X^{BM}_A = \frac{(n_A-1)(1-\alpha_A)}{n_A^2}$.

Proof: The proof follows exactly the same lines as Lemma 1, but with the roles of the Groups reversed. Now A Monopolizes B so $X^{BM}_B = 0$. We skip this proof. ■

▶ **CASE 3**:

Both (13) and (14) hold with equality at some $(x_{kA}, x_{kB}) > (0, 0)$

Lemma **3**

If $-n_A > \alpha_B n_A(n_B - 1) - \alpha_A n_B(n_A - 1) < n_B$, then neither group is Monopolized. In the symmetric within group Nash Equilibrium , $x^{NM}_{ki} = \frac{1}{n_i}(n_j(X^{NM})^2 - (n_j - 1)(1 - \alpha_j)X^{NM})$,$\forall k \in N_i$, $i, j = A, B$ and $i \neq j$, where the combined contest effort of the groups is $X^{NM} = X^{NM}_A + X^{NM}_B = \frac{1+(n_A-1)(1-\alpha_A)+(n_A-1)(1-\alpha_A)}{N}$. The probability of winning for the groups is $(P^{NM}_i, P^{NM}_j) = (\frac{\chi_i}{N}, 1 - \frac{\chi_i}{N})$.

Proof: Firstly, if none of the Groups is to be Monopolized neither Lemma 1 nor Lemma 2 can apply. The antecedent of Lemma 3 follows directly by negation of Lemma 1 and Lemma 2 .

In this case all the F.O.C.'s in (13) and (14) hold with equality.

To figure out the individual efforts in the within group symmetric Nash Equilibrium we sum (13) over $k \in N_A$ to arrive at the following condition:

$$\frac{X_B^{NM}}{(X_A^{NM}+X_B^{NM})^2}+\frac{(1-\alpha_A)(n_A-1)}{X_A^{NM}+X_B^{NM}}=n_A \tag{19}$$

We sum (14) over $k \in N_B$ to arrive at the following condition:

$$\frac{X_A^{NM}}{(X_A^{NM}+X_B^{NM})^2}+\frac{(1-\alpha_B)(n_B-1)}{X_A^{NM}+X_B^{NM}}=n_B \tag{20}$$

Defining total effort in the collective contest as $X^{NM}=X_A^{NM}+X_B^{NM}$ and simplifying equations (19) and (20) we obtain:

$$x_{kA}^{NM}=\frac{1}{n_A}(n_B(X^{NM})^2-(1-\alpha_B)(n_B-1)X^{NM}), \forall k \in N_A \tag{21}$$

and

$$x_{kB}^{NM}=\frac{1}{n_B}(n_A(X^{NM})^2-(1-\alpha_A)(n_A-1)X^{NM}), \forall k \in N_B \tag{22}$$

Equations (21) and (22) are the Nash equilibrium effort levels in a within group symmetric equilibrium when both groups put in positive efforts in the collective contest.

Adding equations (19) and (20) we obtain:

$$X^{NM}=\frac{1+(1-\alpha_A)(n_A-1)+(1-\alpha_B)(n_B-1)}{N} \tag{23}$$

■

Note (21) that $P_A^{NM}=\frac{X_A^{NM}}{X^{NM}}=n_AX^{NM}-(1-\alpha_B)(n_B-1)$

Replacing value of X^{NM} from (23) we get

$$P_A^{NM}=\frac{n_A+n_A(n_B-1)\alpha_B-n_B(n_A-1)\alpha_A}{N}=\frac{\chi_A}{N}$$

Similarly we can find the winning chances for group B.

Proposition 1 follows from Lemma 1, 2 and 3.

9.2 Leader's Optimization Problem

The last sub-section dealt with the individual effort choice problem, taking as given the choices made by the respective group leaders. In this section, we focus on the choice problem of the leaders in the first stage. The leaders are assumed to choose the prize sharing rules simultaneously to maximize group payoffs. The sharing rules are subject to restrictions on competitiveness. So the problem faced by leader of group i is as follows:

$$\begin{aligned} \underset{\alpha_i}{\text{maximize}} \quad & \Pi_i(\alpha_i, \alpha_j) \\ \text{subject to} \quad & \underline{\alpha}_i \leqslant \alpha_i \leqslant 1,\ i = A, B. \end{aligned}$$

Here $\Pi_i(\alpha_i, \alpha_j)$ denotes the payoff of group i. Leader of Group i takes α_j as given. The group payoffs are also a function of the group sizes n_i and n_j, but they are suppressed for notational convenience.

To solve the above problem we set up the Kuhn-Tucker problem for the groups. To set-up the Lagrangian, however, we need to figure out the group payoffs first.

Lemma 4

For $i, j = A, B$,$i \neq j$

a)If Group i is Monopolized then,

$$\Pi_i^{iM} = 0 \ ,\textit{and}\ \Pi_j^{iM} = 1 - X_j^{iM}$$

b) If neither group is monopolized then

$$\Pi_i^{NM} = (1 - X^{NM})(n_j X^{NM} - (n_j - 1)(1 - \alpha_j))$$

where $1 - X^{NM}$ is the total rent ex-post and $n_j X^{NM} - (n_j - 1)(1 - \alpha_j)$ is group i's chance of winning.

Proof: The payoff function of group i can be written as follows:

$$\Pi_i(X_i, X_j) = \frac{X_i}{X_i + X_j} - X_i \tag{24}$$

If Group i is monopolized then from Lemma 1 and Lemma 2 we have, $X_i^{iM} = 0$ and $X_j^{iM} > 0$. Replacing in equation (24) we get part (a) of the Lemma.

If neither group is monopolized the from Lemma 3

$$X_i^{NM} = n_j(X^{NM})^2 - (n_j - 1)(1 - \alpha_j)X^{NM}$$

Replacing in equation (24) we get the expression for the group payoffs in part (b) of the Lemma.

■

Now we can set-up the Optimization Problem that the leaders of the groups face. While setting up the Lagrangian we ignore the Monopolization cases. We ignore the constraints $\alpha_i \leqslant 1$, $i = A, B$. We verify later that they are indeed satisfied in equilibrium.
The Lagrangian of the leader of Group A is as follows:

$$L_A = [1 - X^{NM}][n_B X^{NM} - (n_B - 1)(1 - \alpha_B)] + \lambda_A[\alpha_A - \underline{\alpha}_A] \tag{25}$$

The Lagrangian of the leader of Group B is as follows:

$$L_B = [1 - X^{NM}][n_A X^{NM} - (n_A - 1)(1 - \alpha_A)] + \lambda_B[\alpha_B - \underline{\alpha}_B] \tag{26}$$

λ_i is the Lagrangian multiplier of Group i. For ease of notation let us define $\theta_i = (n_i - 1)(1 - \alpha_i)$, $i \in \{A, B\}$.

The Kuhn -Tucker conditions are as follows:

$$\frac{dL_A}{d\alpha_A} = (n_B - 2n_B X^{NM} + \theta_B)\frac{dX^{NM}}{d\alpha_A} + \lambda_A = 0 \tag{27}$$

$$\frac{dL_B}{d\alpha_B} = (n_A - 2n_A X^{NM} + \theta_A)\frac{dX^{NM}}{d\alpha_B} + \lambda_B = 0 \tag{28}$$

$$\lambda_A \geqslant 0, \quad \alpha_A \geqslant \underline{\alpha}_A, \quad \lambda_A[\alpha_A - \underline{\alpha}_A] = 0 \tag{29}$$

$$\lambda_B \geqslant 0, \quad \alpha_B \geqslant \underline{\alpha}_B, \quad \lambda_B[\alpha_B - \underline{\alpha}_B] = 0 \tag{30}$$

We can use the Kuhn-Tucker conditions to break up the problem into four mutually exclusive cases. Each case is stated as Lemmas. These set of Lemmas help us prove Proposition 2

9.2.1 Neither Group's Constraints Bind

In this case we have $\lambda_A = 0$ and $\lambda_B = 0$.

LEMMA **5**

If neither Group's constraint binds then in Nash Equilibrium $(\alpha_A^*, \alpha_B^*) = (\alpha_A^{NN}, \alpha_B^{NN}) = (\frac{n_A - n_B}{N(n_A-1)}, \frac{n_B - n_A}{N(n_B-1)})$*. The net surplus in the contest in equilibrium is* $S^{NN} = \frac{1}{N}$*. The probabilities of winning are* $(P_A^{NN}, P_B^{NN}) = (\frac{n_B}{N}, \frac{n_A}{N})$*.*

Proof:

Set $\lambda_A = 0$ and $\lambda_B = 0$ in (27) and (28)

It can be easily verified that $\frac{dX^{NM}}{d\alpha_i} = \frac{-(n_i-1)}{N} < 0$ $i = A, B$. Therefore, (27) and (28) reduce to the following conditions:

$$n_B - 2n_B X^{NM} + \theta_B = 0 \tag{31}$$

and

$$n_A - 2n_A X^{NM} + \theta_A = 0 \tag{32}$$

If (27) and (28) are to hold simultaneously then the following equation must hold:

$$n_A \theta_B = n_B \theta_A \tag{33}$$

From Lemma 3 we know that

$$X^{NM} = \frac{1 + \theta_A + \theta_B}{N}$$

Using this fact and (33) in (31), and solving we get :

$$\theta_B = \frac{(N-2)n_B}{N} \tag{34}$$

Replacing θ_B from (34) in (33) we get:

$$\theta_A = \frac{(N-2)n_A}{N} \tag{35}$$

Using the definition of θ_i in (34) and (35), we get that in a Nash Equilibrium

$$(\alpha_A^{NN}, \alpha_B^{NN}) = (\frac{n_A - n_B}{N(n_A - 1)}, \frac{n_B - n_A}{N(n_B - 1)})$$

The net surplus and probabilities of winning can be obtained by replacing the Nash equilibrium values of (α_A, α_B) in part (B) of Proposition 1

■

9.2.2 Group A's Constraint Binds, Group B's Constraint does not Bind

This is the case which corresponds to $\lambda_A \geqslant 0$ and $\lambda_B = 0$

Lemma **6**

If Group A's constraint binds but Groups B's does not then in Nash Equilibrium $(\alpha_A^*, \alpha_B^*) = (\alpha_A^{AB}, \alpha_B^{AB}) = (\underline{\alpha}_A, \frac{(n_B-n_A)(1+(n_A-1)\underline{\alpha}_A)}{2n_A(n_B-1)})$*. The net surplus in the contest in equilibrium is* $S^{AB} = \frac{1+\underline{\alpha}_A(n_A-1)}{2n_A}$*. The probabilities of winning are* $(P_A^{AB}, P_B^{AB}) = (\frac{1-\underline{\alpha}_A(n_A-1)}{2}, \frac{1+\underline{\alpha}_A(n_A-1)}{2})$*.*

Proof:

Set $\lambda_B = 0$ in (28) and noting that $\frac{dX^{NM}}{d\alpha_B} = \frac{-(n_B-1)}{N} < 0$, the following condition is the relevant one

$$n_A - 2n_A X^{NM} + \theta_A = 0 \tag{36}$$

Replacing X^{NM} from Lemma 3 in (36) simplifying we get

$$Nn_A + N\theta_A = 2n_A(1 + \theta_A + \theta_B) \tag{37}$$

Solving for θ_B from (37)

$$\theta_B = \frac{n_A(N-2) + (n_B - n_A)\theta_A}{2n_A} \tag{38}$$

By definition $\theta_B = (n_B - 1)(1 - \alpha_B)$. Applying this definition and the fact that $\alpha_A = \underline{\alpha}_A$ and simplifying the above equation we get

$$\alpha_B^{AB} = \frac{(n_B - n_A)(1 + (n_A - 1)\underline{\alpha}_A)}{2n_A(n_B - 1)} \tag{39}$$

Therefore in this case the in a Nash equilibrium we have

$$(\alpha_A^{AB}, \alpha_B^{AB}) = (\underline{\alpha}_A, \frac{(n_B - n_A)(1 + (n_A - 1)\underline{\alpha}_A)}{2n_A(n_B - 1)})$$

We, however, need to verify that $\lambda_A \geqslant 0$. To do that we use (27). We know that $\frac{dX^{NM}}{d\alpha_A} = \frac{-(n_A-1)}{N} < 0$. Therefore to show that $\lambda_A \geqslant 0$, we need to show that $(n_B - 2n_B X^{NM} + \theta_B) \geqslant 0$. This is satisfied as long as

$$\underline{\alpha}_A \geqslant \frac{n_A - n_B}{N(n_A - 1)} = \alpha_A^{NB}$$

This is the choice made by group A when none of the constraints bind in Lemma 5. This condition delineates the zone where groups A's constraint binds and where it does not in equilibrium.

The net surplus and probabilities of winning can be obtained by replacing the Nash equilibrium values of (α_A, α_B) in part (B) of Proposition 1

■

9.2.3 Group B's Constraint Binds, Group A's Constraint does not Bind

This is the case where we have $\lambda_A = 0$ and $\lambda_B \geqslant 0$

Lemma **7**

If Group B's constraint binds but Groups A's does not then in Nash Equilibrium $(\alpha_A^*, \alpha_B^*) = (\alpha_A^{BB}, \alpha_B^{BB}) = (\frac{(n_A-n_B)(1+(n_B-1)\underline{\alpha}_B)}{2n_B(n_A-1)}, \underline{\alpha}_B)$. *The net surplus in the contest in equilibrium is* $S^{BB} = \frac{1+\underline{\alpha}_B(n_B-1)}{2n_B}$. *The probabilities of winning are* $(P_A^{BB}, P_B^{BB}) = (\frac{1+\underline{\alpha}_B(n_B-1)}{2}, \frac{1-\underline{\alpha}_B(n_B-1)}{2})$.

Proof:

The proof follows exactly the same line as the proof of Lemma 6, but now the relevant first order condition being (27). Therefore, we skip the proof.

■

9.2.4 Both Groups Constraint Binds

This is the case where we must have $\lambda_A \geqslant 0$ and $\lambda_B \geqslant 0$

LEMMA **8**

If both Group A and Group B's constraint binds then in Nash Equilibrium $(\alpha_A^*, \alpha_B^*) = (\underline{\alpha}_A, \underline{\alpha}_B)$. *The net surplus in the contest in equilibrium is* $S^B = \frac{1+(n_A-1)\underline{\alpha}_A+(n_B-1)\underline{\alpha}_B}{N}$. *The probabilities of winning are* $(P_A^B, P_B^B) = (\frac{\chi_A}{N}, \frac{\chi_A}{N})$.

Proof:

In this case $\alpha_A^* = \underline{\alpha}_A$ and $\alpha_B^* = \underline{\alpha}_B$.

However, for this case to be valid we need to verify that $\lambda_A \geqslant 0$ and $\lambda_B \geqslant 0$. In light of the fact that $\frac{dX^{NN}}{d\alpha_i} = \frac{-(n_i-1)}{N} < 0$, we can immediately conclude from (27) and (28) that $\lambda_i \geqslant 0$ as long as $(n_j - 2n_j X^T + \underline{\theta}_j) \geqslant 0$, $i, j = A, B$ and $i \neq j$, where X^T is given in (**??**).

We work with the expression $(n_j - 2n_j X^{NM} + \underline{\theta}_j)$ to find conditions under which it is non-negative. Replacing X^T in the expression we get

$$n_j - 2n_j \frac{1 + \underline{\theta}_i + \underline{\theta}_j}{N} + \underline{\theta}_j \geqslant 0$$

$$\Rightarrow Nn_j - 2n_j - 2n_j\underline{\theta}_i - 2n_j\underline{\theta}_j + N\underline{\theta}_j \geqslant 0$$

$$\Rightarrow (N-2)n_j + (n_i - n_j)(1 - \underline{\alpha}_j)(n_j - 1) - 2n_j(1 - \underline{\alpha}_i)(n_i - 1) \geqslant 0$$

$$\Rightarrow 2n_j(n_i - 1)\underline{\alpha}_i - (n_i - n_j)(n_j - 1)\underline{\alpha}_j \geqslant n_i - n_j$$

Simplifying we get that $\lambda_A \geqslant 0$ and $\lambda_B \geqslant 0$ as long as

$$\underline{\alpha}_B \geqslant \frac{(n_B - n_A)(1 + (n_A - 1)\underline{\alpha}_A)}{2n_A(n_B - 1)} = \alpha_B^{AB} \tag{40}$$

and

$$\underline{\alpha}_A \geqslant \frac{(n_A - n_B)(1 + (n_B - 1)\underline{\alpha}_B)}{2n_B(n_A - 1)} = \alpha_A^{BB} \tag{41}$$

where α_B^{AB} is the equilibrium choice of group B in the case where the constraint of group A binds but group B does not (Lemma 5) and α_A^{BB} is the equilibrium choice of group A in the case where group B's constraint binds but group A's does not (Lemma 6). So, these conditions cleanly delineate the zones of equilibria characterized in Lemma 5, 6 and 7.

The net surplus and probabilities of winning can be obtained by replacing the Nash equilibrium values of (α_A, α_B) in part (B) of Proposition 1

■

Having exhaustively analyzed the cases where neither group is monopolized, now we bring in monopolization to check when a group is monopolized in equilibrium.

Lemma **9**

Group i is monopolized in a Nash equilibrium iff $\underline{\alpha}_i \in [\frac{1}{n_i-1}, 1]$ and $\underline{\alpha}_j \in (-\infty, \alpha_j^M]$. In this case any combination of prize sharing rules (α_i^*, α_j^*), such that $\alpha_i^* \geqslant \underline{\alpha}_i$ and $\alpha_j^* = -\frac{1}{n_j-1} + \frac{(n_i-1)n_j}{(n_j-1)n_i}\alpha_i^*$ is a Nash equilibrium. The net surplus in the contest in equilibrium is $S^{iM} = \frac{\alpha_i(n_i-1)}{n_i}$. The probabilities of winning are $(P_i^{iM}, P_j^{iM}) = (0, 1)$.

Proof:

Let us consider the case where $i = A$. The proof for $i = B$ will be analogous and is skipped.

Now for group A to be monopolized we know from Lemma 1 that the following condition needs to be satisfied

$$\alpha_A n_B(n_A - 1) - \alpha_B n_A(n_B - 1) \geqslant n_A \tag{42}$$

So if group B were to monopolize group A, then given any choice of α_A, group B's best response is to choose

$$\alpha_B = -\frac{1}{n_B - 1} + \frac{(n_A - 1)n_B}{(n_B - 1)n_A}\alpha_A \tag{43}$$

This is so because it is the most egalitarian and hence the least costly way in which group B could monopolize group A. This is obtained by solving for α_B from (42) with an equality. As for choice of of group A we have the following two cases

Case 1:

Suppose in an equilibrium, group A behaves in a hawkish manner, so that $\alpha_A = \underline{\alpha}_A$. To monopolize A, group B will choose from (43).

$$\alpha_B^M = -\frac{1}{n_B - 1} + \frac{(n_A - 1)n_B}{(n_B - 1)n_A}\underline{\alpha}_A \tag{44}$$

Now in this case, group B in equilibrium obtains a payoff of $\Pi_B^{AM} = \frac{\underline{\alpha}_A(n_A-1)}{n_A}$) (see Proposition 3).

If instead it were to deviate to α_B^{AB} it would get $\Pi_B^{AB} = \frac{(1+(n_A-1)\underline{\alpha}_A)^2}{4n_A}$ (see Lemma 6).

But notice that $\Pi_B^{AB} \geqslant \Pi_B^{AM}$. Therefore group B always wants to deviate to α_B^{AB}. This deviation is not possible if $\underline{\alpha}_A > \frac{1}{n_A-1}$ [16], because then $\Pi_A^{AB} < 0$, so that group A is drops out. Given that group A will drop out group B's best response is to choose α_B^M, because $\alpha_B^M > \alpha_B^{AB}$ in this case and choosing α_B^M is the less costly way to monopolize A.

As group A gets zero payoff when monopolized, $\underline{\alpha}_A$ is a best response to α_B^M. Therefore, when $\underline{\alpha}_A > \frac{1}{n_A-1}$, $(\underline{\alpha}_A, \alpha_B^M)$ constitute a Nash equilibrium in which group A is monopolized.

If, however, $\underline{\alpha}_A < \frac{1}{n_A-1}$, then $\Pi_A^{AB} > 0$. Given that it is always optimal for group B to deviate to α_B^{AB}, it will do so and group A will not be monopolized. Therefore, there does not exist a Nash equilibrium in which A is monopolized when $\underline{\alpha}_A < \frac{1}{n_A-1}$.

[16] This is where α_B^{AB} and α_B^M intersect

Case 2:

Group A acts in a dovish manner $\alpha_A > \underline{\alpha}_A$ in equilibrium.

When $\underline{\alpha}_A < \frac{1}{n_A - 1}$, the best response for group B is to choose α_B such that group A is not monopolized. Given that group B will not monopolize group A, the best response for group A do deviate to a hawkish stance, as its payoff is decreasing in α_A. Therefore, there does not exist a Nash equilibrium in which group A is dovish when $\underline{\alpha}_A < \frac{1}{n_A - 1}$.

If $\underline{\alpha}_A \geqslant \frac{1}{n_A - 1}$, nothing which group A does can guarantee it a positive payoff. So group A is indifferent and can choose any $\alpha_A \geqslant \underline{\alpha}_A$. In this case the best group B can do is to choose the least costly way to monopolize A by choosing α_B given in (43).

The fact that group A is indifferent between choices of α_A when it is monopolized in equilibrium, gives rise to multiple Nash equilibria. But, we can get around this issue by assuming that when indifferent group A chooses $\alpha_A = \underline{\alpha}_A$, because this choice is immune to trembles in strategies of group B. ■

Proposition 2 follows directly from Lemma 5, 6, 7, 8, 9. Also look at Figures 1 and 2.

The net surplus and probabilities of winning can be obtained by replacing the Nash equilibrium values of (α_A, α_B) in part (B) of Proposition 1.

■ Proof of Proposition 3

The proof directly follows from Proposition 2 noting that $\Pi_i = P_i S$.

■ Proof of Proposition 4.

Proof: To prove this Proposition we use Figures 1 and break up the proposition into four mutually exclusive cases. [17]

▶ **Case 1:** $\underline{\alpha}_A \geqslant 0$ and $\underline{\alpha}_B < 0$

From Figure 3 it is clear that in this case either group A is Monopolized or we are in the case where Group A's constraint binds but Group B's does not.

[17] Even though GSP has been defined in terms of winning probabilities in the chapter, we proceed by comparing payoffs of the groups, as these are equivalent in our framework.

If group A is monopolized then of course the larger group B wins the contest with probability 1 and GSA applies.

If group A is not monopolized then Lemma 6 applies. We can immediately verify that $\Pi_B^{AB} \geqslant \Pi_A^{AB}$. This inequality holds as long as $\underline{\alpha}_A \geqslant 0$. So again GSA applies.

▶ **Case 2:** $\underline{\alpha}_A \geqslant 0$ and $\underline{\alpha}_B \geqslant 0$

From Figure 1 it is clear that in this case we have many subcases, i.e., group A can be monopolized, group B can be monopolized, both groups constraints may bind and we may also be in situation where Group A's constraint binds but Group B's does not.

But just considering the case where both group's constraint binds helps us to cleanly delineate the parametric zone into zones where GSP or GSA applies. When both group's constraints bind then Lemma 8 applies. It can be easily verified that $\Pi_A^B > \Pi_B^B$ if and only if $\underline{\alpha}_B > \alpha_B^{GSP} = \frac{n_B - n_A}{2n_A(n_B-1)} + \frac{(n_A-1)n_B}{(n_B-1)n_A}\underline{\alpha}_A$.

α_B^{GSP} intersects α_B^{AB} at $\underline{\alpha}_A = 0$ and lies above it at any $\underline{\alpha}_A > 0$. Also, α_B^{GSP} lies entirely above α_B^{AM} at any $\underline{\alpha}_A \geqslant 0$. So, these cases belong where $\underline{\alpha}_B \leqslant \alpha_B^{GSP}$, and therefore GSA should apply in these cases. It can be easily verified from Lemma 6 and Lemma 9, that it is indeed the case. Look at Figure 4.

Also, α_B^{GSP} lies completely below α_A^{BM}. So, the cases in which group B is monopolized belong where $\underline{\alpha}_B > \alpha_B^{GSP}$, and therefore GSP applies.

α_B^{GSP} provides a clear delineation of this parametric zone, i.e., $\underline{\alpha}_A \geqslant 0$ and $\underline{\alpha}_B \geqslant 0$, as far as occurrence of GSP or GSA is concerned.

▶ **Case 3:** $\underline{\alpha}_A < 0$ and $\underline{\alpha}_B < 0$

From Figure 1 it is clear that either we are in the case where Group A's constraint binds but Group B's does not or we are in the case where neither groups constraint binds.

In the case where neither groups constraint binds Lemma 5 applies. It can be immediately verified from the Lemma that $\Pi_A^{NN} > \Pi_B^{NN}$. Therefore, GSP applies in such cases.

In the case where Group A's constraint binds but Group B's does not, Lemma 6 applies. And again it is straightforward to check from the Lemma that $\Pi_B^{AB} < \Pi_A^{AB}$ when $\underline{\alpha}_A < 0$. So, again GSP applies.

▶ **Case 4:** $\underline{\alpha}_A < 0$ and $\underline{\alpha}_B \geqslant 0$

From Figure 1 it is clear that this case has many subcases, i.e., neither group's constraints bind, group B is monopolized, Group A's constraint binds but Group B's does not and also Group B's constraint binds but Group A's does not. In what follows we consider each case one by one.

If we are in the case where group B is monopolized, then group A wins with probability 1 and GSP applies.

If neither groups constraint binds then Lemma 5 applies. It can be immediately verified from the Lemma 5 that $\Pi_A^{NN} > \Pi_B^{NN}$. Therefore, GSP applies in such cases.

If group A's constraint binds but group B's does not then, Lemma 6 applies. It can be easily verified from Lemma 6 that $\Pi_B^{AB} < \Pi_A^{AB}$ when $\underline{\alpha}_A < 0$. Therefore, GSP applies in this case.

If group B's constraint binds but group A's does not then, Lemma 7 applies. It can be easily verified from Lemma 7 that $\Pi_B^{BB} < \Pi_A^{BB}$ when $\underline{\alpha}_B \geqslant 0$. Therefore, GSP applies in this case too.

The last case to consider is the one where both groups constraint binds. We saw that in **Case 2** that GSP applies when $\underline{\alpha}_B > \alpha_B^{GSP}$. When both groups constraints bind in this case, the condition for GSP to occur is still $\underline{\alpha}_B > \alpha_B^{GSP}$ as Lemma 8 still applies. But, we also noted in the proof of **Case 2** that α_B^{GSP} intersects α_B^{AB} at $\underline{\alpha}_A = 0$. In this particular case, α_B^{GSP} lies entirely below α_B^{AB}. For both groups constraints to bind it must be the case that $\underline{\alpha}_B > \alpha_B^{AB}$. But because $\alpha_B^{AB} > \alpha_B^{GSP}$ in this case, it follows that $\underline{\alpha}_B > \alpha_B^{GSP}$. Therefore, GSP applies in this case as well. For visual clarity consider the dotted section of α_B^{GSP} in Figure 4.

Proposition 4 directly follows from the above four cases and can be visualized in Figure 4

■

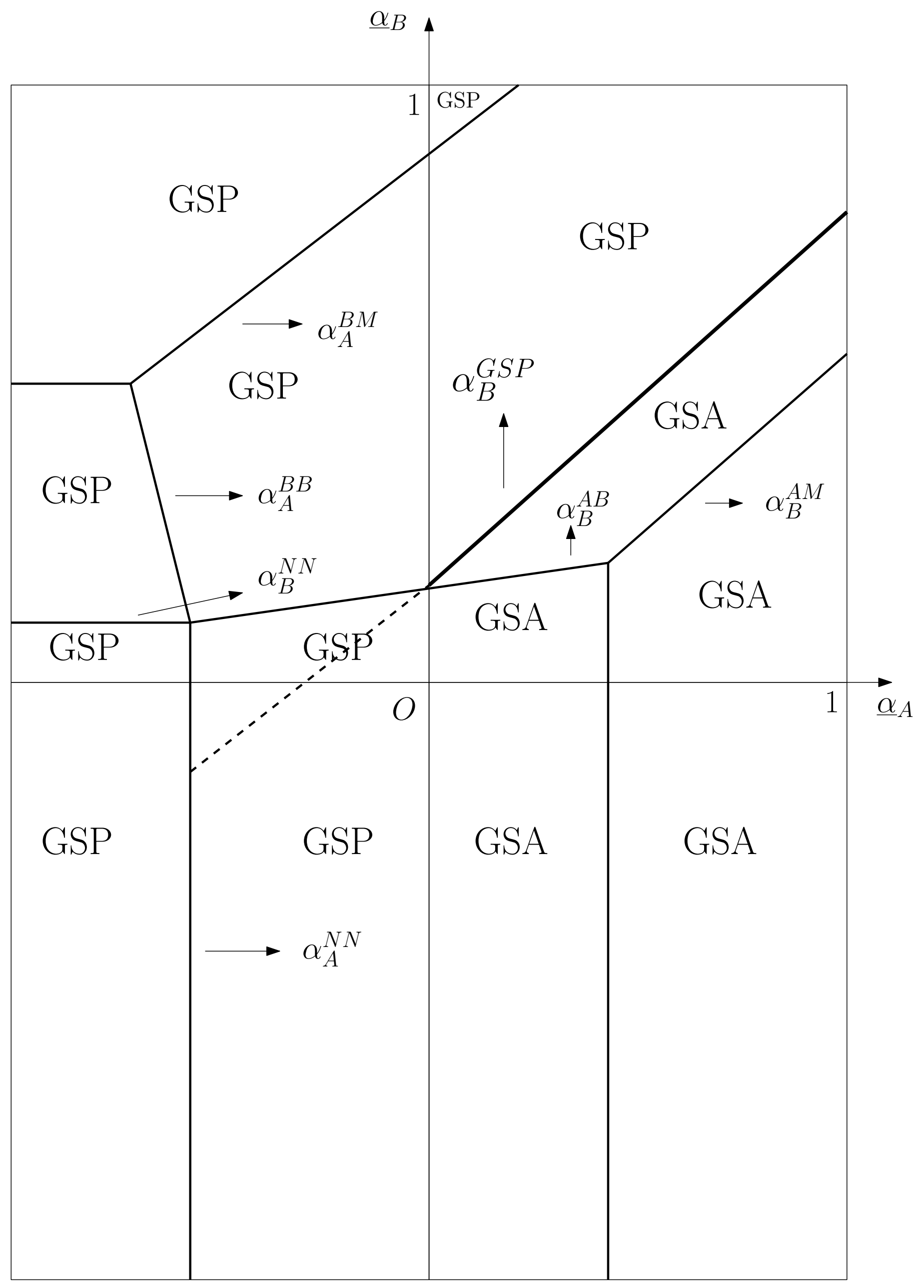

Figure 4: GSP-GSA