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Selling two complementary goods

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Selling two complementary goods *

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Abstract

A seller is selling a pair of complementary goods to an agent. The agent consumes the goods only in a certain *ratio* and freely disposes of excess in either of the goods. The value of the bundle and the ratio are private information of the agent. In this two-dimensional type space model, we characterize the incentive constraints and show that the optimal (expected revenue-maximizing) mechanism is a *ratio-dependent posted price* mechanism for a class of distributions; that is, it has a different posted price for each ratio report. We identify additional sufficient conditions on the joint distribution for a *posted price* to be an optimal mechanism. We also show that the optimal mechanism is a posted price mechanism when the value and the ratio types are independently distributed.

JEL Codes: D82, D40, D42

Keywords: optimal mechanism, complementary goods, multi-dimensional private information, posted-price mechanism.

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1 Introduction

An agent consumes a pair of goods only in a specific ratio of quantities. For instance, a firm needs two inputs in a particular ratio to produce a final product; that is, the firm has a Leontief production function. The ratio in which the firm uses the inputs is its proprietary information. Another example is of a consumer who treats a pair of goods, hardware and software, for example, as perfect complements. The ratio of consumption is private information of the consumer. Such Leontief preferences are ubiquitous in the economy. A monopolistic seller who owns one unit each of the two *divisible* goods is selling to such an agent; what is the revenue-maximizing optimal mechanism in this setting?

We analyse this question as a mechanism design problem. For any allocation of the bundle of goods, the agent evaluates it using the ratio in which she consumes. The agent has quasilinear preferences across such bundles of goods whose quantities are in the desired ratio. The agent's payoff is determined by a *value*, the ratio, and quantity of the bundle she consumes. The value is interpreted as the payoff from consuming one unit of one of the goods combined with the other good in the desired ratio. Both per unit value from the consumption of the bundle of goods and the ratio itself are *private* information of the agent. The central theme of the paper is in finding the revenue-maximizing mechanism in such an environment.

For each report, a mechanism assigns quantities of both the goods, and payment to be made by the agent. Due to the revelation principle, we focus, without loss of generality, on direct mechanisms that are incentive compatible. An agent could potentially misreport both on value and ratio dimensions. Dealing with incentive constraints in multi-dimensional mechanism design problems is difficult (Manelli and Vincent, 2007; Carroll, 2017). We present this natural two-dimensional mechanism design model and show that a simple class of *non-wasteful* mechanisms are optimal under some conditions over the seller's beliefs on the agent's type. Note that we consider a *divisible* goods model, while the optimal mechanism in the indivisible model may involve randomization. (Hart and Reny (2015); Thanassoulis (2004)).

We show that a POSTED PRICE mechanism or a RATIO-DEPENDENT POSTED PRICE mechanism is optimal. The former is a mechanism in which the seller offers one unit of one of the goods and the other good in the desired ratio at some fixed price. In the latter mechanism, each type gets the same bundle as in the former mechanism, but the price depends on the reported ratio. We first show that it is without loss of generality to focus on mechanisms in which allocations to any type are in the desired ratio; that is, the agent, after a truthful report, does not dispose of either of the goods that the mechanism allocates. This result allows us to use Myersonian techniques. We then characterize incentive compatible mechanisms and provide sufficient conditions over the seller's belief on the type-space for simple non-wasteful mechanisms to be optimal. We fully describe these mechanisms over the parameters of the problem. The price function that describes the optimal mechanisms is derived from the virtual valuation of the type's joint distribution.

1.1 Related Literature

Armstrong (1996); Rochet and Chone (1998) analyze the standard multidimensional model with divisible goods while Mcafee and Mcmillan (1988); Manelli and Vincent (2006) among others analyze the problem of indivisible goods. The optimal mechanism is known to be stochastic (that is, allocation of the objects is randomized) for many distributions (Hart and Reny (2015); Thanassoulis (2004)) in the case of the indivisible goods. Manelli and Vincent (2006); Devanur et al. (2020); Bikchandani and Mishra (2020) are among the papers that find sufficient conditions (on type distribution) under which deterministic mechanisms are optimal. Our model considers divisible complementary goods and finds sufficient conditions under which one of the goods is allocated to the maximum quantity available with the seller. This maximum quantity allocation is interpreted as a deterministic mechanism in the standard indivisible goods model (see Pavlov (2011)).

Devanur et al. (2020) consider a model in which there are multiple copies of a good for sale. The agent derives a constant marginal 'value' up to a 'quantity' of the goods and no value beyond the desired quantity. The value and quantity are private information of the agent. They find conditions for deterministic mechanisms to be optimal and focus on the computational complexity of the problem. Our paper differs from theirs in that we consider a pair of heterogeneous goods with a privately known ratio of consumption while they consider homogeneous goods with privately known demand. The two optimization exercises are similar after we prove our first result of 'non-wastefulness.' While they use the 'utility' approach to show that there exists a deterministic mechanism under some conditions, we use the Myersonian approach to characterize the optimal mechanism under a different set of conditions.

Fiat et al. (2016)'s model has two-dimensional private type for a single object. One dimension is for 'value,' which is constant up to a 'deadline' and drops to zero beyond the deadline. The paper characterizes optimal mechanisms, not just focussing on deterministic mechanisms. The agent's utility in their model changes sharply beyond the deadline, while in our model (and in Devanur et al. (2020)'s model), the utility is continuous, as a function of the allocations.

2 The model

A monopolistic seller is selling a pair of divisible goods to an agent. The seller has one unit each of the goods, denoted by $\textsc{good}_1$ and $\textsc{good}_2$, and has no value for them. A consumption bundle for the agent is a tuple (a_1, a_2, t), where $a_1, a_2 \in [0, 1]$ is the allocation quantities of $\textsc{good}_1$ and $\textsc{good}_2$, respectively, and $t \in \mathbb{R}$ is the transfer - the amount *paid* by the agent.

The agent treats the goods as perfect complements, that is any two allocations (a_1, a_2) and (a_1', a_2') with $\min\{\frac{a_1}{k}, a_2\} = \min\{\frac{a_1'}{k}, a_2'\}$ are payoff equivalent, where $k \in K \equiv (0, 1]$ is the ratio of quantities of $\textsc{good}_1$ and $\textsc{good}_2$ that the agent demands. If the agent gets (a_1, a_2) and her desired ratio is k, then she can produce $\min\{\frac{a_1}{k}, a_2\}$ units of the final good, which she values at v per unit. v is drawn from $V \equiv [0, 1]$[1]. Both v and k are private information of the agent, therefore the agent has a "type" $(v, k) \in V \times K$.

The utility derived by agent of type (v, k) from an outcome (a_1, a_2, t) is given by,

$$U_{(v,k)}(a_1, a_2, t) := v \min\left\{\frac{a_1}{k}, a_2\right\} - t.$$

$\textsc{good}_2$ is the *primary* good, whereas $\textsc{good}_1$ is its *complement* which is always consumed lesser in quantity than the former as $k \in (0, 1]$. For instance, consider an agent with type $(v, k) = (\frac{1}{2}, \frac{1}{3})$. From an outcome $(a_1, a_2, t) = \left(\frac{1}{4}, 1, t\right)$, the agent derives a utility of

$$\frac{1}{2}.\min\left\{\frac{\frac{1}{4}}{\frac{1}{3}}, 1\right\} = \frac{1}{2}.\frac{3}{4} - t.$$

The agent produces $\frac{3}{4}$ units of the final good in this example.

We assume that the random variables v, k follow a joint distribution function G with strictly positive density function g. We use g_v, g_k to denote marginal density functions of V and K, respectively. The conditional density of v given k is denoted by $g(v|k)$.

3 Optimal Mechanism

An allocation function $f : V \times K \to [0, 1]^2$ and a payment function $p : V \times K \to \mathbb{R}$ define a direct mechanism (f, p). For any allocation function f, we use subscript notations f_1 and f_2 to denote allocations corresponding to $\textsc{good}_1$ and $\textsc{good}_2$, respectively. Standard revelation principle argument implies that we can focus, without loss of generality, on incentive compatible direct mechanisms.

[1]Our results holds even with $V \equiv [0, \bar{v}]$ for some $\bar{v} > 0$.

Definition 1 *A mechanism* (f,p) *is* **incentive compatible (IC)** *if for all* $(v,k),(v',k') \in V \times K$,

$$U_{(v,k)}(f(v,k),p(v,k)) \geq U_{(v,k)}(f(v',k'),p(v',k'))$$

IC condition ensures that the agent has the incentive to report his type - both value and ratio - truthfully. We also impose a participation constraint; that is, the utility for every type of the agent is at least zero from participating in the mechanism.

Notation. We use $(v,k) \to (v',k')$ to denote the incentive constraint for the type (v,k) to not misreport as type (v',k').

Definition 2 *A mechanism* (f,p) *is* **individually rational (IR)** *if for all* $(v,k) \in V \times K$,

$$U_{(v,k)}(f(v,k),p(v,k)) \geq 0.$$

3.1 Non-wasteful Mechanisms

A mechanism that allocates GOOD$_2$ (primary good) and GOOD$_1$ (complementary good) in the desired ratio is defined as a *non-wasteful* mechanism. We state two simple classes of non-wasteful mechanisms and show that they are IC and IR.

Definition 3 *A mechanism* (f,p) *is* POSTED PRICE *mechanism if there exists a* $\rho^* \in [0,1]$ *such that*

$$(f(v,k),p(v,k)) = \begin{cases} (0,0,0) & \text{if } v \leq \rho^* \\ (k,1,\rho^*) & \text{otherwise.} \end{cases}$$

In a POSTED PRICE mechanism, there exists a price ρ^* such that all the types whose value is less than ρ^* get no good and pay nothing. A type (v,k) with $v > \rho^*$ gets k units of GOOD$_1$, 1 unit of GOOD$_2$, and pays ρ^* to the seller.

Definition 4 *A mechanism* (f,p) *is* RATIO-DEPENDENT POSTED PRICE *mechanism if there exists a function* $\psi : K \to V$ *such that for all* $k' > k$,

$$\psi(k) \leq \psi(k'),$$
$$\frac{k}{k'}\psi(k') \leq \psi(k), \text{and}$$
$$(f(v,k),p(v,k)) = \begin{cases} (0,0,0) & \text{if } v \leq \psi(k) \\ (k,1,\psi(k)) & \text{otherwise.} \end{cases}$$

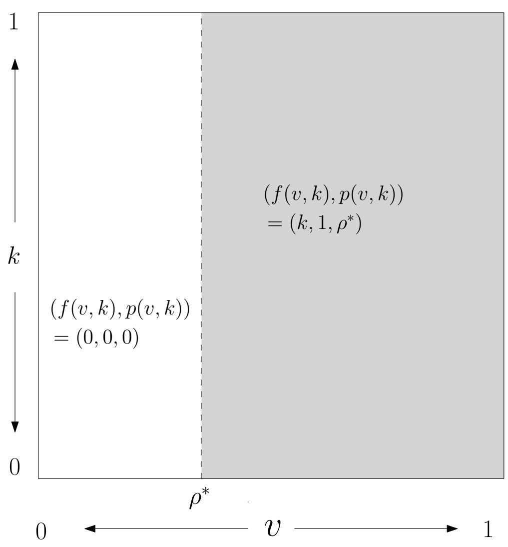

Figure 1: POSTED PRICE

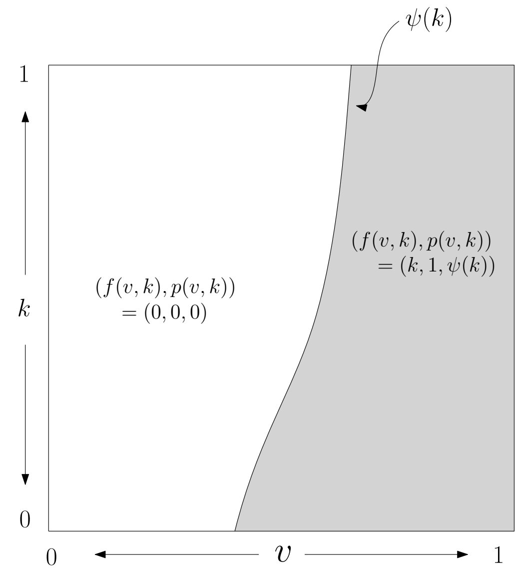

Figure 2: RATIO-DEPENDENT POSTED PRICE

For instance $\psi(k) = (\frac{1}{k+2})^{\frac{1}{k+1}}$ satisfies the conditions that defines a RATIO-DEPENDENT POSTED PRICE mechanism, and this is not a POSTED PRICE mechanism. Observe that the POSTED PRICE mechanism is a special case of the RATIO-DEPENDENT POSTED PRICE mechanism by setting $\psi(k) = \rho$ for all k. Our next proposition shows that the RATIO-DEPENDENT POSTED PRICE mechanism is IC and IR. Therefore, this also proves that the POSTED PRICE mechanism is IC and IR. While the POSTED PRICE mechanism has a unique price in the menu, the RATIO-DEPENDENT POSTED PRICE mechanism has a potentially infinite menu. These mechanisms are easy to describe since the primary good is allocated fully or not allocated at all.

PROPOSITION **1** *A* RATIO-DEPENDENT POSTED PRICE *mechanism is IC and IR.*

Proof: Consider a RATIO-DEPENDENT POSTED PRICE mechanism (f, p) defined by a function ψ. We first show that (f, p) is IR. For any type (v, k),

$$U_{(v.k)}(f(v,k), p(v,k)) = \begin{cases} 0 & \text{if } v \leq \psi(k) \\ v - \psi(k) & \text{otherwise.} \end{cases}$$

Clearly, $U_{(v.k)}(f(v,k), p(v,k)) \geq 0$ and hence (f, p) is IR. We now show that (f, p) is IC. Without loss of generality, consider any two representative types $(v, k), (v', k')$ such that $k' \geq k$. Note that $\frac{k}{k'}\psi(k') \leq \psi(k) \leq \psi(k')$.

$\underline{(v,k) \to (v',k')}$. Note that $\big(f(v',k'), p(v',k')\big)$ is either $(0,0,0)$ or $(k',1,\psi(k'))$, we only need to check deviation to the latter outcome because IR implies (v,k) does not deviate to a type with outcome $(0,0,0)$. We check deviation to the outcome $(k',1,\psi(k'))$ in two cases.
Case 1: $v \leq \psi(k)$. (v,k) has no incentive to deviate to (v',k') because

$$U_{(v,k)}(0,0,0) = 0 \geq v - \psi(k) \geq v - \psi(k') = v\min\{\frac{k'}{k}, 1\} - \psi(k') = U_{(v,k)}(k',1,\psi(k')).$$

Case 2: $v > \psi(k)$.

$$U_{(v,k)}(k,1,\psi(k)) = v - \psi(k) \geq v - \psi(k') = v\min\{\frac{k'}{k}, 1\} - \psi(k') = U_{(v,k)}(k',1,\psi(k')),$$

The second inequality in the first case and the inequality in the second case come from the fact that $\psi(k) \leq \psi(k')$. This means (v,k) has no incentive to deviate to (v',k').

$\underline{(v',k') \to (v,k)}$. Again we only need to check (v',k') deviating to the outcome $(k,1,\psi(k))$.
Case 1: $v' \leq \psi(k')$.

$$U_{(v',k')}(0,0,0) = 0 \geq \psi(k')\frac{k}{k'} - \psi(k) \geq v'\frac{k}{k'} - \psi(k) = v'\min\{\frac{k}{k'}, 1\} - \psi(k) = U_{(v',k')}(k,1,\psi(k)).$$

Case 2: $v' > \psi(k')$.

$$U_{(v',k')}(k',1,\psi(k')) = v' - \psi(k') \geq v'\frac{k}{k'} - \psi(k) = v'\min\{\frac{k}{k'}, 1\} - \psi(k) = U_{(v',k')}(k,1,\psi(k)).$$

The first inequality in the first case comes from the condition that $\psi(k) \geq \frac{k}{k'}\psi(k')$. The inequality in the second case comes from the following argument. $\psi(k) \geq \frac{k}{k'}\psi(k')$ implies $\psi(k')(1-\frac{k}{k'}) \geq \psi(k') - \psi(k)$. Since $v' > \psi(k')$, we have $v'(1-\frac{k}{k'}) \geq \psi(k') - \psi(k)$. Rearranging the terms we get the inequality. ■

3.2 Optimal Mechanism

We now describe our optimal mechanism. The expected (ex-ante) revenue of a mechanism (f,p) is given by

$$\Pi(f,p) := \int_{V\times K} p(v,k)dG(v,k).$$

We say that a mechanism (f,p) is **optimal** if

- (f,p) is IC and IR,
- and $\Pi(f,p) \geq \Pi(f',p')$ for any other IC and IR mechanism (f',p').

We can restrict the class of mechanisms to optimize over due to the following result.

PROPOSITION **2** *For every IC and IR mechanism (f,p) there exists another IC and IR mechanism (f',p') such that*

1. $\Pi(f',p') = \Pi(f,p)$, *and*

2. $f'_1(v,k) = kf'_2(v,k)$ *for all* (v,k). - **non-wasteful allocation**

Omitted proofs are relegated to the Appendix A. Proposition 2 implies that, to find the optimal mechanism it is without loss of generality to focus on the class of mechanisms with the property that allocation of GOOD_1 is k times allocation of GOOD_2. To prove this, we start with an arbitrary IC and IR mechanism (f,p) and construct the desired form mechanism (f',p') while keeping the revenue constant. (f',p') is derived from (f,p) by reducing the allocation of one of the goods so that the allocation ratio is as reported. The payments remain the same. For an insight into why (f',p') is IC, observe that the utility of any type in (f',p') is the same as that in (f,p). For any type, the utility from misreporting to some type in (f',p') is weakly lower than the utility from misreporting to the same type in (f,p). This is due to the fact that allocation of one of the goods in (f',p') lower than that in (f,p) while keeping payment and other good's allocation the same. IC of (f,p) implies IC of (f',p').

These **non-wasteful** mechanisms are denoted by,

$$\mathcal{M} := \{(f,p) : f_1(v,k) = kf_2(v,k) \text{ for all } (v,k)\}.$$

Note: If $(f,p) \in \mathcal{M}$ then $\min\left\{\frac{f_1(v,k)}{k}, f_2(v,k)\right\} = f_2(v,k)$ for all (v,k). In the next Proposition and rest of the paper we use the following fact without explicitly stating: for any k,

$$U_{(v,k)}(f(v',k),p(v',k)) = vf_2(v',k) - p(v',k) \text{ for all } v,v'.$$

This allows us to focus only on one of the allocation function components f_2, and deduce f_1 from it in the final step. However, this does not reduce the problem to a one-dimensional exercise as incentive constraints across the ratio dimension are crucial to the optimal program. The following result makes this clear.

3.2.1 Characterization IC Mechanisms

We characterize the IC mechanisms in the class $\mathcal{M}$.

PROPOSITION **3** $(f,p) \in \mathcal{M}$ *is IC if and only if the following are true for any* (v,k), *(1)* $f_2(v,k) \leq f_2(v',k)$ *for all* $v' > v$,

(2) $p(v,k) = p(0,1) + vf_2(v,k) - \int_0^v f_2(t,k)dt$,

(3) $\int_0^v f_2(t,k')dt \leq \int_0^v f_2(t,k)dt$ *for all* $k' > k$,

(4) $\int_0^{v\frac{k}{k'}} f_2(t,k)dt \leq \int_0^v f_2(t,k')dt$ *for all* $k' > k$.

The conditions (1) and (2) in Proposition 3 correspond to IC constraints between two types on a horizontal line in the type-space (see Figure 3). Mechanisms in $\mathcal{M}$ have the property of reducing the IC constraints on any horizontal line equivalent to that of a one-dimensional problem. This is the same as Myerson (1981)'s IC characterization when restricted to any k. However, Proposition 3 shows that some 'vertical' and 'diagonal' constraints are enough to guarantee the incentive compatibility of the mechanism. Condition (3) corresponds to the vertical constraints, while (4) corresponds to the diagonal constraints. The arrows in Figure 3 indicate the direction in which the incentive constraints need to be satisfied. To see why condition (3) and (4) are necessary for IC, observe that applying conditions (1) and (2) for the types indicated in the figure and then simplifying the IC expression yields the expressions. Interestingly, these 'local' constraints are enough to guarantee global incentive compatibility. Describing optimal mechanisms in multidimensional models is difficult partly because binding constraints cannot be pinned down (Rochet and Chone (1998)). However, due to the incentive constraints characterization, we can do so in this model.

We state two lemmas which we use in our analysis.

Lemma **1** *If a mechanism* (f,p) *is IC, then* $p(0,k) = p(0,1)$ *for all* k.

Proof: For any k, $(0,1) \rightarrow (0,k)$ implies that $-p(0,1) \geq -p(0,k)$ while $(0,k) \rightarrow (0,1)$ implies that $-p(0,k) \geq -p(0,1)$. ■

In line with other models in mechanism design, the following standard result holds in this setting too.

Lemma **2** *An IC mechanism* (f,p) *is individually rational if and only if,*

$$p(0,1) \leq 0.$$

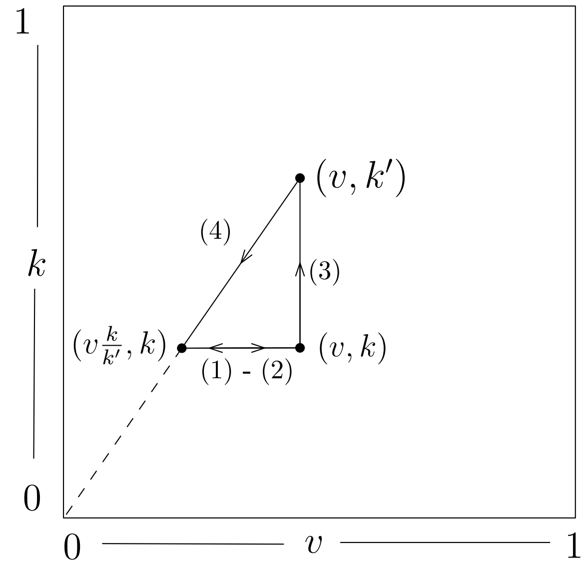

Figure 3: IC Constraints

Proof: Fix an IC mechanism (f, p). Suppose that (f, p) is IR. Consider the type $(0, 1)$. IR implies that $U_{(0,1)}(f(0, 1), p(0, 1)) \geq 0$, this simplifies to $p(0, 1) \leq 0$. To show the other way, fix any (v, k) and observe that $U_{(v,k)}(f(v, k), (v, k)) = v \min\{\frac{f_1(v,k)}{k}, f_2(v, k)\} - p(v, k) \geq v \min\{\frac{f_1(0,k)}{k}, f_2(0, k)\} - p(0, k) \geq -p(0, k) \geq 0$. The first inequality is from the incentive constraint $(v, k) \to (0, k)$, the second from the fact that allocation functions are non-negative. The third is true since $p(0, 1) \leq 0$ implies that $p(0, k) \leq 0$ for all k due to Lemma 1. ■

Using Lemma 1, Lemma 2, and Proposition 3, the equivalent optimal program can now be written as follows,

$$\max_{(f,p)\in\mathcal{M}} \int_0^1 \int_0^1 \left[p(0, 1) + v f_2(v, k) - \int_0^v f_2(t, k)dt\right] g(v, k)dvdk \tag{O}$$

$$f_2(v, k) \leq f_2(v', k) \text{ for all } v < v', k, \tag{C1}$$

$$\int_0^v f_2(t, k')dt \leq \int_0^v f_2(t, k)dt \text{ for all } v, k' > k, \tag{C2}$$

$$\int_0^{v\frac{k}{k'}} f_2(t, k)dt \leq \int_0^v f_2(t, k')dt \text{ for all } v, k' > k, \tag{C3}$$

$$p(0, 1) \leq 0. \tag{C4}$$

Notice that the $p(0, 1)$ appears only in the (C4) constraint and to maximize (O) we set $p(0, 1) = 0$ without changing any other constraints. We rewrite the objective function (O) by changing the order of integration. Also, since f_2 uniquely determines f_1 and p by Propositions 2 and 3, respectively, we suppress these decision variables in the optimal program and rewrite it as follows:

Optimal Program

$$\max_{f_2:V\times K\to[0,1]} \int_0^1 \left[\int_0^1 \left(v - \frac{1-G(v|k)}{g(v|k)}\right) f_2(v,k)g(v|k)dv\right] g_k(k)dk \tag{O}$$

$$f_2(v,k) \le f_2(v',k) \text{ for all } v < v', k, \tag{C1}$$

$$\int_0^v f_2(t,k')dt \le \int_0^v f_2(t,k)dt \text{ for all } v, k' > k, \tag{C2}$$

$$\int_0^{v\frac{k}{k'}} f_2(t,k)dt \le \int_0^v f_2(t,k')dt \text{ for all } v, k' > k. \tag{C3}$$

3.3 Main Results

We impose the following restrictions on G for our next results.

Definition **5** *A distribution G satisfies* Condition A *if for any k, $v(1-G(v|k))$ is strictly concave in v.*

This condition has been used in the literature before (Che and Gale, 2000; Devanur et al., 2020; Mishra and Paramahamsa, 2018). Let,

$$\phi(v,k) := v - \frac{1-G(v|k)}{g(v|k)}.$$

Given a value of k, this is the standard virtual valuation expression. Condition A is equivalent to strictly increasing $\phi(v,k)g(v|k)$ for every k. Notice that $\phi(0,k) < 0$ and $\phi(1,k) > 1$ for all k and that continuity of G ensures continuity of $\phi(v,k)g(v|k)$. Since $g(v|k) > 0$, the solution to $\phi(v,k)g(v|k) = 0$ and $\phi(v,k) = 0$ is the same and unique, for any k. Therefore, whenever Condition A is satisfied, for any k, there exists a unique $v \in (0,1)$ such that $\phi(v,k) = 0$. We denote the value satisfying this equation by $\phi_k^{-1}(0)$.

Definition **6** *A distribution G is said to satisfy* Condition B *if it satisfies* Condition A *and for all $k < k'$ the following is true,*

$$\frac{k}{k'}\phi_{k'}^{-1}(0) \leq \phi_k^{-1}(0) \leq \phi_{k'}^{-1}(0).$$

The uniform distribution satisfies this condition as $\phi_k^{-1}(0) = \phi_{k'}^{-1}(0)$ for all k, k'. We derive the optimal mechanism for a distribution that satisfies this condition after stating the next result.

Theorem **1** *If G satisfies Condition B, then the following* ratio-dependent posted price *mechanism is optimal,*

$$\big(f(v,k), p(v,k)\big) = \begin{cases} (0,0,0) & v \leq \phi_k^{-1}(0) \\ \big(k, 1, \phi_k^{-1}(0)\big) & \textit{otherwise} \end{cases}$$

Proof: Ignoring the constraints (C1), (C2), and (C3), a point-wise maximization (for each k) of the objective function (O) implies that the optimal allocation function f_2 is as in the statement of the theorem, since $\phi(v,k) \leq 0$ for all (v,k) with $v \leq \phi_k^{-1}(0)$ and $\phi(v,k) > 0$ for all (v,k) with $v > \phi_k^{-1}(0)$, due to Condition A. Condition B implies that this mechanism is indeed ratio-dependent posted price mechanism. We have already shown this mechanism to be IC (Proposition 1). Hence, the ignored constraints hold. ■

Example 1. Consider a density function $g(v,k) = \frac{v^k}{\ln 2}$. We evaluate the conditional density to $g(v|k) = v^k(k+1)$. From this we derive the virtual valuation to,

$$\phi(v,k) = v - \frac{1 - v^{k+1}}{v^k(k+1)}.$$

We can show that $\phi(v,k)$ is strictly increasing by first order condition, and $\phi_k^{-1}(0) = (\frac{1}{k+2})^{\frac{1}{k+1}}$ satisfies Condition B. Therefore, the optimal mechanism for this distribution evaluates as,

$$\big(f(v,k), p(v,k)\big) = \begin{cases} (0,0,0) & v \leq (\frac{1}{k+2})^{\frac{1}{k+1}} \\ (k, 1, (\frac{1}{k+2})^{\frac{1}{k+1}}) & \text{otherwise} \end{cases}$$

Observe that, for this result we can replace Condition A with a more standard regularity condition, that $\phi(v,k)$ is strictly increasing in v. For a detailed comparison of the regularity condition with Condition A see Devanur et al. (2020), Section 6.1. We use the current form of Condition A as we require it for our next result.

Definition 7 *A distribution G is satisfies* Condition B′ *if it satisfies* Condition A *and for all $k < k'$ the following is true,*

$$\phi_k^{-1}(0) > \phi_{k'}^{-1}(0).$$

Notice that in Condition B we have $\phi_k^{-1}(0)$ to be strictly decreasing in k whereas the opposite is true in Condition B′.

Theorem **2** *If G satisfies* Condition B′, *then a* posted price *mechanism is optimal.*

Following is an example of a distribution that satisfies Condition B′ and the optimal mechanism.

Example 2. Consider a density function $g(v, k) = \frac{2}{3}(v + 2k)$. The cdf of this distribution is $\frac{vk}{3}(v + 2k)$. We evaluate the conditional density to $g(v|k) = \frac{v+2k}{0.5+2k}$. From this we derive the virtual valuation to,

$$\phi(v, k) = \frac{1.5v^2 + 4kv - 2k - 0.5}{v + 2k}.$$

We can show that $\phi(v, k)$ is strictly increasing by first order condition, and that

$$\phi_k^{-1}(0) = \frac{-4k + \sqrt{16k^2 + 12k + 3}}{3}$$

is decreasing in k. Therefore, the optimal mechanism for this distribution evaluates to,

$$\big(f(v, k), p(v, k)\big) = \begin{cases} (0, 0, 0) & v \le \rho^* \\ (k, 1, \rho^*) & \text{otherwise} \end{cases}$$

$\rho^* = \operatorname{argmax}_p p\big(1 - G_v(p)\big)$, this evaluates to $\rho^* = \frac{\sqrt{13}-2}{3}$.

The following proposition describes the optimal mechanism when the value and ratio random variables are independent. This result does not require any other condition on the type distribution.

Proposition **4** *If $g(v, k) = g_v(v)g_k(k)$, then the following* posted price *mechanism is optimal,*

$$\big(f(v, k), p(v, k)\big) = \begin{cases} (k, 1, p^*) & v \ge p^* \\ (0, 0, 0) & \textit{otherwise} \end{cases}$$

where p^* is any p that maximizes $p\big(1 - G_v(p)\big)$

Proof: We solve the reduced problem by ignoring constraints (C2) and (C3); this can be written as:

Using $g(v|k) = g_v(v)$ we rewrite (O) as,

$$\max_{f_2: V\times K\to[0,1]} \int_0^1 \left[\int_0^1 \left[v - \frac{1-G_v(v)}{g_v(v)}\right] g_v(v) f_2(v,k) dv\right] g_k(k) dk. \tag{O}$$

$$f_2(v,k) \le f_2(v',k) \text{ for all } v < v', k. \tag{C1}$$

We first maximize the objective function point-wise for each k, while satisfying the constraint for that k. To that end, fix some k, and observe that maximizing the term inside large bracket along with the monotonocity constraint is the same as in the standard Myerson's problem for a general distribution. Therefore, the solution of f_2, as described in Myerson (1981), is a step function as follows,

$$f_2(v,k) = \begin{cases} 1 & v \ge p^* \\ 0 & \text{otherwise} \end{cases}$$

p^* is any p that maximizes $p\big(1 - G_v(p)\big)$

Since we have picked an arbitrary k, and this allocation function is independent of k, the point-wise maximization must yield a POSTED PRICE mechanism. We need to verify that the constraints (C2) and (C3) are also satisfied. But since we have shown in Proposition 1 that a POSTED PRICE mechanism is IC mechanism; this fact together with Proposition 3 implies constraints (C2) and (C3) are satisfied. ■

4 Concluding Remarks

Often, models in multidimensional are intractable, even in the two-dimensional case. Even if some of the models are tractable, it is hard to derive a reduced-form solution for the optimal mechanism. In this paper, we consider a two-dimensional private information model with a 'separation' in the dimensions. While one dimension captures the value of the bundle, the other represents the ratio of consumption. This feature helps us solve the problem and provide a reduced-form solution that is simple and intuitive. The POSTED PRICE mechanism can be described by one parameter and involves a finite menu of outcomes. While the RATIO-DEPENDENT POSTED PRICE mechanism involves a potentially infinite size of the menu, it has a simple feature of allocating the primary good fully and the secondary good in the desired ratio.

There are three main directions we intend to extend this work. First, to explore results in a broader class of distributions, and identifying non-wasteful mechanisms beyond RATIO-DEPENDENT POSTED PRICE mechanism. Second, to analyze a multi-good perfect complements model. Third, to consider a scenario in which multiple agents compete for the same pair of complementary goods.

References

Armstrong, M. (1996): "Multiproduct Nonlinear Pricing," *Econometrica*, 64, 51–75.

Bikchandani, S. and D. Mishra (2020): "Selling Two Identical Objects," *working paper.*

Carroll, G. (2017): "Robustness and Separation in Multidimensional Screening," *Econometrica*, 85, 453–488.

Che, Y.-K. and I. Gale (2000): "The optimal mechanism for selling to a budget-constrained buyer," *Journal of Economic Theory*, 92, 198–233.

Devanur, N. R., N. Haghpanah, and A. Psomas (2020): "Optimal multi-unit mechanisms with private demands," *Games and Economic Behavior*, 121, 482–505.

Fiat, A., K. Goldner, A. R. Karlin, and E. Koutsoupias (2016): "The FedEx Problem," *Proceedings of the 2016 ACM Conference on Economics and Computation - EC 16.*

Hart, S. and P. J. Reny (2015): "Maximal revenue with multiple goods: Nonmonotonicity and other observations," *Theoretical Economics*, 10, 893–922.

Manelli, A. M. and D. R. Vincent (2006): "Bundling as an optimal selling mechanism for a multiple-good monopolist," *Journal of Economic Theory*, 127, 1–35.

——— (2007): "Multidimensional mechanism design: Revenue maximization and the multiple-good monopoly," *Journal of Economic Theory*, 137, 153–185.

Mcafee, R. and J. Mcmillan (1988): "Multidimensional incentive compatibility and mechanism design," *Journal of Economic Theory*, 46, 335–354.

Mishra, D. and K. Paramahamsa (2018): "Selling to a naive (agent, manager) pair," *working paper.*

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

Pavlov, G. (2011): "A Property of Solutions to Linear Monopoly Problems," *The B.E. Journal of Theoretical Economics*, 11, 1–18.

Rochet, J.-C. and P. Chone (1998): "Ironing, Sweeping, and Multidimensional Screening," *Econometrica*, 66, 783–826.

Thanassoulis, J. (2004): "Haggling over substitutes," *Journal of Economic Theory*, 117, 217–245.

A Appendix: Omitted Proofs

A.1 Proof of the Proposition 2.

Proof: Fix an IC and IR mechanism (f,p) and define (f',p') as follows,

$$(f'(v,k),p'(v,k)) := \begin{cases} \big(f_1(v,k), \frac{f_1(v,k)}{k}, p(v,k)\big) & \text{if } \frac{f_1(v,k)}{k} \leq f_2(v,k) \\ \big(kf_2(v,k), f_2(v,k), p(v,k)\big) & \text{if } \frac{f_1(v,k)}{k} > f_2(v,k). \end{cases}$$

The new mechanism generates as much revenue as the original mechanism and satisfies the *non-wasteful* allocation condition. Showing that it satisfies IC and IR conditions will prove the proposition. Fix any type (v,k) and to show that this type does not deviate to some other type (u,j), we do this in two cases.

Case 1 - $\frac{f_1(u,j)}{j} \leq f_2(u,j)$.

$$\begin{aligned} U_{(v,k)}(f'(v,k),p'(v,k)) &= U_{(v,k)}(f(v,k),p(v,k)) \\ &\geq U_{(v,k)}(f(u,j),p(u,j)) \\ &= v\min\{\frac{f_1(u,j)}{k}, f_2(u,j)\} - p(u,j) \\ &\geq v\min\{\frac{f_1(u,j)}{k}, \frac{f_1(u,j)}{j}\} - p(u,j) \\ &= U_{(v,k)}(f_1(u,j), \frac{f_1(u,j)}{j}, p(u,j)) \\ &= U_{(v,k)}(f'(u,j),p'(u,j)). \end{aligned}$$

Case 2 - $\frac{f_1(u,j)}{j} > f_2(u,j)$.

$$\begin{aligned} U_{(v,k)}(f'(v,k),p'(v,k)) &= U_{(v,k)}(f(v,k),p(v,k)) \\ &\geq U_{(v,k)}(f(u,j),p(u,j)) \\ &= v\min\{\frac{f_1(u,j)}{k}, f_2(u,j)\} - p(u,j) \\ &\geq v\min\{\frac{jf_2(u,j)}{k}, f_2(u,j)\} - p(u,j) \\ &= U_{(v,k)}(jf_2(u,j), f_2(u,j), p(u,j)) \\ &= U_{(v,k)}(f'(u,j),p'(u,j)). \end{aligned}$$

In both the cases, first inequality is by incentive compatibility of (f,p), second inequality by the condition that defines the particular case, the first and last equations by construction of (f', p'), and the rest by definitions. Using first equations and the fact that (f, p) is IR implies that (f', p') is IR. ■

A.2 Proof of the Proposition 3.

Proof: Let a mechanism $(f, p) \in \mathcal{M}$ be IC, then to show (1) and (2) fix some k. For any $v' > v$, consider the following IC constraints,

$$
\begin{aligned}
(v, k) \to (v', k) &\equiv v f_2(v, k) - p(v, k) \geq v f_2(v', k) - p(v', k) \\
(v', k) \to (v, k) &\equiv v' f_2(v', k) - p(v', k) \geq v' f_2(v, k) - p(v, k).
\end{aligned}
$$

After suppressing k in the above inequalities notice that these are the standard one-dimensional IC constraints between two types v, v'. Therefore, in similar fashion to the one-dimensional problem we get (1) by adding the inequalities. For any k applying Myerson (1981)'s revenue equivalence formula we get

$$
p(v, k) = p(0, k) + v f_2(v, k) - \int_0^v f_2(t, k) dt \text{ for all } v
$$

Applying Lemma 1 to this expression we get (2).

To show (3) and (4) consider any $v, k' > k$. IC constraint $(v, k) \to (v, k')$ implies that,

$$
\begin{aligned}
& U_{(v,k)}(f(v, k), p(v, k)) \geq U_{(v,k)}(f(v, k'), p(v, k')) \\
\implies & v f_2(v, k) - p(v, k) \geq v \min\{\frac{f_1(v, k')}{k}, f_2(v, k')\} - p(v, k') \\
& = v \min\{\frac{k' f_2(v, k')}{k}, f_2(v, k')\} - p(v, k') \\
& = v f_2(v, k') - p(v, k') \\
\implies & \int_0^v f_2(t, k) dt \geq \int_0^v f_2(t, k') dt
\end{aligned}
$$

IC constraint $(v,k') \to (v\frac{k}{k'},k)$ implies that,

$$
\begin{aligned}
& U_{(v,k')}(f(v,k'),p(v,k')) \geq U_{(v,k')}(f(v\frac{k}{k'},k),p(v\frac{k}{k'},k)) \\
\implies \; & vf_2(v,k') - p(v,k') \geq v\min\{\frac{f_1(v\frac{k}{k'},k)}{k'}, f_2(v\frac{k}{k'},k)\} - p(v\frac{k}{k'},k) \\
& = v\min\{\frac{k}{k'}f_2(v\frac{k}{k'},k), f_2(v\frac{k}{k'},k)\} - p(v\frac{k}{k'},k) \\
& = v\frac{k}{k'}f_2(v\frac{k}{k'},k) - p(v\frac{k}{k'},k) \\
& = U_{(v\frac{k}{k'},k)}(f(v\frac{k}{k'},k),p(v\frac{k}{k'},k)) \\
\implies \; & \int_0^v f_2(t,k')dt \geq \int_0^{v\frac{k}{k'}} f_2(t,k)dt
\end{aligned}
$$

The first equality in both the constraints uses the fact that $(f,p) \in \mathcal{M}$. The second implication uses the necessary condition (2) of this Proposition.

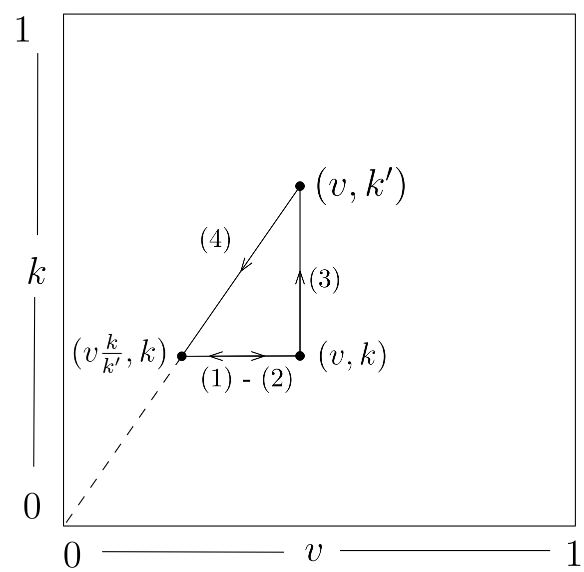

Figure 4: IC Constraints

For the only if part, fix any k and notice that IC constraints of the type $(v,k) \to (v',k')$ when $k = k'$ are satisfied by conditions (1) and (2) as this is equivalent to standard one-dimensional one agent model. Therefore, it is enough to show that any type (v,k) does not deviate to a (v',k') in the following two cases.

Case 1: $k' > k$**.**

$$\begin{aligned}
U_{(v,k)}(f(v,k),p(v,k)) &= vf_2(v,k) - p(v,k) \\
&= \int_0^v f_2(t,k)dt - p(0,1) \\
&\geq \int_0^v f_2(t,k')dt - p(0,1) \\
&= vf_2(v,k') - p(v,k') \\
&\geq vf_2(v',k') - p(v',k') \\
&= v\min\{\frac{k'}{k}f_2(v',k'), f_2(v',k')\} - p(v',k') \\
&= U_{(v,k)}(f(v',k'),p(v',k'))
\end{aligned}$$

The second and third equation uses condition (2). The second inequality is from IC constraint $(v,k') \to (v',k')$ which in turn come from conditions (1) and (2) as argued already. The first inequality is from condition (3).

Case 2: $k' < k$**.**

$$\begin{aligned}
U_{(v,k)}(f(v,k),p(v,k)) &= vf_2(v,k) - p(v,k) \\
&= \int_0^v f_2(t,k)dt - p(0,1) \\
&\geq \int_0^{v\frac{k'}{k}} f_2(t,k')dt - p(0,1) \\
&= v\frac{k'}{k}f_2(v\frac{k'}{k},k') - p(v\frac{k'}{k},k') \\
&\geq v\frac{k'}{k}f_2(v',k') - p(v',k') \\
&= v\min\{\frac{k'}{k}f_2(v',k'), f_2(v',k')\} - p(v',k') \\
&= U_{(v,k)}(f(v',k'),p(v',k'))
\end{aligned}$$

The second and third equation uses condition (2), the second inequality is from IC constraint $(v\frac{k'}{k},k') \to (v',k')$ which in turn come from conditions (1) and (2) as argued already. The first inequality is from condition (4). ■

A.3 Proof of the Theorem 2.

We solve for the optimal mechanism by ignoring the constraint (C3). We show that the optimal in this reduced problem is a POSTED PRICE mechanism. We first prove the following

Lemma towards this.

Lemma **3** *If G satisfies* Condition A *then for every mechanism $(f,p) \in \mathcal{M}$ that satisfies constraints (C1), (C2), then the mechanism $(f',p') \in \mathcal{M}$ defined by,*

$$f_2'(v,k) = \begin{cases} 0 & \text{if } v \leq 1 - \int_0^1 f_2(t,k)dt \\ 1 & \text{otherwise.} \end{cases}$$

satisfies constraints (C1), (C2) and generates more(weakly) expected revenue than (f,p).

Proof: It is straightforward to see that constraint (C1) is satisfied. For (C2), observe that, for any (v,k),

$$\int_0^v f_2'(t,k)dt = \begin{cases} 0 & \text{if } v \leq 1 - \int_0^1 f_2(t,k)dt \\ v - 1 + \int_0^1 f(t,k)dt & \text{otherwise.} \end{cases} \tag{1}$$

Fix any $k' > k$, and since (f,p) satisfies constraint (C2) we have,

$$\int_0^1 f_2(t,k')dt \leq \int_0^1 f_2(t,k)dt. \tag{2}$$

If $v \leq 1 - \int_0^1 f_2(t,k')dt$, then $\int_0^v f_2'(t,k')dt = 0 \leq \int_0^v f_2'(t,k)dt$, as $f_2'(v,k) \geq 0 \ \forall (v,k)$.

Else if $v > 1 - \int_0^1 f_2(t,k')dt$, then $v > 1 - \int_0^1 f_2(t,k)dt$ by equation 2. Therefore, $\int_0^v f_2'(t,k')dt = v - 1 + \int_0^1 f_2(t,k') \leq v - 1 + \int_0^1 f_2(t,k) = \int_0^v f_2'(t,k)dt$. The inequality is by equation 2. The equations are by expression 1.

Now we show that (f',p') generates weakly more expected revenue than (f,p). Fix any k. Denote $\beta_{(f,p,k)} := 1 - \int_0^1 f_2(t,k)dt$ and consider the difference in expected revenue of the two mechanisms,

$$\begin{aligned} \int_0^1 \phi(v,k)g(v|k)\big(f_2'(v,k) - f_2(v,k)\big)dv &= \int_{\beta_{(f,p,k)}}^1 \phi(v,k)g(v|k)\big(f_2'(v,k) - f_2(v,k)\big)dv \\ &\quad - \int_0^{\beta_{(f,p,k)}} \phi(v,k)g(v|k)f_2(v,k)dv \\ &\geq \phi(\beta_{(f,p,k)},k)g(\beta_{(f,p,k)}|k) \int_{\beta_{(f,p,k)}}^1 \big(f_2'(v,k) - f_2(v)\big)dv \\ &\quad - \phi(\beta_{(f,p,k)},k)g(\beta_{(f,p,k)}|k) \int_0^{\beta_{(f,p,k)}} f_2(v,k)dv \\ &= \phi(\beta_{(f,p,k)},k)g(\beta_{(f,p,k)}|k)\Big(\int_0^1 (f_2'(v,k) - f_2(v,k))dv\Big) \\ &= 0 \end{aligned}$$

The equations use the definition of (f', p') and rearranging of terms, the inequality is from the fact that $\phi(v,k)g(v|k)$ is increasing. Since we have shown this for an arbitrary k therefore expected revenue from (f', p') is greater(weakly) than (f, p). ■

Proof of Theorem 2.
Lemma 3 implies that, without loss of generality, we can focus on mechanisms $(f, p) \in \mathcal{M}$ such that there exists $\rho(k)$ **increasing** in k and,

$$f_2(v,k) = \begin{cases} 0 & \text{if } v \leq \rho(k) \\ 1 & \text{otherwise.} \end{cases}$$

ρ is increasing because (C2) is satisfied in Lemma 3, and due to the definition of f' in Lemma 3. We will show that we can improve such a mechanism to a POSTED PRICE mechanism. Fix any such mechanism (f, p) and note that CONDITION B′ implies $\phi_k^{-1}(0) > \phi_{k'}^{-1}(0)\ \forall k' > k$. Consider the following three mutually exclusive and exhaustive cases:

1. $\rho(1) \leq \phi_1^{-1}(0)$. Consider the following mechanism (f', p') defined by,

   $$f_2'(v,k) = \begin{cases} 0 & \text{if } v \leq \rho(1) \\ 1 & \text{otherwise.} \end{cases}$$

   Fix any k, note that $\rho(k) \leq \rho(1)$. If $v \leq \rho(k)$ or $v > \rho(1)$ then $f_2'(v,k) = f_2(v,k)$. $\rho(1) \leq \phi_1^{-1}(0) \leq \phi_k^{-1}(0)$ implies $\phi(\rho(1),k)g(\rho(1)|k) \leq \phi(\phi_k^{-1}(0),k)g(\phi_k^{-1}(0)|k) = 0$ since $\phi(v,k)\rho(v|k)$ is increasing in v. This also implies $\int_{\rho(k)}^{\rho(1)} f_2(v,k)\phi(v,k)g(v|k)dv \leq 0$. Noticing $\int_{\rho(k)}^{\rho(1)} f_2'(v,k)\phi(v,k)g(v|k)dv = 0$ by construction implies revenue in (f', p') is more than that of (f, p).

   With some abuse of notation , we use $\rho(0^+)$ to denote $\lim_{k\to 0^+}\rho(k)$, and $\phi_{0^+}^{-1}(0)$ to denote $\lim_{k\to 0^+}\phi_k^{-1}(0)$.

2. $\rho(1) > \phi_1^{-1}(0)$ and $\rho(0^+) < \phi_{0^+}^{-1}(0)$. ρ is increasing in k, and $\phi_k^{-1}(0)$ is strictly decreasing in k and continuous, hence the function $\rho(k) - \phi_k^{-1}(0)$ is strictly increasing (and continuous a.e.). Therefore, there exists a unique k^* such that $\rho(k) > \phi_k^{-1}(0)\ \forall k > k^*$ and $\rho(k) < \phi_k^{-1}(0)\ \forall k < k^*$. Let $v^* := \rho(k^*)$. Define a POSTED PRICE mechanism (f', p') as follows,

   $$f_2'(v,k) = \begin{cases} 0 & \text{if } v \leq v^* \\ 1 & \text{otherwise.} \end{cases}$$

   We show that (f', p') generates more expected revenue than (f, p) for every k in two following cases.

(a) Fix any $k > k^*$. Note that $v^* \leq \rho(k)$. If $v \leq v^*$ or $v > \rho(k)$ then $f_2'(v,k) = f_2(v,k)$. Since $\phi_k^{-1}(0) \leq \phi_{k^*}^{-1}(0) = v^*$ and $\phi(v,k)g(v|k)$ increasing in v we have $\phi(v,k)g(v|k) > 0$ for all $v > v^*$. Therefore, $\int_{v^*}^{\rho(k)} \big(f_2'(v,k) - f_2(v,k)\big)\phi(v,k)g(v|k)dv \geq 0$ since $f_2'(v,k) = 1$ in this range.

(b) Fix any $k < k^*$. Note that $v^* \geq \rho(k)$. If $v > v^*$ or $v \leq \rho(k)$ then $f_2'(v,k) = f_2(v,k)$. Since $\phi_k^{-1}(0) \geq \phi_{k^*}^{-1}(0) = v^*$ and $\phi(v,k)g(v|k)$ increasing in v we have $\phi(v,k)g(v|k) < 0$ for all $v < v^*$. Therefore, $\int_{\rho(k)}^{v^*} \big(f_2'(v,k) - f_2(v,k)\big)\phi(v,k)g(v|k)dv \geq 0$ since $f_2'(v,k) = 0$ in this range.

3. $\rho(0^+) \geq \phi_{0^+}^{-1}(0)$. Consider the following mechanism (f', p') defined by,

$$f_2'(v,k) = \begin{cases} 0 & \text{if } v \leq \rho(0^+) \\ 1 & \text{otherwise.} \end{cases}$$

Fix any k and note that $\rho(k) \geq \rho(0^+)$. If $v \leq \rho(0^+)$ then $f_2'(v,k) = f_2(v,k)$. Since $\phi_k^{-1}(0) \leq \phi_{0^+}^{-1}(0) \leq \rho(0^+)$ and $\phi(v,k)g(v|k)$ increasing in v we have $\phi(v,k)g(v|k) > 0$ for all $v > \rho(0^+)$. Therefore, $\int_{\rho(0^+)}^{1} \big(f_2'(v,k) - f_2(v,k)\big)\phi(v,k)g(v|k)dv \geq 0$ since $f_2'(v,k) = 1$ in this range.

In each of the above three cases, we have shown that the revenue is higher in a POSTED PRICE mechanism for an arbitrary k. Therefore, a POSTED PRICE mechanism is optimal in the reduced problem we considered. This also implies it is the optimal mechanism since we have shown that a POSTED PRICE mechanism satisfies all the constraints, including the ignored constraint (C3).