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Selling to a principal and a budget-constrained agent*

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Abstract

We analyze a model of selling a single object to a principal-agent pair who want to acquire the object for a firm. The principal and the agent have different assessments of the object's value to the firm. The agent is budget-constrained while the principal is not. The agent participates in the mechanism, but she can (strategically) approach the principal for decision-making. We derive the revenue-maximizing mechanism in a two-dimensional type space (values of the agent and the principal). We show that below a threshold budget, a mechanism involving two posted prices and three outcomes (one of which involves randomization) is the optimal mechanism for the seller. Otherwise, a single posted price mechanism is optimal.

JEL Classification number: D82

Keywords: budget constraint, posted price, multidimensional mechanism design, behavioral mechanism design

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1 Introduction

A seller is selling an object to a firm. An agent (the manager of the firm) participates in a mechanism to acquire the object. The agent's payment cannot exceed a budget. However, the agent can approach the principal (the board of directors), who is not budget-constrained, for decision-making.

The object is of value to the firm, where the principal and the agent are shareholders. So, they both want to maximize the firm's payoff from acquiring the object. However, they evaluate the value of the object to the firm differently. In particular, we assume that the agent finds more value in the object than the principal. The difference in valuation may be because the agent gets some additional personal value when the firm acquires the object or because the agent is more informed about the potential uses of the object. We assume that the value to the principal is common knowledge among the principal and the agent, but the value to the agent is privately known to her.[1] The seller does not know the values of the principal and the agent but observes the agent's budget constraint. The agent's budget constraint may be reflected by the liquidity constraint of the firm, which may be verified from publicly available information.

Our agent approaches the principal for decision-making if the best outcome from the menu of the mechanism in her *budget set* is worse than the principal's best outcome from the menu of the mechanism. Otherwise, the agent chooses the best outcome from the menu of the mechanism in her budget set.

What is the expected revenue-maximizing mechanism of the seller? The communication between the principal and the agent (when the budget constraint binds for the agent) has different implications on incentive constraints than in the usual mechanism design problems. We show that a revelation principle holds and a new class of (direct) mechanisms, which we call POST-2 mechanisms, are incentive compatible.

In a POST-2 mechanism, the seller posts two prices: κ_1, κ_2, both above the budget b. If the agent has value less than κ_1, the object is not allocated (and zero payment is made).

[1] The common knowledge of the value of the principal among the principal-agent pair may be because the agent knows some common attributes of the object that the principal uses to evaluate the object. On the other hand, the principal does not know other uses of the object or the personal value of the object to the agent. The agent cannot persuade the principal about the extra value of the object.

Else, a fraction $\frac{b}{\kappa_1}$ of the object is allocated at a price equal to the budget b. Further, if the principal has value more than κ_2 (and the agent has value more than κ_1), the remaining fraction of the object $\left(1 - \frac{b}{\kappa_1}\right)$ is allocated at a price $\left(1 - \frac{b}{\kappa_1}\right)\kappa_2$. This mechanism is akin to a two-part tariff mechanism.

A simpler class of mechanisms ignores the value of the principal and considers only the value of the agent. By definition, such a mechanism cannot charge more than the budget. A POST-1 mechanism is such a posted price mechanism with a price $\kappa \leq b$.

Our main result says that there is a threshold b^* such that for all budgets $b < b^*$, the optimal mechanism is a POST-2 mechanism and for all $b \geq b^*$, the optimal mechanism is a POST-1 mechanism. The threshold corresponds to the optimal posted-price of an *unconstrained agent* whose values are drawn using the marginal distribution of the current model, i.e., b^* is the optimal solution to $\max x(1 - F_1(x))$, where F_1 is the marginal distribution of the value of the agent. For all our results, we assume that $x(1 - F_1(x))$ is strictly concave, an assumption satisfied by many distributions (which also allow for correlation among values of the principal and the agent). This allows us to identify b^* uniquely. Under stronger conditions on distributions, we give a more precise description of the optimal mechanism.

These results are in contrast with the single object mechanism design literature, where a deterministic posted-price mechanism is optimal (Mussa and Rosen, 1978). The deterministic optimality result usually does not extend to multidimensional mechanism design problems. Even for two-dimensional mechanism problems, the menu in the optimal mechanism may contain an infinite set of outcomes (Hart and Nisan, 2019). On the other hand, ours is a two-dimensional mechanism design problem, and the optimal mechanism has three outcomes in the menu. The particular nature of decision-making allows tractability in our two-dimensional model and results in a simple solution.

Besides the agent and the board of directors example highlighted throughout the paper, there are other settings where such decision-making seems plausible: a sports team owner (principal) and the team manager (agent) trying to recruit a new player; the dean and a department hiring a new faculty; parents (principal) and their child (agent) buying a product. Our results shed light on the nature of the optimal mechanism in these settings.

1.1 An illustration

Suppose the values of the agent (v_1) and the principal (v_2) are distributed in the triangle: $\{(v_1, v_2) \in [0,1]^2 : v_1 \geq v_2\}$. Consider a simple posted-price mechanism $p > b$ such that the object is allocated at price p if the principal has value more than p. Otherwise, the object is not allocated and zero payment is made. The left triangle in Figure 1 illustrates the mechanism. This mechanism is incentive compatible (IC) since the agent and the principle prefer getting the object when $v_1 \geq v_2 \geq p$. The agent approaches the principal for decision-making in this case (since $p > b$) and chooses not to buy otherwise.

However, the seller can improve expected revenue from this mechanism by the following mechanism. The seller does not allocate the object if the value of the agent is less than p. Types (v_1, v_2) with $v_1 \geq p$ but $v_2 < p$ get the object with probability $\frac{b}{p}$ and pay b. Types (v_1, v_2) with $v_1 \geq v_2 \geq p$ get the object with probability 1 and pay p. So, the seller allocates the object more often in this mechanism and collects more revenue from types (v_1, v_2) with $v_1 \geq p$ and $v_2 < p$. The right triangle in Figure 1 illustrates this mechanism.

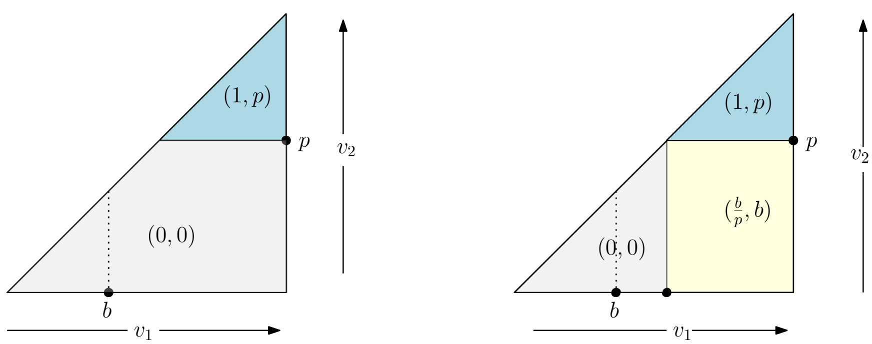

Figure 1: Illustration of new IC mechanisms

Why is this mechanism incentive compatible? There are three types of principal-agent pairs corresponding to the three outcomes in the menu of the mechanism. When $v_2 \leq v_1 < p$, both the agent and the principal prefer the outcome of not getting the object and zero payment to other outcomes in the menu. Similarly, when $p \leq v_2 \leq v_1$, both the agent and the principal prefer getting the object at a price p to the other outcomes in the menu. In this case, the agent approaches the principal for decision-making (since $p > b$).

The types (v_1, v_2) with $v_1 \geq p > v_2$ require a careful look. At any such type (v_1, v_2), the agent's best outcome in her *budget set* is to get the object with probability $\frac{b}{p}$ at price b, since $\frac{b}{p}(v_1 - p) \geq 0$. However, the agent prefers the outcome outside her budget set to this outcome: $v_1 - p \geq \frac{b}{p}(v_1 - p)$ with strict inequality holding if $v_1 > p$. But she knows that the best outcome for the principal is to not get the object and pay zero: $v_2 - p < \frac{b}{p}(v_2 - p) < 0$. So, the agent is strategic and chooses the best outcome in her budget set. This makes the mechanism incentive compatible.

It is worth noting that when $v_2 < p \leq v_1$, the principal gets a negative payoff in the mechanism. Our interpretation is that the principal incurs a cost in making decisions. On average, considering this cost, the principal's ex-ante payoff is positive. Hence, the principal is happy to delegate decision-making to the agent with a budget constraint. We do not model this explicitly but discuss models in the literature that study such contracts between the principal and the agent. In section 5, we illustrate with an example that the principal's ex-ante payoff is positive in our optimal mechanism.

The seller is able to exploit this by offering an additional outcome in the menu of the mechanism that the agent will find optimal to accept at some types. This allows the seller to extract more revenue. Though there are other mechanisms that can allow the seller to extract more revenue, we show that an optimal mechanism is a simple one like in Figure 1. Thus, our result says that there is an upper bound (achieved by our optimal mechanism) on the revenue extracted by the seller in these decision making environments.

1.2 Layout of the paper

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 has our main results characterizing the optimal mechanism. Section 4 relates our model and results to the literature. Finally, we end with some discussions of our modeling assumptions in Section 5. All our missing proofs and the discussions on our revelation principle are in an appendix at the end.

2 The model

We now formally introduce our model. There is a single object which is sold by a seller to a firm. An agent and her principal are the decision-makers for the firm. We describe the preferences of the agent and the principal and the sequence of decison-making in our model.

We index the agent by 1 and the principal by 2 in the paper.

Preferences. Let Z be the set of all outcomes, i.e., $Z := \{(a,t) : a \in [0,1], t \in \mathbb{R}\}$, where a is allocation probability and t is the payment made to the seller. Let b be the budget of the agent. That is, the agent cannot pay more than b. Throughout the paper, we will assume that b is common knowledge among the agent, the principal and the seller. For every $X \subseteq Z$, define $X_b := \{(a,t) \in X : t \leq b\}$ to be the *budget set* of the agent.

We now introduce some notation to define how the principal-agent pair chooses outcomes from a given set of outcomes. Let $v_i \in [0, \beta]$ be any valuation, where $\beta \in \mathbb{R}_{++}$. Define for every valuation v_i and every $X \subseteq Z$,

$$Ch(X; v_i) = \{(a,t) \in X : v_i a - t \geq v_i a' - t' \ \forall\ (a', t') \in X\}$$

Note that Ch is a standard choice correspondence satisfying independence of irrelevant alternatives. That is, if $X' \subseteq X$ and $Ch(X; v_i) \subseteq X'$, then $Ch(X'; v_i) = Ch(X; v_i)$.

For any pair of sets of outcomes $X, Y \subseteq Z$, we say

$$X \trianglerighteq_{v_i} Y \text{ if } v_i a - t \geq v_i a' - t' \qquad \forall\ (a,t) \in X,\ \forall\ (a', t') \in Y$$

We write $X \triangleright_{v_i} Y$ if the above inequality is strict at least for one pair of outcomes. If $X \trianglerighteq_{v_i} Y$ and $Y \trianglerighteq_{v_i} X$, then we say $X \sim_{v_i} Y$. Note that for any $v_i \in [0, \beta]$, the relations $\triangleright_{v_i}$ and $\trianglerighteq_{v_i}$ are transitive but incomplete. These relations will be used to define choices of our principal-agent pair.

The type is a pair of valuations $v \equiv (v_1, v_2)$, where v_1 is the value of the agent and v_2 is the value of the principal. We assume that the value of the agent is higher than that of the principal. So, the type space is

$$V = \{(v_1, v_2) \in [0, \beta]^2 : v_1 \geq v_2\}$$

For a vector of valuations $v = (v_1, v_2)$, the choice from any $X \subseteq Z$ is given by

$$Ch(X; v) = \begin{cases} Ch(X; v_2) & \text{if } Ch(X; v_2) \triangleright_{v_1} Ch(X_b; v_1) \\ Ch(X_b; v_1) & \text{otherwise} \end{cases}$$

Note that if $Ch(X; v_2) \triangleright_{v_1} Ch(X_b; v_1)$ is not true, then either $Ch(X; v_2) \sim_{v_1} Ch(X_b; v_1)$ or there exists an outcome $(a', t') \in Ch(X; v_2)$ such that $v_1 a - t > v_1 a' - t'$ for all $(a, t) \in$

$Ch(X_b; v_1)$.[2]

The choice correspondence $Ch(X; v)$ describes how choice is made by the principal-agent pair from a set of outcomes X. In particular, if v is the type vector, then the agent first evaluates her choice correspondence from the budget set: $Ch(X_b; v_1)$. She then evaluates the (unconstrained) choice correspondece of the principal: $Ch(X; v_2)$. The evaluation of $Ch(X; v_2)$ critically uses the fact that the agent knows the value v_2 of the principal. Now, the agent compares $Ch(X; v_2)$ and $Ch(X_b; v_1)$ using her own value v_1. If the unconstrained correspondence of the principal is strictly better than the constrained correspondence of the agent according to $\rhd_{v_1}$, she approaches the principal for decision-making. In that case, we set $Ch(X; v)$ equal to $Ch(X; v_2)$. Else, the agent makes the decision and her constrained choice correspondence $Ch(X_b; v_1)$ equals $Ch(X; v)$. The choice correspondence $Ch(X; v)$ denotes how choices are made in our model. It is not difficult to see that $Ch(X; v)$ need not satisfy independence of irrelevant alternatives.

Mechanism. A mechanism is a pair (q, p), where $q : V \to [0, 1]$ is the allocation rule and $p : V \to \mathbb{R}$ is the payment rule. The range of the mechanism is $R(q, p) := \{(q(v), p(v)) : v \in V\}$. A mechanism (q, p) is **incentive compatible (IC)** if

$$\big(q(v), p(v)\big) \in Ch(R(q, p); v)$$

This notion of IC is standard in the literature of behavioral mechanism design (de Clippel, 2014). Since the choice correspondence Ch may fail independence of irrelevant alternatives, it is not clear if restricting attention to direct mechanisms is without loss of generality. For instance, Saran (2011) shows that the revelation principle may fail in models with *behavioral* agents. In Appendix C, we show that a version of the revelation principle holds in our setting, and hence, it is without loss of generality to focus attention to such direct mechanisms.

A mechanism (q, p) is **individually rational (IR)** if $(0, 0) \in R(q, p)$.

It is important to emphasise the timing of the game induced by the direct mechanism.

- The seller announces the mechanism (q, p). Let $X \equiv R(q, p)$ be its range.
- The agent learns her value v_1 and the principal's value v_2.

[2] In the above definition $Ch(X; v)$ refers to a two-dimensional $v \equiv (v_1, v_2)$ but the second argument of $Ch(X; v_2)$ (and $Ch(X_b; v_1)$) is one-dimensional. This abuse of notation saves us from introducing a new notation.

- If $Ch(X; v_2) \rhd_{v_1} Ch(X_b; v_1)$, then
  - the agent approaches the principal for decision-making
  - the principal learns his type v_2 and chooses an outcome in $Ch(X; v_2)$
- If $Ch(X; v_2) \rhd_{v_1} Ch(X_b; v_1)$ is *not true*, then the agent chooses an outcome in $Ch(X_b; v_1)$.

As highlighted earlier in the introduction, there are couple of features of this model that make it tractable. First, the agent learns her own value and that of the principal. This can be interpreted in two ways. In the first interpretation, the agent knows all possible uses of the object to the firm but also knows the possible uses of the object that the principal (board of directors) knows. An alternate interpretation is that the value of the principal v_2 is the actual value of the object to the firm, which is common knowledge between the principal and the agent. But the agent gets additional payoff (for instance, reputation payoffs) from acquiring the object which is privately known to her. For instance, suppose the object has a value v_2 to the firm. Then, the principal gets a payoff of $\alpha_2(v_2 - p)$ by acquiring the object at price p, where α_2 is the share of the principal in the firm. Thus, v_2 will be the value of the object to the principal. The agent gets a payoff of $\alpha_1(v_2 - p) + \delta_1$, where α_1 is the share of the agent in the firm and δ_1 is the additional payoff of the agent. Hence, we can interpret $v_1 := v_2 + \frac{\delta_1}{\alpha_1}$ as the value to the agent.

Second, there is no formal contract between the principal and the agent about how the agent will behave in the mechanism. Models studied in Burkett (2015, 2016) look at such contracts between the principal and the agent given a mechanism. In our model, the principal is happy to let the agent make decisions in the mechanism as long as the liquidity constraint of the firm is not violated.

Finally, the only communication between the principal and the agent in our model is when the agent approaches the principal for decision-making. In the interpretation where v_1 and v_2 are assessments of values by the agent and the principal respectively, and the agent knows both of them, the agent does not *persuade* the principal about the value of the object to the firm. That is, there is no deliberation between the principal and the agent about the actual value of the object to the firm. Models of mechanism design with such a communication phase is studied in Malenko and Tsoy (2019). In the second interpretation, where v_2 is treated as the actual value of the object to the firm and $v_1 - v_2$ is the additional private value of the agent, our model assumes that the principal cannot question the decision making of the agent ex-post.

Prior. The joint density function of $v \equiv (v_1, v_2)$ is f with support V. As the support of v_2 depends on the realized value of v_1 (since each v satisfies $v_1 \geq v_2$), the values v_1 and v_2 are not independent. We will denote the cdf of v as F. For any IC and IR mechanism (q, p), the **expected revenue** is given by

$$\textsc{Rev}(q, p) = \int_V p(v) f(v) dv$$

A mechanism (q^*, p^*) is **optimal** if it is IC and IR and for every other IC and IR mechanism (q, p), we have $\textsc{Rev}(q^*, p^*) \geq \textsc{Rev}(q, p)$.

We will denote the *marginal distribution* of agent's valuation as F_1 (which admits a density f_1) and principal's valuation as F_2 (which admits a density f_2). The density functions f_1 and f_2 will be assumed to be positive and differentiable in V.

3 Optimal mechanism

We describe our main results in this section. We start by describing two simple classes of mechanisms. Our main result will show that the optimal mechanism belongs to one of these classes of mechanisms. The first class contains posted-price mechanisms for the agent.

Definition **1** *A mechanism (q, p) is a* post-1 *mechanism if there exists $\kappa_1 \in [0, b]$ such that for every $v \in V$, we have*

$$(q(v), p(v)) = \begin{cases} (0, 0) & \text{if } v_1 \leq \kappa_1 \\ (1, \kappa_1) & \text{otherwise} \end{cases}$$

A post-1 mechanism is IC and IR. It only considers valuation of the agent. The following class of mechanisms considers the valuations of the agent and the principal.

Definition **2** *A mechanism (q, p) is a* post-2 *mechanism if there exist $\kappa_1, \kappa_2 \in [b, \beta]$ such*

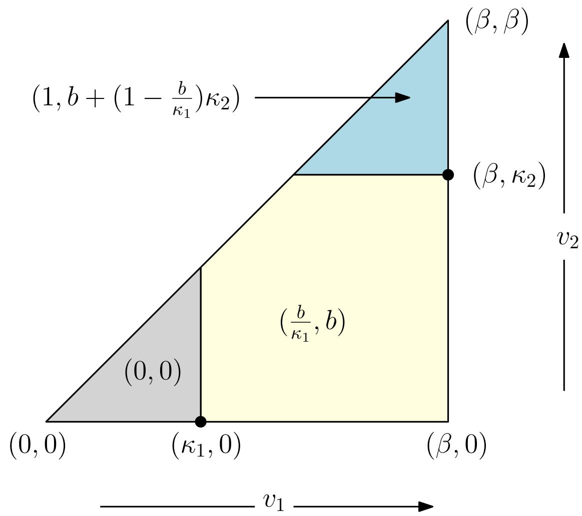

Figure 2: A POST-2 mechanism

that $\kappa_1 \leq \kappa_2$ *and for every* $v \in V$*, we have*

$$(q(v), p(v)) = \begin{cases} (0,0) & \textit{if } v_1 < \kappa_1 \\ \Big(1, b + \kappa_2\big(1 - \frac{b}{\kappa_1}\big)\Big) & \textit{if } v_2 > \kappa_2 \\ \big(\frac{b}{\kappa_1}, b\big) & \textit{otherwise} \end{cases}$$

Figure 2 shows a POST-2 mechanism. A POST-2 mechanism allows for randomization only when $v_2 \leq \kappa_2$ and $v_1 \geq \kappa_1$. In this region the object is allocated with a *fixed* probability $\frac{b}{\kappa_1}$. This ensures that the range of a POST-2 mechanism contains at most three outcomes. The following lemma (proof is in Appendix A) establishes that a POST-2 mechanism is IC and IR.

LEMMA **1** *A* POST-2 *mechanism is IC and IR.*

The main result of the paper is the following.

THEOREM **1** *Suppose* $x(1 - F_1(x))$ *is strictly concave. Then, there exists an optimal mechanism which is either a* POST-1 *or a* POST-2 *mechanism.*

PROOF SKETCH. The proof is quite involved and is presented in the Appendix B. We give a brief sketch here. If we optimize over the class of mechanisms in which all types pay less than b, then standard techniques lead to the optimality of a POST-1 mechanism. In the other case, we start with an arbitrary IC and IR mechanism in which all types pay more than b. We divide the type space into three regions (see Figure 3 in the Appendix B) and construct the restriction of the mechanism into each of these three regions. We provide upper bounds on the revenue of the mechanism in each of these three regions. Finally, we put together these upper bounds and show that a POST-2 mechanism achieves this upper bound. The method of coming up with this partition of the type space into three parts is somewhat involved. It is described in detail in Appendix B.

Theorem 1, and all subsequent results, will maintain the distributional assumption that $x(1-F_1(x))$ is strictly concave. Notice that $x(1-F_1(x))$ is the expected revenue of the seller to sell the object to the (unconstrained) agent (with marginal distribution F_1) at a posted price x. This condition implies single-crossing of virtual value condition used in the single object auction design literature. Since it is only a condition on the marginal distribution of values of the agent, it allows for correlation among the values of the agent and the principal.

When the budget is high, the seller must expect that the agent takes decisions at most of the types. In that case, a POST-1 mechanism may be optimal. Similarly, if the budget is low, the decision-making is given to the principal for more types. So, a POST-2 mechanism may be optimal. The next theorem shows that this intuition holds in general and provides a precise threshold on budget below which POST-2 is optimal and above which POST-1 is optimal.

For each $i \in \{1,2\}$, let $\widetilde{\kappa}_i$ be defined as a maximizer of

$$\max_{x \in [0,\beta]} x(1 - F_i(x))$$

This maximizer will be unique under appropriate conditions on F_i. In particular, when $x(1 - F_1(x))$ is strictly concave, $\widetilde{\kappa}_1$ is uniquely defined. We show that $\widetilde{\kappa}_1$ is the threshold budget level, below which the optimal mechanism is a POST-2 mechanism and above which the optimal mechanism is a POST-1 mechanism.

THEOREM **2** *Suppose $x(1 - F_1(x))$ is strictly concave. Then, the following statements hold.*

1. *If $b < \widetilde{\kappa}_1$, a* POST-2 *mechanism is optimal.*
2. *If $b \geq \widetilde{\kappa}_1$, a* POST-1 *mechanism is optimal.*

Proof: Proof of (1). Since $b < \widetilde{\kappa}_1$ and $x(1-F_1(x))$ is strictly concave, the optimal post-1 mechanism charges b and allocates the object to all types v with $v_1 \geq b$. This generates an expected revenue of $b(1-F_1(b))$. Now, consider the post-2 mechanism with $\kappa_1 = \kappa_2 = b$. This generates an expected revenue of $b(1-F_1(b))$. Hence, the optimal post-2 mechanism is an optimal mechanism.

Proof of (2). Suppose $b \geq \widetilde{\kappa}_1$. Consider an arbitrary post-2 mechanism with prices κ_1, κ_2. Notice that $b \leq \kappa_1 \leq \kappa_2$. Expected revenue from this mechanism is

$$b(1-F_1(\kappa_1)) + (1-\frac{b}{\kappa_1})\kappa_2(1-F_2(\kappa_2)) \leq \frac{b}{\kappa_1}\kappa_1(1-F_1(\kappa_1)) + (1-\frac{b}{\kappa_1})\kappa_2(1-F_1(\kappa_2)) \quad (1)$$

where the inequality follows since F_1 first-order stochastic dominates F_2 (since $v_1 \geq v_2$). If $\kappa_1 = b$, the RHS of (1) is $b(1-F_1(b))$, which is the revenue of a post-1 mechanism with price b. Hence, a post-1 mechanism generates higher expected revenue than this post-2 mechanism.

Now assume $\kappa_1 < b$. This implies the RHS of (1) is convex combination of $\kappa_1(1-F_1(\kappa_1))$ and $\kappa_2(1-F_1(\kappa_2))$. Define

$$\widehat{\kappa} := \frac{b}{\kappa_1}\kappa_1 + (1-\frac{b}{\kappa_1})\kappa_2$$

By strict concavity of $x(1-F_1(x))$, we get that the RHS of (1) is less than or equal to

$$\widehat{\kappa}(1-F_1(\widehat{\kappa})) \leq \widetilde{\kappa}_1(1-F_1(\widetilde{\kappa}_1)) \quad (2)$$

where the inequality follows from the definition of $\widetilde{\kappa}_1$. Since $\widetilde{\kappa}_1 \leq b$, the RHS of (2) is the expected revenue of the optimal post-1 mechanism. This implies that optimal post-1 mechanism generates strictly higher expected revenue than the post-2 mechanism. ■

3.1 Description of optimal mechanisms

We now give more precise descriptions of the optimal mechanism. To remind, a post-2 mechanism is described by two cutoffs, κ_1, κ_2 such that $b \leq \kappa_1 \leq \kappa_2 \leq \beta$. We are interested in the following two classes of post-2 mechanisms. A post-2 mechanism (κ_1, κ_2) is

- an **interior** post-2 mechanism if $b < \kappa_1 < \kappa_2 < \beta$;

- a **uniform** POST-2 mechanism if $b \le \kappa_1 = \kappa_2 \le \beta$.

We are mainly interested to know if an interior or a uniform POST-2 mechanism is optimal when a POST-2 mechanism is optimal. This will require stronger conditions on distributions than assumed in Theorems 1 and 2. We say the marginal distribution F_2 of the value of the principal satisfies **single crossing (SC)** if $x - \frac{1-F_2(x)}{f_2(x)}$ crosses zero exactly one. Notice that $x - \frac{1-F_2(x)}{f_2(x)}$ is negative when $x = 0$ and positive when $x = \beta$. Further, $xf_2(x) - (1 - F_2(x))$ is the negative of the derivative of $x(1 - F_2(x))$. So, the point at which $x - \frac{1-F_2(x)}{f_2(x)}$ equals zero is $\widetilde{\kappa}_2$ and the sign of $x - \frac{1-F_2(x)}{f_2(x)}$ indicates whether $x(1 - F_2(x))$ is increasing or decreasing at x. In particular, below $\widetilde{\kappa}_2$, we have $x(1 - F_2(x))$ is increasing and above $\widetilde{\kappa}_2$, it is decreasing. We will use these facts in our remaining results.

Our first result says that if the optimal POST-2 mechanism is an interior POST-2 mechanism it is a solution to a pair of equations.

PROPOSITION 1 *Suppose $x(1 - F_1(x))$ is strictly concave and F_2 satisfies SC. If the optimal* POST-2 *mechanism (κ_1^*, κ_2^*) is an interior* POST-2 *mechanism, then κ_2^* is the solution to*

$$x = \frac{1 - F_2(x)}{f_2(x)}$$

and κ_1^* is the solution to

$$x^2 f_1(x) = \kappa_2^*(1 - F_2(\kappa_2^*))$$

Proof: The expected revenue generated from an arbitrary POST-2 mechanism (κ_1, κ_2) is given by

$$b(1 - F_1(\kappa_1)) + (1 - \frac{b}{\kappa_1})\kappa_2(1 - F_2(\kappa_2)) = b(1 - F_1(\kappa_1)) + (1 - \frac{b}{\kappa_1})\text{REV}(\kappa_2), \tag{3}$$

where $\text{REV}(\kappa_2) := \kappa_2(1 - F_2(\kappa_2))$. So, (κ_1^*, κ_2^*) maximizes expression (3) under the constraint that $b \le \kappa_1 \le \kappa_2 \le \beta$.

Fixing κ_1^*, by (3), κ_2^* must be a solution to $\max_{x \in [\kappa_1^*, \beta]} x(1 - F_2(x))$. Since F_2 satisfies SC, then the expression $x(1 - F_2(x))$ is increasing till $\widetilde{\kappa}_2$ and decreasing after that. Hence, either $\kappa_2^* = \kappa_1^*$ or $\kappa_2^* = \widetilde{\kappa}_2$. Since $\kappa_1^* < \kappa_2^* < \beta$, we conclude that $\kappa_2^* = \widetilde{\kappa}_2$.

We now denote $\text{REV}_2 := \kappa_2^*(1 - F_2(\kappa_2^*))$. Hence, the expected revenue of any POST-2 mechanism (κ_1, κ_2^*) is given by

$$b(1 - F_1(\kappa_1)) + (1 - \frac{b}{\kappa_1})\text{REV}_2 \tag{4}$$

Since $b < \kappa_1^* < \kappa_2^*$, we conclude that κ_1^* must satisfy first order conditions of the expression in (3). Differentiating with respect to κ_1, the expression in (4), we get

$$-bf_1(\kappa_1) + \frac{b}{(\kappa_1)^2}\textsc{Rev}_2 = \frac{b}{(\kappa_1)^2}\Big(\textsc{Rev}_2 - (\kappa_1)^2 f_1(\kappa_1)\Big) \tag{5}$$

We argue that $x^2 f_1(x)$ is strictly increasing. This is because its derivative $2xf_1(x) + x^2\frac{df_1}{dx} = x(2f_1(x) + x\frac{df_1}{dx}) > 0$ by strict concavity of $x(1-F_1(x))$. Hence, expression (5) is positive till κ_1 is such that $\textsc{Rev}_2 = (\kappa_1)^2 f_1(\kappa_1)$ and negative after that. This means there is a unique optimum to the revenue expression in (4), which is given by the solution to $x^2 f_1(x) = \textsc{Rev}_2$. ■

The following theorem identifies sufficient conditions under which an optimal mechanism is a uniform POST-2 mechanism. The distributional assumptions stated below play an important role.

We say joint distribution F satisfies **monotone likelihood ratio property (MLRP)** if $\frac{f_1(x)}{f_2(x)}$ is increasing in x, where f_i is the density function of F_i for each $i \in \{1, 2\}$. The MLRP property implies that F_1 first-order stochastic-dominates F_2 (which comes for free in our model). This implies that $\widetilde{\kappa}_1 > \widetilde{\kappa}_2$.

Under these distributional assumptions, the theorem below characterizes the optimal POST-2 mechanism whenever the optimal POST-2 mechanism generates strictly more revenue than the optimal POST-1 mechanism.

THEOREM **3** *Suppose F satisfies the MLRP property, $x(1-F_1(x))$ is strictly concave, and F_2 satisfies SC. Then, the following are true if the optimal* POST-2 *mechanism generates strictly higher expected revenue than the optimal* POST-1 *mechanism.*

1. *If $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} \le 1$, then the optimal mechanism is a uniform* POST-2 *mechanism.*
2. *If $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} > 1$, then the optimal mechanism is an interior* POST-2 *mechanism.*

Further, if $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} \le 1$, the optimal mechanism is a uniform POST-2 *mechanism (κ_1, κ_2) with $\kappa_1 = \kappa_2 = \kappa^*$ such that $\kappa^* \ge \widetilde{\kappa}_2$ and is a solution to the equation*

$$xf_2(x) - (1 - F_2(x)) = b\Big(f_2(x) - f_1(x)\Big)$$

Proof: Consider the optimal POST-2 mechanism given by (κ_1^*, κ_2^*). If $b = \kappa_1^*$, the expected revenue of the optimal POST-2 mechanism is $b(1 - F_1(b))$, which is also the expected revenue from the POST-1 mechanism with price b. Since the optimal POST-2 mechanism generates strictly higher expected revenue than the optimal POST-1 mechanism, we conclude that $b < \kappa_1^*$. A similar argument implies that $\kappa_2^* < \beta$. Therefore, Theorem 1 and conditions of this theorem imply that either a uniform POST-2 mechanism with $b < \kappa_1^* = \kappa_2^* < \beta$ or an interior POST-2 mechanism is optimal.

We now prove two claims before proceeding with the proof.

CLAIM **1** *If an optimal* POST-2 *mechanism is a uniform* POST-2 *mechanism with* $\kappa_1 = \kappa_2 = \kappa^*$*, then* $\kappa^* \geq \widetilde{\kappa}_2$*.*

Proof: Assume for contradiction $\kappa^* < \widetilde{\kappa}_2$. Then, the expected revenue from this mechanism is

$$b(1 - F_1(\kappa^*)) + (1 - \frac{b}{\kappa^*})\kappa^*(1 - F_2(\kappa^*)) < b(1 - F_1(\kappa^*)) + (1 - \frac{b}{\kappa^*})\widetilde{\kappa}_2(1 - F_2(\widetilde{\kappa}_2)),$$

where the inequality follows from the definition of $\widetilde{\kappa}_2$. But the RHS is the expected revenue from the interior POST-2 mechanism $(\kappa_1 = \kappa^*, \kappa_2 = \widetilde{\kappa}_2)$. This contradicts the optimality of the uniform POST-2 mechanism. ■

The next claim proves a part of (1).

CLAIM **2** *If an optimal* POST-2 *mechanism is a uniform* POST-2 *mechanism with* $\kappa_1 = \kappa_2 = \kappa^*$*, then* κ^* *is the solution to*

$$xf_2(x) - (1 - F_2(x)) = b\Big(f_2(x) - f_1(x)\Big),$$

and $\frac{f_1(\kappa^*)}{f_2(\kappa^*)} \leq 1$.

Proof: Since $\kappa^* \in (b, \beta)$, it must be a solution to the first order condition. The expected revenue from an arbitrary uniform POST-2 mechanism $\kappa = \kappa_1 = \kappa_2$ is

$$b(1 - F_1(\kappa)) + (\kappa - b)(1 - F_2(\kappa))$$

The first order condition at κ^* is

$$-bf_1(\kappa^*) + (1 - F_2(\kappa^*)) - (\kappa^* - b)f_2(\kappa^*) = 0$$
$$\iff \kappa^* - \frac{1 - F_2(\kappa^*)}{f_2(\kappa^*)} = b\Big(1 - \frac{f_1(\kappa^*)}{f_2(\kappa^*)}\Big) \tag{6}$$

Using Claim 1, $\kappa^* \geq \widetilde{\kappa}_2$. Since F_2 satisfies SC, LHS of (6) is non-negative. Hence, $\frac{f_1(\kappa^*)}{f_2(\kappa^*)} \leq 1$. Hence, the value of κ^* is the solution to the first order condition (6). ■

Now, we proceed to the proofs of (1) and (2).

Proof of (1). Suppose $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} \leq 1$. Assume for contradiction that an interior POST-2 mechanism (κ_1^*, κ_2^*) is an optimal mechanism. By Proposition 1, $\kappa_2^* = \widetilde{\kappa}_2$ and

$$(\kappa_1^*)^2 f_1(\kappa_1^*) = \widetilde{\kappa}_2(1 - F_2(\widetilde{\kappa}_2)) = (\widetilde{\kappa}_2)^2 f_2(\widetilde{\kappa}_2),$$

where the second equality comes from the optimality of $\widetilde{\kappa}_2$ to $\max x(1-F_2(x))$. Since $x^2 f_1(x)$ is strictly increasing and $\widetilde{\kappa}_2 > \kappa_1^*$, we conclude that

$$(\widetilde{\kappa}_2)^2 f_1(\widetilde{\kappa}_2) > (\widetilde{\kappa}_2)^2 f_2(\widetilde{\kappa}_2)$$

As a result, we get

$$\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} > 1$$

which is a contradiction to our assumption. Hence, a uniform POST-2 mechanism is an optimal mechanism.

Proof of (2). Assume for contradiction that a uniform POST-2 mechanism is optimal. Since $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} > 1$, by MLRP and Claim 1, $\frac{f_1(\kappa^*)}{f_2(\kappa^*)} > 1$. This contradicts Claim 2. Hence, an optimal POST-2 mechanism must be an interior POST-2 mechanism. ■

Theorem 3 assumes that the optimal POST-2 mechanism generates *strictly* higher payoff than the optimal POST-1 mechanism. Theorem 2 showed that if $b < \widetilde{\kappa}_1$, optimal POST-2 mechanism generates *weakly* higher payoff than the optimal POST-1 mechanism. The theorem below only assumes $b < \widetilde{\kappa}_1$ (ensuring optimality of POST-2 mechanism due to Theorem 2). But for different ranges of b, it provides sufficient conditions on distributions under which a uniform POST-2 mechanism is the optimal mechanism.

Theorem 4 *Suppose F satisfies the MLRP, $x(1-F_1(x))$ is strictly concave, F_2 satisfies SC, and $b < \widetilde{\kappa}_1$. Then, the following statements hold.*

1. *If $b < \widetilde{\kappa}_2$ and $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} \leq 1$, a uniform* POST-2 *mechanism with price $\kappa^* > b$ that solves the following equation is optimal:*

$$xf_2(x) - (1 - F_2(x)) = b\Big(f_2(x) - f_1(x)\Big)$$

2. *If $\widetilde{\kappa}_2 \leq b$ and $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} > 1$, a uniform* POST-2 *mechanism with price $\kappa^* = b$ is optimal.*

3. *If $\widetilde{\kappa}_2 \leq b$ and $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} \leq 1 < \frac{f_1(b)}{f_2(b)}$, a uniform* POST-2 *mechanism with price $\kappa^* = b$ is optimal.*

Proof: We now prove the three parts of Theorem 4. By Theorem 1, the optimal mechanism is either a POST-1 or POST-2 mechanism. By Theorem 2, the optimal POST-2 mechanism generates weakly higher expected revenue than the optimal POST-1 mechanism (since $b < \widetilde{\kappa}_1$).

Proof of (1). Since $b < \widetilde{\kappa}_1$, the optimal POST-1 mechanism has a price of b with expected revenue $b(1 - F_1(b))$. Now, since $b < \widetilde{\kappa}_2$ and MLRP holds, by $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} \leq 1$, we have

$$f_1(b) \leq f_2(b) \tag{7}$$

Now, consider a uniform POST-2 mechanism and the derivative of its expected revenue from (6) at $\kappa^* = b$:

$$-bf_1(b) + (1 - F_2(b)) \geq -bf_2(b) + (1 - F_2(b)) > 0$$

where the first inequality follows from (7) and the second follows from the fact that F_2 satisfies SC and $b < \widetilde{\kappa}_2$. So, the expected revenue is increasing in κ^* at b. Hence, the optimal uniform POST-2 mechanism has $\kappa^* > b$ and generates more expected revenue than $b(1 - F_1(b))$, the optimal POST-1 mechanism. By Theorem 3, the result then follows.

Proof of (2). We show that the expected revenues are the same in the optimal POST-1 and optimal POST-2 mechanism. Assume for contradiction that the optimal POST-2 mechanism generates strictly higher expected payoff than the optimal POST-1 mechanism. Since $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} > 1$ the optimal mechanism is an interior POST-2 mechanism by Theorem 3. This implies that if the optimal interior POST-2 mechanism is $\kappa_1 = \kappa_1^*$ and $\kappa_2 = \kappa_2^*$, we have $b < \kappa_1^* < \kappa_2^* < \beta$. By Proposition 1, $\kappa_2^* = \widetilde{\kappa}_2$. This implies $b < \widetilde{\kappa}_2$, which contradicts the assumption of (2).

Hence, the optimal POST-2 mechanism must generate the same payoff as the optimal POST-1 mechanism, which is $b(1 - F_1(b))$. This is also the revenue of the uniform POST-2 mechanism with price $\kappa^* = b$. Thus, the optimal POST-2 mechanism is a uniform POST-2 mechanism with price b and generates the same expected revenue as the optimal POST-1 mechanism.

PROOF OF (3). We show that the expected revenues are the same in the optimal POST-1 and optimal POST-2 mechanism. Assume for contradiction that the optimal POST-2 mechanism generates strictly higher expected payoff than the optimal POST-1 mechanism. Since $\frac{f_1(\widetilde{\kappa}_2)}{f_2(\widetilde{\kappa}_2)} \leq 1$, by Theorem 3, the optimal mechanism is a uniform POST-2 mechanism. By Claim 2, $\frac{f_1(\kappa^*)}{f_2(\kappa^*)} \leq 1$. Since $\frac{f_1(b)}{f_2(b)} > 1$ and $\kappa^* > b$, MLRP implies that $\frac{f_1(\kappa^*)}{f_2(\kappa^*)} > 1$, a contradiction. Hence, the optimal POST-2 mechanism must generate the same payoff as the optimal POST-1 mechanism, which is $b(1-F_1(b))$. This is also the revenue of the uniform POST-2 mechanism with price $\kappa^* = b$. Thus, the optimal POST-2 mechanism is a uniform POST-2 mechanism with price b and generates the same expected revenue as the optimal POST-1 mechanism. ■

If a uniform price mechanism has price $\kappa^* = b$, it is also a POST-1 mechanism with price b. Hence, under conditions (2) and (3) of Theorem 4, a POST-1 mechanism is also optimal.

REMARK. The sufficient conditions in our results can be satisfied for a variety of distributions. As an illustration, consider a cdf G with density g on $[0, \beta]$. Let x_1 and x_2 be two draws from $[0, \beta]$ using G. We assign $v_1 := \max(x_1, x_2)$ and $v_2 := \min(x_1, x_2)$. In this case $F_1(x) = [G(x)]^2$ and $F_2(x) = 1-[1-G(x)]^2$. We claim that *if $xg(x)$ is increasing* then all the conditions imposed in Theorem 4 (and all other theorems) on distributions are satisfied. To see this, note that $f_1(x) = 2g(x)G(x)$ and $f_2(x) = 2g(x)(1 - G(x))$. Hence, $\frac{f_1(x)}{f_2(x)} = \frac{G(x)}{1-G(x)}$, which is strictly increasing. Hence, MLRP holds. Next,

$$x - \frac{1 - F_2(x)}{f_2(x)} = x - \frac{[1 - G(x)]^2}{2g(x)(1 - G(x))} = x - \frac{1 - G(x)}{2g(x)}$$

Since $xg(x)$ is increasing, the above expression crosses zero exactly once. Finally, for strict concavity of $x(1 - F_1(x))$, we calculate its derivative: $-xf_1(x) + 1 - F_1(x) = -2xg(x)G(x) +$

$1 - [G(x)]^2$. Since $xg(x)$ is increasing, the derivative is strictly decreasing, establishing strict concavity of $x(1 - G(x))$. This shows that all the conditions of our theorems hold in this class of distributions (this includes the uniform distribution and some family of Beta distributions). Depending on the exact nature of distribution (which determines the value of $\widetilde{\kappa}_i$ for each i), and the budget, we can be more precise about the optimal mechanism.

4 Relation to the literature

Our paper is related to a couple of strands of literature in mechanism design. Before describing them, we relate our work to two papers that seem most related to our work. The first is the work of Burkett (2016), who studies a principal-agent model where the agent is participating in an auction mechanism with a third party. In their model, the third party has proposed a mechanism for selling a single good. Given this third-party mechanism, the principal announces another mechanism (termed as *contract*) to the agent. The sole purpose of the contract is to determine the amount the agent will bid in the third-party mechanism. In their model, the agent's value of the good is the *only* private information - the principal's value can be determined from that of the agent's. The main result in this paper is that the optimal contract for the principal is a "budget-constraint" contract. This optimal contract specifies a cap on the report of each type of the agent to the third-party mechanism and involves no side-payments between the principal and the agent. [3]

In our model, the values of the principal and the agent can be completely different (at a technical level, Burkett (2016) has a one-dimensional mechanism design problem, whereas ours is a two-dimensional mechanism design problem). Further, we do not model decision-making by our principal-agent pair via a contract. In other words, the sequential decision-making in our model makes it quite different from Burkett (2015, 2016).

A unique feature of our model is that the agent and the principal have a different value of the object to the firm. The agent (who is better informed) does not persuade the principal to change his value. This is different from Malenko and Tsoy (2019), who study a model where a single good is sold to a set of buyers, and each buyer is advised by a unique advisor with a bias. Before the start of the auction, there is a communication from the advisor to

[3] In a related paper, Burkett (2015) considers first-price and second-price auctions and compares their revenue and efficiency properties when a seller faces such principal-agent pairs.

the buyer, which influences how much the buyer bids in the auction.

Multidimensional mechanism design. The type space of our agent is two-dimensional. It is well known that the problem of finding an optimal mechanism for selling multiple goods (even to a single buyer) is difficult. A long list of papers have shown the difficulties involved in extending the one-dimensional results in Mussa and Rosen (1978); Myerson (1981); Riley and Zeckhauser (1983) to multidimensional framework - see Armstrong (2000); Manelli and Vincent (2007) as examples. Even when the seller has just *two* objects, and there is just one buyer with additive valuations (i.e., value for both the objects is the sum of values of both the objects), the optimal mechanism is difficult to describe (Manelli and Vincent, 2007; Daskalakis et al., 2017; Hart and Nisan, 2017). Indeed, strong conditions on prior are required to ensure that the optimal mechanism is deterministic (Pavlov, 2011; Bikhchandani and Mishra, 2022). This has inspired researchers to consider *approximately* optimal mechanisms (Chawla et al., 2007, 2010; Hart and Nisan, 2017) or additional robustness criteria for design (Carroll, 2017). Compared to these problems, our two-dimensional mechanism design problem becomes tractable because of the nature of incentive constraints, which in turn is a consequence of the preference of the agent. Gonczarowski et al. (2021) study a model of single object sale where the value of the object and an outside option are private information of the agent. They give sufficient conditions under which a posted price mechanism is optimal. In our model, the principal's decision leads to another option for the agent. However, this option is dependent on the mechanism of the seller. Further, the usual outside option $(0, 0)$ exists in our model.

Mechanism design with budget constraints. In our model, the agent is budget-constrained, but the principal is not. In the standard model, when there is a single object and the buyer(s) is budget constrained, the space of mechanisms is restricted to be such that payment is no more than the budget. This feasibility requirement on the mechanisms essentially translates to violating the quasilinearity assumption of the buyer's preference for prices above the budget. This introduces additional complications for finding the optimal mechanism (Laffont and Robert, 1996; Che and Gale, 2000; Pai and Vohra, 2014). When the budget is private information, the problem becomes even more complicated - see Che and Gale (2000) for a description of the optimal mechanism for the single buyer case and Pai and Vohra (2014) for a description of the optimal mechanism for the multiple buyers case. All these mechanisms involve randomization, but the nature of randomization is quite different from ours. This is because the source of randomization in all these papers is either due to budget being private information (hence, part of the type, as in Che and Gale (2000); Pai and Vohra (2014)) or because of multiple agents with budget being common knowledge (as in Laffont and Robert (1996); Pai and Vohra (2014)). Indeed, with a single agent and public budget, the optimal mechanism in a standard single object allocation model is a posted price mechanism. This is in contrast with our result where we get a randomized optimal mechanism even with the budget being common knowledge. Also, the set of menus in the optimal mechanism in the standard single object auction with budget constraint may have more than three outcomes. Further, the outcomes in the menu of these optimal mechanisms are not as simple as our POST-2 mechanism. Finally, like us, these papers assume that the budget is exogenously determined by the agent. If the buyer can choose his budget constraint, then Baisa and Rabinovich (2016) shows that the optimal mechanism in a multiple buyers setting allocates the object efficiently whenever it is allocated - this is in contrast to the exogenous budget case (Laffont and Robert, 1996; Pai and Vohra, 2014). Li (2021) studies a model of financially constrained agents buying a single object when the mechanism designer can inspect the budget at a cost (both value and budget are private information). The optimal mechanism is significantly complicated and involves inspection.

5 Discussions

We have considered a model where a principal and an agent acquire an object for the firm jointly. However, the principal has delegated the participation in the mechanism to the agent with a budget constraint. The principal does not question the agent as long as the agent does not violate the budget constraint. However, the agent can come back to the principal if the payment exceeds the budget, in which case the principal takes decisions. Besides the agent and the board of directors example highlighted in the paper, there are other settings where such decision-making may be plausible. Our results highlight the nuanced, but still simple, nature of the optimal mechanism in such settings.

As we saw in the example in Section 1.1, the ex-post payoff of the principal in the optimal mechanism can be negative. However, the ex-ante payoff the principal in the optimal mechanism may be positive as the example below illustrates. We consider an example with

$\beta = 1$ and uniform distribution with budget constraint $b = 0.25$. In this case, the optimal mechanism is a POST-2 mechanism with $k_1 = k_2 = 0.39$. The principal's payoff when agent chooses $(\frac{0.25}{0.39}, 0.25)$ is given by

$$\int_0^{0.39} 2\big(y\frac{0.25}{0.39} - 0.25\big)(1 - 0.39)dy = -0.059$$

The principal's payoff when he chooses $(1, 0.39)$ is given by

$$\int_{0.39}^{1}\int_{y}^{1} 2\big(y - 0.39\big)dxdy = 0.075$$

Therefore, the principal's ex-ante payoff from the mechanism is positive.

We make a few comments about possible extensions. First, we have assumed that the value of the agent is more than that of the principal. Under stronger conditions on distributions, we believe that our main results can be extended when we relax this assumption. Second, we assume that the budget is observed by the seller. Relaxing this assumption significantly complicates the model. Some partial characterizations of optimal mechanisms seem possible when the budget is also a private information.

A Proof of Lemma 1

Proof: Let (q, p) be a POST-2 mechanism defined by $\kappa_1, \kappa_2 \in [0, \beta]$ such that $\kappa_1 \leq \kappa_2$. If $\kappa_1 = b$, the POST-2 mechanism collapses to a POST-1 mechanism, which is IC and IR. Similarly, if $\kappa_2 = \beta$, we have a modified POST-1 mechanism where all types with $v_1 \geq \kappa_1$ are given the object with probability $\frac{b}{\kappa_1}$ at price b and others do not get the object and pay zero. Again, this is an IC and IR mechanism.

So, we consider the case $b < \kappa_1 \leq \kappa_2 < \beta$. In this case, the range of the POST-2 mechanism contains three outcomes: $(0, 0), (1, b + \kappa_2(1 - \frac{b}{\kappa_1}))$, and $(\frac{b}{\kappa_1}, b)$. Two of these outcomes are within the budget. Take a type $v \equiv (v_1, v_2)$. We consider three cases. Denote $X := R(q, p)$.

CASE 1. $v_2 < \kappa_2$. Then

$$v_2 - b - \kappa_2 + b\frac{\kappa_2}{\kappa_1} = (v_2 - \kappa_2)\big(1 - \frac{b}{\kappa_1}\big) + v_2\frac{b}{\kappa_1} - b < v_2\frac{b}{\kappa_1} - b \tag{8}$$

where the inequality follows from $b < \kappa_1$ and $v_2 < \kappa_2$. This implies $Ch(X; v_2) \subseteq X_b$ since $(1, b + \kappa_2(1 - \frac{b}{\kappa_1})) \notin Ch(X; v_2)$. Therefore, $Ch(X; v_2) \rhd_{v_1} Ch(X_b; v_1)$ does not hold. Hence, $Ch(X; v) = Ch(X_b; v_1)$ whenever $v_2 < \kappa_2$.

If $v_1 < \kappa_1$, then $Ch(X_b; v_1) = \{(0, 0)\}$ since $v_1 \frac{b}{\kappa_1} - b < 0$. Hence, $(q(v), p(v)) = (0, 0) \in Ch(X_b; v_1) = Ch(X; v)$.

If $v_1 \geq \kappa_1$, then $(\frac{b}{\kappa_1}, b) \in Ch(X_b; v_1)$ since $v_1 \frac{b}{\kappa_1} - b \geq 0$. Hence, $(q(v), p(v)) = (\frac{b}{\kappa_1}, b) \in Ch(X_b; v_1) = Ch(X; v)$.

Case 2. $v_1 \geq v_2 \geq \kappa_2 \geq \kappa_1$ and $v_1 > \kappa_2$. Since $v_i \geq \kappa_2$

$$v_i - b - \kappa_2 + b\frac{\kappa_2}{\kappa_1} = (v_i - \kappa_2)(1 - \frac{b}{\kappa_1}) + v_i\frac{b}{\kappa_1} - b \geq v_i\frac{b}{\kappa_1} - b \geq 0 \tag{9}$$

where the first inequality follows from $v_i \geq \kappa_2$ and $\kappa_1 > b$, the second inequality follows from $v_i \geq \kappa_1$.

Hence, we get $(1, b + \kappa_2(1 - \frac{b}{\kappa_1})) \in Ch(X; v_i)$ in this case. Since $v_1 > \kappa_2 \geq \kappa_1$ and $\kappa_1 > b$, both inequalities in (9) are strict for $v_i = v_1$. Hence, $Ch(X_b; v_1) = \{(\frac{b}{\kappa_1}, b)\}$. Therefore, $Ch(X; v_2) \rhd_{v_1} Ch(X_b; v_1)$ holds and $Ch(X; v) = Ch(X; v_2)$ in this case.

Now, when $v_2 > \kappa_2$, the first inequality is strict in (9) for $v_i = v_2$. As a result, $Ch(X; v_2) = \{(1, b + \kappa_2(1 - \frac{b}{\kappa_1}))\}$. This implies that $(q(v), p(v)) = (1, b + \kappa_2(1 - \frac{b}{\kappa_1})) \in Ch(X; v_2) = Ch(X; v)$. When $v_2 = \kappa_2$, the first inequality in (9) is an equality. Hence, $(q(v), p(v)) = (\frac{b}{\kappa_1}, b) \in Ch(X; v_2) = Ch(X; v)$.

Case 3. $v_1 = v_2 = \kappa_2$. Then $Ch(X; v_2) \sim_{v_1} Ch(X_b; v_1)$ holds. Hence, $(q(v), p(v)) = (\frac{b}{\kappa_1}, b) \in Ch(X_b; v_1) = Ch(X; v)$. ■

B Proof of Theorem 1

We begin the proof of this theorem with some preliminary lemmas.

Lemma **2** *Suppose $v_i, v_i' \in [0, \beta]$ with $v_i' > v_i$. For any $X \subseteq Z$, if $(a', t') \in Ch(X; v_i')$ and $(a, t) \in Ch(X; v_i)$, then $a \leq a'$ and $t \leq t'$.*

Proof: Since $(a', t') \in Ch(X; v_i')$ and $(a, t) \in Ch(X; v_i)$, we have

$$a'v_i' - t' \geq av_i' - t$$
$$av_i - t \geq a'v_i - t'$$

Adding gives $(a' - a)(v_i' - v_i) \geq 0$. Since $v_i' > v_i$, we have $a' \geq a$. Then, the second inequality gives $(a' - a)v_i \leq t' - t$. Since $a' \geq a$, we get $t' \geq t$. ■

Lemma **3** *If (q, p) is IC and IR, then there exists $\kappa_{(q,p)} \in (0, \beta]$ such that for each $v_i \in [0, \beta]$ and each $(a, t) \in Ch(R(q, p); v_i)$, we have*

$$v_i < \kappa_{(q,p)} \Longrightarrow t \leq b$$
$$v_i > \kappa_{(q,p)} \Longrightarrow t > b$$

Proof: By IR, if $(a, t) \in Ch(R(q, p); 0)$ then $t \leq 0$. Let $V^- := \{v_i \in [0, \beta] : t \leq b \ \forall \ (a, t) \in Ch(R(q, p); v_i)\}$. Then, $0 \in V^-$. Let $\kappa_{(q,p)} = \sup_{v_i \in V^-} v_i$. Note here that $\kappa_{(q,p)}$ does not depend on i. By Lemma 2, the result then follows. ■

For any mechanism (q, p), define the following set of types.

$$V^+(q, p) = \{v : p(v) > b\}$$

The proof of Theorem 1 consists of considering two classes of mechanisms, one where $V^+(q, p)$ has zero Lebesgue measure and the other where it has non-zero Lebesgue measure. In particular, consider the following partitioning of mechanisms.

$$M^- = \{(q, p) \text{ is IC and IR} : V^+(q, p) \text{ has zero Lebesgue measure}\}$$
$$M^+ = \{(q, p) \text{ is IC and IR} : (q, p) \notin M^-\}$$

B.1 post-2 is optimal in M^+

Lemma **4** *If $(q, p) \in M^+$ then for all v,*

$$\begin{aligned} p(v) &\leq b \qquad \text{if } v_2 < \kappa_{(q,p)} \\ p(v) &> b \qquad \text{if } v_2 > \kappa_{(q,p)} \end{aligned}$$

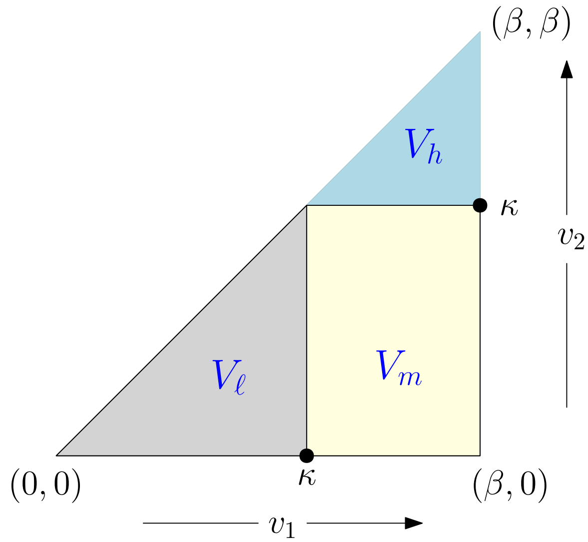

Figure 3: Partitioning of type space V

Proof: Denote $X \equiv R(q,p)$. Lemma 3 implies that for all v with $v_2 < \kappa_{(q,p)}$ we have $Ch(X; v_2) \subseteq X_b$. Therefore, $Ch(X; v) = Ch(X_b; v_1)$. Incentive compatibility then implies that $p(v) \leq b$. Lemma 3 also implies that for all v with $v_2 > \kappa_{(q,p)}$ we have $Ch(X; v_2) \subseteq X \setminus X_b$. Pick any $(a,t) \in Ch(X; v_2)$ and $(a',t') \in X_b$ and note that $v_2 a - t > v_2 a' - t'$. Using $v_1 \geq v_2$ and $t > t'$ we derive $v_1 a - t > v_1 a' - t'$. Since this is true for all $(a,t) \in Ch(X; v_2)$ and $(a',t') \in X_b$, we conclude $Ch(X; v_2) \rhd_{v_1} Ch(X_b; v_1)$. Therefore, $Ch(X; v) = Ch(X; v_2) \subseteq X \setminus X_b$. Incentive compatibility then implies that $p(v) > b$. ■

We now fix (q,p) and show that there exists a POST-2 mechanism that generates more expected revenue than (q,p). For simplicity, we will drop (q,p) subscript from $\kappa_{(q,p)}$. We will partition the type space $V = \{v \in [0,\beta]^2 : v_1 \geq v_2\}$ into three parts as (see Figure 3):

$$V_\ell = \{v \in V : v_1 \leq \kappa\}$$
$$V_h = \{v \in V : v_2 \geq \kappa\}$$
$$V_m = V \setminus (V_1 \cup V_2)$$

Note that $V_h \cap V_\ell = \{(\kappa,\kappa)\}$. Hence, V_ℓ, V_h, V_m do not represent an exact partitioning of V, but this will not influence our analysis.

Now, consider the following type: $(\kappa, 0)$ and define $q^\dagger$ as the unique solution to

$$\kappa q^\dagger - b = \kappa q(\kappa, 0) - p(\kappa, 0) \tag{10}$$

Since $p(\kappa,0) \leq b$, we get

$$q^\dagger = q(\kappa,0) + \frac{b - p(\kappa,0)}{\kappa} \geq q(\kappa,0) \tag{11}$$

Now, consider the type $\kappa^\epsilon \equiv (\kappa+\epsilon, \kappa+\epsilon)$, where $\epsilon > 0$ but sufficiently small. By Lemma 4, $p(\kappa^\epsilon) > b$. Incentive compatibility of (q,p) implies $(q(\kappa^\epsilon), p(\kappa^\epsilon)) \in Ch(R(q,p); \kappa+\epsilon)$. Hence,

$$\begin{aligned}
(\kappa+\epsilon)q(\kappa^\epsilon) - p(\kappa^\epsilon) &\geq (\kappa+\epsilon)q(\kappa,0) - p(\kappa,0) = \epsilon q(\kappa,0) + \kappa q^\dagger - b \\
\iff q^\dagger &\leq q(\kappa^\epsilon) - \frac{p(\kappa^\epsilon) - b}{\kappa} + \frac{\epsilon}{\kappa}(q(\kappa^\epsilon) - q(\kappa,0)) \\
&\leq q(\kappa^\epsilon) + \frac{\epsilon}{\kappa}(q(\kappa^\epsilon) - q(\kappa,0))
\end{aligned}$$

where the last inequality uses $p(\kappa^\epsilon) > b$. As $\epsilon \to 0$, we get (using (11))

$$q(\kappa,0) \leq q^\dagger \leq \lim_{\epsilon \to 0} q(\kappa^\epsilon) \tag{12}$$

This shows that $q^\dagger \in [0,1]$ is a valid allocation probability. We also observe the following about utilities of types. Take any $v \equiv (v_1, v_2)$ with $v_2 > \kappa$. By IC and the fact that $p(v) > b$, we get

$$u(v) := v_2 q(v) - p(v) \geq v_2 q(\kappa,0) - p(\kappa,0) \geq \kappa q(\kappa,0) - p(\kappa,0) = \kappa q^\dagger - b$$

where the last equality follows from the definition of $q^\dagger$. Hence, for all such v, we have

$$u(v) \geq \kappa q^\dagger - b \tag{13}$$

Now, $q^\dagger$ will be used to determine an upper bound on expected revenue of (q,p). We will show that this upper bound can be achieved by a POST-2 mechanism. We derive this upper bound by deriving upper bounds for expected revenue from V_ℓ, V_h, and V_m separately.

B.2 Upper bounding V_ℓ

For any $v \in V_\ell$ with $v_1 < \kappa$, we have $(q(v), p(v)) \in Ch(R(q,p); v_1)$ by Lemma 3. Also, $(q(\kappa,0), p(\kappa,0)) \in Ch(R(q,p); \kappa)$. Hence, for any $v_1 < \kappa$, IC constraints in V_ℓ imply

$$\begin{aligned}
v_1 q(v) - p(v) &\geq v_1 q(\kappa,0) - p(\kappa,0) \\
\kappa q(\kappa,0) - p(\kappa,0) &\geq \kappa q(v) - p(v)
\end{aligned}$$

The first inequality together with equation 11 and the fact that $p(\kappa, 0) \leq b$ implies for all v with $v_1 < \kappa$,

$$v_1 q(v) - p(v) \geq v_1 q^\dagger - b \tag{14}$$

while the second inequality together with equation 11 implies

$$\kappa q^\dagger - b \geq \kappa q(v) - p(v) \tag{15}$$

We define a mechanism $(\tilde{q}, \tilde{p})$ on V_ℓ by setting $(\tilde{q}(v), \tilde{p}(v)) = (q^\dagger, b)$ when $v_1 = \kappa$ and $(\tilde{q}(v), \tilde{p}(v)) = (q(v), p(v))$ otherwise. Note that IC constraints in V_ℓ involve only the agent's type. Inequalities (14) and (15) and the fact that (q, p) is IC and IR, imply that $(\tilde{q}(v), \tilde{p}(v))$ is IC and IR in V_ℓ.

Define M^ℓ to be the set of all IC and IR mechanisms on V_ℓ such that $(\tilde{q}(v), \tilde{p}(v)) = (q^\dagger, b)$ for all v with $v_1 = \kappa$. The optimal mechanism in M^ℓ gives an upper bound on the expected revenue of (q, p) in V_ℓ.

We derive the optimal mechanism in M^ℓ in some steps. First, we show a lemma which further restricts the class of mechanisms we need to consider inside M^ℓ.

Lemma **5** *For every $(\tilde{q}, \tilde{p}) \in M^\ell$, there exists a mechanism $(\tilde{q}', \tilde{p}') \in M^\ell$ such that*

$$\begin{aligned} \tilde{p}'(v_1, v_2) &= \tilde{p}'(v_1, v_2') && \forall\ v_1, v_2, v_2' \\ \tilde{q}'(v_1, v_2) &= \tilde{q}'(v_1, v_2') && \forall\ v_1, v_2, v_2' \\ \tilde{p}'(v) &\geq p(v) && \forall\ v \in V \end{aligned}$$

Proof: Pick any $(\tilde{q}, \tilde{p}) \in M^\ell$. By definition of $(\tilde{q}, \tilde{p})$, $(\tilde{q}(v), \tilde{p}(v)) \in Ch(R(\tilde{q}, \tilde{p}); v_1)$. For every v_1, let $Ch^*(R(\tilde{q}, \tilde{p}); v_1)$ be the set of outcomes defined as:

$$Ch^*(R(\tilde{q}, \tilde{p}); v_1) := \{(a, t) \in Ch(R(\tilde{q}, \tilde{p}); v_1) : t \geq t'\ \forall\ (a', t') \in Ch(R(\tilde{q}, \tilde{p}); v_1)\}$$

By Claim 5 (in Appendix C), $Ch^*(R(\tilde{q}, \tilde{p}); v_1)$ is non-empty. Note that if $v_1 \neq 0$, then for any $(a, t), (a', t') \in Ch^*(R(\tilde{q}, \tilde{p}); v_1)$, we have $t = t'$ and $av_1 - t = a'v_1 - t'$. But $t = t'$ implies $a = a'$. Hence, $Ch^*(R(\tilde{q}, \tilde{p}); v_1)$ is a singleton if $v_1 \neq 0$.

Hence, we construct another mechanism $(\tilde{q}', \tilde{p}')$ such that $(\tilde{q}'(v), \tilde{p}'(v))$ is assigned the unique element in $Ch^*(R(\tilde{q}, \tilde{p}); v_1)$ if $v_1 \neq 0$. When $v_1 = 0$, we choose $(a, t) \in Ch^*(R(\tilde{q}, \tilde{p}); 0)$

such that a is minimum across all outcomes in $Ch^*(R(\tilde{q},\tilde{p});0)$ and set $(\tilde{q}'(v),\tilde{p}'(v))=(a,t)$. Since $(\tilde{q},\tilde{p})$ is IC and IR, $(\tilde{q}',\tilde{p}')$ is also IC and IR. Since $(\tilde{q},\tilde{p})\in M^{\ell}$, $\tilde{q}(v)\leq q^{\dagger}$ and $\tilde{p}(v)\leq b$ for all $v\in V^{\ell}$. Hence, by construction, $\tilde{q}'(v)\leq q^{\dagger}$ and $\tilde{p}'(v)\leq b$. As a result, $(\tilde{q}',\tilde{p}')\in M^{\ell}$. Further, by construction, $\tilde{p}'(v)\geq\tilde{p}(v)$ for all $v\in V^{\ell}$. Also, by construction, $\tilde{p}'(v_1,v_2)=\tilde{p}'(v_1,v_2')$ for all $(v_1,v_2),(v_1,v_2')\in V^{\ell}$. ■

Due to Lemma 5, we conclude that an optimal mechanism in M^{ℓ} must belong to the following class of mechanisms:

$$M^{\ell\ell}=\{(\tilde{q},\tilde{p})\in M^{\ell}:\tilde{p}(v_1,v_2)=\tilde{p}(v_1,v_2'),\tilde{q}(v_1,v_2)=\tilde{q}(v_1,v_2')\ \forall\ (v_1,v_2),(v_1,v_2')\in V^{\ell}\}$$

The expected revenue from any mechanism $(\tilde{q},\tilde{p})\in M^{\ell\ell}$ is given by

$$\textsc{Rev}(\tilde{q},\tilde{p})=\int\limits_{V^{\ell}}\tilde{p}(v)f(v)dv=\int\limits_{0}^{\kappa}\tilde{p}(v_1,v_2)\int\limits_{0}^{v_1}f(v_1,v_2)dv_2dv_1=\int\limits_{0}^{\kappa}\tilde{p}(v_1,v_2)f_1(v_1)dv_1$$

where f_1 is the density of the marginal distribution of v_1. Since $(\tilde{q},\tilde{p})\in M^{\ell\ell}$, we let $\pi(v_1):=\tilde{p}(v_1,v_2)$ for all $(v_1,v_2)\in V^{\ell}$. Hence, the revenue expression simplifies to

$$\textsc{Rev}(\tilde{q},\tilde{p})=\int\limits_{0}^{\kappa}\pi(v_1)f_1(v_1)dv_1$$

Now, define for every $v_i\in[0,\kappa]$, $\alpha(v_1):=\tilde{q}(v_1,v_2)$ and

$$\tilde{u}(v_1):=\alpha(v_1)v_1-\pi(v_1)$$

Since the IC constraints involve type of agent (v_1), standard envelope theorem arguments imply for all $v_1\in[0,\kappa]$,

$$\tilde{u}(v_1)=\tilde{u}(0)+\int\limits_{0}^{v_1}\alpha(x)dx$$

and $\frac{d\tilde{u}}{dx}=\alpha(x)$ for almost all $x\in[0,\kappa]$. Using this and rewriting the revenue expression, we get

$$\textsc{Rev}(\tilde{q},\tilde{p})=\int\limits_{0}^{\kappa}\pi(v_1)f_1(v_1)dv_1$$

$$= \int_0^\kappa \big[\frac{d\tilde{u}}{dv_1} v_1 - \tilde{u}(v_1)\big] f_1(v_1) dv_1$$

$$= \kappa \tilde{u}(\kappa) f_1(\kappa) - \int_0^\kappa \Big(v_1 \frac{df_1}{dv_1} + 2 f_1(v_1)\Big) \tilde{u}(v_1) dv_1 \tag{16}$$

We now define another mechanism (q^*, p^*) from $(\tilde{q}, \tilde{p})$. For every $v_1 \in [0, \kappa]$, we set $\alpha^*(v_1) := \alpha(\kappa)$ and $\pi^*(v_1) := \pi(\kappa)$ if $\alpha(\kappa)v_1 - \pi(\kappa) \geq 0$ and assign $(\alpha^*(v_1), \pi^*(v_1)) := (0, 0)$ otherwise. Extend this to entire V^ℓ in the usual way using Lemma 5.

Clearly, this defines an IC and IR mechanism in $M^{\ell\ell}$. So, for $v_1 < \frac{\pi(\kappa)}{\alpha(\kappa)} = \frac{b}{q^\dagger}$, we get $u^*(v_1) = 0$. If $v_1 \geq \frac{b}{q^\dagger}$, we have $u^*(v_1) = v_1\alpha(\kappa) - \pi(\kappa) \leq \tilde{u}(v_1)$, where the inequality follows from IC of $(\tilde{q}, \tilde{p})$. As a result, $u^*(v_1) \leq \tilde{u}(v_1)$ for all $v_1 \leq \kappa$ with equality holding for $v_1 = \kappa$. Since $v_1(1 - F_1(v_1))$ is concave, we know that $v_1 \frac{df_1}{dv_1} + 2f_1(v_1) \geq 0$ for all v_1. Using the expression in (16), we conclude that $\text{Rev}(q^*, p^*) \geq \text{Rev}(\tilde{q}, \tilde{p})$.

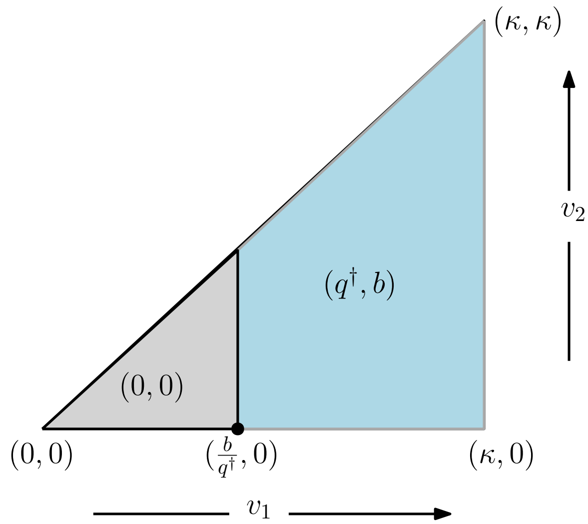

Figure 4: Optimal mechanism in M^ℓ for type space V_ℓ

Since $(\alpha(\kappa), \pi(\kappa)) = (q^\dagger, b)$ by construction, an optimal mechanism in $M^{\ell\ell}$ is described by a cutoff $\frac{b}{q^\dagger}$ such that for all types with $v_1 \in [\frac{b}{q^\dagger}, \kappa]$, the object is allocated with probability $q^\dagger$ and payment b. The object is not allocated and payment is zero for all other types. This is shown in Figure 4.

B.3 Upper bounding V_h

Let $V_h^- := V_h \setminus \{(v_1, \kappa) : v_1 \geq \kappa\}$. Consider the restriction of (q,p) to V_h^-, which is an IC and IR mechanism defined on V_h^-. For all $v \in V_h^-$, by definition of V_h, we have $p(v) > b$. This also implies that $(q(v), p(v)) \in Ch(R(q,p); v_2)$. This implies that for any $v \equiv (v_1, v_2), v' \equiv (v_1', v_2') \in V_h^-$, with $v_2' > v_2$, we must have

$$v_2' q(v') - p(v') \geq v_2' q(v) - p(v)$$
$$v_2 q(v) - p(v) \geq v_2 q(v') - p(v')$$

Adding them gives $q(v') \geq q(v)$. Using (12), we get $q(v) \geq q^\dagger$ for all $v \in V_h^-$. Further, by IC of (q,p) for any $v \in V_h^-$, we have $v_2 q(v) - p(v) \geq kq^\dagger - b$ by (13).

We can then extend the restriction of (q,p) to V_h^- to the entire V_h by taking convergent sequences of $q(v), p(v)$ in V_h^-. This will define a new IC and IR mechanism $(\tilde{q}, \tilde{p})$ on V_h which satisfies three properties: (i) $\tilde{p}(v) \geq b$, (ii) $\tilde{q}(v) \geq q^\dagger$ and (iii) $\tilde{u}(v) := v_2 \tilde{q}(v) - \tilde{p}(v) \geq \kappa q^\dagger - b$ for all $v \in V_h$.

Hence, define M^h to be the set of all IC and IR mechanisms on V_h such that each $(\tilde{q}, \tilde{p}) \in M^h$ satisfies $\tilde{q}(v) \geq q^\dagger$ and $\tilde{u}(v) \geq \kappa q^\dagger - b$ for all $v \in V_h$.[4] The optimal mechanism in M^h gives an upper bound on the expected revenue of (q,p) in V_h.

Since the IC constraints are determined by v_2 for any $(v_1, v_2) \in V_h$, analogous to Lemma 5, we can assume that for every mechanism $(\tilde{q}, \tilde{p}) \in M^h$,

$$\tilde{q}(v_1, v_2) = \tilde{q}(v_1', v_2) \qquad \forall\ (v_1, v_2), (v_1', v_2) \in V_h$$
$$\tilde{p}(v_1, v_2) = \tilde{p}(v_1', v_2) \qquad \forall\ (v_1, v_2), (v_1', v_2) \in V_h$$

Hence, we can set $\alpha(v_2) = \tilde{q}(v_1, v_2), \pi(v_2) = \tilde{p}(v_2)$ and $u(v_2) = \tilde{u}(v_1, v_2)$ for all $v_2 \in [\kappa, \beta]$. IC constraints define a one dimensional mechanism design problem of allocating a single object in the type space $[\kappa, \beta]$ with IR constraints replaced by $u(v_2) \geq \kappa q^\dagger - b$. Further, the allocation probabilities lie in $[q^\dagger, 1]$. We know from Manelli and Vincent (2007) and Borgers (2015) that the extreme points of such optimization problems assign extreme allocation probabilities to each type, i.e., object is either given with probability $q^\dagger$ or 1 at every type and the lowest type is assigned utility equal to $\kappa q^\dagger - b$.

[4] We have ignored the constraint $\tilde{p}(v) \geq b$ and hence, considered a larger set of mechanisms.

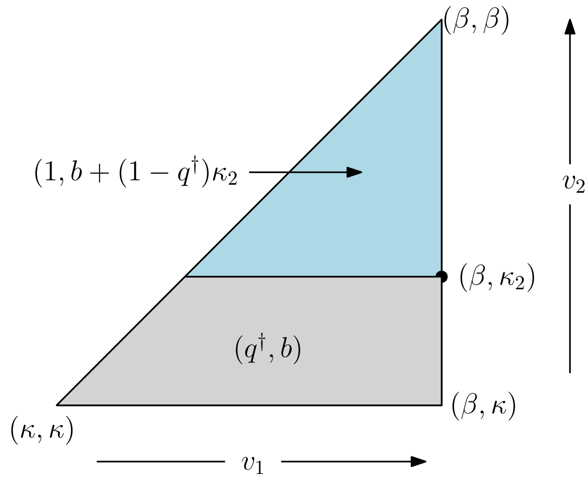

Figure 5: Optimal mechanism in M^h for type space V_h

Hence, the expected revenue maximizing mechanism (q^*, p^*) in M^h must involve a cutoff $\kappa_2 \in [\kappa, \beta]$, such that for all $(v_1, v_2) \in V_h$ with $v_2 \geq \kappa_2$, we have $q^*(v_1, v_2) = 1$ and for all $(v_1, v_2) \in V_h$ with $v_2 < \kappa_2$, we have $q^*(v_1, v_2) = q^\dagger$. Using the revenue equivalence (envelope theorem) arguments and the fact that lowest possible utility is $\kappa q^\dagger - b$, we conclude that $p^*(v) = b$ for all $v \in V_h$ with $v_2 < \kappa_2$ and $p^*(v) = b + (1 - q^\dagger)\kappa_2$ for all $v \in V_h$ with $v_2 \geq \kappa_2$. This is shown in Figure 5.

The expected revenue from such a mechanism is

$$b(1 - F_2(\kappa)) + (1 - F_2(\kappa_2))(1 - q^\dagger)\kappa_2 \tag{17}$$

where F_2 is the marginal cdf of v_2. Hence, an upper bound on the expected revenue of (q, p) from the region V_h is given by a κ_2^* such that an allocation probability $q^\dagger$ at price b is given to all types in V_h, but the additional allocation probability $(1 - q^\dagger)$ is given to types v with $v_2 \geq \kappa_2^*$ at price $(1 - q^\dagger)\kappa_2^*$. The value of κ_2^* solves

$$\max_{\kappa_2 \in [\kappa, \beta]} \kappa_2(1 - F_2(\kappa_2)) \tag{18}$$

B.4 Upper bounding V_m and proof of Theorem 1

Now, consider the following *post-2* mechanism (q^*, p^*):

$$(q^*(v), p^*(v)) = \begin{cases} (0,0) & \text{if } v_1 < \frac{b}{q^\dagger} \\ \left(1, b + \kappa_2^*\big(1 - q^\dagger\big)\right) & \text{if } v_2 > \kappa_2^* \\ \big(q^\dagger, b\big) & \text{otherwise} \end{cases}$$

where κ_2^* is as defined in (18). This is shown in Figure 6.

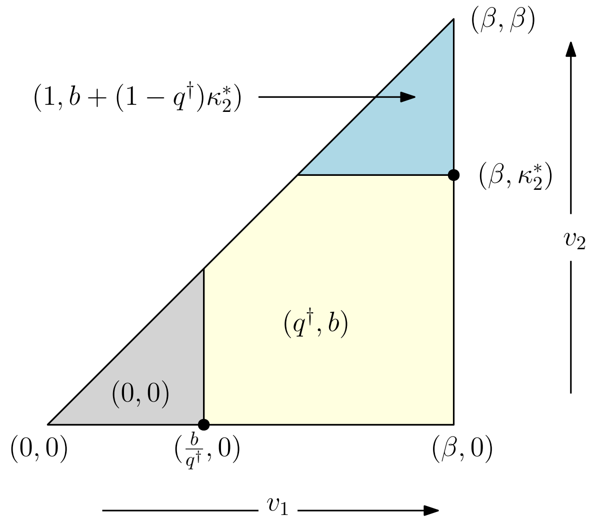

Figure 6: A POST-2 mechanism that revenue dominates (q, p)

Clearly, the POST-2 mechanism (q^*, p^*) coincides with the mechanisms achieving the upper bound in regions V_ℓ and V_h. The types in the V_m region do not pay more than b in the mechanism (q, p). Hence, this POST-2 mechanism generates greater expected revenue than (q, p). This concludes the proof.

B.5 POST-1 IS OPTIMAL IN M^-

LEMMA **6** *For any type $v \equiv (v_1, v_2)$ and any $X \subseteq Z$, if $(a', t') \in Ch(X; v_2)$ and $t' > b$, then for any $(a, t) \in X$ with $t \leq b$, we have $v_1 a' - t' \geq v_1 a - t$ and the inequality is strict if $v_1 > v_2$.*

Proof: Since $(a', t') \in Ch(X; v_2)$ and $t' > b \geq t$ we have $(a' - a)v_2 \geq t' - t > 0$. This implies $a' > a$. Since $v_1 \geq v_2$, we get $(a' - a)v_1 \geq t' - t$ with strict inequality holding if $v_1 > v_2$. ■

Lemma 7 *If $(q,p) \in M^-$, then there exists another $(q',p') \in M^-$ such that for every v we have $p'(v) \leq b$ for all v and $(q(v),p(v)) = (q'(v),p'(v))$ for all $v \neq (\beta,\beta)$.*

Proof: Let $R(q,p) = X$. For any $v \equiv (v_1,v_2)$ with $v_2 < \beta$, suppose $p(v) > b$. By Lemma 3, for all $v_2' > v_2$, for any $(a',t') \in Ch(X;v_2')$ we have $t > b$.

Take any $v' \equiv (v_1',v_2') \in V$ and suppose $(a,t) \in Ch(X_b;v_1')$. By Lemma 6, if $v_1' > v_2'$, we have $v_1'a' - t' > v_1'a - t$. Hence, for all $v' \equiv (v_1',v_2')$ such that $v_1' > v_2' > v_2$, we have $Ch(X;v') = Ch(X;v_2)$. Hence, $p(v') > b$. Since $v_2 < \beta$, this shows that the set of v' with $p(v') > b$ has positive Lebesgue measure. This contradicts the fact that $(q,p) \in M^-$.

Hence, we conclude that for any $v \equiv (v_1,v_2)$ with $p(v) > b$, we must have $v_1 = v_2 = \beta$. Then, consider $X' := X \setminus \{q(\beta,\beta), p(\beta,\beta)\}$. Denote by $\bar{X}'$ the closure of X'. Pick some $(a,t) \in Ch(\bar{X}',\beta)$ and define $q'(\beta,\beta) := a$ and $p'(\beta,\beta) = t$. By definition, $t \leq b$. Further, let $(q'(v),p'(v)) = (q(v),p(v))$ for all $v \neq (\beta,\beta)$. It is easily verified that (q',p') is an IC and IR mechanism (since (q,p) is IC and IR). Further, $p'(v) \leq b$ for all v. ■

Hence, we can assume that the mechanism (q,p) satisfies $p(v) \leq b$ for all v. By mimicking the arguments in Section B.2, we can conclude that the expected revenue of (q,p) is dominated by an IC and IR mechanism $(\tilde{q},\tilde{p})$ such that for some $\kappa_1 \in [0,\beta]$, we have $\tilde{q}(v) = q(\beta,0)$ and $\tilde{p}(v) = p(\beta,0) = \kappa_1 q(\beta,0)$ for all v with $v_1 \geq \kappa_1$. And $(\tilde{q}(v),\tilde{p}(v)) \equiv (0,0)$ for all other v.

So, every optimal mechanism in M^- is characterized by a posted-price κ_1 and an allocation probability α, where the object is allocated with this probability at price $\alpha\kappa_1$ if the value of the agent is above the posted price. The optimal such posted price is the solution to the following program:

$$\max_{\kappa_1,\alpha} \kappa_1\alpha(1 - F_1(\kappa_1))$$
$$\text{subject to } \alpha\kappa_1 \leq b, \quad \alpha \in [0,1]$$

We argue that the optimal solution to this program must have $\alpha = 1$. To see this, let κ_1^* be the unique solution to the following optimization

$$\max_{\kappa_1 \in [0,b]} \kappa_1(1 - F_1(\kappa_1)).$$

The fact that this optimization program has a unique solution follows from the fact that $x - xF_1(x)$ is strictly concave. Hence, the revenue from the solution when $\alpha = 1$ is $\kappa_1^*(1 - F_1(\kappa_1^*))$. Now, suppose the optimal solution has $\hat{\kappa}_1$ and $\hat{\alpha}_1$. Note that the $\hat{\kappa}_1\hat{\alpha} \leq b$. So, define $\tilde{\kappa}_1 = \hat{\kappa}_1\hat{\alpha} \leq b$. By definition,

$$\begin{aligned}\kappa_1^*(1 - F_1(\kappa_1^*)) &\geq \tilde{\kappa}_1(1 - F_1(\tilde{\kappa}_1)) \\ &= \hat{\kappa}_1\hat{\alpha}(1 - F_1(\hat{\kappa}_1\hat{\alpha})) \\ &\geq \hat{\kappa}_1\hat{\alpha}(1 - F_1(\hat{\kappa}_1)),\end{aligned}$$

where the final inequality used the fact that $F_1(\hat{\kappa}_1\hat{\alpha}) \leq F_1(\hat{\kappa}_1)$. This implies that the optimal solution must have $\alpha = 1$ and κ_1 must be the unique solution to $\kappa_1(1 - F_1(\kappa_1))$ with the constraint $\kappa_1 \in [0, b]$. Hence, the optimal solution in M^- must be a POST-1 mechanism, where the posted price is a unique solution to

$$\max_{\kappa_1 \in [0,b]} \kappa_1(1 - F_1(\kappa_1)).$$

C Revelation principle

In this appendix, we establish a revelation principle which allows us to work with direct mechanism. A mechanism is specified by a message space M and an outcome function $\mu : M \to Z$. Define the range of the mechanism (M, μ) as

$$R(M, \mu) = \{\mu(m) : m \in M\}$$

Suppose the type space is $V \subseteq [0, \beta]^2$. A strategy in mechanism (M, μ) is a map $s : V \to M$. Strategy s is an **equilibrium** if for every v,

$$\mu(s(v)) \in Ch(R(M, \mu); v)$$

The direct mechanism (V, μ^*) is **incentive compatible** if for each v,

$$\mu^*(v) \in Ch(R(V, \mu^*); v)$$

For any pair of distinct outcomes (a, t) and (a', t'), we say (a, t) transfer-dominates (a', t'), written as $(a, t) \succ_{tr} (a', t')$ if $t > t'$.

Theorem **5** *If s is an equilibrium in (M, μ), then there exists an incentive compatible direct mechanism (V, μ^*) such that $\mu^*(v) = \mu(s(v))$ or $\mu^*(v) \succ_{tr} \mu(s(v))$ for all v.*

The proof of this theorem uses a series of claims. The following claim, which we write without proof, is useful.

Claim **3** *Suppose $X \subseteq Y \subseteq Z$ and $v \in [0, \beta]$ such that $Ch(Y; v) \cap X$ is non-empty, then $Ch(X; v) \subseteq Ch(Y; v)$.*

For any $X \subseteq Z$, let $X^* := \{(a, t) \in X : (a, t) \in Ch(X; v) \text{ for some } v\}$.

Claim **4** *Suppose $X \subseteq Z$ is such that $Ch(X; v)$ is non-empty for all v. Then,*

$$Ch(X; v) = Ch(X^*; v) \qquad \forall\, v$$

Proof: Consider any $v \equiv (v_1, v_2)$ and $X \subseteq Z$ such that $Ch(X; v)$ is non-empty. This implies that X^* is also non-empty. We consider three cases.

Case 1. $Ch(X; v_2) \rhd_{v_1} Ch(X_b; v_1)$. Note that in this case, $Ch(X; v) = Ch(X; v_2)$. Hence, $Ch(X; v_2) \subseteq X^*$. Since $X^* \subseteq X$, we get $Ch(X^*; v_2) = Ch(X; v_2)$. Now, $Ch(X_b; v_1) \trianglerighteq_{v_1} Ch(X_b^*; v_1)$ since $X_b^* \subseteq X_b$. Then, $Ch(X; v_2) \rhd_{v_1} Ch(X_b; v_1)$ implies that $Ch(X^*; v_2) \rhd_{v_1} Ch(X_b^*; v_1)$. Hence, $Ch(X^*; v) = Ch(X^*; v_2) = Ch(X; v_2) = Ch(X; v)$.

Case 2. $Ch(X_b; v_1) \sim_{v_1} Ch(X; v_2)$. Then $Ch(X; v) = Ch(X_b, v_1)$ which implies $Ch(X_b; v_1) \subseteq X^*$. Hence $Ch(X_b; v_1) \subseteq X_b^*$. This implies that $Ch(X_b^*; v_1) = Ch(X_b; v_1)$ since $X_b^* \subseteq X_b$. Now, $X^* \subseteq X$ implies that $Ch(X; v_2) \trianglerighteq_{v_1} Ch(X^*; v_2)$. Then $Ch(X_b; v_1) \sim_{v_1} Ch(X; v_2)$ implies $Ch(X_b^*; v_1) \trianglerighteq_{v_1} Ch(X^*; v_2)$. Therefore, $Ch(X^*; v_2) \rhd_{v_1} Ch(X_b^*; v_1)$ cannot hold. Hence, $Ch(X^*; v) = Ch(X_b^*; v_1) = Ch(X_b; v_1) = Ch(X; v)$.

Case 3. There exists $(a', t') \in Ch(X; v_2)$ such that $v_1 a - t > v_1 a' - t'$ for every $(a, t) \in Ch(X_b; v_1)$. Fix any $(a, t) \in Ch(X_b; v_1)$. If $t' > t$ then $v_1 a - t > v_1 a' - t'$ implies $v_2 a - t > v_2 a' - t'$ since $v_1 \geq v_2$. This contradicts $(a', t') \in Ch(X; v_2)$. Therefore, $t' \leq t \leq b$ which implies $(a', t') \in Ch(X_b; v_2)$. Now, consider type $v' = (v_2, v_2)$ and observe that

$Ch(X;v') = Ch(X_b;v_2)$. Therefore, $Ch(X_b;v_2) \subseteq X^*$ which then implies $(a',t') \in X^*$ since $(a',t') \in Ch(X_b;v_2)$. Hence, $(a',t') \in Ch(X^*;v_2)$ since $X^* \subseteq X$.

Since $Ch(X_b^*;v_1) = Ch(X_b;v_1)$ as in Case 2, we conclude $(a',t') \in Ch(X^*;v_2)$ such that $v_1 a - t > v_1 a' - t'$ for every $(a,t) \in Ch(X_b^*;v_1)$. Then $Ch(X^*;v) = Ch(X_b^*;v_1) = Ch(X_b;v_1) = Ch(X;v)$.

■

For every $X \subseteq Z$, an outcome (a,t) is **tr-max** in X if $(a,t) \succ_{tr} (a',t')$ for all $(a',t') \in X \setminus (a,t)$. Consider any value $v \in [0,\beta]$. If $v \neq 0$ and a tr-max outcome exists in $Ch(X;v)$, then it is unique. If $v = 0$, there may be more than one tr-max outcome in $Ch(X;v)$. In that case, we assign one of them as tr-max arbitrarily. We denote this tr-max outcome at every v and every X as $Ch^*(X;v)$:

$$Ch^*(X;v) = (a,t) \qquad \text{if } (a,t) \text{ is tr-max in } Ch(X;v)$$

The following claim is useful.

Claim **5** *Suppose $X \subseteq Z$ and $v \in [0,\beta]$ be such that $Ch(X;v)$ is non-empty. Then, $Ch^*(X;v)$ exists.*

Proof: If $Ch(X;v)$ is non-empty, then let $(a,t) \in Ch(X;v)$. Let $u = av - t$. Now, consider the following maximization program:

$$\max_{(a',t')} t' \;\text{ s.t. }\; va' - t' = u$$

Each outcome in $Ch^*(X,v)$ must be a solution to this. But this is equivalent to solving

$$\max_{a' \in [0,1]} [va' - u]$$

Since this is a maximization of a linear function over a compact set, an optimal solution exists. This shows that $Ch^*(X;v)$ exists. ■

We can now define $Ch^*(X;v)$ analogous to $Ch(X;v)$:

$$Ch^*(X;v) = \begin{cases} Ch^*(X;v_2) & \text{if } Ch^*(X;v_2) \rhd_{v_1} Ch^*(X_b;v_1) \\ Ch^*(X_b;v_1) & \text{otherwise} \end{cases}$$

where we abuse notation to write $Ch^*(X;v_2) \rhd_{v_1} Ch^*(X_b;v_1)$ instead of $\{Ch^*(X;v_2)\} \rhd_{v_1} \{Ch^*(X_b;v_1)\}$. Note that if $Ch^*(X;v_2) \rhd_{v_1} Ch^*(X_b;v_1)$ is not true, then $Ch^*(X_b;v_1) \unrhd_{v_1} Ch^*(X;v_2)$ is true.

We can now state a claim analogous to Claim 4. For every $X \subseteq Z$, define $\widehat{X}$ as

$$\widehat{X} := \{Ch^*(X;v) : v \in V\}$$

Claim **6** *Suppose $X \subseteq Z$ is such that $Ch(X;v)$ is non-empty for all v. Then,*

$$Ch^*(X;v) = Ch^*(\widehat{X};v) \qquad \forall\ v$$

Proof: If $X \subseteq Z$ is such that $Ch(X;v)$ is non-empty for all v, by Claim 5, $Ch^*(X;v)$ exists for all v. Pick v and consider the two possible cases.

Case 1. $Ch^*(X;v_2) \rhd_{v_1} Ch^*(X_b;v_1)$. Then, $Ch^*(X;v) = Ch^*(X;v_2)$. Hence, $Ch^*(X;v_2) \in \widehat{X}$. Since $\widehat{X} \subseteq X$, we get $Ch^*(\widehat{X};v_2) = Ch^*(X;v_2) = Ch^*(X;v)$. Finally, $Ch^*(X_b;v_1) \unrhd_{v_1} Ch^*(\widehat{X}_b;v_1)$ since $\widehat{X}_b \subseteq X_b$. So, we have $Ch^*(X;v_2) \rhd_{v_1} Ch^*(X_b;v_1) \unrhd_{v_1} Ch^*(\widehat{X}_b;v_1)$. Hence, $Ch^*(\widehat{X};v_2) = Ch^*(X;v_2) \rhd_{v_1} Ch^*(\widehat{X}_b;v_1)$. As a result,

$$Ch^*(\widehat{X};v) = Ch^*(\widehat{X};v_2) = Ch^*(X;v_2) = Ch^*(X;v)$$

Case 2. $Ch^*(X_b;v_1) \unrhd_{v_1} Ch^*(X;v_2)$. Then, $Ch^*(X;v) = Ch^*(X_b;v_1)$. Now, let $Ch^*(X_b;v_1) = \{(a,t)\}$ and $Ch^*(X;v_2) = \{(a',t')\}$ and note that $t \leq b$. Condition of the case implies $v_1 a - t \geq v_1 a' - t'$. $Ch^*(X;v_2) = \{(a',t')\}$ implies $v_2 a' - t' \geq v_2 a - t$. Adding the inequalities and using $v_1 \geq v_2$ we derive $a' \leq a$. The second inequality then implies $t' \leq t \leq b$. Therefore, $Ch^*(X;v_2) = Ch^*(X_b;v_2)$. Then consider a type $v' = (v_2,v_2)$ and observe that $Ch^*(X;v') = Ch^*(X;v_2)$. Hence, $Ch^*(X;v_2) \in \widehat{X}$. Since $\widehat{X} \subseteq X$ we then have $Ch^*(X;v_2) = Ch^*(\widehat{X};v_2)$.

Also, $Ch^*(X_b;v_1) = Ch^*(X;v)$ implies $Ch^*(X_b;v_1) \in \widehat{X}$. Hence, $Ch^*(X_b;v_1) \in \widehat{X}_b$. This implies that $Ch^*(X_b;v_1) = Ch^*(\widehat{X}_b;v_1)$ since $\widehat{X}_b \subseteq X_b$. As a result, we have $Ch^*(\widehat{X}_b;v_1) = Ch^*(X_b;v_1) \unrhd_{v_1} Ch^*(X;v_2) = Ch^*(\widehat{X};v_2)$. Hence, $Ch^*(\widehat{X};v) = Ch^*(\widehat{X}_b;v_1) = Ch^*(X_b;v_1) = Ch^*(X;v)$. ■

Proof of Theorem 5.

Proof: Let $X := \{Ch(R(M,\mu);v) : v \in V\}$. Since s is an equilibrium of (M,μ), $\mu(s(v)) \in Ch(R(M,\mu);v)$. Hence, $Ch(R(M,\mu);v)$ is non-empty for each v. By Claim 4, $Ch(R(M,\mu);v) = Ch(X;v)$. So, for each v, $Ch(X;v)$ is non-empty, and by Claim 5, $Ch^*(X;v)$ exists.

Now, for every $v \in V$, define

$$\mu^*(v) := Ch^*(X;v)$$

Since s is an equilibrium in (M,μ), for every v, we have $\mu(s(v)) \in Ch(R(M,\mu);v) = Ch(X;v)$. Since $\mu^*(v) = Ch^*(X;v)$, we have $\mu^*(v) = \mu(s(v))$ or $\mu^*(v) \succ_{tr} \mu(s(v))$.

Finally, $R(V,\mu^*) = \{\mu^*(v) : v \in V\} = \{Ch^*(X;v) : v \in V\} = \widehat{X}$. By Claim 6, for all $v \in V$, we have $Ch^*(X;v) = Ch^*(\widehat{X};v)$, which further implies that

$$\mu^*(v) = Ch^*(\widehat{X};v) \in Ch(\widehat{X};v),$$

which is the required incentive compatibility constraint. ■

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