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Selling Two Identical Objects*

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Abstract

It is well-known that optimal (i.e., revenue-maximizing) selling mechanisms in multidimensional type spaces may involve randomization. We study mechanisms for selling two identical, indivisible objects to a single buyer. We analyze two settings: (i) decreasing marginal values (DMV) and (ii) increasing marginal values (IMV). Thus, the two marginal values of the buyer are not independent. We obtain sufficient conditions on the distribution of buyer values for the existence of an optimal mechanism that is deterministic.

In the DMV model, we show that under a well-known condition, it is optimal to sell the first unit deterministically. Under the same sufficient condition, a bundling mechanism (which is deterministic) is optimal in the IMV model. Under a stronger sufficient condition, a deterministic mechanism is optimal in the DMV model.

Our results apply to heterogenous objects when there is a specified sequence in which the two objects must be sold.

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1 Introduction

We consider optimal, i.e., expected revenue maximizing, mechanisms for selling two identical units of an object to a buyer. The buyer's type (values for the units) is two dimensional and privately known to the buyer. We focus on two cases: decreasing marginal values and increasing marginal values. Thus, the buyer's values for the units are not independent.

The assumption of homogenous objects reduces the dimensionality of the price space and therefore the dimensionality of random allocation rules (compared to heterogenous objects). While this represents a simplification of the problem of finding an optimal mechanism, the dependence of values in our paper increases complexity. Moreover, the natural requirement that the first unit is sold before the second unit adds a constraint to the feasible region.

A general solution to the optimal mechanism design problem for the sale of multiple indivisible products is unknown. Unlike the single product case, the optimal mechanism for selling two or more products may involve randomization (see Thanassoulis (2004), Manelli and Vincent (2006), Pycia (2006), and Hart and Reny (2015)). Our objective is to find sufficient conditions under which a deterministic mechanism is optimal among all mechanisms for selling two identical units, including random mechanisms.

We assume that the seller can commit to a mechanism. Implicit in this assumption is that the buyer can verify that the seller followed the mechanism that she committed to. Verification is easier for a deterministic mechanism as it does not require access to a randomization device that both parties regard as authentic. Perhaps this is a reason for the limited use of randomized selling methods.[1]

With homogenous objects, there is a natural order of transactions: the second unit can be sold only after the first unit is sold. Our results apply to the sale of two heterogenous objects when one of the two objects can be sold only after the other object is sold.[2] For instance, the warranty on a product is only sold to a buyer who purchases the product. Another example is when a seller offers two versions of a product, basic or premium. The premium version of a product can be viewed as consisting of the basic version plus an upgrade. That is, the upgrade is sold only if the basic product is also sold.

[1]Random selling methods – called opaque selling – are used by travel websites such as Hotwire and Priceline.

[2]See Armstrong (2016) for a discussion of this issue.

A function of buyer marginal valuations, $\Phi(v_1, v_2)$, plays a key role in the analysis.[3] The function Φ acts as a guidepost for making revenue improvements to any incentive compatibile and individual rational mechanism. If Φ satisfies certain single-crossing conditions, then incentive compatibility and individual rationality is maintained in the improved mechanism. The function Φ depends only on the distribution of types and not on any specific mechanism.

With decreasing marginal values, we show that if Φ satisfies single-crossing in the horizontal direction (which corresponds to changes in v_1 only), then there exists an optimal selling mechanism in which the first unit is sold deterministically. We refer to a mechanism in which the first unit is sold with probability 0 or 1 as a *line mechanism*. Line mechanisms are completely described by the payment for the first unit and the probability of allocating the second unit to types on the vertical line $(1, v_2)$, $0 \leq v_2 \leq a$. The resulting simplification allows us to precisely describe the restrictions placed on line mechanisms by optimality. Next, if Φ satisfies single-crossing in the vertical direction (which corresponds to changes in v_2 only) then there is an optimal mechanism which is *semi-deterministic*, i.e., a line mechanism with at most one probabilistic value for allocating the second unit. Finally, if Φ satisfies diagonal single-crossing (along the diagonal boundary of the support of the distribution) then there exists an optimal mechanism that is deterministic.

Our results for increasing marginal values are under weaker conditions, in that single-crossing of Φ in the horizontal direction is sufficient for the existence of an optimal mechanism that is deterministic. In this optimal mechanism, the two units are bundled together and sold at a take-it-or-leave-it price.

While decreasing marginal values is a common assumption, there are scenarios where marginal values are increasing. For instance, if the buyer is unfamiliar with the product and incurs a learning cost before using it, the marginal value for the second unit may be higher than the marginal value for the first unit. Alternatively, if there is a fixed cost of production, then the model resembles increasing marginal values. As described later, increasing marginal values also obtain when a buyer might consume the good over one or two periods.

We provide a class of distributions for decreasing marginal values, called the ordered decreasing values model, which satisfies our single-crossing assumptions on Φ. Similarly, an ordered increasing values model satisfies the single-crossing condition on Φ.

To our knowledge, the function Φ is new to this literature. However, horizontal single-

[3] The buyer's marginal valuations (or type) for the two units are $v_1 \in [0, 1]$ and $v_2 \in [0, a]$.

crossing of Φ is equivalent to a sufficient condition introduced by McAfee and McMillan (1988). A version of Φ may be useful in proving the optimality of deterministic mechanisms in other settings, such as the sale of heterogenous objects.

Related Literature: Early work on mechanism design with multidimensional types includes Rochet (1987), McAfee and McMillan (1988), Wilson (1993), Armstrong (1996), and Rochet and Choné (1998). As these papers focused primarily on divisible products, existence of deterministic mechanisms was not an issue.

Thanassoulis (2004), Manelli and Vincent (2006), Manelli and Vincent (2007), Pycia (2006), Pavlov (2011), and Hart and Reny (2015) investigate the sale of indivisible, heterogenous objects with independent values that are additive. As already noted, it may be optimal to randomize in this setting. Moreover, as Hart and Reny (2015) show, revenue may not be monotone in the distribution of the buyer's type. Correlation between marginal values adds another layer of complexity and may increase the desirability of randomization. In a model with two heterogenous goods and correlated values, Hart and Nisan (2019) show that mechanisms of bounded menu size, such as deterministic mechanisms, may yield a negligible fraction of the optimal revenue.

The sale of homogenous objects is analyzed by Malakhov and Vohra (2009) and by Devanur et al. (2020). These models differ from ours in that buyers have the same privately known value for all units, but the number of units desired is privately known. Pavlov (2020) investigates a model in which buyers buy one of two heterogenous objects, each of which has two units with the same value to the buyer.

Apart from Malakhov and Vohra (2009) and Devanur et al. (2020), two other papers obtain sufficient conditions for the existence of optimal mechanisms that are deterministic. Manelli and Vincent (2006) obtain sufficient conditions in a model with two heterogenous goods with independent, additive values. Haghpanah and Hartline (2020) obtain sufficient conditions for bundling to be optimal in a general model.

When there are two or more buyers, Chen et al. (2019) provide sufficient conditions for the existence of optimal Bayesian incentive compatible mechanisms that are deterministic. However, these conditions do not apply to our setting, where there is one buyer, or to dominant strategy incentive compatible mechanisms. Daskalakis et al. (2017) and Kleiner and Manelli (2019) characterize optimality for a multi-product monopolist using duality theory.

The rest of the paper is organized as follows. We investigate the decreasing marginal values model in Section 2. After presenting an example in which it is optimal to randomize over the first unit, we show in Section 2.1 that it is optimal to sell the first unit deterministically. Line mechanisms are characterized in Section 2.2 and the existence of an optimal mechanism that is deterministic is established in Section 2.3. Necessary conditions for a deterministic optimal mechanism are presented in Section 2.4. A special case of decreasing marginal values, the ordered decreasing model, is introduced in Section 2.5. Our results for increasing marginal values are in Section 3. All proofs are in an Appendix.

2 Decreasing Marginal Values

We begin with an example with identical objects and decreasing marginal values in which it is optimal to randomize.[4]

Example 1 There are two units and the buyer has three possible types, A, B and C. The valuations of the three types and the probability distribution over types are provided below:[5]

Type	(v_1, v_2)	**Probability**
A	$(3, 1)$	0.5
B	$(4, 3)$	0.25
C	$(10, 1)$	0.25

The candidates for a deterministic optimal mechanism are:[6]

Bundling. The three candidate optimal prices for the two units bundled together are 4, 7, and 11. The highest expected revenue (of 4) is achieved at bundle price 4. All types buy the two units at this price.

Unbundling. The prices of the two units, p_1 and p_2, are strictly positive. The most that can be earned by selling only to type C is $\frac{11}{4}$. The most that can be earned by

[4]Examples of optimal random mechanisms in the literature involve heterogenous objects.

[5]The example can be modified to a type space with continuous density as modeled in this paper.

[6]For simplicity, we assume that when a buyer is indifferent between two options, she chooses the option the seller prefers. That is, the mechanism is seller favorable as defined in Hart and Reny (2015).

selling only to types B and C is $\frac{7}{2}$. Therefore, consider prices at which type A buys either one or two units. If type A buys exactly one unit then $p_1 \leq 3$, $p_2 > 1$ and maximum revenue is 3.75 with $p_1 = 3$, $p_2 = 3$. If type A buys two units then $p_1 \leq 3$, $p_2 \leq 1$ and maximum revenue is 4, which yields the bundling outcome.

Thus, the maximum expected revenue from a deterministic mechanism is 4. Consider the following random mechanism.

Price	Package
2	$\left(\frac{2}{3}, 0\right)$, i.e., sell unit 1 with probability $\frac{2}{3}$ and unit 2 with probability 0
$6\frac{1}{3}$	(1,1), i.e., sell both units with probability 1

With this menu, type A chooses the first option and pays 2. Types B and C choose the second option and pay $6\frac{1}{3}$. This generates an expected revenue of

$$2 \times 0.5 + \frac{19}{3} \times 0.5 = \frac{25}{6} > 4.$$

Thus, randomization yields higher expected revenue than any deterministic mechanism. $\square$

The Model

We describe a model with decreasing marginal values over two identical units of an indivisible object. The buyer's (marginal) value for the ith unit is v_i, $i = 1, 2$. The joint density function of $v = (v_1, v_2)$ is $f(v)$, which has support

$$D \equiv \{(v_1, v_2) \in [0,1] \times [0,a] : v_2 \leq a\, v_1\}$$

Marginal values are decreasing if $a \leq 1$, although this assumption is not necessary for any of our results. We assume that density $f(\cdot)$ is strictly positive on its support and is absolutely continuous. As the support of the marginal distribution of v_2 depends on the realized value of v_1, the values v_1 and v_2 are not independent.

An *allocation rule* is a function $q = (q_1, q_2)$, where $q_i : D \to [0,1]$, $i = 1, 2$ is the (unconditional) probability that the ith unit is sold to the buyer. If buyer type (v_1, v_2) obtains a second unit, then this buyer must also obtain the first unit. Therefore, the following feasibility constraint is implied:

$$q_1(v) \geq q_2(v), \qquad \forall v$$

A *transfer* is a function $t : D \to \Re$, a payment by the buyer to the seller. A *mechanism* is (q, t).

The payoff of a buyer who truthfully reports v is

$$u(v) \equiv v \cdot q(v) - t(v)$$

A mechanism (q, t) is *individually rational* if $u(v) \geq 0$ for all v; it is *incentive compatible* if

$$u(v) \;\geq\; u(v') + (v - v') \cdot q(v'), \qquad \forall v, v'$$

It is well-known (see Börgers (2015), for instance) that a necessary and sufficient condition for incentive compatibility is that $u(v)$ is a convex function and

$$q_i(v) = \frac{\partial u(v)}{\partial v_i}, \qquad \text{a.e.,} \;\; i = 1, 2$$

Thus

$$t(v) = \triangledown u(v) \cdot v - u(v), \qquad \text{a.e.}$$

The seller's expected revenue is

$$\textsc{Rev}(q, t) \;\equiv\; \mathrm{E}[t(v)] \;\;=\;\; \int_D \Big[\triangledown\, u(v) \cdot v - u(v) \Big] f(v) dv \tag{1}$$

The integral of the first term in the integrand is the expected welfare from the mechanism. Subtracting the expected payoff of the buyer yields the seller's expected revenue.

A mechanism (q^*, t^*) is *optimal* if it is incentive compatible (IC) and individually rational (IR), and for any other IC and IR mechanism (q', t') we have

$$\textsc{Rev}(q^*, t^*) \;\geq\; \textsc{Rev}(q', t')$$

It is easy to show that in any optimal mechanism (q^*, t^*)

$$q^*(v) = (0, 0) \quad \Longrightarrow \quad t^*(v) = 0$$

Thus, in an optimal mechanism, the payoff of a buyer type who received zero units is zero.

A mechanism (q, t) is *deterministic* if its allocation rule is deterministic, i.e., $q_i(v) \in \{0, 1\}$ for all v and i. If a mechanism is not deterministic, it is *random*. A random mechanism (or allocation rule) is a lottery over deterministic mechanisms (or allocation rules).

Let $\mathcal{Q}$ be the set of IC and IR mechanisms. If mechanisms (q^a, t^a) and (q^b, t^b) are IC and IR then so is $\lambda(q^a, t^a) + (1 - \lambda)(q^b, t^b)$, $\lambda \in [0, 1]$. Thus, $\mathcal{Q}$ is a convex set. The set $\mathcal{Q}$

is compact.[7] Therefore, as the expected revenue is a continuous, linear functional of q, it is maximized at an extreme point of $\mathcal{Q}$. When two or more indivisible objects are for sale, the extreme points of $\mathcal{Q}$ may be random mechanisms. This contrasts with the sale of one object to one buyer, where all extreme points of the set of IC, IR mechanisms are deterministic. Hence, a deterministic optimal mechanism always exists when a single object is sold to a buyer but a random mechanism might be optimal if two or more objects are sold.

Heterogenous objects

Our results apply to the sale of two heterogenous objects when one of the two objects can be sold only after the other object is sold. Here are two scenarios.

- After a decision to purchase a product, the buyer might be offered a related product or service. For instance, after purchasing a new car the buyer might also purchase an extended warranty. Such *add-on sales* fit our model.
- A seller who offers two versions of a product, basic or premium. The buyer's value for the basic product is v_1 and for the premium product is $v_1 + v_2$.[8]

2.1 The First Unit is Sold Deterministically

In Proposition 1 below, we show that under a sufficient condition on the density, there exists an optimal mechanism in which the first unit is sold deterministically. The following lemma is required for the proposition. The proof, which is in the Appendix, is similar to that of a result in McAfee and McMillan (1988); however, our assumption of decreasing marginal values yields a simpler expression for expected revenue.

Lemma 1 *The seller's expected revenue from an IC and IR mechanism (q,t) is*

$$\textsc{Rev}(q,t) \;=\; \int_0^a u(1,v_2)f(1,v_2)dv_2 - \int_0^a \int_{\frac{v_2}{a}}^1 u(v_1,v_2)\Big[3f(v_1,v_2) + (v_1,v_2)\cdot\bigtriangledown f((v_1,v_2))\Big]dv_1dv_2$$

[7]See Manelli and Vincent (2007) for a proof of compactness of the set of mechanisms for the sale of heterogenous objects. A similar proof applies for the case of homogenous objects considered in this paper.

[8]The additional value a buyer places on a premium product might be more than the value for a basic product, $v_2 > v_1$. This is admissible in our model as we do not assume $a \leq 1$.

The following condition on density, introduced by McAfee and McMillan (1988), is often invoked in the multidimensional mechanism design literature.

The density f satisfies **Condition SC-H** *if* $3f(v) + v \cdot \bigtriangledown f(v) \geq 0$ *for all* $v \in D$.

In the single-object case, Condition SC-H becomes $2f(v) + v\frac{df(v)}{dv} \geq 0$, which (i) is equivalent to the assumption that the expected revenue, $v[1 - F(v)]$, is concave and (ii) implies that Myerson's virtual value function satisfies single crossing.[9] As shown next, under SC-H the first unit is allocated deterministically in an optimal mechanism.[10]

Proposition 1 *If the density function f satisfies Condition SC-H, then there exists an optimal mechanism (q, t) in which $q_1(v) \in \{0, 1\}$ for all v.*

Lemma 1 implies that if, for any IC and IR mechanism (q, t), it is possible to decrease $u(v_1, v_2)$ when $v_1 < 1$, without decreasing $u(1, v_2)$ and Condition SC-H is satisfied, then expected revenue increases. Proposition 1 is proved by making such decoupled changes in u in a mechanism in which the first unit is allocated randomly to some buyer types, thereby creating a new IC and IR mechanism with greater expected revenue. The proof is similar to the proof of Proposition 2 in Pavlov (2011), who showed that in the unit-demand case and in the additive, heterogenous objects case, there is an optimal mechanism in which any positive allocation belongs to the upper boundary of the feasible allocation set.

REMARK 1: Proposition 1 holds if the seller's costs $c = (c_1, c_2)$ are positive with $1 > c_2/a \geq c_1 \geq 0$ and Condition SC-H is modified to $3f(v) + (v - c) \cdot \bigtriangledown f(v) \geq 0$, $\forall v \in D$.

Thus, under SC-H we may restrict our search for optimal mechanisms to those that allocate the first unit deterministically. This reduces the dimensionality of the problem as potentially optimal mechanisms are specified by a price for the first unit and an allocation rule for types $(1, v_2)$ only. We refer to such potentially optimal allocation mechanisms as line mechanisms.

[9]Condition SC-H is one of three conditions we impose on a function Φ that is defined in Section 2.3; SC-H is a single-crossing assumption in the horizontal direction on Φ. Together, the three conditions imply the existence of an optimal mechanism that is deterministic.

[10]If the inequality in SC-H is strict, then in any optimal mechanism $q_1(v) \in \{0, 1\}$ for almost all v.

2.2 Line Mechanisms

Let $Y \equiv \{(1, v_2) : v_2 \in [0, a]\}$ be the one-dimensional subset of the type space along the v_2-axis.

Definition 1 *A mechanism (q, t) is a* **line mechanism** *if*

i. its restriction to Y is IC and IR

ii. for every $(1, v_2) \in Y$, $q_1(1, v_2) = 1$

iii. for every $v \equiv (v_1, v_2) \in D \setminus Y$,

$$\Big(q_1(v), q_2(v), t(v)\Big) = \begin{cases} (0, 0, 0), & \text{if } v_1 + v_2 q_2(1, v_2) < t(1, v_2) \\ (1, q_2(1, v_2), t(1, v_2)), & \text{otherwise.} \end{cases}$$

The first unit is allocated deterministically in a line mechanism, with types $(1, v_2)$ obtaining the first unit with probability one. A type (v_1, v_2) is allocated $(q_1(1, v_2) = 1,\ q_2(1, v_2),\ t(1, v_2))$ if it is IR; otherwise type (v_1, v_2) gets $(0, 0, 0)$. Thus,

$$\begin{aligned} u(v_1, v_2) &= \max\Big[0, v_1 + v_2 q_2(1, v_2) - t(1, v_2)\Big] \\ &= \max\Big[0, u(1, v_2) - (1 - v_1)\Big] \end{aligned} \tag{2}$$

Lemma 2 *Every line mechanism is IC and IR on D.*

For any IC and IR mechanism, the proof of Proposition 1 constructs a line mechanism which generates at least as much expected revenue. Hence, we have the following corollary to Proposition 1.

Corollary 1 *If the density function f satisfies Condition SC-H, then there is an optimal mechanism that is a line mechanism.*

Corollary 1 simplifies the problem significantly, as a line mechanism is completely described by $t(1, 0)$, the payment by type $(1, 0)$ (i.e., the price for the first unit), and $q_2(1, v_2)$, the allocation rule for the second unit for types with $v_1 = 1$. However, the problem does not become one-dimensional. Two line mechanisms with the same allocation rule on Y will have different allocation rules on D if the prices for the first unit are different in the two mechanisms. Moreover, the set of line mechanisms is not convex.

The Structure of Line Mechanisms

For any line mechanism (q, t), define

$$Z_0(q,t) := \{(v_1, v_2) : u(1, v_2) - (1 - v_1) < 0\}$$

Eq. (2) implies that the set of buyer types who do not receive any units in the line mechanism is $Z_0(q, t)$. The closure of $Z_0(q, t)$ consists of (v_1, v_2) such that $u(v_1, v_2) = 0$. A line mechanism is shown in Figure 1.

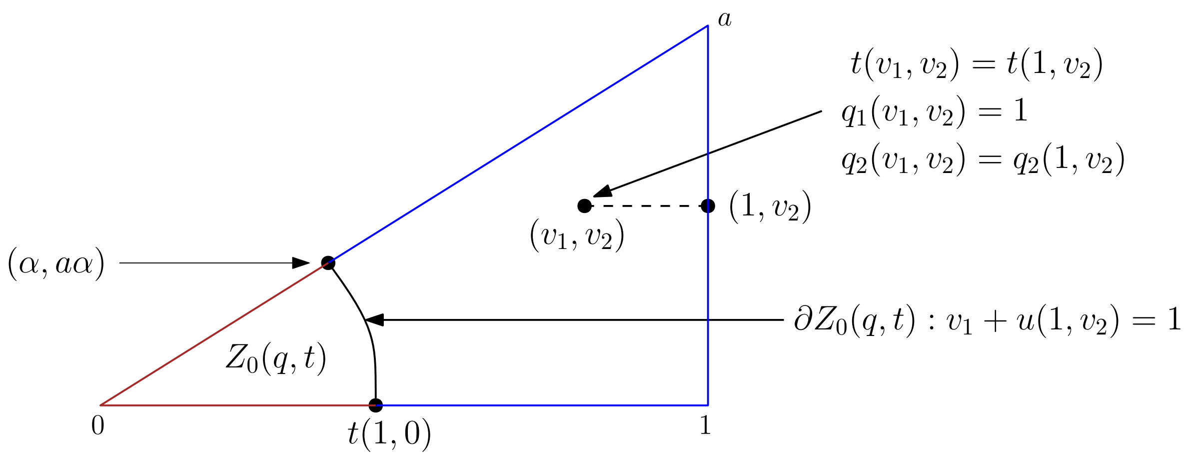


Figure 1: A line mechanism

For any line mechanism (q, t), define

$$\alpha \;\equiv\; \{x \in [0,1] : u(1, ax) - (1 - x) = 0\} \tag{3}$$

That α exists and is unique follows from $1 + u(1, a) \geq 1$, $u(1, 0) \leq 1$, and the fact that $x + u(1, ax)$ is strictly increasing and continuous in x. The dependence of α on (q, t) is suppressed in the notation. From (3) we have

$$\alpha = 1 - u(1, a\alpha) \leq 1 - u(1, 0) = t(1, 0) \tag{4}$$

The upper boundary of $Z_0(q, t)$ is

$$\partial Z_0(q,t) := \{(v_1, v_2) : u(1, v_2) - (1 - v_1) = 0\}$$

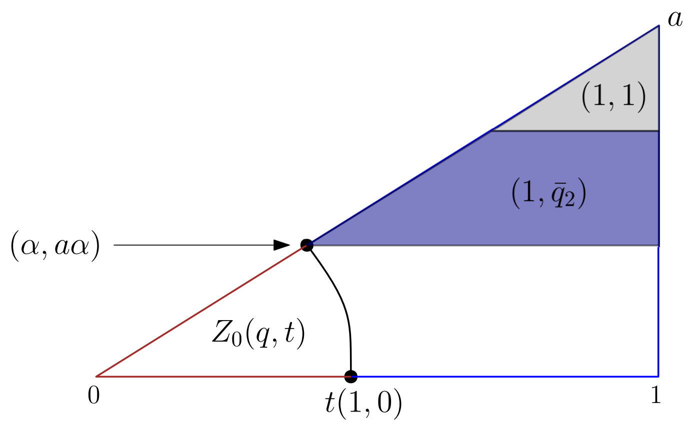


Figure 2: A constrained line mechanism

Note that $\partial Z_0(q,t)$ is a curve with slope $-\frac{1}{q_2(1,v_2)}$ (see Lemma 8 in Appendix A.2 for a proof) that connects the points $(t(1,0),0)$ and $(\alpha, a\alpha)$. A deterministic mechanism is a line mechanism in which $\partial Z_0(q,t)$ is piecewise linear with (at most) two line segments, one vertical and the other with slope -1.

The next result further limits the search for an optimal mechanism to a subset of line mechanisms defined next. For a line mechanism (q,t), let

$$\bar{q}_2 := \begin{cases} \sup_{v_2 < a\alpha} \left[q_2(1,v_2) \right], & \text{if } \ \alpha > 0 \\ 0, & \text{if } \ \alpha = 0 \end{cases}$$

Definition 2 *A line mechanism (q,t) is a* **constrained line mechanism** *if*
either (i) $q_2(1,a\alpha) = \bar{q}_2$ and for all $v_2 > a\alpha$, $q_2(1,v_2) \in \{\bar{q}_2, 1\}$
or (ii) $q_2(1,a\alpha) = 1$.

If $\bar{q}_2 = 1$ then (i) and (ii) mean the same thing. Thus, in a constrained line mechanism for $v_2 \geq a\alpha$ the probability of allocating the second unit takes at most two values, one of which may be less than 1 (see Figure 2); if there is a discontinuity in $q_2(1,v_2)$ at $v_2 = a\alpha$, then $q_2(1,a\alpha) = 1$. However, for $v_2 < a\alpha$ any (incentive compatible) value for q_2 is possible.

Lemma 3 *If the density function f satisfies Condition SC-H, then there exists an optimal mechanism which is a constrained line mechanism.*

Lemma 3 is proved by showing that there is an optimal mechanism which is an extreme point of a convex, compact subset of line mechanisms. As we are maximizing a linear function on this subset, the maximum is attained at an extreme point. Every extreme point of the subset is a constrained line mechanism.

REMARK 2: From Definition 2, we conclude that if in a constrained line mechanism $\bar{q}_2 = 0$ then the mechanism is deterministic as $q_2(1, v_2) = 0,\ \forall v_2 < a\alpha$ and $q_2(1, v_2) \in \{0, 1\},\ \forall v_2 \geq a\alpha$. Therefore, in the sequel we restrict attention to constrained line mechanisms in which $\bar{q}_2 > 0$. This, and the definition of $\bar{q}_2$, implies $\alpha > 0$ and therefore (4) implies $t(1, 0) > 0$.

2.3 Optimality of Deterministic Mechanisms

In this section, we provide conditions under which there is an optimal mechanism that is deterministic.

It is useful to split the expected revenue of a constrained line mechanism (q, t) into two parts,

$$\text{REV}(q,t) = \text{REV}^{\alpha-}(q,t) + \text{REV}^{\alpha+}(q,t)$$

where $\text{REV}^{\alpha-}(q,t)$ is the expected revenue from types $v_2 \leq a\alpha$ and $\text{REV}^{\alpha+}(q,t)$ is the expected revenue from types $v_2 > a\alpha$. Define for every (v_1, v_2),

$$\Phi(v_1, v_2) \quad := \quad f(1, v_2) - \int_{v_1}^{1} \Big[3f(x, v_2) + (x, v_2) \cdot \nabla f(x, v_2)\Big] dx$$

Note that the function Φ depends only on f and not on any mechanism. The role of Φ is discussed after the next lemma.

Lemma 4 *If (q, t) is a constrained line mechanism, then*

$$\text{REV}(q,t) = \text{REV}^{\alpha-}(q,t) + \text{REV}^{\alpha+}(q,t),$$

where

$$\text{REV}^{\alpha-}(q,t) := \int_{0}^{a\alpha} \int_{1-u(1,v_2)}^{1} \Phi(v_1, v_2) dv_1 dv_2 \tag{5}$$

$$\text{REV}^{\alpha+}(q,t) := \int_{a\alpha}^{a} \int_{\frac{v_2}{a}}^{1} \Phi(v_1, v_2) dv_1 dv_2 - \int_{a\alpha}^{a} (1 - \frac{v_2}{a} - u(1, v_2)) \Phi(\frac{v_2}{a}, v_2) dv_2 \tag{6}$$

As noted immediately after Proposition 1, decreasing $u(v_1, v_2)$ when $v_1 < 1$ and increasing $u(1, v_2)$ increases expected revenue (provided Condition SC-H is satisfied). These decoupled changes in u are not possible in constrained line mechanisms. Whenever $u(1, v_2)$ is increased, $u(v_1, v_2)$ either increases or stays the same (see (2)). Thus, in a constrained line mechanism the net change in expected revenue by increasing $u(1, v_2)$, and the consequent increase in $u(v_1, v_2)$, may be positive or negative. This trade-off is captured by the function Φ (which we reiterate is independent of the mechanism and is only a function of the density).

To see the role of Φ, consider a constrained line mechanism (q, t) with buyer payoff $u(1, \cdot)$ for types in Y. First, consider $v_2 \leq a\alpha$. Differentiating $\text{Rev}^{\alpha-}(q, t)$ with respect to $u(1, v_2)$, we see from (5) that if

$$\Phi(1 - u(1, v_2), v_2) > 0$$

then increasing $u(1, v_2)$ increases expected revenue. This is the process of "straightening" described later. If, instead,

$$\Phi(1 - u(1, v_2), v_2) < 0$$

then decreasing $u(1, v_2)$ increases expected revenue. This is the process of "covering" a mechanism described later. The single-crossing property SC-V, introduced below, allows changes in $u(1, v_2)$ for a range of $v_2 \leq a\alpha$ in a manner that preserves incentive compatibility.

Similarly, differentiating $\text{Rev}^{\alpha+}(q, t)$ with respect to $u(1, v_2)$, we see from (6) that for $v_2 > a\alpha$ if $u(1, v_2)$ is increased [decreased] when $\Phi(\frac{v_2}{a}, v_2) > 0$ [$\Phi(\frac{v_2}{a}, v_2) < 0$], then the expected revenue increases. The single-crossing property SC-D, introduced below, allows changes in $u(1, v_2)$ for a range of $v_2 > a\alpha$ in a manner that preserves incentive compatibility.

Thus, Φ indicates the direction of revenue improvements, if any, for an arbitrary mechanism. Consider the following single-crossing properties of Φ in the horizontal, vertical, and diagonal directions in the type space:

Definition 3 *The density function f satisfies* **Condition SC** *if*

SC-H: Φ is increasing in v_1

SC-V: for every v_1, $\Phi(v_1, \cdot)$ crosses zero at most once (from above). That is, for all (v_1, v_2)

$$\Big[\Phi(v_1, v_2) > 0\Big] \implies \Big[\Phi(v_1, v_2') > 0, \ \forall\ v_2' < v_2\Big]$$

SC-D: for every v_2, $\int_{v_2}^{a} \Phi(\frac{y}{a}, y)dy$ crosses zero at most once (from below). That is, for all v_2

$$\Big[\int\limits_{v_2}^{a} \Phi(\frac{y}{a}, y)dy \geq 0\Big] \implies \Big[\int\limits_{v_2'}^{a} \Phi(\frac{y}{a}, y)dy \geq 0, \; \forall \; v_2' > v_2\Big]$$

Note that SC-H is equivalent to $3f(v) + v \cdot \nabla f(v) \geq 0$ for all v. Further, SC-V is implied if Φ is decreasing in v_2 and SC-D is implied if $\Phi(\frac{y}{a}, y)$ satisfies single crossing. In Section 2.5, we provide a class of distributions that satisfy Condition SC.

These restrictions on the prior yield our main result.

Theorem 1 *If the density function f satisfies Condition SC, then there is an optimal mechanism that is deterministic.*

The proof consists of two steps.

- Step 1. SC-H implies that there is an optimal mechanism that is a constrained line mechanism (Lemma 3). In a constrained line mechanism, $q_2(1, v_2)$ takes at most two values for $v_2 \geq a\alpha$ but may take any number of values for $v_2 < a\alpha$. Under SC-H and SC-V, Proposition 2 in Section 2.3.1 shows that there is an optimal mechanism which is *semi-deterministic*; that is, a constrained line mechanism in which $q_2(1, v_2)$ takes at most three values for $v_2 \in [0, a]$, and only one of these three values is strictly between 0 and 1.[11]

- Step 2. If SC holds, a deterministic line mechanism is optimal in the class of semi-deterministic line mechanisms, completing the proof of Theorem 1.

In the next section, we explain Step 1 in some detail. The proof of Step 2 is in Appendix A.3.

2.3.1 Optimality of Semi-deterministic Mechanisms

We show that under SC-H and SC-V there exists an optimal mechanism that is semi-deterministic.

[11]In a deterministic mechanism, for any v_2, $q_2(1, v_2)$ is either 0 or 1.

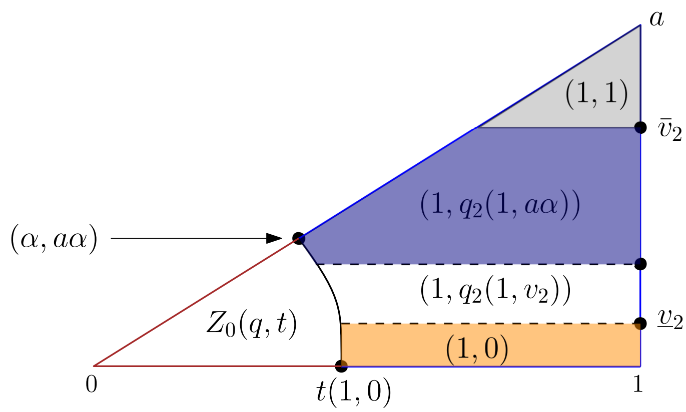


Figure 3: $\underline{v}_2$ and $\bar{v}_2$ in a constrained line mechanism

For a constrained line mechanism (q,t), define

$$\begin{aligned}\underline{v}_2 &:= \inf\{v_2 \in [0,1] : q_2(1,v_2) > 0\} \\ \bar{v}_2 &:= \sup\{v_2 \in [0,1] : q_2(1,v_2) < 1\}\end{aligned} \tag{7}$$

As IC implies that $q_2(1,v_2)$ is increasing in v_2, we have $\underline{v}_2 \le \bar{v}_2$ with equality only if (q,t) is a deterministic mechanism. Figure 3 illustrates $\underline{v}_2$ and $\bar{v}_2$ for the case $\underline{v}_2 < a\alpha < \bar{v}_2$.

Consider the following definition:

Definition 4 *A constrained line mechanism* (q^s,t^s) **straightens** *another constrained line mechanism* (q,t) *at* $\underline{v}_2^s \in (\underline{v}_2, a\alpha]$[12] *if*

$$\begin{aligned}u^s(1,v_2) &= u(1,\underline{v}_2^s), &\quad \forall\, v_2 \le \underline{v}_2^s \\ u^s(1,v_2) &= u(1,v_2), &\quad \forall\, v_2 \ge \underline{v}_2^s.\end{aligned}$$

Note that (q^s,t^s) is completely specified by $u^s(1,\cdot)$. Moreover,

$$\begin{aligned}t^s(1,0) &= 1 - u(1,\underline{v}_2^s) \\ q_2^s(1,v_2) &= \begin{cases}0, & \text{if } v_2 < \underline{v}_2^s \\ q_2(1,v_2), & \text{if } v_2 \ge \underline{v}_2^s\end{cases}\end{aligned}$$

as illustrated in Figure 4. By construction, $\alpha = \alpha^s$ and $u^s(1,v_2) \ge u(1,v_2)$ for all $v_2 < \underline{v}_2^s$. In a straightening, the payoff of types $(1,v_2)$, $v_2 < \underline{v}_2^s$ increases compared to the payoff in (q,t). Therefore, the price of the first unit is lower, $t^s(1,0) < t(1,0)$, and a buyer with $v_2 < v_2^s$ is never allocated a second unit. Consequently, $Z_0(q^s,t^s) \subsetneq Z_0(q,t)$.

[12]By Remark 2, we may assume that for small positive ϵ, $q_2(1,a\alpha-\epsilon) > 0$. Therefore, $\underline{v}_2 < a\alpha$.

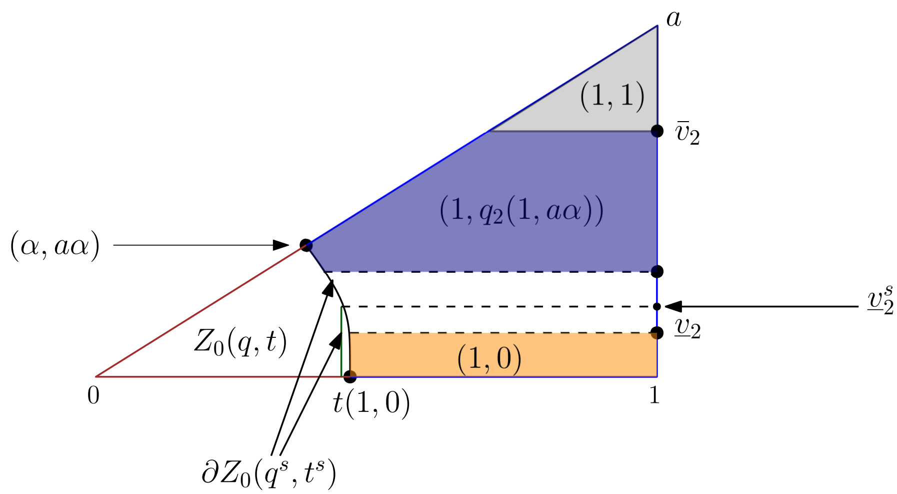


Figure 4: Straightening a line mechanism

Lemma 5 *Suppose that the density function f satisfies Conditions SC-H and SC-V. Consider a constrained line mechanism (q,t). If*

$$\Phi(t(1,0), \underline{v}_2) > 0,$$

then there exists a straightening (q^s, t^s) of (q,t), such that $\textsc{Rev}(q^s, t^s) > \textsc{Rev}(q,t)$.

Consider the following definition.

Definition 5 *A constrained line mechanism (q,t) is* **semi-deterministic** *if*

$$\Big(q_1(v), q_2(v)\Big) \in \Big\{(0,0), (1,0), (1, q_2(1, a\alpha)), (1,1)\Big\} \qquad \forall\ v \in D$$

Thus, in a semi-deterministic mechanism the menu size is no more than four.

By Lemma 5, $\Phi(t(1,0), \underline{v}_2) \leq 0$ is a necessary condition for an optimal constrained line mechanism (q,t). Under this condition, we show that the revenue of any constrained line mechanism is no more than the revenue of its semi-deterministic cover, defined below.

Definition 6 *A mechanism (q^c, t^c) is a* **cover** *of a constrained line mechanism (q,t) if (q^c, t^c) is semi-deterministic and*

$$\begin{aligned} \alpha^c &= \alpha \qquad (8)\\ t^c(1,0) &= t(1,0) \end{aligned}$$

$$
\begin{aligned}
u^c(1,v_2) &\leq u(1,v_2), & \forall\, v_2 < a\alpha \\
u^c(1,v_2) &= u(1,v_2), & \forall\, v_2 \geq a\alpha
\end{aligned} \tag{9}
$$

In a cover, the payoff of types $(1, v_2)$, $v_2 < a\alpha$ decreases compared to the payoff in (q, t). Several implications of the definition are worth noting. First, $t^c(1, 0) = t(1, 0)$ implies $u^c(1, 0) = u(1, 0)$. Second, (2) and (9) imply that $u^c(v_1, v_2) = u(v_1, v_2)$ for all $v_2 \geq a\alpha$. Therefore, (8) and (9) imply $q_2^c(1, a\alpha) = q_2(1, a\alpha)$. As (q^c, t^c) is semi-deterministic, we have $q_2^c(1, v_2) \in \{0, q_2(1, a\alpha), 1\}$ for all v_2. Thus, if $q_2(1, a\alpha) = 1$, then its cover is deterministic.

Figure 5 shows the boundaries of type sets where 0, 1, $1 + q_2(1, a\alpha)$ and 2 units are sold in the cover (q^c, t^c) of a constrained line mechanism (q, t). Observe that (q^c, t^c) is the most generous out of all semi-deterministic mechanisms that never allocate a unit when (q, t) does not. That is, among all semi-deterministic mechanisms (q', t') such that if $q_i(v) = 0$ for any v, then $q_i'(v) = 0$, we have $q_i^c(v) \geq q_i'(v)$.

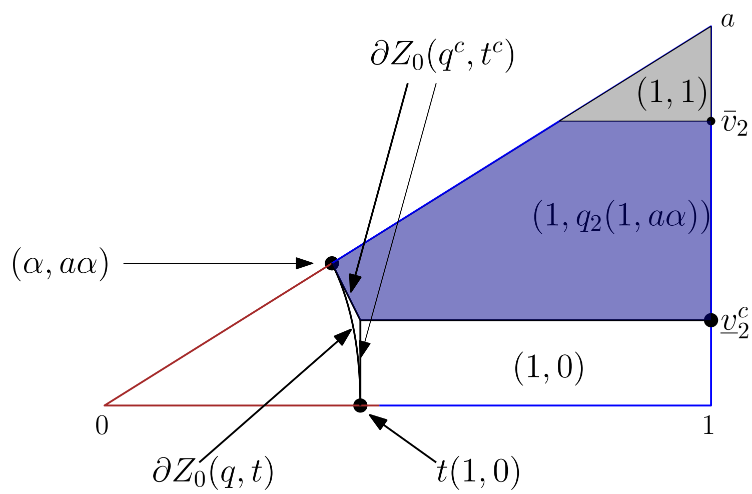


Figure 5: Cover of a constrained line mechanism

The cover, (q^c, t^c), of a constrained line mechanism, (q, t), is constructed from $Z_0(q, t)$ using two line segments: (i) a straight line with slope $-\frac{1}{q_2(1,a\alpha)}$ at $(\alpha, a\alpha)$ and (ii) a vertical line at $(t(1, 0), 0)$. The convexity of $Z_0(q, t)$ (see Lemma 8 in Appendix A.2) and the fact that the slope of $Z_0(q, t)$ equals $-\frac{1}{q_2(1,v_2)}$, which is less than or equal to $-\frac{1}{q_2(1,a\alpha)}$, ensures that $Z_0(q, t) \subseteq Z_0(q^c, t^c)$ and therefore $u(1, v_2) \geq u^c(1, v_2), \forall v_2 < a\alpha$. This construction gives us

$$
(q_1^c(1,v_2), q_2^c(1,v_2)) \;:=\; \begin{cases} (1,0), & \text{if } v_2 < \underline{v}_2^c \\ (1, q_2(1,a\alpha)), & \text{if } \underline{v}_2^c \leq v_2 < \bar{v}_2 \\ (1,1), & \text{otherwise.} \end{cases}
$$

where $\underline{v}_2^c := \frac{1}{q_2(1,a\alpha)}\Big[(1+aq_2(1,a\alpha))\alpha - t(1,0)\Big]$. Note that $\bar{v}_2^c = \bar{v}_2$.

Lemma 6 *Suppose that the density function f satisfies Conditions SC-H and SC-V. Consider a constrained line mechanism (q,t). If*

$$\Phi(t(1,0), \underline{v}_2) \quad \leq \quad 0, \tag{10}$$

then $\textsc{Rev}(q^c,t^c) \geq \textsc{Rev}(q,t)$ *where (q^c,t^c) is the cover of (q,t).*

This leads to the main result of this section.

Proposition 2 *Suppose that the density function f satisfies Conditions SC-H and SC-V. Then there exists an optimal mechanism that is semi-deterministic.*

Proof: Condition SC-H and Lemma 3 imply that there is an optimal mechanism (q,t) which is a constrained line mechanism. Therefore, Lemma 5 implies that (10) is satisfied for (q,t). Let (q^c,t^c) be the (semi-deterministic) cover of (q,t). Hence, by Lemma 6

$$\textsc{Rev}(q^c,t^c) \quad \geq \quad \textsc{Rev}(q,t) \tag{11}$$

Hence, (q^c,t^c) is an optimal mechanism that is semi-deterministic. ■

Proposition 2 is used in the proof of Theorem 1 in Appendix A.3.

2.4 Necessary Conditions for a Deterministic Optimal Mechanism

We provide necessary conditions for a deterministic mechanism to be optimal in the class of all deterministic mechanisms. If Condition SC is satisfied, then these conditions are necessary for optimality of a deterministic mechanism in the class of all mechanisms.

A deterministic mechanism is described by prices p_1 and p_2 for the two units. If at optimality $ap_1 \leq p_2$ then the derivation of optimal prices is straightforward. The optimal price p_i^*, $i = 1,2$ is the price that maximizes the profit of a seller who sells one unit to a buyer whose valuation for the object has probability density equal to the marginal density of v_i.

Therefore, assume that $ap_1 > p_2$ and that p_1, p_2 are in the interior of the domain D, i.e., $p_1 \in (0,1),\ p_2 \in (0, ap_1)$.[13] The following necessary conditions are implied.

Proposition 3 *If (p_1^*, p_2^*) are optimal prices in the interior of the domain D, then*

$$\int_0^{p_2^*} \Phi(p_1^*, v_2) dv_2 = 0 \tag{12}$$

$$\int_{p_2^*}^{a\alpha^*} \Phi((1+a)\alpha^* - v_2, v_2) dv_2 + \int_{a\alpha^*}^{a} \Phi(\frac{v_2}{a}, v_2) dv_2 = 0 \tag{13}$$

Further, $\Phi(p_1^*, p_2^*) \leq 0$ and $\Phi(p_1^*, 0) \geq 0$.

2.5 Ordered Decreasing Values Model

We describe a model in which valuations are based on the order statistics of two draws from the same distribution. Let X_1, X_2 be two i.i.d. random variables with cdf $G(\cdot)$ and density function $g(\cdot)$ that is strictly positive on its support $[0,1]$. Let

$$v_1 = \max\{X_1, X_2\}, \quad v_2 = a \min\{X_1, X_2\}$$

Thus $av_1 \geq v_2$. We call this an **ordered decreasing values** model. Note that

$$f(v_1, v_2) = \frac{2}{a} g(v_1) g(\frac{v_2}{a}), \qquad 1 \geq v_1 \geq \frac{v_2}{a} \geq 0$$

This model is a natural generalization of the maximum game of Bulow and Klemperer (2002) to two objects. In Bergemann et al. (2020)'s interpretation of the maximum game, the X_i's represent the values from the different ways of using the object; the buyer will put the object to its best possible use. A similar interpretation applies to the ordered decreasing values model, where, if the buyer obtains one unit of the object, she will deploy it in its best usage and if she obtains two units, she will deploy them in the two best usages.

Another interpretation is that the buyer in the ordered decreasing values model is an intermediary who resells the units to two final consumers. The seller does not have access

[13] The assumption that $p_1 \leq 1$ is without loss of generality because for every deterministic mechanism (p_1, p_2) with $p_1 > 1$, the prices $(\hat{p}_1, \hat{p}_2)$, with $\hat{p}_1 = 1$ and $\hat{p}_2 = p_1 + p_2 - 1$, yield the same expected revenue as (p_1, p_2).

to the final consumers and can only sell the units to the intermediary. The final consumers have unit demand, their values are i.i.d. and known to the intermediary. If the intermediary purchases only one unit, she will resell it to the final consumer with a higher value.[14]

Let $\eta(x) := \frac{x}{g(x)} \frac{dg(x)}{dx}$ be the elasticity of g. For every (v_1, v_2), define

$$W(v_1, v_2) := v_1 - \frac{1 - G(v_1)}{g(v_1)} \left[2 + \eta(\frac{v_2}{a})\right]$$
$$W_{min}(v_2) := v_2 - \frac{1 - G_{min}(v_2)}{g_{min}(v_2)},$$

where $G_{min}(v_2) = 1 - (1 - G(v_2))^2$ is the cumulative distribution function of the minimum of two independent random variables drawn from G.

Definition 7 *For every v_1, $W(v_1, \cdot)$* **crosses zero at most once (from above)** *if for every v_2*

$$\left[W(v_1, v_2) > 0\right] \Longrightarrow \left[W(v_1, v_2') > 0 \ \forall \ v_2' < v_2\right]$$

W_{min} **crosses zero at most once (from below)** *if for every v_2*[15]

$$\left[W_{min}(v_2) \geq 0\right] \Longrightarrow \left[W_{min}(v_2') \geq 0 \ \forall \ v_2' > v_2\right]$$

The following proposition gives sufficient conditions for the ordered decreasing model to satisfy Condition SC.

Proposition 4 *In an ordered decreasing model,*

SC-H is satisfied if and only if $\eta(x) \geq -\frac{3}{2}$.

SC-V is satisfied if and only if $W(v_1, \cdot)$ crosses zero at most once for all v_1.

SC-D is satisfied if and only if W_{min} crosses zero at most once.

Examples of densities that satisfy the sufficient conditions of Proposition 4 include the uniform family $g(x) = \alpha x^{\alpha - 1}$ with $\alpha > 0$, $g(x) = \frac{e^x}{e-1}$, and some Beta distributions.

[14]These two interpretations assume that $a = 1$.

[15]Note that this is the condition that density g_{min} satisfies the usual regularity condition. It is satisfied if g has increasing hazard rate.

Example 2 Uniform Distribution

We describe the optimal mechanism for a uniform distribution on the domain $D = [0,1] \times [0,a]$. The density is

$$f(v_1, v_2) \quad = \quad \begin{cases} \frac{2}{a}, & \text{if } 1 \geq v_1 \geq \frac{v_2}{a} \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

It may be verified that there is an optimal solution (p_1, p_2) where prices satisfy $ap_1 > p_2$. For a uniform distribution, $\Phi(v_1, v_2) = \frac{6v_1 - 4}{a}$ which does not depend on v_2. Thus, $\Phi(2/3, v_2) = 0$ for any value of v_2. Consequently $p_1^* = 2/3$ is the only solution to (12) and it does not depend on the value of a. The unique prices in the interior of D that satisfy necessary conditions (12) and (13) for an internal optimal solution are[16]

$$p_1^* = \frac{2}{3}, \quad p_2^* = \frac{1}{3}\left(2a - \sqrt{a(1+a)}\right) \qquad \text{Unbundled prices}$$

A direct calculation reveals that the unique prices on the boundary of D that are a candidate for an optimal solution are

$$p_1^* = \sqrt{\frac{1+a}{3}}, \quad p_2^* = 0 \qquad \text{Bundle price}$$

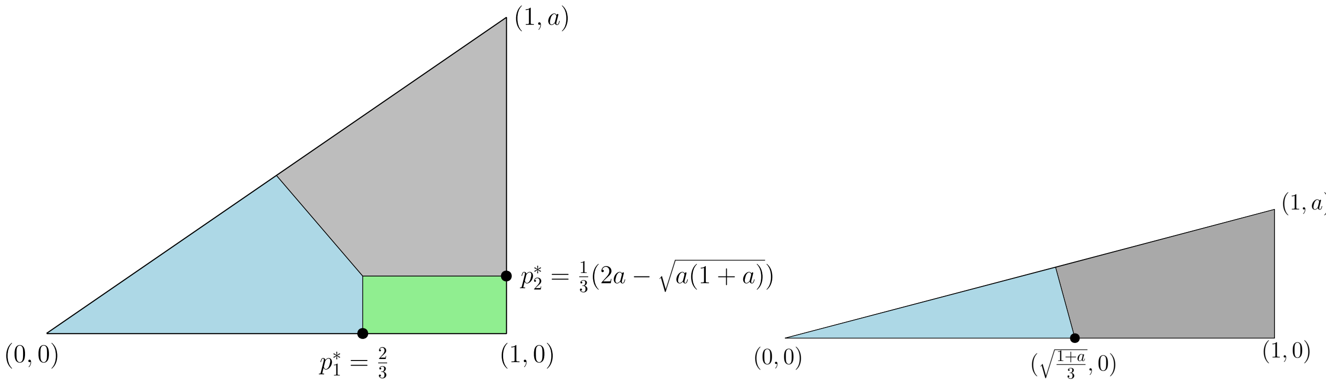
As we are maximizing a continuous function on a compact set, an optimal solution exists, Therefore, one of these two sets of prices is optimal. A calculation reveals that if $a > \frac{1}{3}$ then the optimal prices (i.e., optimal mechanism among all deterministic mechanisms) are the unbundled prices above. If, instead, $a < \frac{1}{3}$ then it is optimal to sell the two units as a bundle at the price $\sqrt{\frac{1+a}{3}}$.

In the limit as $a \to 0$, the buyer has positive value for one object only with density $f(v_1) = 2v_1$. The limit of the optimal bundling price as $a \to 0$ is $\sqrt{\frac{1}{3}}$, which is the optimal price for selling one object to a buyer with density $f(v_1) = 2v_1$.

The optimal prices are shown in Figure 6a for $a \geq \frac{1}{3}$ and in Figure 6b for $a < \frac{1}{3}$.

That there is no random mechanism that yields greater expected revenue than these deterministic mechanisms follows from our results. First, note that the uniform model is an ordered decreasing values model with $v_1 = \max\{X_1, X_2\}$ and $v_2 = a \min\{X_1, X_2\}$, where X_i are i.i.d. uniform on $[0,1]$. The uniform density on $[0,1]$ has elasticity 0 and has increasing

[16] Armstrong (2016) shows that these are optimal prices for the case $a = 1$.



(a) $a \geq \frac{1}{3}$.

(b) $a < \frac{1}{3}$.

Figure 6: Optimal mechanism for uniform distribution.

hazard rate. Thus, the single-crossing conditions in Definition 7 are satisfied. By Proposition 4, Condition SC is satisfied and by Theorem 1 there is a deterministic mechanism that is optimal. □

3 Increasing Marginal Values

We begin with an example with identical objects and increasing marginal values in which it is optimal to randomize.

Example 3 The probability distribution of values is in the table below:

Type	(v_1, v_2)	**Probability**
A	$(6, 8)$	0.1
B	$(3, 12)$	0.9

There are two candidates for an optimal deterministic mechanism:

D1. Bundle the two units at a bundle price to 14. Each type buys the bundle, yielding an expected revenue of 14.

D2. Offer unit 1 at price 6 (for type A) and the bundle at price 15 (for type B). This yields a revenue of $6 \times 0.1 + 15 \times 0.9 = 14.1 > 14$.

Hence, D2 is the optimal deterministic mechanism with a revenue of 14.1.

Price	Package
12	$(1, 0.75)$, i.e., unit 1 with probability 1 and unit 2 with probability 0.75
15	(1,1), i.e. both units with probability 1

Consider the following randomized mechanism.
Type A chooses the first item in the menu and pays 12. Type B is indifferent between both items and chooses the bundle at price 15. The expected revenue is $12 \times 0.1 + 15 \times 0.9 = 14.7 > 14.1$. Hence, there is no optimal mechanism that is deterministic. $\square$

The departure from the earlier model is that the domain of v is the following triangle

$$D := \{(v_1, v_2) \in [0, a] \times [0, 1] : v_1 \leq av_2\}.$$

The definitions of a mechanism and its properties remain as in section 2. In particular, the constraint $q_1(v) \geq q_2(v),\ \forall v$ is imposed. The density function f has support D and is assumed to be absolutely continuous.

The counterpart of Proposition 1 is the following.

Proposition 5 *If the density function f satisfies Condition SC-H, then there exists an optimal mechanism in which $q_1(v) = q_2(v)$ for all v.*

There is an optimal mechanism in which probability of selling each unit is the same.[17] In other words, the seller bundles the two units and sells them as one object. Hence, Riley and Zeckhauser (1983) and Myerson (1981) imply the following:

Theorem 2 *If the density function f satisfies Condition SC-H, then it is optimal to bundle the two units and sell them at a take-it-or-leave-it price.*

Thus, a deterministic mechanism is optimal in this model under weaker conditions than in the model with decreasing marginal values.

Let T be the cdf and τ the density of $w \equiv v_1 + v_2$. If T is regular in the sense of Myerson (1981), then the optimal bundle price solves $B = \frac{1-T(B)}{\tau(B)}$.

[17]If there are n units, then the conclusion of Proposition 5 generalizes to "there exists an optimal mechanism in which $q_{n-1}(v) = q_n(v)$ for all v."

Ordered Increasing Values Model

This is the counterpart of the order decreasing values model. Let X_1, X_2 be two i.i.d. random variables with density $g(\cdot)$ that is strictly positive on its support $[0, 1]$. Let $v_1 = a\min\{X_1, X_2\}$ and $v_2 = \max\{X_1, X_2\}$. Thus $v_1 \leq av_2$. The following is a sufficient condition for the regularity of T.

Lemma 7 *Let v_1 and v_2 be from an ordered increasing values model. If g, the density of X_i, has increasing hazard rate and $\eta(x) \geq -\frac{3}{2}$, then T is regular.*[18]

Hence, Theorem 2 implies that the optimal mechanism for an ordered increasing values model is to the sell the two units as a bundle under the conditions on g specified in Lemma 7.

A Two-period Model

As with decreasing marginal values, the results here also apply when there is an order in which two objects can be sold. Consider a two-period model in which a durable product may be sold either at the beginning of the first period or at the beginning of the second period. If the product is sold in the first period, the buyer consumes it in both periods. If, instead, it is bought in the second period, then only second-period consumption is possible.

It is convenient to number time by the number of periods left, including the current period. Thus, the *first period is period* 2 *and second period is period* 1. If the buyer purchases the product in period 2, she also consumes it in periods 2 and 1.

The buyer's values are v_1 for consumption in period 1 (the latter period) and v_2 for consumption in period 2 (the earlier period). The restriction $v_1 \leq v_2$ follows from discounting. Values for both the periods are known to the buyer at the beginning (no dynamics).

An allocation rule Q determines two things: $Q_1(v)$, the probability of selling the product in period 1 (the latter period), and $Q_2(v)$, the probability of selling the product in period 2 (the earlier period), with the natural restriction that $Q_1(v)+Q_2(v) \leq 1$. The expected value to buyer type (v_1, v_2) from this allocation rule is

$$(v_1 + v_2)Q_2(v) + v_1Q_1(v) = v_1[Q_1(v) + Q_2(v)] + v_2Q_2(v)$$

Define $q_1(v) := Q_1(v)+Q_2(v)$ and $q_2(v) := Q_2(v)$. So, $q_1(v)$ is the probability with which the buyer consumes the product in the first period only and $q_2(v)$ is the probability with which the buyer consumes the product in both periods, with $q_1(v), q_2(v) \in [0, 1]$ and $q_1(v) \geq q_2(v)$.

[18] Recall that η is the elasticity of g.

4 Discussion

There are several directions we hope to explore in future work. An obvious one is generalizing the results to more than two units. Proposition 1 generalizes to the sale of $n > 2$ units as follows. Suppose that there are n units for sale with $D = \{(v_1, v_2, \ldots, v_n) | 0 \leq v_1 \leq 1,\ 0 \leq v_i \leq a_i v_{i-1},\ i \geq 2\}$. If the inequality in Condition SC-H is changed to

$$(n+1)f(v) + v \cdot \bigtriangledown f(v) \geq 0, \qquad \text{for all } v \in D,$$

then it is optimal to sell the first unit deterministically. A generalization of condition SC would be required to obtain a fully deterministic optimal mechanism.

Another direction to build on our results would be to obtain optimal dominant-strategy incentive compatible auctions. As noted on Remark 1, Proposition 1 generalizes to allow for positive seller costs. With a modification of the definition of Φ to include seller costs, Theorem 1 also generalizes. With two buyers, the seller's cost for providing a unit to buyer A is the lost revenue from buyer B. This may be useful in constructing optimal auctions.

The strategy of proofs developed in this paper may be useful in other models. Our preliminary investigations indicate that the approach used here can be adapted to some settings with heterogenous objects.

A Appendix

A.1 Proofs of Section 2.1

Proof of Lemma 1: From (1), the seller's expected revenue is

$$\begin{aligned}
\textsc{Rev}(q,t) &= \int_D \Big[\bigtriangledown u(v)\cdot v - u(v)\Big] f(v)dv \\
&= \int_0^a \int_{\frac{v_2}{a}}^1 \Big[\bigtriangledown u(v_1,v_2)\cdot(v_1,v_2) - u(v_1,v_2)\Big] f(v_1,v_2)dv_1 dv_2 \\
&= \int_0^1 \int_0^{av_1} \Big[\bigtriangledown u(v_1,v_2)\cdot(v_1,v_2) - u(v_1,v_2)\Big] f(v_1,v_2)dv_2 dv_1
\end{aligned}$$

Observe that

$$\begin{aligned}
\int_{\frac{v_2}{a}}^{1} \frac{\partial u(v)}{\partial v_1} v_1 f(v)dv_1 &= v_1 u(v) f(v)\Big|_{\frac{v_2}{a}}^{1} - \int_{\frac{v_2}{a}}^{1} u(v)\Big[f(v) + v_1\frac{\partial f(v)}{\partial v_1}\Big]dv_1 \\
&= u(1,v_2)f(1,v_2) - \frac{v_2}{a}u(\frac{v_2}{a},v_2)f(\frac{v_2}{a},v_2) \\
&\quad - \int_{\frac{v_2}{a}}^{1} u(v)\Big[f(v) + v_1\frac{\partial f(v)}{\partial v_1}\Big]dv_1
\end{aligned}$$

$$\begin{aligned}
\implies \int_0^a \int_{\frac{v_2}{a}}^{1} \frac{\partial u(v)}{\partial v_1} v_1 f(v)dv_1 dv_2 &= \int_0^a u(1,v_2)f(1,v_2)dv_2 - \int_0^a \frac{v_2}{a}u(\frac{v_2}{a},v_2)f(\frac{v_2}{a},v_2)dv_2 \\
&\quad - \int_0^a \int_{\frac{v_2}{a}}^{1} u(v)\Big[f(v) + v_1\frac{\partial f(v)}{\partial v_1}\Big]dv_1 dv_2
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_0^{av_1} \frac{\partial u(v)}{\partial v_2} v_2 f(v)dv_2 &= v_2 u(v) f(v)\Big|_0^{av_1} - \int_0^{av_1} u(v)\Big[f(v) + v_2\frac{\partial f(v)}{\partial v_2}\Big]dv_2 \\
&= av_1 u(v_1,av_1)f(v_1,av_1) - \int_0^{av_1} u(v)\Big[f(v) + v_2\frac{\partial f(v)}{\partial v_2}\Big]dv_2
\end{aligned}$$

$$\implies \int_0^1 \int_0^{av_1} \frac{\partial u(v)}{\partial v_2} v_2 f(v)dv_2 dv_1 = \int_0^1 av_1 u(v_1,av_1)f(v_1,av_1)dv_1$$

$$- \int_0^1 \int_0^{av_1} u(v) \Big[f(v) + v_2 \frac{\partial f(v)}{\partial v_2} \Big] dv_2 dv_1$$

By a change of variable $v_2 = av_1$, we have

$$\int_0^1 av_1 u(v_1, av_1) f(v_1, av_1) dv_1 = \int_0^a \frac{v_2}{a} u(\frac{v_2}{a}, v_2) f(\frac{v_2}{a}, v_2) dv_2$$

Thus,

$$\int_D [\triangledown u(v) \cdot v] f(v) dv = \int_0^a u(1, v_2) f(1, v_2) dv_2 - \int_0^a \int_{\frac{v_2}{a}}^1 u(v) \Big[2f(v) + v \cdot \triangledown f(v) \Big] dv_2 dv_1$$

and

$$\textsc{Rev}(q,t) = \int_0^a u(1, v_2) f(1, v_2) dv_2 - \int_0^a \int_{\frac{v_2}{a}}^1 u(v) \Big[3f(v) + v \cdot \triangledown f(v) \Big] dv_1 dv_2$$

■

Proof of Proposition 1: Condition SC-H and Lemma 1 imply that if u is modified to $\hat{u}$ (while maintaining IC and IR) such that

$$\hat{u}(1, v_2) \geq u(1, v_2), \quad \forall\, v_2 \quad \text{and} \quad \hat{u}(v_1, v_2) \leq u(v_1, v_2), \quad \forall (v_1, v_2) \text{ s.t. } v_1 < 1 \tag{14}$$

then $\textsc{Rev}(\hat{q}, \hat{t}) \geq \textsc{Rev}(q, t)$.

Let (q, t) be any IC and IR mechanism. WLOG, assume that $q(0, 0) = (0, 0),\ t(0, 0) = 0$. Let $Y = \{(1, v_2) : v_2 \leq a\}$. Define

$$\begin{aligned} \hat{q}_1(1, v_2) &= 1, \qquad \hat{q}_2(1, v_2) = q_2(1, v_2) \\ \hat{t}(1, v_2) &= t(1, v_2) + (1 - q_1(1, v_2)) \end{aligned}$$

and $\hat{q}(0, 0) = (0, 0),\ \hat{t}(0, 0) = 0$. In the mechanism $(\hat{q}, \hat{t})$, the probability of getting the first unit is increased to 1 for types $(1, v_2)$ and the payment increased so as to leave such types indifferent between (q, t) and $(\hat{q}, \hat{t})$. Extend $(\hat{q}, \hat{t})$ from $Y \cup \{(0, 0)\}$ to $v \in D \setminus [Y \cup \{(0, 0)\}]$ as follows:

$$\Big(\hat{q}_1(v), \hat{q}_2(v), \hat{t}(v) \Big) = \begin{cases} (0, 0, 0), & \text{if } v_1 + v_2 \hat{q}_2(1, v_2) < \hat{t}(1, v_2) \\ (1, \hat{q}_2(1, v_2), \hat{t}(1, v_2)), & \text{otherwise.} \end{cases} \tag{15}$$

So, the range of $(\hat{q}, \hat{t})$ is $\{(0,0,0)\}$ and the outcomes for types $(1, v_2) \in Y$. Clearly, $(\hat{q}, \hat{t})$ is IR on $D \setminus Y$.

In the mechanism $(\hat{q}, \hat{t})$, type $(1, v_2)$ obtains payoff equal to that in (q, t) as

$$\begin{aligned} \hat{u}(1, v_2) &= (1, v_2) \cdot \hat{q}(1, v_2) - \hat{t}(1, v_2) \\ &= (1, v_2) \cdot q(1, v_2) + (1 - q_1(1, v_2)) - [t(1, v_2) + (1 - q_1(1, v_2))] \\ &= u(1, v_2) \end{aligned}$$

Thus, $(\hat{q}, \hat{t})$ is IR on Y. That $(\hat{q}, \hat{t})$ is IC on Y follows from

$$\hat{u}(1, v_2) - \hat{u}(1, v_2') \;=\; u(1, v_2) - u(1, v_2') \;\geq\; (v_2 - v_2')q_2(1, v_2') \;=\; (v_2 - v_2')\hat{q}_2(1, v_2')$$

where the inequality follows from IC of (q, t).

We use the fact that $(\hat{q}, \hat{t})$ is IC on Y to prove that $(\hat{q}, \hat{t})$ is IC on $D \setminus Y$. Consider any type $(v_1, v_2) \in D \setminus Y$. The payoff to this type from outcome $(1, \hat{q}_2(1, v_2'), \hat{t}(1, v_2'))$ is

$$\begin{aligned} v_1 + v_2\hat{q}_2(1, v_2') - \hat{t}(1, v_2') &= (v_1 - 1) + (v_2 - v_2')\hat{q}_2(1, v_2') + 1 + v_2'\hat{q}_2(1, v_2') - \hat{t}(1, v_2') \\ &= (v_1 - 1) + (v_2 - v_2')\hat{q}_2(1, v_2') + \hat{u}(1, v_2') \\ &\leq (v_1 - 1) + \hat{u}(1, v_2) \\ &= v_1 + v_2\hat{q}_2(1, v_2) - \hat{t}(1, v_2), \end{aligned}$$

where the inequality follows since $(\hat{q}, \hat{t})$ is IC for any $(1, v_2) \in Y$. But $v_1 + v_2\hat{q}_2(1, v_2) - \hat{t}(1, v_2)$ is the payoff of type (v_1, v_2) from the outcome $(1, \hat{q}_2(1, v_2), \hat{t}(1, v_2))$. Hence, the payoff of type (v_1, v_2) is maximized at the outcome $(1, \hat{q}_2(1, v_2), \hat{t}(1, v_2))$. The payoff from this outcome is $v_1 + v_2q_2(1, v_2) - t(1, v_2)$.

To summarize, if $v_1 + v_2\hat{q}_2(1, v_2) < \hat{t}(1, v_2)$, then type (v_1, v_2) strictly prefers $(0, 0, 0)$ to all other outcomes in the range of $(\hat{q}, \hat{t})$; otherwise, this type's payoff is maximized at the outcome $(1, \hat{q}_2(1, v_2), \hat{t}(1, v_2))$. From (15) we see that $(\hat{q}, \hat{t})$ is IC on $D \setminus Y$.

Finally, the payoff of type $(v_1, v_2) \in D \setminus Y$ that is allocated $(1, \hat{q}_2(1, v_2), \hat{t}(1, v_2))$ in the mechanism $(\hat{q}, \hat{t})$ is

$$\begin{aligned} \hat{u}(v_1, v_2) &= \hat{u}(1, v_2) - (1 - v_1) \\ &= u(1, v_2) - (1 - v_1) \\ &\leq u(v_1, v_2) + (1 - v_1)q_1(1, v_2) - (1 - v_1) \\ &= u(v_1, v_2) - (1 - v_1)(1 - q_1(1, v_2)) \end{aligned}$$

$$\leq u(v_1, v_2)$$

where the first inequality follows from the IC of (q, t) and the second from $v_1 < 1$. If, instead, $(q_1(v_1, v_2), q_2(v_1, v_2), t(v_1, v_2)) = (0, 0, 0)$ then $\hat{u}(v_1, v_2) = 0 \leq u(v_1, v_2)$ by IR of (q, t).

Hence, $\hat{u}(1, v_2) = u(1, v_2)$ for all $(1, v_2) \in Y$ and $\hat{u}(v) \leq u(v)$ for all $v \in D \setminus Y$. As the conditions in (14) are satisfied, we conclude that $\text{Rev}(\hat{q}, \hat{t}) \geq \text{Rev}(q, t)$. Therefore, as (q, t) was arbitrary, there is an optimal mechanism in which the allocation of the first unit is deterministic. ■

A.2 Proofs of Section 2.2

Proof of Lemma 2: Fix a line mechanism (q, t). By definition, $q_1(1, \cdot) = 1$ and (q, t) is IC and IR on Y. The rest of the proof is identical to the second part of the proof of Proposition 1. ■

The following lemma is needed in the sequel.

Lemma 8 *For any line mechanism (q, t), the set $Z_0(q, t)$ satisfies the following properties.*

i. $Z_0(q, t)$ is convex.

ii. Further, $\alpha \leq t(1, 0) \leq 1$. If $t(1, 0) = \alpha$, then $q_2(1, y) = 0$ for all $y \in [0, a\alpha)$.

iii. The slope of the boundary $\partial Z_0(q, t)$ is $-\frac{1}{q_2(1, v_2)}$.

Proof:
i. Take $v, v' \in Z_0(q, t)$ and let $v'' = \lambda v + (1 - \lambda)v'$ for some $\lambda \in (0, 1)$. Then,

$$\begin{aligned}
v_1'' + u(1, v_2'') &= \lambda v_1 + (1 - \lambda)v_1' + u\big(1, \lambda v_2 + (1 - \lambda)v_2'\big) \\
&\leq \lambda v_1 + (1 - \lambda)v_1' + \lambda u(1, v_2) + (1 - \lambda)u(1, v_2') \\
&= \lambda(v_1 + u(1, v_2)) + (1 - \lambda)(v_1' + u(1, v_2')) \\
&< 1
\end{aligned}$$

where the first inequality follows from the fact that u is convex and the second from the fact that $v, v' \in Z_0(q, t)$. Therefore, $v'' \in Z_0(q, t)$.

ii. That $\alpha \leq t(1,0)$ follows from (4) and $t(1,0) \leq 1$ follows from IR as $u(1,0) = 1 - t(1,0) \geq 0$. If $\alpha = t(1,0)$, then $u(1,0) = u(1,a\alpha) = u(1,0) + \int_0^{a\alpha} q_2(1,y)$. As q_2 is non-negative, we must have $q_2(1,y) = 0$ for all $y \in [0, a\alpha)$.

iii. Differentiating along the boundary, $v_1 + u(1,v_2) = 1$, we get

$$\begin{aligned} 1 + \frac{\partial u(1,v_2)}{\partial v_2}\frac{dv_2}{dv_1} &= 1 + q_2(1,v_2)\frac{dv_2}{dv_1} = 0 \\ \implies \quad \frac{dv_2}{dv_1} &= -\frac{1}{q_2(1,v_2)} \end{aligned}$$

■

Proof of Lemma 3: We know that the buyer's payoff u from any IC, IR mechanism (q,t) satisfies $\nabla u = (q_1, q_2)$ a.e. WLOG we restrict attention to mechanisms with $u(0,0) = 0$. Therefore,

$$u(v_1,v_2) = \int_0^{v_1} q_1(s_1,0)ds_1 + \int_0^{v_2} q_2(v_1,s_2)ds_2$$

Thus, (1) implies that the expected revenue functional is linear in the allocation rule q.

Let (q^*, t^*) be a line mechanism that is optimal. We know from Corrolary 1 that such a mechanism exists. Let $\mathcal{Q}^{\alpha^*}$ be the set of line allocation rules that use q_1^* for allocating the first unit and, for $v_2 < a\alpha^*$, use q_2^* for allocating the second unit. That is,

$$\mathcal{Q}^{\alpha^*} := \{q' : q_1'(v) = q_1^*(v) \text{ for all } v \text{ and } q_2'(v_1,v_2) = q_2^*(v_1,v_2) \text{ for all } (v_1,v_2) \text{ such that } v_2 < a\alpha^*\}$$

Hence, for every line mechanism (q', t') such that $q' \in \mathcal{Q}^{\alpha^*}$, we have

$$t'(1,0) = t^*(1,0), \quad \alpha' = \alpha^*, \quad \text{and} \quad u'(1,v_2) = u^*(1,v_2),\ \forall\ v_2 \leq a\alpha^*$$

Recall that $\bar{q}_2 = \sup_{v_2 < a\alpha^*}\left[q_2^*(1,v_2)\right]$. Let $\mathcal{Q}_2^{\alpha^*}$ be the set of all increasing functions $q_2'(1,\cdot)$ defined on $[a\alpha^*, a]$ such that $q_2'(1,a\alpha^*) \geq \bar{q}_2$ and $q_2'(1,a) \leq 1$. Any $q' \in \mathcal{Q}^{\alpha^*}$ maps to a $q_2' \in \mathcal{Q}_2^{\alpha^*}$ and vice versa. Moreover, as (q^*, t^*) maximizes expected revenue in the class of all IC and IR mechanisms, q_2^* must maximize expected revenue in $\mathcal{Q}_2^{\alpha^*}$.[19]

The set $\mathcal{Q}_2^{\alpha^*}$ is convex as the convex combination of two increasing functions is increasing. Moreover, $\mathcal{Q}_2^{\alpha^*}$ is compact in the L^1-norm (see Börgers (2015), p. 16). As noted above, the

[19]That is, the mechanism corresponding to q_2^* must maximize expected revenue in the subset of mechanisms corresponding to $\mathcal{Q}_2^{\alpha^*}$.

expected revenue functional is linear in q and therefore it is also linear in $q_2' \in \mathcal{Q}_2^{\alpha^*}$. Hence, the problem of maximizing expected revenue on the set $\mathcal{Q}_2^{\alpha^*}$ has a solution at an extreme point of $\mathcal{Q}_2^{\alpha^*}$. WLOG we may select the optimal (q^*, t^*) to be such that q_2^* is an extreme point of $\mathcal{Q}_2^{\alpha^*}$.

We argue that every extreme point $q_2 \in \mathcal{Q}_2^{\alpha^*}$ satisfies $q_2(1, v_2) \in \{\bar{q}_2, 1\}$ for all $v_2 \geq a\alpha^*$. Assume, instead, that $q_2(1, v_2) \in (\bar{q}_2, 1)$ for some $v_2 \geq a\alpha^*$. Define two line allocation rules $\hat{q}_2, \tilde{q}_2 \in \mathcal{Q}_2^{\alpha^*}$ as follows

$$\hat{q}_2(1, v_2) = \begin{cases} 2q_2(1, v_2) - \bar{q}_2, & \text{if } \frac{1}{2}(1 + \bar{q}_2) \geq q_2(1, v_2) \\ 1, & \text{if } q_2(1, v_2) > \frac{1}{2}(1 + \bar{q}_2) \end{cases}$$

$$\tilde{q}_2(1, v_2) = \begin{cases} \bar{q}_2, & \text{if } \frac{1}{2}(1 + \bar{q}_2) \geq q_2(1, v_2) \\ 2q_2(1, v_2) - 1, & \text{if } q_2(1, v_2) > \frac{1}{2}(1 + \bar{q}_2) \end{cases}$$

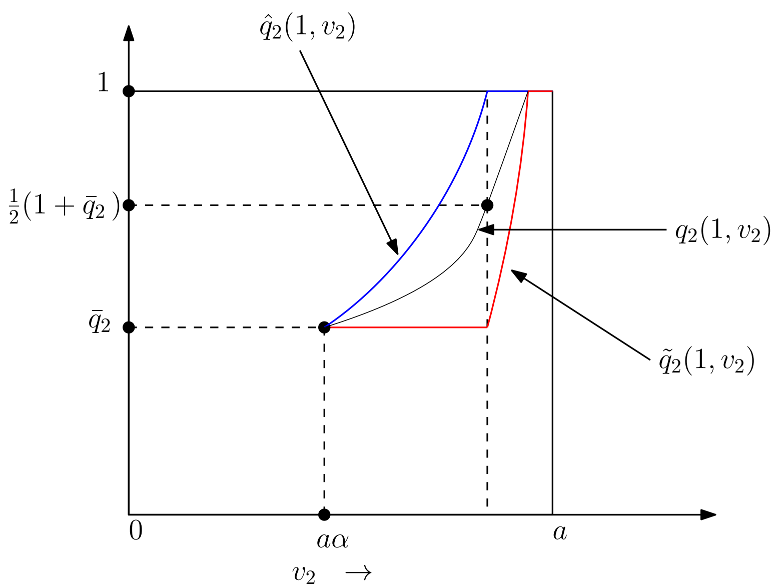


Figure 7: Line allocation rules $\tilde{q}_2$ and $\hat{q}_2$

Both $\hat{q}_2(1, v_2)$ and $\tilde{q}_2(1, v_2)$ are increasing in v_2 and take values between $\bar{q}_2$ and 1 (see Figure 7). Hence, $\hat{q}_2, \tilde{q}_2 \in \mathcal{Q}_2^{\alpha^*}$. As $q_2(1, v_2) \in (\bar{q}_2, 1)$ for some $v_2 \geq a\alpha^*$, we have $\hat{q}_2 \neq \tilde{q}_2 \neq q_2$. As $q_2 = \frac{1}{2}[\hat{q}_2 + \tilde{q}_2]$, q_2 cannot be an extreme point of $\mathcal{Q}_2^{\alpha^*}$.

Note that if $\bar{q}_2 = 0$ then all extreme points of $\mathcal{Q}_2^{\alpha^*}$ are deterministic mechanisms. ■

A.3 Proofs of Section 2.3

Proof of Lemma 4: By Lemma 1,

$$\textsc{Rev}(q,t) = \int_0^a u(1,v_2)f(1,v_2)dv_2 + \int_0^a \int_{\frac{v_2}{a}}^1 u(v_1,v_2)\frac{\partial\psi(v_1,v_2)}{\partial v_1}dv_1dv_2,$$

where $\psi(v_1,v_2) := \int_{v_1}^1 \left[3f(x,v_2) + (x,v_2)\cdot\nabla f(x,v_2)\right]dx$.

In a line mechanism,[20] $u(v_1,v_2) = \max\left[0, v_1 - (1-u(1,v_2))\right]$. In particular,

$$u(v_1,v_2) = \begin{cases} 0, & \text{if } v_1 < \big(1-u(1,v_2)\big) \text{ and } v_2 \le a\alpha \\ v_1 - (1-u(1,v_2)), & \text{otherwise} \end{cases}$$

As a result,

$$\begin{aligned}\textsc{Rev}(q,t) = &\int_0^a u(1,v_2)f(1,v_2)dv_2 \\ &+ \int_0^{a\alpha}\int_{1-u(1,v_2)}^1 u(v_1,v_2)\frac{\partial\psi(v_1,v_2)}{\partial v_1}dv_1dv_2 + \int_{a\alpha}^a\int_{\frac{v_2}{a}}^1 u(v_1,v_2)\frac{\partial\psi(v_1,v_2)}{\partial v_1}dv_1dv_2\end{aligned} \tag{16}$$

We simplify each of the three terms in (16) below.

The first term can be written as

$$\begin{aligned}\int_0^a u(1,v_2)f(1,v_2)dv_2 &= \int_0^{a\alpha}\int_{1-u(1,v_2)}^1 f(1,v_2)dv_1dv_2 + \int_{a\alpha}^a u(1,v_2)f(1,v_2)dv_2 \\ &= \int_0^{a\alpha}\int_{1-u(1,v_2)}^1 f(1,v_2)dv_1dv_2 - \int_{a\alpha}^a \Big(1-\frac{v_2}{a}-u(1,v_2)\Big)f(1,v_2)dv_2 \\ &\quad + \int_{a\alpha}^a (1-\frac{v_2}{a})f(1,v_2)dv_2 \\ &= \int_0^{a\alpha}\int_{1-u(1,v_2)}^1 f(1,v_2)dv_1dv_2 - \int_{a\alpha}^a \Big(1-\frac{v_2}{a}-u(1,v_2)\Big)f(1,v_2)dv_2\end{aligned}$$

[20] Although the lemma is stated for constrained line mechanism, it is true for any line mechanism.

$$+\int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1} f(1,v_2)dv_1dv_2$$

For the second term, use $u(v_1,v_2) = v_1 - (1-u(1,v_2))$ and $\psi(1,v_2) = 0$ for all v_2 to obtain

$$\int_{0}^{a\alpha}\int_{1-u(1,v_2)}^{1} u(v_1,v_2)\frac{\partial\psi(v_1,v_2)}{\partial v_1}dv_1dv_2$$

$$= \int_{0}^{a\alpha}\int_{1-u(1,v_2)}^{1} v_1\frac{\partial\psi(v_1,v_2)}{\partial v_1}dv_1dv_2 - \int_{0}^{a\alpha}(1-u(1,v_2))\int_{1-u(1,v_2)}^{1}\frac{\partial\psi(v_1,v_2)}{\partial v_1}dv_1dv_2$$

$$= \int_{0}^{a\alpha}\Big[v_1\psi(v_1,v_2)\Big]_{1-u(1,v_2)}^{1}dv_2 - \int_{0}^{a\alpha}\int_{1-u(1,v_2)}^{1}\psi(v_1,v_2)dv_1dv_2 + \int_{0}^{a\alpha}(1-u(1,v_2))\psi(1-u(1,v_2),v_2)dv_2$$

$$= -\int_{0}^{a\alpha}\int_{1-u(1,v_2)}^{1}\psi(v_1,v_2)dv_1dv_2$$

For the third term, again use $u(v_1,v_2) = v_1 - (1-u(1,v_2))$ to obtain

$$\int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1} u(v_1,v_2)\frac{\partial\psi(v_1,v_2)}{\partial v_1}dv_1dv_2$$

$$= \int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1} v_1\frac{\partial\psi(v_1,v_2)}{\partial v_1}dv_1dv_2 - \int_{a\alpha}^{a}(1-u(1,v_2))\int_{\frac{v_2}{a}}^{1}\frac{\partial\psi(v_1,v_2)}{\partial v_1}dv_1dv_2$$

$$= \int_{a\alpha}^{a}\Big[v_1\psi(v_1,v_2)\Big]_{\frac{v_2}{a}}^{1}dv_2 - \int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1}\psi(v_1,v_2)dv_1dv_2 + \int_{a\alpha}^{a}(1-u(1,v_2))\psi(\frac{v_2}{a},v_2)dv_2$$

$$= \int_{a\alpha}^{a}\Big(1-\frac{v_2}{a}-u(1,v_2)\Big)\psi(\frac{v_2}{a},v_2)dv_2 - \int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1}\psi(v_1,v_2)dv_1dv_2$$

For all (v_1,v_2), we have $\Phi(v_1,v_2) = f(1,v_2) - \psi(v_1,v_2)$. Therefore, inserting the three terms in (16), we get

$$\textsc{Rev}(q,t) = \int_{0}^{a\alpha}\int_{1-u(1,v_2)}^{1}\Phi(v_1,v_2)dv_1dv_2 + \int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1}\Phi(v_1,v_2)dv_1dv_2 - \int_{a\alpha}^{a}(1-\frac{v_2}{a}-u(1,v_2))\Phi(\frac{v_2}{a},v_2)dv_2$$

$$= \quad \textsc{Rev}^{\alpha-}(q,t) \quad + \quad \textsc{Rev}^{\alpha+}(q,t)$$

■

Proof of Lemma 5: If two constrained line mechanisms (q,t) and (q',t') are identical for $v_2 \geq a\alpha$ then $\alpha = \alpha'$, and $u(1,v_2) = u'(1,v_2)$, $\forall v_2 \geq a\alpha$. It follows from (6) that

$$\textsc{Rev}^{\alpha+}(q,t) := \textsc{Rev}^{\alpha'+}(q',t') \tag{17}$$

By assumption, $\Phi(1-u(1,0),\underline{v}_2) = \Phi(t(1,0),\underline{v}_2) > 0$. The continuity of Φ and of u implies that $\Phi(1-u(1,\underline{v}_2^s),\underline{v}_2^s) > 0$, where $\underline{v}_2^s := \underline{v}_2 + \epsilon$ and $\epsilon > 0$ is small. As noted in Remark 2, we assume $\bar{q}_2 = \sup\limits_{v<a\alpha}[q_2(1,v_2)] > 0$ without loss of generality. Therefore, (7) implies that $\underline{v}_2 < a\alpha$ and we may take ϵ small enough such that $\underline{v}_2^s < a\alpha$. By SC-V,

$$\Phi(1-u(1,\underline{v}_2^s),v_2) \;\; > \;\; 0, \qquad \forall v_2 \in [0,\underline{v}_2^s] \tag{18}$$

Let (q^s,t^s) be a straightening of (q,t) at $\underline{v}_2^s$. Thus, $\alpha = \alpha^s$ and $u^s(1,v_2) \geq u(1,v_2)$, $\forall v_2 < \underline{v}_2^s$. Lemma 4, eq. (17), and $u^s(1,v_2) = u(1,v_2)$, $\forall v_2 \geq \underline{v}_2^s$ imply

$$\begin{aligned}
\textsc{Rev}(q^s,t^s) - \textsc{Rev}(q,t) &= \textsc{Rev}^{\alpha-}(q^s,t^s) - \textsc{Rev}^{\alpha^s-}(q,t) \\
&= \int_0^{\underline{v}_2^s} \int_{1-u^s(1,v_2)}^{1-u(1,v_2)} \Phi(v_1,v_2)dv_1dv_2 \\
&= \int_0^{\underline{v}_2^s} \int_{1-u(1,\underline{v}_2^s)}^{1-u(1,v_2)} \Phi(v_1,v_2)dv_1dv_2 \\
&\geq \int_0^{\underline{v}_2^s} \int_{1-u(1,\underline{v}_2^s)}^{1-u(1,v_2)} \Phi(1-u(1,\underline{v}_2^s),v_2)dv_1dv_2 \\
&> 0,
\end{aligned}$$

where the second equality follows from $u^s(1,v_2) = u(1,\underline{v}_2^s)$ for all $v_2 \leq \underline{v}_2^s$, the first inequality from SC-H, and the second inequality from (18). ■

Proof of Lemma 6: Observe that (10) and SC-V imply that

$$\Phi(t(1,0),v_2) \;\; \leq \;\; 0, \qquad \forall v_2 \geq \underline{v}_2 \tag{19}$$

Let (q^c, t^c) be the cover of (q,t). By the covering property, $\alpha = \alpha^c$ and $u(1, v_2) = u^c(1, v_2)$, $\forall v_2 \geq a\alpha$. Therefore, (17) implies $\text{Rev}^{\alpha+}(q,t) = \text{Rev}^{\alpha^c+}(q^c, t^c)$. This, together with Lemma 4, implies

$$\begin{aligned}
\text{Rev}(q,t) - \text{Rev}(q^c,t^c) &= \text{Rev}^{\alpha-}(q,t) - \text{Rev}^{\alpha^c-}(q^c,t^c) \\
&= \int_{\underline{v}_2}^{a\alpha} \int_{1-u(1,v_2)}^{1-u^c(1,v_2)} \Phi(v_1, v_2) dv_1 dv_2 \\
&\leq \int_{\underline{v}_2}^{a\alpha} \int_{1-u(1,v_2)}^{1-u^c(1,v_2)} \Phi(1 - u^c(1,v_2), v_2) dv_1 dv_2 \\
&= \int_{\underline{v}_2}^{a\alpha} \big[u(1,v_2) - u^c(1,v_2)\big] \Phi(1 - u^c(1,v_2), v_2) dv_2 \\
&\leq \int_{\underline{v}_2}^{a\alpha} \big[u(1,v_2) - u^c(1,v_2)\big] \Phi(1 - u^c(1,0), v_2) dv_2 \\
&= \int_{\underline{v}_2}^{a\alpha} \big[u(1,v_2) - u^c(1,v_2)\big] \Phi(t(1,0), v_2) dv_2 \\
&\leq 0,
\end{aligned}$$

where the first inequality follows from SC-H, the second inequality from $u(1, v_2) \geq u^c(1, v_2)$, $\forall v_2 < a\alpha$, SC-H and $1 - u^c(1, v_2) \leq 1 - u^c(1, 0)$, $\forall v_2$, and the third inequality from $u(1, v_2) \geq u^c(1, v_2)$, $\forall v_2 < a\alpha$ and (19). ■

Proof of Theorem 1: As f satisfies SC-H and SC-V, by Proposition 2 there is an optimal mechanism which is semi-deterministic: $(q,t) \equiv (\underline{t} \equiv t(1,0), q_2(1,a\alpha), \underline{v}_2, \bar{v}_2)$. If $q_2(1, a\alpha) = 0$ or 1, then (q,t) is deterministic. Therefore, assume that $q_2(1, a\alpha) \in (0,1)$. Figure 8 shows such a semi-deterministic mechanism.

In this semi-deterministic mechanism,

$$u(1,v_2) = \begin{cases} 1 - \underline{t}, & \text{if } v_2 \leq \underline{v}_2 \\ 1 + (v_2 - \underline{v}_2) q_2(1,a\alpha) - \underline{t}, & \text{if } v_2 \in [\underline{v}_2, a\alpha] \\ 1 - \alpha + \int_{a\alpha}^{v_2} q_2(1,y)dy, & \text{if } v_2 \in [a\alpha, a], \end{cases} \tag{20}$$

where $q_2(1, y) = q_2(1, a\alpha)$ if $y \in [a\alpha, \bar{v}_2)$ and $q_2(1, y) = 1$ if $y \in [\bar{v}_2, a]$.

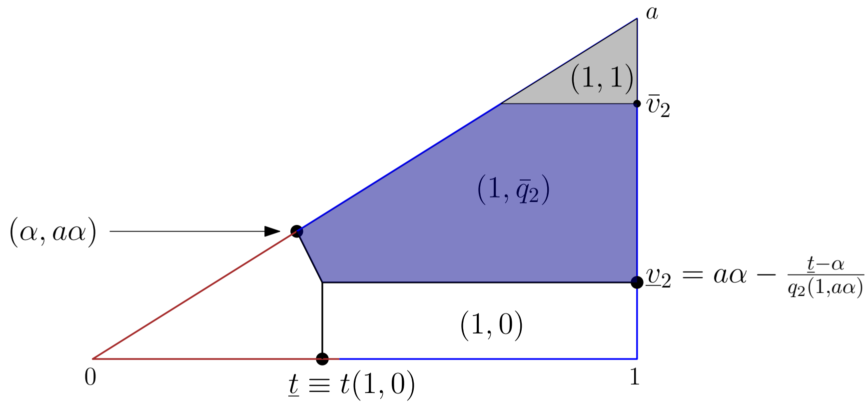


Figure 8: A semi-deterministic mechanism

From Lemma 4, we have

$$
\begin{aligned}
\textsc{Rev}(q,t) &= \int_0^{a\alpha}\int_{1-u(1,v_2)}^{1} \Phi(v_1,v_2)dv_1dv_2 + \int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1}\Phi(v_1,v_2)dv_1dv_2 - \int_{a\alpha}^{a}(1-\frac{v_2}{a}-u(1,v_2))\Phi(\frac{v_2}{a},v_2)dv_2 \\
&= \int_0^{\underline{v}_2}\int_{\underline{t}}^{1}\Phi(v_1,v_2)dv_1dv_2 + \int_{\underline{v}_2}^{a\alpha}\int_{\underline{t}-(v_2-\underline{v}_2)q_2(1,a\alpha)}^{1}\Phi(v_1,v_2)dv_1dv_2 + \int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1}\Phi(v_1,v_2)dv_1dv_2 \\
&\quad - \int_{a\alpha}^{a}\Big(1-\frac{v_2}{a}\Big)\Phi(\frac{v_2}{a},v_2)dv_2 + (1-\alpha)\int_{a\alpha}^{a}\Phi(\frac{v_2}{a},v_2)dv_2 + \int_{a\alpha}^{a}\Big[\int_{a\alpha}^{v_2}q_2(1,y)dy\Big]\Phi(\frac{v_2}{a},v_2)dv_2 \\
&\qquad\qquad [\text{Inserting } u(1,v_2) = 1-\alpha+\int_{a\alpha}^{v_2}q_2(1,y)dy \text{ from (20)}] \\
&= \int_0^{\underline{v}_2}\int_{\underline{t}}^{1}\Phi(v_1,v_2)dv_1dv_2 + \int_{\underline{v}_2}^{a\alpha}\int_{\underline{t}-(v_2-\underline{v}_2)q_2(1,a\alpha)}^{1}\Phi(v_1,v_2)dv_1dv_2 + \int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1}\Phi(v_1,v_2)dv_1dv_2 \\
&\quad - \int_{a\alpha}^{a}\Big(1-\frac{v_2}{a}\Big)\Phi(\frac{v_2}{a},v_2)dv_2 + (1-\alpha)\int_{a\alpha}^{a}\Phi(\frac{v_2}{a},v_2)dv_2 + \int_{a\alpha}^{a}\Big[\int_{v_2}^{a}\Phi(\frac{y}{a},y)dy\Big]q_2(1,v_2)dv_2 \\
&\qquad\qquad [\text{Changing the order of integration in the last term}] \\
&= \int_0^{\underline{v}_2}\int_{\underline{t}}^{1}\Phi(v_1,v_2)dv_1dv_2 + \int_{\underline{v}_2}^{a\alpha}\int_{\underline{t}-(v_2-\underline{v}_2)q_2(1,a\alpha)}^{1}\Phi(v_1,v_2)dv_1dv_2 + \int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1}\Phi(v_1,v_2)dv_1dv_2
\end{aligned}
$$

$$- \int_{a\alpha}^{a} \Big(1 - \frac{v_2}{a}\Big)\Phi(\frac{v_2}{a}, v_2)dv_2 + (1-\alpha)\int_{a\alpha}^{a} \Phi(\frac{v_2}{a}, v_2)dv_2 + q_2(1, a\alpha)\int_{a\alpha}^{\bar{v}_2} \Big[\int_{v_2}^{a} \Phi(\frac{y}{a}, y)dy\Big]dv_2$$

$$+ \int_{\bar{v}_2}^{a} \Big[\int_{v_2}^{a} \Phi(\frac{y}{a}, y)dy\Big]dv_2 \tag{21}$$

As noted at the beginning of the proof, $q_2(1, a\alpha) < 1$. Then $\bar{v}_2 > a\alpha$. Differentiate $\textsc{Rev}(q,t)$ with respect to $q_2(1, a\alpha)$, changing α but not changing $\bar{v}_2, \underline{v}_2$, and $\underline{t}$. As $q_2(1, a\alpha) <\ 1$, we know from Definition 2 that $q_2(1, a\alpha) = \bar{q}_2$. Note that $\frac{d\alpha}{d\bar{q}_2} = -\frac{a\alpha - \underline{v}_2}{1 + a\bar{q}_2}$. Using this and (21), we have

$$\begin{aligned}
\frac{\partial \textsc{Rev}(q,t)}{\partial \bar{q}_2} &= a\frac{d\alpha}{d\bar{q}_2}\int_{\alpha}^{1} \Phi(v_1, a\alpha)dv_1 + \int_{\underline{v}_2}^{a\alpha}(v_2 - \underline{v}_2)\Phi\Big(\underline{t} - (v_2 - \underline{v}_2 q_2(1, a\alpha), v_2\Big)dv_2 \\
&\quad - a\frac{d\alpha}{d\bar{q}_2}\int_{\alpha}^{1} \Phi(v_1, a\alpha)dv_1 + a\frac{d\alpha}{d\bar{q}_2}(1-\alpha)\Phi(\alpha, a\alpha) - a\frac{d\alpha}{d\bar{q}_2}(1-\alpha)\Phi(\alpha, a\alpha) \\
&\quad - \frac{d\alpha}{d\bar{q}_2}\int_{a\alpha}^{a} \Phi(\frac{v_2}{a}, v_2)dv_2 - aq_2(1, a\alpha)\frac{d\alpha}{d\bar{q}_2}\int_{a\alpha}^{a} \Phi(\frac{v_2}{a}, v_2)dv_2 + \int_{a\alpha}^{\bar{v}_2} \Big[\int_{v_2}^{a} \Phi(\frac{y}{a}, y)dy\Big]dv_2 \\
&= \int_{\underline{v}_2}^{a\alpha}(v_2 - \underline{v}_2)\Phi\Big(\underline{t} - (v_2 - \underline{v}_2)q_2(1, a\alpha), v_2\Big)dv_2 + (a\alpha - \underline{v}_2)\int_{a\alpha}^{a} \Phi(\frac{v_2}{a}, v_2)dv_2 \\
&\quad + \int_{a\alpha}^{\bar{v}_2} \Big[\int_{v_2}^{a} \Phi(\frac{y}{a}, y)dy\Big]dv_2
\end{aligned}$$

The first-order condition $\frac{\partial \textsc{Rev}(q,t)}{\partial \bar{q}_2} = 0$ implies

$$\int_{a\alpha}^{\bar{v}_2} \Big[\int_{v_2}^{a} \Phi(\frac{y}{a}, y)dy\Big]dv_2 = -\int_{\underline{v}_2}^{a\alpha}(v_2 - \underline{v}_2)\Phi\Big(\underline{t} - (v_2 - \underline{v}_2)q_2(1, a\alpha), v_2\Big)dv_2 - (a\alpha - \underline{v}_2)\int_{a\alpha}^{a} \Phi(\frac{v_2}{a}, v_2)dv_2 \tag{22}$$

Consider two cases.

Case 1. Suppose $\int_{a\alpha}^{a} \Phi(\frac{v_2}{a}, v_2)dv_2 \geq 0$. Then, SC-D implies that $\int_{v_2}^{a} \Phi(\frac{y}{a}, y)dy \geq 0$ for all

$v_2 > a\alpha$. Hence, we get

$$\int_{a\alpha}^{\bar{v}_2} \Big[\int_{v_2}^{a} \Phi(\frac{y}{a}, y) dy \Big] dv_2 \geq 0 \tag{23}$$

Case 2. Suppose $\int_{a\alpha}^{a} \Phi(\frac{v_2}{a}, v_2) dv_2 < 0$. Lemma 5 implies that $\Phi(\underline{t}, \underline{v}_2) \leq 0$. Hence, SC-H and SC-V imply that $\Phi\Big(\underline{t} - (v_2 - \underline{v}_2) q_2(1, a\alpha), v_2\Big) \leq 0$ for all $v_2 \in [\underline{v}_2, a\alpha]$. This implies that the right-hand side of (22) is positive. Hence, we have

$$\int_{a\alpha}^{\bar{v}_2} \Big[\int_{v_2}^{a} \Phi(\frac{y}{a}, y) dy \Big] dv_2 > 0$$

So, in both cases, (23) holds. Define a new mechanism (q', t') from (q, t) by increasing $q'_2(1, v_2)$ from $q_2(1, a\alpha) < 1$ to 1 for all $v_2 \in [a\alpha, \bar{v}_2)$. Thus, $\bar{v}'_2 = a\alpha$ and everything else, including $\alpha, \underline{v}_2$, and $\underline{t}$, remains as in (q, t). This only changes the last two terms of (21) and hence, we have

$$\text{Rev}(q', t') - \text{Rev}(q, t) = \int_{a\alpha}^{\bar{v}_2} \Big[\int_{v_2}^{a} \Phi(\frac{y}{a}, y) dy \Big] dv_2 - q_2(1, a\alpha) \int_{a\alpha}^{\bar{v}_2} \Big[\int_{v_2}^{a} \Phi(\frac{y}{a}, y) dy \Big] dv_2 \;\geq\; 0,$$

where the inequality follows from (23). Thus, the revenue from (q', t') is no less than that from (q, t). Further, $q'_2(1, a\alpha) = 1$. Therefore, the cover of the mechanism (q', t') is a deterministic mechanism. As (q, t) is an optimal semi-deterministic mechanism by assumption, Lemma 5 implies that $\Phi(\underline{t}, \underline{v}_2) \leq 0$. Hence, by Lemma 6, there is a deterministic mechanism (the cover of (q', t')) which generates at least as much revenue as (q', t') which in turn generates at least as much revenue as the optimal mechanism (q, t). ■

A.4 Proof of Section 2.4

Proof of Proposition 3: A deterministic mechanism is a constrained line mechanism. Therefore, use (5) and (6) to obtain the expected revenue from prices (p_1, p_2):

$$\text{Rev}(p_1, p_2) \;\; = \;\; \int_{0}^{p_2} \int_{p_1}^{1} \Phi(v_1, v_2) dv_1 dv_2 + \int_{p_2}^{a\alpha} \int_{(1+a)\alpha - v_2}^{1} \Phi(v_1, v_2) dv_1 dv_2$$

$$+\int_{a\alpha}^{a}\int_{\frac{v_2}{a}}^{1}\Phi(v_1,v_2)dv_1dv_2+\frac{1+a}{a}\int_{a\alpha}^{a}(v_2-a\alpha)\Phi(\frac{v_2}{a},v_2)dv_2$$

where we use $u(1,v_2)=1+v_2-(1+a)\alpha$ for $v_2\geq a\alpha$.

As $(1+a)\alpha=p_1+p_2$, we have

$$\frac{d\alpha}{dp_1}=\frac{d\alpha}{dp_2}=\frac{1}{1+a}$$

The derivatives are well-defined as p_1 and p_2 are in the interior of the domain. The first-order conditions at optimal prices (p_1^*,p_2^*) are

$$\frac{\partial \textsc{Rev}(q,t)}{\partial p_1} = -\int_{0}^{p_2^*}\Phi(p_1,v_2)dv_2-\int_{p_2^*}^{a\alpha^*}\Phi((1+a)\alpha-v_2,v_2)dv_2-\int_{a\alpha^*}^{a}\Phi(\frac{v_2}{a},v_2)dv_2 = 0$$

$$\frac{\partial \textsc{Rev}(q,t)}{\partial p_2} = -\int_{p_2^*}^{a\alpha^*}\Phi((1+a)\alpha^*-v_2,v_2)dv_2-\int_{a\alpha^*}^{a}\Phi(\frac{v_2}{a},v_2)dv_2 = 0$$

The second equation is (13). Inserting it in the first equation above yields (12).

That $\Phi(p_1^*,p_2^*)\leq 0$ follows from Lemma 5 as $p_1^*=t(1,0)$ and $p_2^*=\underline{v}_2$.

Finally, suppose that $\Phi(p_1^*,0)<0$. Then by SC-V, $\Phi(p_1^*,v_2)\leq 0$ for all v_2. By continuity, there is an $\epsilon>0$ such that $\Phi(p_1^*,v_2)<0$ for all $v_2<\epsilon$. Consequently, (12) is not satisfied. Hence, $\Phi(p_1^*,0)\geq 0$. ■

A.5 Proof of Section 2.5

Proof of Proposition 4: SC-H: For any (v_1,v_2),

$$\begin{aligned}
a[3f(v)+v\cdot\nabla f(v)] &= 6g(v_1)g(\frac{v_2}{a})+2v_1\frac{dg(v_1)}{dv_1}g(\frac{v_2}{a})+2v_2\frac{dg(\frac{v_2}{a})}{dv_2}g(v_1)\\
&= 2g(v_1)g(\frac{v_2}{a})\Big[3+\frac{v_1}{g(v_1)}\frac{dg(v_1)}{dv_1}+\frac{v_2}{g(\frac{v_2}{a})}\frac{dg(\frac{v_2}{a})}{dv_2}\Big]\\
&= 2g(v_1)g(\frac{v_2}{a})\Big[3+\eta(v_1)+\eta(\frac{v_2}{a})\Big] \qquad (24)
\end{aligned}$$

Hence, if $\eta(x)\geq -\frac{3}{2}$ for all x, then SC-H holds. If, instead, $\eta(x)<-\frac{3}{2}$ for some x, then SC-H is violated at $v_1=\frac{v_2}{a}=x$.

SC-V: For any (v_1, v_2),

$$
\begin{aligned}
a\Phi(v_1, v_2) &= 2g(1)g(\frac{v_2}{a}) - 2g(\frac{v_2}{a})\int_{v_1}^{1}\Big[3g(x) + x\frac{dg(x)}{dx} + \eta(\frac{v_2}{a})g(x)\Big]dx \\
&= 2v_1 g(v_1)g(\frac{v_2}{a}) - 2g(\frac{v_2}{a})\Big(1 - G(v_1)\Big)\Big(2 + \eta(\frac{v_2}{a})\Big) \\
&= 2g(v_1)g(\frac{v_2}{a})\left[v_1 - \frac{1 - G(v_1)}{g(v_1)}\Big(2 + \eta(\frac{v_2}{a})\Big)\right] \\
&= 2g(v_1)g(\frac{v_2}{a})W(v_1, v_2) \qquad (25)
\end{aligned}
$$

Thus, SC-V is equivalent to $W(v_1, \cdot)$ crosses zero at most once for all v_1.

SC-D: For any $y \in [0, a]$,

$$
a\Phi(\frac{y}{a}, y) = 2g(\frac{y}{a})g(\frac{y}{a})\left[\frac{y}{a} - \frac{1 - G(\frac{y}{a})}{g(\frac{y}{a})}\Big(2 + \eta(\frac{y}{a})\Big)\right],
$$

Denoting $y' := \frac{y}{a}$, this simplifies to

$$
\begin{aligned}
a\Phi(\frac{y}{a}, y) &= 2[g(y')]^2\left[y' - \frac{1 - G(y')}{g(y')}\Big(2 + \frac{1}{g(y')}y'\frac{dg(y')}{dy'}\Big)\right] \\
&= 2y'[g(y')]^2 - 4(1 - G(y'))g(y') - 2y'(1 - G(y'))\frac{dg(y')}{dy'} \\
&= -2\frac{d[y'g(y')(1 - G(y'))]}{dy'} + \frac{d(1 - G(y'))^2}{dy'}
\end{aligned}
$$

Hence, we get for all $v_2 \in [0, a]$

$$
\begin{aligned}
a\int_{v_2}^{a}\Phi(\frac{y}{a}, y)dy &= 2v_2 g(v_2)(1 - G(v_2)) - (1 - G(v_2))^2 \\
&= v_2 g_{min}(v_2) - (1 - G_{min}(v_2)) \\
&= g_{min}(v_2)\Big[v_2 - \frac{1 - G_{min}(v_2)}{g_{min}(v_2)}\Big] \\
&= g_{min}(v_2)W_{min}(v_2)
\end{aligned}
$$

Thus, SC-D is equivalent to W_{min} crosses zero at most once. ■

A.6 Proofs of Section 3

Proof of Proposition 5: Given an IC and IR mechanism (q, t), construct another mechanism $(\hat{q}, \hat{t})$ as follows. Let $X := \{(v_1, 1) : v_1 \in [0, a]\}$. We first define $(\hat{q}, \hat{t})$ on X. For each $v_1 \in [0, a]$, let

$$\begin{aligned} \hat{q}_1(v_1, 1) &:= q_1(v_1, 1), \qquad \hat{q}_2(v_1, 1) := q_1(v_1, 1) \\ \hat{t}(v_1, 1) &:= t(v_1, 1) + \big[q_1(v_1, 1) - q_2(v_1, 1)\big] \end{aligned}$$

Thus, we keep the allocation probability of the first unit unchanged and increase the allocation probability of the second unit to the maximum feasible. For every $v \in D \setminus X$, set $\hat{q}_2(v) := \hat{q}_1(v)$ and

$$\Big(\hat{q}_1(v), \hat{t}(v)\Big) := \begin{cases} (0, 0), & \text{if } (v_1 + v_2)\hat{q}_1(0, 1) < t(0, 1) \\ \big(\hat{q}_1(0, 1), \hat{t}(0, 1)\big), & \text{if } v_1 + v_2 < 1, (v_1 + v_2)\hat{q}_1(0, 1) \geq \hat{t}(0, 1) \\ \big(\hat{q}_1(v_1 + v_2 - 1, 1), \hat{t}(v_1 + v_2 - 1)\big), & \text{otherwise} \end{cases}$$

We first show that $(\hat{q}, \hat{t})$ restricted to X is IC and IR. Note that for all $(v_1, 1) \in X$,

$$\hat{u}(v_1, 1) = (v_1 + 1)q_1(v_1, 1) - t(v_1, 1) - \big[q_1(v_1, 1) - q_2(v_1, 1)\big] = u(v_1, 1) \tag{26}$$

Hence, IR of (q, t) implies IR of $(\hat{q}, \hat{t})$ restricted to X. Similarly, IC of (q, t) implies for every $(v_1, 1), (v_1', 1) \in X$, we have

$$\hat{u}(v_1, 1) - \hat{u}(v_1', 1) = u(v_1, 1) - u(v_1', 1) \geq q_1(v_1', 1)(v_1 - v_1') = \hat{q}_1(v_1', 1)(v_1 - v_1')$$

Next, we show that $(\hat{q}, \hat{t})$ is IC and IR of on $D \setminus X$. Note that the range of this mechanism is $(0, 0, 0)$ and outcomes on X. The payoff of type $(v_1, v_2) \in D \setminus X$ from the outcome for type $(v_1', 1) \in X$ is

$$\begin{aligned} (v_1 + v_2)\hat{q}_1(v_1', 1) - \hat{t}(v_1', 1) &= (v_1 + v_2 - v_1' - 1)\hat{q}_1(v_1', 1) + \hat{u}(v_1', 1) \qquad (27) \\ &= (v_1 + v_2 - 1)\hat{q}_1(v_1', 1) - v_1'\hat{q}_1(v_1', 1) + \hat{u}(0, 1) + \int_0^{v_1'} \hat{q}_1(x, 1)dx \\ &= (v_1 + v_2 - 1)\hat{q}_1(v_1', 1) + \hat{u}(0, 1) - \int_0^{v_1'} \big[\hat{q}_1(v_1', 1) - \hat{q}_1(x, 1)\big]dx \qquad (28) \end{aligned}$$

where the second equality follows from IC of $(\hat{q}, \hat{t})$ on X. Now, consider two cases for $(v_1, v_2) \in D \setminus X$.

Case 1: If $v_1 + v_2 - 1 < 0$, then eq. (28) is maximized at $v_1' = 0$ (as IC of (q,t) implies that $\hat{q}_1(v_1', 1) = q_1(v_1', 1)$ is increasing in v_1'). The payoff of type (v_1, v_2) from the outcome for type $(0, 1)$ is

$$(v_1 + v_2 - 1)\hat{q}_1(0, 1) + \hat{u}(0, 1) = (v_1 + v_2)\hat{q}_1(0, 1) - \hat{t}(0, 1)$$

If $(v_1+v_2)\hat{q}_1(0, 1) < \hat{t}(0, 1)$, type (v_1, v_2) prefers the outcome $(0, 0, 0)$ to the outcome for type $(0, 1)$. If $v_1 + v_2 < 1$ and $(v_1 + v_2)\hat{q}_1(0, 1) \geq \hat{t}(0, 1)$, then type (v_1, v_2) prefers outcome for $(0, 1)$ to every other outcome on X and $(0, 0, 0)$. Thus, if $v_1 + v_2 < 1$, type $(v_1, v_2) \in D \setminus X$ cannot manipulate $(\hat{q}, \hat{t})$ and IR also holds.

Case 2: Consider the case when $v_1 + v_2 \geq 1$. With $\bar{v}_1 \equiv v_1 + v_2 - 1$, we have

$$\hat{u}(\bar{v}_1, 1) = u(\bar{v}_1, 1) \geq u(v_1', 1) + (\bar{v}_1 - v_1')\hat{q}_1(v_1', 1) \qquad \forall\ v_1' \in [0, a],$$

where the equality follows from (26) and the inequality follows from IC of (q, t). From (27), $u(v_1', 1) + (\bar{v}_1 - v_1')\hat{q}_1(v_1', 1)$ is the utility of type (v_1, v_2) from the outcome for type $(v_1', 1)$. Hence, the outcome for $(\bar{v}_1, 1)$ maximizes the utility of (v_1, v_2) among all outcomes of types in X. Thus, (v_1, v_2) cannot manipulate $(\hat{q}, \hat{t})$.

So, $(\hat{q}, \hat{t})$ is IC and IR.

Finally, for every $(v_1, v_2) \in D \setminus X$ if $\hat{u}(v_1, v_2) = 0$ then $u(v_1, v_2) \geq \hat{u}(v_1, v_2)$ due to IR of (q, t). Else, the outcome of (v_1, v_2) is the same as the outcome of type $(v_1', 1) \in X$, where $v_1' = \max(v_1 + v_2 - 1, 0)$. IC of (q, t) implies

$$\begin{aligned} u(v_1, v_2) &\geq u(v_1', 1) + (v_1 - v_1')q_1(v_1', 1) + (v_2 - 1)q_2(v_1', 1) \\ &\geq u(v_1', 1) + (v_1 - v_1')q_1(v_1', 1) + (v_2 - 1)q_1(v_1', 1) \\ &= (v_1 + v_2 - v_1' - 1)\hat{q}_1(v_1', 1) + \hat{u}(v_1', 1) \\ &= \hat{u}(v_1, v_2), \end{aligned}$$

where the second inequality follows from $q_1(v_1', 1) \geq q_2(v_1', 1)$ and $v_2 < 1$, the first equality from (26), and the second equality from (27).

Hence, we have proved

$$\hat{u}(v_1, v_2) \quad = \quad u(v_1, v_2) \qquad \forall\ (v_1, v_2) \in X$$

$$\hat{u}(v_1, v_2) \quad \leq \quad u(v_1, v_2) \qquad \forall \ (v_1, v_2) \in D \setminus X \tag{29}$$

Using integration by parts, as in the proof of Lemma 1, one can show that:

$$\textsc{Rev}(q,t) = \int_0^a u(v_1,1)f(v_1,1)dv_1 - \int_0^1 \int_0^{av_2} u(v)\Big[3f(v) + v\cdot\nabla f(v)\Big]dv_1dv_2 \tag{30}$$

Since f satisfies SC-H, (29) and (30) imply that $\textsc{Rev}(\hat{q},\hat{t}) \geq \textsc{Rev}(q,t)$. ■

Proof of Lemma 7: With $v_1 \leq av_2$, we have

$$f(v_1, v_2) \;\; = \;\; \frac{2}{a}g(\frac{v_1}{a})g(v_2)$$

Analogous to the derivation of (24), we have

$$a[3f(v_1,v_2) + (v_1,v_2)\cdot\nabla f(v_1,v_2)] = 6g(\frac{v_1}{a})g(v_2)\Big[3 + \eta(\frac{v_1}{a}) + \eta(v_2)\Big]$$

Hence, if $\eta(x) \geq -\frac{3}{2}$ for all x, then SC-H is satisfied.

Theorem 4.5.8 in Barlow and Proschan (1975) implies that v_1, v_2 have increasing hazard rates. That $v_1 + v_2 = X_1 + X_2$ has increasing hazard rate follows from Corollary 1.B.39 in Shaked and Shanthikumar (2007). Consequently, $w = v_1 + v_2$ is regular. ■

References

Armstrong, M. (1996): "Multiproduct Nonlinear Pricing," *Econometrica*, 64, 51–75.

——— (2016): "Nonlinear Pricing," *Annual Review of Economics*, 8, 583–614.

Barlow, R. E. and F. Proschan (1975): *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart, New York.

Bergemann, D., B. Brooks, and S. Morris (2020): "Countering the Winner's Curse: Optimal Auction Design in a Common Value Model," *forthcoming, Theoretical Economics.*

Börgers, T. (2015): *An Introduction to the Theory of Mechanism Design*, Oxford University Press, USA.

Bulow, J. and P. Klemperer (2002): "Prices and the Winner's Curse," *RAND Journal of Economics*, 33, 1–21.

Chen, Y.-C., W. He, J. Li, and Y. Sun (2019): "Equivalence of Stochastic and Deterministic Mechanisms," *Econometrica*, 87, 1367–1390.

Daskalakis, C., A. Deckelbaum, and C. Tzamos (2017): "Strong Duality for a Multiple-Good Monopolist," *Econometrica*, 85, 735–767.

Devanur, N. R., N. Haghpanah, and A. Psomas (2020): "Optimal Multi-unit Mechanisms with Private Demands," *Games and Economic Behavior*, 121, 482–505.

Haghpanah, N. and J. Hartline (2020): "When is Pure Bundling Optimal?" *forthcoming, Review of Economic Studies.*

Hart, S. and N. Nisan (2019): "Selling Multiple Correlated Goods: Revenue Maximization and Menu-size Complexity," *Journal of Economic Theory*, 183, 991–1029.

Hart, S. and P. J. Reny (2015): "Maximal Revenue with Multiple Goods: Nonmonotonicity and other Observations," *Theoretical Economics*, 10, 893–922.

Kleiner, A. and A. Manelli (2019): "Strong Duality in Monopoly Pricing," *Econometrica*, 87, 1391–1396.

Malakhov, A. and R. V. Vohra (2009): "An Optimal Auction for Capacity Constrained Bidders: A Network Perspective," *Economic Theory*, 39, 113–128.

Manelli, A. M. and D. R. Vincent (2006): "Bundling as an Optimal Selling Mechanism for a Multiple-good Monopolist," *Journal of Economic Theory*, 127, 1–35.

——— (2007): "Multidimensional Mechanism Design: Revenue Maximization and the Multiple-good Monopoly," *Journal of Economic Theory*, 137, 153–185.

McAfee, R. P. and J. McMillan (1988): "Multidimensional Incentive Compatibility and Mechanism Design," *Journal of Economic Theory*, 46, 335–354.

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

Pavlov, G. (2011): "A Property of Solutions to Linear Monopoly Problems," *The BE Journal of Theoretical Economics*, 11.

——— (2020): "Selling Two Units of a Customizable Good," Working paper, University of Western Ontario.

Pycia, M. (2006): "Stochastic vs Deterministic Mechanisms in Multidimensional Screening," MPhil. Thesis, Mimeo available at http://pycia.bol.ucla.edu/pycia-multidimensional-screening.pdf.

Riley, J. and R. Zeckhauser (1983): "Optimal Selling Strategies: When to Haggle, When to Hold Firm," *The Quarterly Journal of Economics*, 98, 267–289.

Rochet, J.-C. (1987): "A Necessary and Sufficient Condition for Rationalizability in a Quasi-linear Context," *Journal of Mathematical Economics*, 16, 191–200.

Rochet, J.-C. and P. Choné (1998): "Ironing, Sweeping, and Multidimensional Screening," *Econometrica*, 783–826.

Shaked, M. and J. G. Shanthikumar (2007): *Stochastic Orders*, Springer.

Thanassoulis, J. (2004): "Haggling over Substitutes," *Journal of Economic Theory*, 117, 217–245.

Wilson, R. B. (1993): *Nonlinear Pricing*, Oxford University Press.