Multilevel Multidimensional Consistent Aggregators *

Mihir Bhattacharya †

December 26, 2014

Abstract

This paper examines the structure of *consistent*, multidimensional, multilevel aggregators in two distinct models- one where the set of evaluations is the unit interval and the other, where it is finite. In the evaluations model, we characterize a class of separable rules called component-wise α-median rules. In the finite model, separability is no longer guaranteed. In addition to consistency, stronger notions of unanimity and anonymity are required to characterize a class of separable rules called Bipartite Rules.

*I am thankful to Arunava Sen for providing feedback to the paper at each and every stage. I am thankful to Debasis Mishra for suggestions and comments.

†Indian Statistical Institute. Email: mihir8r@isid.ac.in

1 Introduction

It is well known that political parties can manipulate or gerrymander voting results by dividing and redistributing voters among districts. This phenomenon has been observed at regional and national levels in the U.S, Canada, India, the United Kingdom, Germany, Australia and France.[1] An important consideration in the design of voting rules is to ensure that they are immune to such manipulation. The specific property of voting rules or aggregators that guarantees this form of non-manipulability has been called *consistency*. There are several papers that have studied the structure of consistent voting rules satisfying various versions of consistency. Virtually all these papers have considered models where voters express opinions about a *single* alternative which have to be aggregated into a social opinion about that alternative. Our goal in this paper is to investigate the consistency of voting rules in models where voter opions over *several* alternatives have to be aggregated.

Multidimensional voting models arise naturally in many contexts. Consider the case where there is a finite set of public projects that is under consideration by the Government. Not all projects are feasible because of resource constraints. The Government therefore needs to aggregate the opinions of all voters over all projects by means of a voting rule. We note that multidimensional voting models have been received a great deal of attention in social choice and positive political theory - see Austen-Smith and Banks (2000), (2005) for an extensive review of the literature.

We consider a model of aggregation where voter opinions have to be aggregated over several alternatives. Each voter submits an *evaluation* for each alternative (or component) indicating the intensity with which she likes the alternative.[2] The set of permissible evaluations for any alternative is the closed unit interval. An *aggregator* considers an *arbitrary* collection of voter evaluations and transforms them into an aggregate opinion.

Voters can be divided into mutually exclusive subgroups. This could be based, for example, on geographical regions/districts or political constituencies. The aggregator generates an aggregate for each subgroup. It can also be used to aggregate subgroup opinions into an opinion for the whole population. Consistency requires the same opinion for the population to emerge (for every possible configuration of voter opinions) irrespective of the way the population is split into subgroups. This paper examines the implications of consistency on aggregators.

We characterize *component-wise α median* rules. These rules are *separable*, i.e. the outcome for an alternative depends only on voter opinions for that alternative. Moreover, the outcome for each alternative is the median of the minimum utility (across voters), the maximum utility (across voters) and a fixed but arbitrary number α_j for each alternative j.

Consistent voting rules have also been analyzed in Chambers (2008), Chambers (2009)

[1] Katz (1998) and Samuels and Snyder (2001) provide empirical evidence of gerrymandering in different electoral systems and countries respectively.

[2] Macé (2013) provides another model of aggregation over evaluations.

and Nermuth (1994).[3] Both the Chambers' papers consider a different notion of consistency where the sub-group aggregate opinion is replicated the same number of times as the number of voters in that subgroup. This notion is inspired by the electoral college voting system in U.S Presidential elections. Our notion of consistency is similar to that in Nermuth (1994). Both the Nermuth and Chambers papers consider a single alternative voting model.

Our result is a generalization of the result of Fung and Fu (1975) who prove an α-median characterization result for the one alternative case. There are significant difficulties involved in the extension to the multidimensional case due to its additional richness. However, these are resolved using the same set of axioms as in Fung and Fu (1975) defined suitably for the multidimensional model.

The separability result depends critically on the structure of the model, in particular on the fact that the set of possible evaluations is a continuum. The result no longer holds if the set of evaluations is finite. We investigate this issue in a special "finite" model. This is a model where there are m alternatives and voter/social opinions pertain to the selection of set of these candidates. The set of possible evaluations for a candidate is either 0 or 1 indicating disapproval and approval respectively. We characterize a class of separable rules called *Bipartite Rules* by consistency and some stronger versions of some of the axioms used for the earlier result. The Bipartite Rule partitions the set of alternatives into two sets (independently of opinions). Alternatives in the first set are assigned value 1 unless all voters disapprove, while alternatives in the second are never selected unless they are unanimously approved.

The paper is organized as follows. We discuss the Evaluations model formally and the notion of consistency in Section 2.1 A discussion of the other axioms is contained in Section 2.2. Section 2.3 presents the component-wise α-median result and its proof followed by a discussion in Section 2.4. Section 3 considers the finite set selection model while Section 4 concludes.

2 The Evaluation Model

The set of components or alternatives is X with $|X| = m$. The set of voters is $\mathbf{N} = \{1, 2, ..., n\}$. Each voter submits an evaluation for each candidate. The set of evaluations is normalized without loss of generality to be the set $[0, 1]$. A voter submits $v_i \in [0, 1]^m$ and we denote the set $[0, 1]^m$ by A. A vote *profile* $v \in A^n$ is the set of voter evaluations $v = (v_1, \ldots, v_n)$. A component $v_{ij} \in [0, 1]$ can be interpreted as the evaluation by voter i for alternative j.

A district or a group is a non-empty set $N \subset \mathbf{N}$. A vote profile is a collection of v_i for all voters $i \in N$ such that $N \subseteq \mathbf{N}$. A vote profile v_S is the restriction of v to the set of voters $S \subseteq \mathbf{N}$. An *aggregator* is a function $f : \cup_{N \in \mathbf{N}} A^n \to A$ which aggregates vote profiles for

[3] Perote-Peña (2005), Bervoets and Merlin (2012) and Plott (1973) analyze models that are similar in spirit to ours with related notions of consistency.

any district or subset $\mathbf{N}$. Some examples of aggregators are given below.

- A *constant* aggregator, $f^c : A^n \to A$ for every profile v outputs a fixed set of evaluations $\boldsymbol{c}$ in A,

$$f^c(v) = \boldsymbol{c} \quad \forall v \in A^n \;\; \forall N \in \mathbf{N}.$$

- A *mean* aggregator, $f^{\text{mean}} : A^n \to A$ outputs the arithmetic mean of the evaluation values for each alternative,

$$f_j^{\text{mean}}(v) = \frac{\sum_i v_{ij}}{N} \quad \forall j \in X \;\; v \in A^n \;\; \forall N \in \mathbf{N}.$$

- The *median* denoted by med(.) of a set of K numbers is $\frac{K}{2}^{th}$ lowest evaluation when K is even, or the $\frac{K+1}{2}^{th}$ lowest evaluation if K is odd. A *median* aggregator selects the median for each component, $f^{\text{med}} : A^n \to A$,

$$f_j^{\text{med}}(v) = \text{med}_{i \in N}(v_{ij}) \quad \forall j \in X \;\; \forall v \in A^n \;\; \forall N \in \mathbf{N}.$$

- A *min* aggregator, $f^{\min} : A^n \to A$ outputs the minimum evaluation from the set of numbers submitted by the voters for each alternative.

$$f_j^{\min}(v) = \min_{i \in N}(v_{ij}) \quad \forall j \in X \;\; \forall v \in A^n \;\; \forall N \in \mathbf{N}.$$

  A *max* aggregator can be similarly defined.

- An aggregator $f^\alpha : A^n \to A$ is *component-wise α-median* aggregator if $\exists \; \alpha \in A$ such that,

$$f_j^\alpha(v) = \text{med}(\min_{i \in N}(v_i), \alpha_j, \max_{i \in N}(v_i)) \quad \forall j \in X \;\; \forall v \in A^n \;\; \forall N \in \mathbf{N}.$$

  For each alternative j the aggregator f^α picks median of the following three numbers- the smallest and greatest among the set of evaluations submitted by all the voters and the j^{th} component of α.

  Component-wise α-median aggregators are generalizations of the min and max aggregators. The min and max rules are α-median rules with $\alpha = 0$ and $\alpha = 1$ respectively.

- Let $\succ$ be a strict ordering on components. Pick an arbitrary $v \in A^n$. The *lexicographic-minimum* or *L-min* voter for v, N is a voter whose evaluation for the $\succ$-max alternative is lowest. If there is more than one such voter, break ties by picking a voter whose evaluation for the next-highest alternative according to $\succ$ is lowest and so on. The L-min rule at v, N picks the evaluation vector of the L-min voter, i.e $f^{\text{L}-\min}(v) = v_{\text{L-min}}$. The L-max rule can be defined analogously.

It is worth drawing attention to the feature of the rules above. The constant, median, mean, min and max aggregators are component-separable rules i.e they aggregate the outcome for each component or alternative separately. The left-aligned and L-min aggregators are not component-separable.

2.1 Consistency of Aggregators

Definition **1 (Consistency)** An aggregator f satisfies *consistency* if for all $N \in \mathbf{N}$, for all partitions $\{N_1, N_2, ..., N_K\}$ of N and all $v \in A$,

$$f(v) = f\left(f(v_{N_1}), f(v_{N_1}), ..., f(v_{N_K})\right).$$

A vote profile v can be aggregated directly by f. It can also be aggregated indirectly as follows. The profile v can be split into the opinions of subgroups $(v_{N_1}, \ldots, v_{N_K})$. Since f is defined for arbitrary collections of opinions, f can be applied to each sub-collection $v_{N_1}, \ldots, v_{N_K}$. This yields a K sized opinion profile on which f can be applied again. If f is consistent, the direct and indirect procedures generate the same outcome.

Consistency prevents manipulation by re-assigning voters to subgroups. It is a strong requirement as many of the aggregators described earlier do not satisfy it.

1. (**Median**) The median aggregator violates consistency. Let $N = \{1, 2, 3\}$ and $m = 2$. Considering the partition $I = \{\{1, 2\}, \{3\}\}$ of $\mathbf{N}$ we have,

$$\begin{aligned} f^{\text{med}}\left(f^{\text{med}}\begin{pmatrix} 0.4 & 0.1 \\ 0.3 & 0.8 \end{pmatrix}, \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix}\right) &= f^{\text{med}}\begin{pmatrix} 0.1 & 0.7 \\ 0.3 & 0.4 \end{pmatrix} \\ &= \begin{pmatrix} 0.1 \\ 0.3 \end{pmatrix} \\ &\neq \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix} \\ &= f^{\text{med}}\begin{pmatrix} 0.4 & 0.1 & 0.7 \\ 0.3 & 0.8 & 0.4 \end{pmatrix}. \end{aligned}$$

   Note that our definition of the median aggregator picks the "lower median evaluation" in societies with an even number of voters. The violation of consistency by the median rule does not depend on this assumption.

2. (**Mean**) The mean aggregator also violates consistency.

Consider the same example and partition as before. We have,

$$\begin{aligned} f^{\text{mean}}\left(f^{\text{mean}}\begin{pmatrix} 0.4 & 0.1 \\ 0.3 & 0.8 \end{pmatrix}, \begin{matrix} 0.7 \\ 0.4 \end{matrix}\right) &= f^{\text{mean}}\begin{pmatrix} 0.25 & 0.7 \\ 0.55 & 0.4 \end{pmatrix} \\ &= \begin{pmatrix} 0.475 \\ 0.475 \end{pmatrix} \\ &\neq \begin{pmatrix} 0.4 \\ 0.5 \end{pmatrix} \\ &= f^{\text{mean}}\begin{pmatrix} 0.4 & 0.1 & 0.7 \\ 0.3 & 0.8 & 0.4 \end{pmatrix}. \end{aligned}$$

On the other hand, the constant rule, the min rule, the L-min rule and their max counterparts and the component-wise α-median rules satisfy consistency. The consistency of the constant aggregator is trivial. The consistency of the component-wise α-median rule is demonstrated in the proof of the theorem. We show the consistency of the min and L-min rule below.

1. (**Min**) Pick an arbitrary profile v and alternative j. Suppose that the miminum evaluation for j in v is $\underline{v}_j$. Let $\underline{v}_j = v_{ij}$. Consider an arbitrary partition $I = \{N_1, \ldots, N_K\}$ of $\mathbf{N}$ and suppose $i \in N_k$. Then $f_j^{\min}(v_{N_k}) = \underline{v}_j$ and $f_j^{\min}(v_{N_k}) \leq f_j^{\min}(v_{N_{k'}})$ for all $N_{k'} \in I$. Therefore, $f_j^{\min}(f^{\min}(v_{N_1}), \ldots, f^{\min}(v_{N_K})) = \min_{N_{k'} \in I}\{f_j^{\min}(v_{N_{k'}})\} = \underline{v}_j = f_j^{\min}(v)$.

   Similarly max rules also satisfy consistency. So do aggregators that pick the minimum for some alternatives and the maximum for others.

2. (**L-min**) Let j be the alternative that is $\succ$ maximal. Let v be an arbitrary profile. The argument for the min rule for alternative j suffices to show that the L-min aggregator is consistent.

2.2 Further Axioms

In addition to consistency, we impose certain axioms.

Definition **2 (Anonymity)** An aggregator f is *anonymous* if for all $N \in \mathbf{N}$ for all $v, v' \in A^n$ and for all bijections $\Pi_i : N \to N$,

$$\left[v_i = v'_{\Pi(i)} \quad \text{for all} \quad i \in N\right] \Rightarrow \left[f(v) = f(v')\right].$$

An aggregator satisfies anonymity if it is invariant with respect to changes in the identities of voters. All the aggregators mentioned above are anonymous. Its easy to construct aggregators that are non-anonymous, for instance, by constructing a "dictator" for every subset of $\mathbf{N}$. Consider the case when $\mathbf{N} = \{1, 2, 3\}$. Let 1 be the dictator for $\{1, 2\}$ and $\{1, 2, 3\}$, 2 be dictator for $\{2, 3\}$ and 3 be the dictator for $\{1, 3\}$. The outcome at any collection of voter opinions is the evaluation vector of the dictator for that subset of voters.

Definition 3 (Unanimity) An aggregator f is *unanimous* if for all $N \in \mathbf{N}$ for all $v \in A^n$ and any $j \in X$,

$$\left[v_i = \bar{v} \quad \text{for all} \quad i \in N\right] \Rightarrow \left[f(v) = \bar{v}\right].$$

An aggregator that satisfies unanimity respects consensus. Our notion of unanimity is, therefore, very weak. Note that the unanimity condition does not apply if all voters are unanimous over a *subset* of the alternatives. All the aggregators mentioned earlier except the constant aggregator are unanimous.

(**Continuity**) An aggregator specifies a collection of maps that aggregates arbitrary sets of m-dimensional voter opinions into an aggregate opinion i.e it is a collection of maps $f : \mathbb{R}^{ml} \to \mathbb{R}^m$ where $l = 1, \ldots, n$. The aggregator satisfies continuity if each of these maps is continuous in the usual sense.

All aggregators discussed earlier except the L-min aggregator satisfy continuity. The violation by L-min is shown below.

Let the set of voters be $N = \{1, 2\}$ and $m = 2$. Let v^t, $t = 2, 3 \ldots$ be a sequence of profiles such that $v_1^t = \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix}$ and $v_2^t = \begin{pmatrix} 0.7 - \frac{1}{t} \\ 0.1 \end{pmatrix}$, $t = 2, 3 \ldots$. Clearly, $v^t \to \begin{pmatrix} 0.7 & 0.7 \\ 0.4 & 0.1 \end{pmatrix} = \bar{v}$ and $f^{\text{L-min}}(v^t) = \begin{pmatrix} 0.7 - \frac{1}{t} \\ 0.4 \end{pmatrix}$ for all t. Therefore, $f^{\text{L-min}}(v^t) \to \begin{pmatrix} 0.7 \\ 0.4 \end{pmatrix}$. However, $f^{\text{L-min}}(\bar{v}) = \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix}$.

The next axiom uses the order structure on the set A.

Definition 4 (Monotonicity) An aggregator f is *monotonic* if for all $N \in \mathbf{N}$, for all $v, v' \in A^n$,

$$\left[v_{ij} \geq v'_{ij} \text{ for all } i, j\right] \Rightarrow \left[f_j(v) \geq f_j(v') \text{ for all } j\right].$$

Fix an arbitrary collection of voters. Suppose *all* voters in this collection weakly increase their evaluations of *all* alternatives. Then the aggregate opinion outputted by a monotonic aggregator must weakly increase for all alternatives. This is clearly a weak condition and aggregators described earlier, satisfy the axiom. It is of course, easy to construct an aggregator that does not satisfy the axiom.

2.3 The Main Result

The main result shows that the component-wise α-median aggregators are characterized by the axioms of consistency, unanimity, anonymity, monotonicity and continuity.

Theorem 1 An aggregator satisfies *consistency*, *unanimity*, *anonymity*, *monotonicity* and *continuity* if and only if it is a *component-wise α-median aggregator*.

Proof: It is easy to verify that component-wise α-median aggregators satisy anonymity, unanimity, continuity and monotonicity. We show that is satisfies consistency.

Consistency: Let f^α be a component-wise α-median aggregator. In view of the separability of component-wise aggregators it clearly suffices to show that it satisfies consistency for any arbitrary alternative.

Pick a profile $v \in A$ and alternative j. Then $f_j^\alpha(v) \in \{\min_{i\in N}(v_{ij}), \alpha_j, \max_{i\in N}(v_{ij})\}$. If $f_j^\alpha(v) = \min_{i\in N}(v_{ij})$ i.e $\alpha_j < \min_{i\in N}$, consistency follows from the same argument used to show that the min aggergator is consistent. Likewise, the arguments used to show that the max aggregator is consistent can be used to show that f^α is consistent when $f^\alpha(v) = \max_{i\in N}(v_{ij})$ i.e $\alpha_j > \max_{i\in N}(v_{ij})$.

If $\alpha_j \in [\min_{i\in N}(v_i), \max_{i\in N}(v_i)]$, then $f_j^\alpha(v) = \alpha_j$. Let $I = \{N_i, \ldots, N_K\}$ be any partition of N. There exists a set $N_k \in I$ such that for some $i \in N_k$, $v_{ij} \le \alpha_j$. By definition, $f_j^\alpha(v_{N_k}) \le \alpha_j$. Therefore, $\alpha_j \ge \min_{N_l \in I}\left(f_j^\alpha(v_{N_k})\right)$. Similarly, there exists $N_{k'} \in I$ such that for some $i' \in N_{k'}$, $v_{i'j} \ge \alpha_j$. By definition, $f_j^\alpha(v_{N_{k'}}) \ge \alpha_j$. Therefore, $\alpha_j \le \max_{N_l\in I}\left(f_j^\alpha(v_{N_l})\right)$. By the above arguments we have, $\alpha_j \in [\min_{N_l\in I}\left(f_j^\alpha(v_{N_l})\right), \max_{N_l\in I}\left(f_j^\alpha(v_{N_l})\right)]$. By definition, $f_j^\alpha(f^\alpha(v_{N_1}), \ldots, f^\alpha(v_{N_K})) = \alpha_j$. Therefore, component-wise α-median aggregators are consistent.

Let f be an aggregator which satisfies consistency, unanimity, anonymity, monotonicity and continuity. Observe that f is actually a collection of rules $\{f^k\}$, $k = 1, \ldots, |\mathbf{N}|$ where f^k is an aggregator for any k-size collection of voter opinions. The next lemma shows that f can be constructed by a repeated application of the function f^2.

Lemma 1 Let $N = \{i_1, \ldots, i_n\} \subseteq \mathbf{N}$ and let $v_{i_k} \in A$ for $k = 1, \ldots n$. Then $f^n(v_{i_1}, ..., v_{i_n}) = f^2(\ldots f^2(f^2(v_{i_1}, v_{i_2}), v_{i_3}) \ldots v_{i_n})$.

This lemma follows directly by the application of consistency. For instance, if $N = \{1, 2, 3, 4\}$, then

$$f^4(v_1, v_2, v_3, v_4) = f^2(f^3(v_1, v_2, v_3), v_4) = f^2(f^2(f^2(v_1, v_2), v_3), v_4).$$

By Lemma 1 we can restrict attention to f^2.

Applying Lemma 1 we can restrict attention to the two voter aggregator f^2. From now onwards, we simply write f in place of f^2 for simplicity of notation. In some cases, we will revert back to f^2 where necessary. We introduce the notion of two evaluations being *ordered*. Let $v, v' \in A^n$, $N \in \mathbf{N}$. If either $v_j \ge v'_j$ or $v_j \le v'_j$ for all $j \in X$ then v is ordered with v'. We define a Box as follows. Let v_i, v_k be a pair of voter opinions. Then

$$Box(v_i, v_k) = \left\{ v_t \in A^2 \;\middle|\; v_{ij} \in \left[\min_{i\in N}(v_{ij}), \max_{i\in N}(v_{kj})\right] \quad \forall j \in X \right\}.$$

Lemma 2 Let $v_i, v_k \in A$. Then $f(v_i, v_k) \in Box(v_i, v_k)$.

Proof: We consider two cases.

- **Case 1:** v_i and v_k are ordered. Assume w.l.o.g. $v_i \geq v_k$. The case where $v_k \geq v_i$ can be dealt with by using a symmetric argument. Applying monotonicity,

$$f(v_i, v_k) \geq f(v_k, v_k) = v_k.$$

  The last inequality holds due to unanimity. Similarly,

$$f(v_i, v_k) \leq f(v_i, v_i) = v_i.$$

  Therefore $f(v_i, v_k) \in Box(v_i, v_k)$.

- **Case 2:** Case 1 does not hold. Let $\underline{v}$ be such that,

$$\underline{v}_j = \min(v_{ij}, v_{kj}) \quad \forall j \in X.$$

  Similarly, let $\overline{v}$ is such that,

$$\overline{v}_j = \max(v_{ij}, v_{kj}) \quad \forall j \in X.$$

  Note that $Box(v_i, v_k) = Box(\underline{v}, \overline{v})$. By monotonicity,

$$f(v_i, v_k) \geq f(\underline{v}, \underline{v}) = \underline{v}.$$

  Similarly,

$$f(v_i, v_k) \leq f(\overline{v}, \overline{v}) = \overline{v}.$$

  Therefore, $f(v_i, v_k) \in Box(v_i, v_k)$.

■

The next lemma is illustrated in Figure 1.

Lemma **3** Let $v_i, v_k \in A$ be ordered (assume w.l.o.g. $v_i \leq v_k$) and $f(v_i, v_k) = v_t$. Then for all $v_r, v_u \in A$ such that $v_r \in Box(v_i, v_t)$ and $v_u \in Box(v_t, v_k)$,

$$f(v_r, v_u) = v_t, \quad f(v_r, v_t) = v_t, \quad f(v_t, v_u) = v_t \quad .$$

Proof: By Lemma 1 and unanimity,

$$f(v_i, v_t) = f^2(v_i, f^2(v_i, v_k)) = f^3(v_i, v_i, v_k)$$

$$= f^2(f^2(v_i, v_i), v_k) = f(v_i, v_k) = v_t.$$

By an analogous argument $f(v_t, v_k) = v_t$. By monotonicity,

$$f(v_r, v_u) \leq f(v_t, v_u) \leq f(v_t, v_k) = v_t.$$

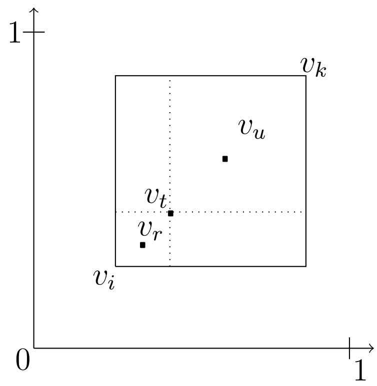

Figure 1: Illustration for Lemma 3

Similarly,

$$f(v_r, v_u) \geq f(v_i, v_u) \geq f(v_i, v_t) = v_t.$$

Therefore $f(v_r, v_u) = v_t$. Again by monotonicity,

$$f(v_r, v_t) \leq f(v_t, v_k) = v_t.$$

Also,

$$f(v_r, v_t) \geq f(v_i, v_t) = v_t.$$

Therefore $f(v_r, v_t) = v_t$. By a similar argument it follows that $f(v_t, v_u) = v_t$. ∎

Lemma 4 Let v_i, v_k, v_i', v_k' be such that (i) v_i is ordered with v_k, v_i' is ordered with v_k' (ii) $f^2(v_i, v_k) = v_t \in \text{int}Box(v_i, v_k)$[4] and (iii) $f(v_i', v_k') = v_t' \in \text{int}Box(v_i', v_k')$. Then $v_i' < v_t$ and $v_k > v_t'$ both cannot hold.

Proof: We prove this by contradiction. So suppose $v_i' < v_t$ and $v_k > v_t'$ hold. Then by applying Lemma 3 on $Box(v_i, v_k)$ we have $f(v_t, v_t') = v_t$ and by applying Lemma 3 on $Box(v_i', v_k')$ we have $f(v_t, v_t') = v_t'$. This is a contradiction. Therefore both $v_i' < v_t$ and $v_k > v_t'$ cannot be true. ∎

Let $v_t \in A$. The box $MBox(v_t) = Box(\bar{v}_i, \bar{v}_k)$ is *a maximal* box for v_t if there does not exist $v_i' < v_i$ and $v_k' > v_k$ such that $f(v_i', v_k') = v_t$. Suppose v_t is in the range of f. Then $MBox(v_t)$ exists by the virtue of continuity of f. Note that a maximal set may not be unique. By similar arguments as in Lemma 3 we can prove the following Lemma.

Lemma 5 Let $MBox(v_t)$ be a maximal box for v_t. Let $v_r, v_u \in A$ such that $v_r \in Box(\overline{v}_i, v_t)$ and $v_u \in Box(v_t, \overline{v}_k)$. Then,

[4] $\text{int}Box(.)$ denotes the interior of $Box(.)$.

(i) $f(v_r, v_u) = v_t, \quad f(v_r, v_t) = v_t, \quad f(v_t, v_u) = v_t.$

(ii) Let $\{v_t^q\}_{q=1}^{\infty}$ be a sequence such that $\lim_{n\to\infty} v_t^q = v_t$. Then,

$$\lim_{n\to\infty} MBox(v_t^q) = MBox(v_t).$$

Proof: The first part of the proof is proved analogously as in the previous Lemma 3. The second part is an implication of the continuity of f^2. ■

LEMMA **6** Let v_i, v_k and v_i', v_k', v_t' be such that (i) v_i is ordered with v_k, v_i' is ordered with v_k' and v_t is ordered with v_t' (ii)$f(v_i, v_k) = v_t \in \text{int}Box(v_i, v_k)$ and (iii) $f(v_i', v_k') = v_t' \in \text{int}Box(v_i', v_k')$. Then $\exists$ v_i'', v_k'' and v_t'' such that (a) $v_i'', v_k'', v_t'' \in Box(v_t, v_t')$ (b) $f(v_i'', v_k'') = v_t''$ and $v_t'' \in \text{int}Box(v_i'', v_k'')$ (c) $v_t'' \notin \{v_t, v_t'\}$.

Proof: W.l.o.g. let $v_i \le v_k$, $v_i' \le v_k'$ and $v_t \le v_t'$. By Lemma 4 we have

$$Box(v_i, v_k) \cap Box(v_t, v_t') \neq \emptyset \;\text{ and }\; Box(v_i, v_k)^C \cap Box(v_t, v_t') \neq \emptyset.$$[5]

$$\text{or }\; Box(v_i', v_k') \cap Box(v_t, v_t') \neq \emptyset \;\text{ and }\; Box(v_i', v_k')^C \cap Box(v_t, v_t') \neq \emptyset.$$

Therefore assume w.l.o.g.

$$Box(v_i', v_k') \cap Box(v_t, v_t') \neq \emptyset \;\text{ and }\; Box(v_i', v_k')^C \cap Box(v_t, v_t') \neq \emptyset. \quad (\#)$$

Pick $v_r \in Box(v_i, v_t)$ and $v_u \in Box(v_t', v_k')$. By applying Lemma 3 to $Box(v_i, v_k)$ and $Box(v_i', v_k')$ we have $f(v_r, v_u) \ge f(v_i, v_t) = v_t$ and $f(v_r, v_u) \le f(v_t', v_k') = v_t'$ respectively. If $f(v_r, v_u) \notin \{v_t, v_t'\}$ then the Lemma holds with $v_i'' = v_r, v_k'' = v_u$ and $v_t'' = f(v_r, v_u)$. So suppose $f(v_r, v_u) \in \{v_t, v_t'\}$. We consider two cases.

Case 1: $f(v_r, v_u') = v_t$. Consider an increasing sequence $\{v_r^q\}$ such that $\lim_{n\to\infty} v_r^q = v_t'$. In view of (#) there exists a q such that v_r^q is on the boundary of $Box(v_i', v_k')$ and is in $Box(v_t, v_t')$. By continuity $\lim_{n\to\infty} f(v_t^r, v_t') = v_t'$. By choosing a point q' sufficiently close to v_t^r we can satisfy the conditions of the Lemma.

Case 2: $f(v_r, v_u) = v_t'$. Suppose $v_k \ge v_t'$. By applying Lemma 3 to $Box(v_i, v_k)$ and $Box(v_i', v_k')$ we have $f(v_r, v_u) = v_t$ and $f(v_r, v_r) = v_t'$. This is a contradiction. Hence $v_k \le v_t'$. Now by repeating the arguments in Case 1 the Lemma holds with $v_i'' = v_r, v_k'' = v_u$ and $v_t'' = f(v_r, v_u)$. ■

The next Lemma states that there exists at most one element in the range of f which is in the interior of its relevant box.

LEMMA **7** There do not exist v_i, v_k, v_i', v_k' such that (i) v_i is ordered with v_k and v_i' is ordered with v_k' (ii) $f(v_i, v_k) \in \text{int}Box(v_i, v_k)$ (iii) $f(v_i', v_k') \in \text{int}Box(v_i', v_k')$ and (iv) $f(v_i, v_k) \neq f(v_i', v_k')$.

[5] A^C is the complement of set A.

Proof: We prove the Lemma by contradiction i.e there exist v_i, v_k, v_i', v_k' as specified in the statement of Lemma 7. Let $f(v_i, v_k) = v_t$ and $f(v_i', v_k') = v_t'$.

1. **Case 1:** Suppose v_t, v_t' are ordered. Assume w.l.o.g. $v_t \leq v_t'$. By Lemma 5 there exists v_i'', v_k'' such that $f(v_i'', v_k'') = v_t''$ and $v_t'' \in \text{int}Box(v_i'', v_k'')$. In fact, by applying the Lemma repeatedly we can contruct a sequence $\{v_t^q\}_{q=1}^{\infty}$ such that $f(v_i^q, v_k^q) = v_t^q \in \text{int}Box(v_i^q, v_k^q)$ for all q and $\lim v_t^q = v_t'$.

   Let $\{\tilde{v_t}^q\}_{q=1}^{\infty}$, be a subsequence of $\{v_t^q\}$ such that $\tilde{v_t}^q \in \text{M}Box(v_t)$ for all q. Note that $Box(v_{i'}, v_t') \cap Box(\tilde{v_i}^q, \tilde{v_t}^q) \neq \emptyset$. We claim that $\tilde{v_k}^q \geq v_t'$ cannot hold. Suppose contrariwise that $\tilde{v_k}^q \geq v_t'$. Pick $v_r \in Box(v_i', v_t') \cap Box(\tilde{v_i}^q, \tilde{v_t}^q)$ and $v_u \in Box(v_t', v_k') \cap Box(\tilde{v_t}^q, \tilde{v_k}^q)$. Applying Lemma 3 to the boxes $Box(\tilde{v_i}^q, \tilde{v_t}^q)$ and $Box(v_i', v_k')$ we have $f(v_r, v_u) = \tilde{v_t}^q$ and $f(v_r, v_u) = v_t'$ respectively. This is a contradiction. Therefore $\tilde{v_k}^q \geq v_t'$ cannot hold and we have

$$\lim_{n\to\infty} \tilde{v_t}^q = v_t' \Rightarrow \lim_{n\to\infty} \tilde{v_k}^q = v_t'.$$

   Since $\tilde{v_t}^q \to v_t'$ we know by Lemma 4 that $\lim_{n\to\infty} MBox(\tilde{v_t}^q) = MBox(v_t')$. Hence $\lim_{n\to\infty} \text{M}Box(\tilde{v_t}^q) = \text{M}Box(v_t') = Box(\bar{v_i}^q, v_t')$ where $\bar{v_i}^q = \lim_{n\to\infty} \tilde{v_i}^q$, i.e $v_t' \notin \text{int}MBox(v_t') = Box(\bar{v_i}^q, v_t')$. However $v_t' \in \text{int}Box(v_i', v_k')$ implies $v_t' \in \text{int}MBox(v_t')$ by assumption. Thus we have a contradiction.

2. **Case 2:** v_t and v_t' are not ordered. Pick $v_r \in Box(\mathbf{0}, v_t) \cap Box(\mathbf{0}, v_t)$.[6] By Case 1 $f(v_r, v_t) \notin \text{int}Box(v_r, v_t)$ and $f(v_r, v_t') \notin \text{int}Box(v_r, v_t')$.

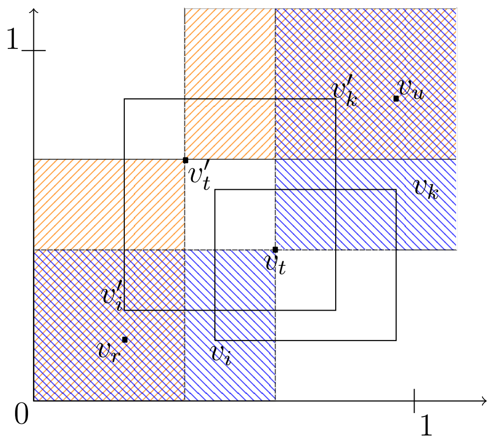

Figure 2: Illustration for Case 2

We claim that $f(v_r, v_t) = v_t$. Suppose this is false. By virtue of the fact that $f(v_r, v_t) \notin \text{int}Box(v_r, v_t)$, $f^2(v_r, v_t)$ must lie on the boundary of $Box(v_r, v_t)$ but not equal to v_t. By

[6]Recall that $\mathbf{0} = (0, 0, ..., 0) \in A$ and $\mathbf{1} = (1, 1, ..., 1) \in A$.

constructing a sequence $\{v_r^q\}_{q=1}^{\infty} \to v_t$ and using arguments from Lemma 5 we obtain a contradiction. Therefore $f(v_r, v_t) = v_t$. By an identical argument $f(v_r, v_t') = v_t'$.

Pick $v_u \in Box(v_t, \mathbf{1}) \cap Box(v_t', \mathbf{1})$. Using the same arguments as in the previous paragraph, we have $f(v_u, v_t) = v_t$ and $f(v_u, v_t') = v_t'$. Applying Lemma 3 and monotonicity,

$$f(v_r, v_u) \geq f(v_r, v_t) = v_t.$$

$$f(v_r, v_u) \leq f(v_t, v_u) = v_t.$$

Therefore $f(v_r, v_u) = v_t$. However, the same argument with v_t' substituted for v_t yields $f(v_r, v_u) = v_t'$. We have a contradiction. ■

LEMMA **8** Let v_i, v_k be ordered and $f(v_i, v_k) = v_t$. Then

$$\big[v_r, v_u \in Box(v_i, v_t), v_r \leq v_u\big] \Rightarrow \big[f(v_r, v_u) = v_u\big].$$

$$\big[v_r, v_u \in Box(v_t, v_k), v_r \leq v_u\big] \Rightarrow \big[f(v_r, v_u) = v_r\big].$$

Proof: Suppose $v_r, v_u \in Box(v_i, v_t)$, $v_r \leq v_u$ and $f(v_r, v_u) \neq v_u$. Suppose $f(v_r, v_u) = v_t'$. By Lemma 7, $v_t' \notin \text{int}Box(v_r, v_u)$. By applying Lemma 3 on $Box(v_r, v_u)$ we have $f(v_t', v_s) = v_t'$ for all $v_s \in Box(v_r, v_t')$. Similarly, by applying Lemma 3 on $Box(v_i, v_t)$ we have $f(v_r, v_t) = v_t$. This implies that there exists $v_k' \geq v_u$ such that $f(v_r, v_k') > v_t'$ and $f(v_r, v_k') \in Box(v_r, v_t')$. By applying Lemma 3 on $Box(v_r, v_t')$ we have $f(f(v_r, v_k'), v_t') = v_t'$. However, by Lemma 3 on $Box(f(v_t, v_k'), v_k')$ we have $f(v_t', f(v_r, v_k')) = f(v_r, v_k')$. This is a contradiction. Therefore, $f(v_r, v_u) = v_u$.

The case where $v_r \leq v_u$ with $v_r, v_u \in Box(v_t, v_k)$ can be proved by an argument similar to the one above. ■

LEMMA **9** Pick any $v_i, v_k \in A$. Then $f(v_i, v_k) = f(\underline{v}, \overline{v})$ where $\underline{v}$ and $\overline{v}$ are as defined before.

Proof: There is nothing to prove in the case where v_i and v_k are ordered. Therefore assume that v_i, v_k are not ordered. Let $f(\underline{v}, \overline{v}) = v_t$. For each $j \in X$ we have $v_t^j, v_t'^j$ such that

(i) $v_{tj}^j = v_{tj}, \; v_{tj'}^j = \min(v_{ij'}, v_{kj'}) \;\; \forall j' \in X.$

(ii) $v_{tj}'^j = v_{tj}, \; v_{tj'}'^j = \max(v_{ij'}, v_{kj'}) \;\; \forall j' \in X.$

Note that $v_t^j \in Box(\underline{v}, v_t)$ and $v_t'^j \in Box(v_t, \overline{v})$ for all j. By applying Lemma 8 to $Box(\underline{v}, v_t)$ and $Box(v_t, \overline{v})$ and using monotonicity we have

$$f(v_i, v_k) \geq f(v_t^j, v_t') = v_t^j.$$

$$f(v_i, v_k) \leq f(v_t'^j, v_t') = v_t'^j.$$

This implies $f(v_i, v_k) = f(\underline{v}, \overline{v}) = v_t$. ■

As an implication of Lemma 9 we can restrict attention to any ordered pair v_i, v_k. Our final Lemma proves the theorem.

Lemma **10** There exists $\alpha \in A$ such that for all $v_i, v_k \in A$

$$f_j(v_i, v_k) = \text{med}(\min_{i \in N}\{v_{ij}\}, \max_{i \in N}\{v_{ij}\}, \alpha_j) \quad \forall j \ \in X.$$

Proof: Let $f(\mathbf{0}, \mathbf{1}) = v_t^*$. We show that f is an α-median rule with $\alpha = v_t^*$. Let $v_i, v_k \in A^n$. By Lemma 9 we only need to consider the case where they are ordered. W.l.o.g. assume $v_i \leq v_k$.

1. **Case 1:** Suppose v_i, v_k are both ordered with respect to v_t^*. We show that f is an α-median rule with $\alpha = v_t^*$. By Lemma 3, $f(v_i, v_k) = v_t^*$ for all $v_i \in Box(\mathbf{0}, v_t^*)$ and $v_k \in Box(v_t^*, \mathbf{1})$. By Lemma 8, $f(v_k, \mathbf{1}) = v_k$ for all $v_k \in Box(v_t^*, \mathbf{1})$ and $f(\mathbf{0}, v_i) = v_i$ for all $v_i \in Box(\mathbf{0}, v_t^*)$. By Lemma 8 and 9,

$$f(v_i, v_k) = f(\underline{v}, \overline{v}) = \overline{v} \quad \forall \ v_i, v_k \in Box(\mathbf{0}, v_t^*).$$

$$f^2(v_i, v_k) = f^2(\underline{v}, \overline{v}) = \underline{v} \quad \forall \ v_i, v_k \in Box(v_t^*, \mathbf{1}).$$

Therefore v_t^* is the α-median for all v_i and v_k ordered such that either $v_i, v_k \in Box(\mathbf{0}, v_t^*)$ or $v_i, v_k \in Box(v_t^*, \mathbf{1})$. If $v_i, v_k \in Box(\mathbf{0}, v_t^*)$ are not ordered then by Lemma 8 and 9, $f^2(v_i, v_k) = \overline{v}$. Similarly if $v_i, v_k \in Box(v_t^*, \mathbf{1})$ are not ordered then by applying Lemma 8 and 9, $f^2(v_i, v_k) = \underline{v}$. Therefore, in both the cases f picks the component-wise α-median for $j \in X$ with $\alpha = v_t^*$.

2. **Case 2:** Suppose v_i is ordered with v_t^* but v_k is not ordered with v_t^*. Pick $v_t^\gamma \in Box(v_i, v_k) \cap Box(\mathbf{0}, v_t^*)$ such that $v_{tj}^\gamma = \text{med}(v_{ij}, v_{kj}, \alpha_j)$ for all $j \in X$. By Lemma 3 and 8 and monotonicity, $f(\mathbf{0}, v_t^*) = v_t^\gamma \leq f(v_i, v_k)$ and $f(v_t^*, \mathbf{1}) = v_t^* \leq f(v_i, v_k)$. This implies $f(v_i, v_k) = v_t^\gamma$. The same arguments hold for the case when v_i is not ordered with v_t^* but v_k is ordered with v_t^*.

3. **Case 3:** Neither v_i nor v_k is ordered with respect to v_t^*. Pick $\underline{v}_i, \underline{v}_k, \overline{v}_i, \overline{v}_k$ such that $\underline{v}_{ij} = \min(v_{ij}, v_{tj}^*)$, $\underline{v}_{kj} = \min(v_{kj}, v_{tj}^*)$, $\overline{v}_{ij} = \max(v_{ij}, v_{tj}^*)$ and $\overline{v}_{kj} = \max(v_{kj}, v_{tj}^*)$. By applying Lemma 8 to $Box(\mathbf{0}, v_t^*)$ and $Box(v_t^*, \mathbf{1})$ and using monotonicity ,

$$f(\underline{v}_i, \underline{v}_k) = \underline{v}_k \leq f(v_i, v_k).$$

$$f(\overline{v}_i, \overline{v}_k) = \overline{v}_i \geq f(v_i, v_k).$$

This implies $f(v_i, v_k) = \text{med}(v_{ij}, v_{kj}, \alpha_j)$.

■

Let $f(\mathbf{0}, \mathbf{1}) = v_t^*$ such that $v_t^* \in A$. We have proved that f^2 is a component-wise α-median aggregator with $\alpha = v_t^*$. Note that f^k is also *a* component-wise aggregator i.e the aggregation over an alternative is independent of the opinions over other alternatives. We show that f^k is a component-wise α-median rule for $k = 1, 2, ..., n$.

Let $v \in A^k$, $k \in N$ be a profile. We show that

$$f_j(v) = \text{med}(\min_{i=1,...,k} v_{ij}, \max_{i=1,...,k} v_{ij}, \alpha_j)$$

for all $j \in X$. There are several cases to consider. Pick $j \in X$. Suppose $v_{ij} \leq \alpha_j$ for all $i \in N$. Since f^2 is a component-wise α-median aggregator $f^2(v_{ij}, v_{i'j}) = \max(v_{ij}, v_{i'j})$ for all i, i'. Therefore,

$$\begin{aligned} f^k(v_{1j}, ..., v_{kj}) &= f^2(...f^2(f^2(v_{1j}, v_{2j}), ..., v_{kj}) \\ &= \max(... \max(\max(v_{1j}, v_{2j})..., v_{kj}) \\ &= \max(v_{1j}, ..., v_{kj}) \\ &= f^k(v_{1j}, ..., v_{kj}) \\ &= \text{med}(\min_i(v_{ij}), \max_i(v_{ij}), \alpha_j). \end{aligned}$$

Suppose $v_{ij} \geq \alpha_j$ for all $i \in N$. An argument analogous to the previous one gives $f^k(v_{1j}, ..., v_{kj}) = \min(v_{1j}, ..., v_{kj}) = \text{med}(\min_i(v_{ij}), \max_i(v_{ij}), \alpha_j)$.

Finally consider the case where $\alpha_j \in (\min_i(v_{ij}), \max_i(v_{ij}))$. Let

$$f^2(v_{1j}, v_{2j}) = z^1.$$

$$f^2(f^2(v_{1j}, v_{2j}), v_{3j}) = z^2.$$

$$\vdots$$

$$f^2(...(f^2(v_{1j}, v_{2j}), ...v_{kj})) = z^{k-1}.$$

In view of the nature of f^2 there must exist q such that $z^q = \alpha_j$ and $z^{q'} = \alpha_j$ for all $q' \geq q$. Therefore $f^k(v_{1j}, ..., v_{kj}) = \text{med}(\min_i(v_{ij}), \max_i(v_{ij}), \alpha_j)$. This completes the proof. ■

2.4 Discussion

Theorem 1 generalizes the Fung and Fu (1975) result from the one dimensional to the multidimensional case. The structure of the proof broadly follows that of Fung and Fu (1975). However, the generalization of specific arguments is not straightforward since several "new" cases can arise regarding the location of the evaluation vectors chosen for aggregation.

2.5 Independence of axioms

We show that the axioms used in Theorem 1 are independent. We consider each axiom in turn and show that there exists an aggregator that satisfies the other axioms.

Consistency: The median aggregator satisfies all the axioms except consistency.

Unanimity: Constant aggregators satisfy all the axioms except unanimity.

Anonymity: We define an aggregator that specifies a dictator for every subset of the voters and outputs the vector of evaluations of the dictator for all profile. We proceed as follows. Let $\underline{i}(N) = \min_{i \in N} \#i$. Then f^D is a sequential dictator aggregator if $f^D = v_{\underline{i}(N)}$ for all $N \in \mathbf{N}$ for all $v \in A^n$.

The aggregator is consistent as we show below. Consider a profile $v \in A^n$. Then by definition of the aggregator, $f^D(v_1, \ldots, v_n) = v_1$. Consider any partition $I = \{N_1, \ldots, N_K\}$. By applying the rule to the sub-groups we have,

$$f^D(f^D(v_{N_1}), \ldots, f^D(v_{N_K})) = f(v_{\underline{i}(N_1)}, \ldots, v_{\underline{i}(N_K)}) = v_{\underline{i}(N)} = f^D(v) = v_1.$$

The sequential dictatorship clearly violates anonymity.

Continuity: We have shown earlier that the L-min aggregator satisfies all the axioms other than continuity.

Monotonicity: We define an aggregator for the case when the number of alternatives is two. The construction can be easily generalized to an arbitrary number of alternatives.

Define f^2 as follows. Pick $\bar{v} \in A$ with $\bar{v}_2 > 0$. The aggregator will be separable. For the first component, f^2 picks the smaller of the first component of the two voter evaluations, i.e $f_1(v_i, v_k) = \min(v_{i1}, v_{k1})$ for all $v_i, v_k \in A$. For the second component, there are three cases:

(i) $\max(v_{i2}, v_{k2}) \leq \bar{v}_2$. Then $f_2(v_i, v_k) = \max(v_{i2}, v_{k2})$.

(ii) $\min(v_{i2}, v_{k2}) \geq \bar{v}_2$. Then $f_2(v_i, v_k) = \min(v_{i2}, v_{k2})$.

(iii) $\min(v_{i2}, v_{k2}) < \bar{v}_2$ and $\max(v_{i2}, v_{k2}) > \bar{v}_2$. Then

$$f_1(v_i, v_k) = \max\big(\min(v_{i2}, v_{k2}), \bar{v}_2 - |\bar{v}_2 - \max(v_{i2}, v_{k2})|\big).$$

The aggregator f^k, $k \in \{1, \ldots, n\}$ can be obtained from f^2 in the following way. For any $v \in A^k$,

$$f(v_1, \ldots, v_k) = \max\big(\min_i v_{i2}, \bar{v}_2 - |\bar{v}_2 - \max_i v_{i2}|\big).$$

We show that the rule is not monotonic. In Figure 3 $v_r \in A$ satisfies $v_{rj} < \bar{v}_j$, $j \in \{1, 2\}$. For the profile $v = (v_r, \bar{v})$ we have $f(v_r, \bar{v}) = \bar{v}$. Pick v_u such that $v_{uj} > \bar{v}_j$, $j \in \{1, 2\}$ and $f(v_r, v_u) = v_t$ where $v_{t2} < \bar{v}_2$. Therefore, the rule violates monotonicity.

The aggregator is consistent for any profile $v \in A^n$. Suppose $\bar{i}$ such that $v_{\bar{i}2} = \max_{i \in N} v_{i2}$. Consider a partition $I = \{N_1, \ldots, N_K\}$. Suppose $\bar{i} \in N_k$ for some $k \in \{1, \ldots, K\}$. Note that $\min_{i \in N_k} v_{i2} \geq \min_{i \in N_k} v_{i2}$. Therefore,

$$\max\big(\min_{i \in N_k} v_{i2}, \bar{v}_2 - |\bar{v}_2 - \max_{i \in N_k} v_{i2}|\big) \geq \max\big(\min_{i \in N_{k'}} v_{i2}, \bar{v}_2 - |\bar{v}_2 - \max_{i \in N_{k'}} v_{i2}|\big)$$

for all $N_{k'} \in I$. Therefore,

$$f(f(v_{N_1}, \ldots, f(v_{N_K})) = \max\big(\min_{i \in N} v_{i2}, \bar{v}_2 - |\bar{v}_2 - v_{\bar{i}2}|\big) = f(v).$$

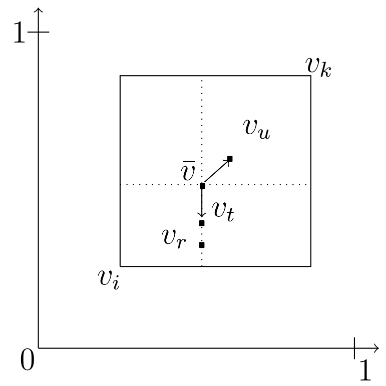

Figure 3: Violation of monotonicity

Figure 4 shows the continuity of the aggregator. Continuity is an issue only for sequences of the following kind: $\{(v_1^q, v_2^q)\}$, $q = 1, 2\ldots$ such that (i) $v_1^q = v_{rj} \leq \hat{v}_j$ for all q and for $j \in \{1, 2\}$ and (ii) $\{v_2^q\} \to \hat{v}$. In this case, $f(v_1^q, v_2^q) \to v_t$ and $f(v_r, \hat{v}) = v_t$ so that f is continuous.

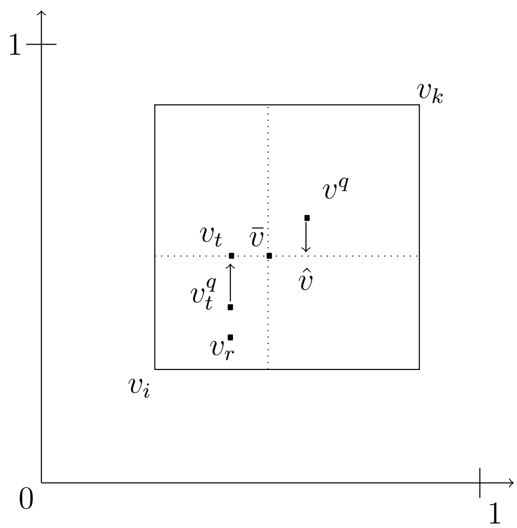

Figure 4: Continuity of the aggregator

3 The Finite Case: Aggregating Sets of Alternatives

In the previous model, voters submited a utility number for each alternative with utilities normalized to lie in the set $[0, 1]$. In this section we depart radically from this model and consider a model where voters have *binary* choices over each alternative. They can either declare 0 for an alternative indicating disapproval or 1 indicating approval. The aggregation rule takes tuples of voter opinions as inputs and outputs an aggregate binary opinion for each candidate. This is therefore a model of the aggregation of sets. Our goal is to study the role of consistency in this framework. [7]

Our first observation is that Theorem 1 no longer holds in this setting. For example, the L-min rule satisfies all the axioms of Theorem 1 since continuity holds vacuously. In particular, separability across components in the aggregation rule is no longer guaranteed. We shall impose further axioms that are natural in this context to show that the aggregation rule must be constant over a large class of profiles. We show that an aggregator satisfies consistency, *component unanimity* and *component anonymity* if and only if it is a *Bipartite Rule*. These aggregators pick the same set of alternatives for "almost" all vote profiles. These aggregators pick a fixed set of alternatives unless voters unanimously approve that alternative and always reject an alternative unless voters unanimously reject its selection. We proceed to details.

The set of candidates or alternatives is X with $|X| = m$. The set of voters is $\mathbf{N} = \{1, 2, ..., n\}$. A voter submits $v_i \in \{0,1\}^m$ an we denote the set $\{0,1\}^m$ by A. A component $v_{ij} = 0$ indicates that voter does not approve of j while a value of 1 indicates approval.

A district or a group is a non-empty set $N \subset \mathbf{N}$. A vote profile is a collection of v_i for all voters $i \in N$ such that $N \subseteq \mathbf{N}$. A vote profile v_S is the restriction of v to a vote profile for voters in $S \subseteq \mathbf{N}$. An *aggregator* is a function $f : \cup_{N \in \mathbf{N}} A^n \to A$ which aggregates voter profiles for any district or subset $\mathbf{N}$.

Several aggregators introduced in Section 2 are not well-defined in this model. These include the median and the mean aggregators. Component-wise α-medians rules are also not well-defined unless α_j is either 0 or 1. The min., left-aligned, constant and L-min. aggregators are well-defined in this setting.

We now turn to axioms. The main axiom as before will be consistency which is defined exactly as before. Monotonicity is no longer required and continuity holds vacuously. However, some new axioms are introduced.

Definition 5 **(Component unanimity)** An aggregator f satisfies *component unanimity* if for all $j \in X$, $N \in \mathbf{N}$ and $v \in A^n$,

$$\left[v_{ij} = \bar{v}_j \ \ \forall i \in N\right] \Rightarrow \left[f_j(v) = \bar{v}_j.\right]$$

The axiom requires the aggregator to select alternatives that have been approved unanimously and reject alternatives that have been rejected unanimously. Aggregators that satisfy component unanimity are the min, max and L-min. Constant rules violate this condition.

Definition 6 **(Component anonymity)** An aggregator f satisfies *component anonymity* if for all $N \in \mathbf{N}$ for all bijections $\sigma_{ij} : N \times K \to N$ and all $j \in X$ $v, v' \in A$,

$$\left[v_{ij} = v'_{\sigma(ij)j} \ \text{ for all } i \in N\right] \Rightarrow \left[f_j(v) = f_j(v')\right].$$

[7] There is a fairly extensive literature on the aggregation of sets of alternatives - see for instance, Barberà et al. (1991), Plott (1973), Goodin and List (2006), Kasher and Rubinstein (1997).

Component anonymity requires the component outcome to be invariant to premutations of opinions an alternative j. The min, max and constant aggregators satisfy this condition. The following piece of notation will be used for the next definition. Let $W(v) = \{j | \; v_{ij} = 1 \text{ for all } i \in N\}$ and $L(v) = \{j | \; v_{ij} = 0 \text{ for all } i \in N\}$.

Definition **7 (Bipartite Rule)** An aggregator f^{BR} is a *Bipartite Rule* if there exists a partition $\{F, F^C\}$ of X such that

(i) $\Big[j \in F\Big] \Rightarrow \Big[f_j^{\text{BF}}(v) = 1 \text{ for all } v \text{ such that } j \notin L(v)\Big]$.

(ii) $\Big[j \in F^c\Big] \Rightarrow \Big[f_j^{\text{BR}}(v) = 0 \text{ for all } v \text{ such that } j \notin W(v)\Big]$.

Bipartite Rule divides the set of alternatives X into favoured (F) and non-favoured sets (F^C). Alternatives in the favoured set are always selected by the aggregator unless all voters reject it. An alternative in the non-favoured set does not get selected unless all voters approve.

Bipartite Rules satisfy component unanimity and component anonymity. These aggregators are consistent and separable. To see that they are consistent suppose $v \in A^n$ and $I = \{N_1, \ldots, N_k\}$ is a partition of N. Let $j \in X$ be any alternative. There exists a set $N_k \in I$ such that if there is no unanimous decision over j in the profile for n voters then there is no unanimity over j in v_{N_k}. This implies $f_j^{\text{BR}}(v_{N_k}) = f_j^{\text{BR}}(v)$. Therefore, $f(f(v_{N_1}), \ldots, f(v_{N_K})) = f(v)$.

Bipartite Rules are constant over a "large" number of vote profiles. If the number of voters is large, the set of profiles where voters are unanimous over a component is "small". Consequently, a Bipartite Rule will be "nearly" constant.

Remark. Note that Bipartite Rules are a type of component-wise α-median rule with $\alpha_j = 1$ or 0 for each alternative.

Example **1** The set of voters $N = \{1, 2, 3\}$ and set of alternatives $X = \{a, b, c, d\}$. Let f^{Q} be a Bipartite Rule with the set of favoured alternative $F = \{a, c\}$ and the set of non-favoured alternatives be $F^c = \{b, d\}$. Then,

$$f^{\text{BR}} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Our next result is a characterization of Bipartite Rules.

3.1 The Result

Theorem 2 An aggregator satisfies *consistency*, *component unanimity* and *component anonymity* if and only if it is a *Bipartite Rule*.

Proof: Suppose an aggregator satisfies consistency, component unanimity and component anonymity. Define the order $(\preceq)$ on A^n as follows.

$$v_i \preceq v_k \text{ if } f(v_i, v_k) = v_i \quad \text{for all } v_i, v_k \in A.$$

We show that the order $(\preceq)$ is a partial order i.e it satisfies the following three properties.

(i) Reflexivity: Pick any $v_i \in A$. By component unanimity, $f(v_i, v_i) = v_i$. Therefore, $v_i \preceq v_i$ for all $v_i \in A$.

(ii) Anti-symmetry: Suppose $v_i, v_k \in A$ such that $v_i \preceq v_k$ and $v_k \preceq v_i$. Then by definition, $f(v_i, v_k) = v_i$ and $f(v_k, v_i) = v_k$. By component anonymity, $f(v_i, v_k) = f(v_k, v_i) = v_i = v_k$.

(iii) Transitivity: Suppose $v_i, v_k, v_t \in A$ such that $v_i \preceq v_k$ and $v_k \preceq v_t$. By definition, $f(v_i, v_k) = v_i$ and $f(v_k, v_t) = v_k$. Therefore, by consistency and component unanimity,

$$f(v_i, v_t) = f^2(f(v_i, v_k), f(v_k, v_t)) = f^4(v_i, v_k, v_k, v_t).$$

$$= f^3(v_i, f(v_k, v_k), v_t) = f^3(v_i, v_k, v_t) = f^2(v_i, f(v_k, v_t)) = f(v_i, v_k) = v_t.$$

Therefore, the ordering $(\preceq)$ is a partial order. We claim the following. Suppose $v_i \preceq v_k$ for some $v_i, v_k \in A^n$. Then $f(v_i, v_t) \preceq f(v_k, v_t)$ for all $v_t \in A^2$.
By consistency and component unanimity we have,

$$f^2(f(v_i, v_k), f(v_k, v_t) = f^4(v_i, v_k, v_k, v_t) = f^3(v_i, v_k, v_t).$$

$$= f^2(f(v_i, v_k), v_t) = f(v_i, v_t).$$

Therefore, the aggregator is increasing in the order $(\preceq)$. We claim that $f(v_i, v_k) \preceq v_i$ and $f(v_i, v_k) \preceq v_k$. By consistency, component anonymity and component unanimity,

$$f^2(f(v_i, v_k), v_i) = f^3(v_i, v_k, v_i) = f^2(f(v_i, v_i), v_k) = f(v_i, v_k).$$

Therefore, by the definition of $(\preceq)$ we have $f(v_i, v_k) \preceq v_i$. Similarly, we can show that $f(v_i, v_k) \preceq v_k$. Therefore, the aggregator outputs a vector of evaluations which is a lower bound according to $(\preceq)$. We finally show that the aggregator must select the unique greatest lower bound vector of opinions for any pair of voter opinions.

Suppose $f(v_i, v_k) = v_t$. We have shown that v_t must be a upper bound of v_i and v_k. We claim that v_t is the unique greatest lower bound. We prove this by contradiction. Suppose v'_t is another lower bound. By definition,

$$f(v_i, v'_t) = v'_t \text{ and } f(v_k, v'_t) = v'_t.$$

Therefore, by consistency,

$$f(v_i, v'_t) = f^2(v_i, f(v_k, v'_t)) = f^3(v_i, v_k, v'_t).$$

$$= f^2(f(v_i, v_k), v'_t) = f(v_t, v'_t).$$

Since $f(v_i, v'_t) = v'_t$ we have $f(v_t, v'_t) = v'_t$. Therefore, $v'_t \preceq v_t = f(v_i, v_k)$. Therefore, $f(v_i, v_k)$ is the unique greatest lower bound of v_i and v_k.

We show that the aggregator is invariant to permutations of opinions over an alternative.[8] We claim the following. Let $\pi : N \times X \to N$ be a bijection. Suppose $v,\ v' \in A^2$ such that $v'_{ij} = v_{\pi(ij)j}$ for some $j \in X$ and $v'_{ij'} = v_{ij'}$ for all $j' \in X, j' \neq j$. Then $f(v) = f(v')$.

We prove the above claim by contradiction. Consider a profile $v \in A^2$ and an alternative $j \in X$. The claim is trivially true if $j \in L(v) \cup W(v)$. Suppose $j \notin L(v) \cup W(v)$. Let $v = (v_i, v_k)$ and $v' = (v'_i, v'_k)$ such that $v_{ij} = v'_{kj}$, $v_{kj} = v_{ij}$, $v'_{ij'} = v_{ij'}$ and $v'_{kj'} = v_{kj'}$ for all $j' \neq j$, $f_j(v) = f_j(v')$ and $f_{j'}(v) \neq f_{j'}(v')$. Therefore, the bijection π is such that $\pi(i,j) = k$ and $\pi(k,j) = i$ and $\pi(i',j') = i'$ for all $i' \in N$ and $j' \in X$, $j' \neq j$.

We claim that $f(v)$ must be ordered with $f(v')$. Suppose contrariwise, that $f(v)$ is not ordered with $f(v')$. Then $f(f(v), f(v')) = v''$ where $v'' \notin \{f(v), f(v')\}$. W.l.o.g assume that $v''_{j'} = f_{j'}(v)$. By definition of $(\preceq)$, we have $f_{j'}(v_i, v'_i) = f_{j'}(v')$. This is a violation of component unanimity. Therefore, $f(v)$ is ordered with $f(v')$.

W.l.o.g suppose $f(v) \preceq f(v')$. By the definition of $(\preceq)$, we have $f_{j'}(v_i, f(v')) = f_{j'}(v)$. This is a contradiction to component anonymity since by our construction $f_{j'}(v) = f_{j'}(v') = 1 - f_{j'}(v)$. Similar arguments can be made when $f(v') \preceq f(v)$. The final claim proves separability.

We claim the following. For all $v, v' \in A^2$, $\left[v_j = v'_j\right] \Rightarrow \left[f_j(v) = f_j(v')\right]$ for all $j \in X$.

Let $\bar{v} = (v_1, v_2) \in A^2$ be a profile such that $\bar{v}_{1j} + \bar{v}_{2j} = 1$. To prove the claim it is sufficient to show that for all $v \in A^2$, $\left[v_{1j} + v_{2j} = 1\right] \Rightarrow \left[f_j(v) = f_j(\bar{v})\right]$ for all $j \in X$. So pick any $j \in X$ and $v \in A^2$ such that $v_{1j} + v_{2j} = 1$. By definition $f^4(\bar{v}, \bar{v}) = f^2(f^2(\bar{v}), f^2(\bar{v}))$. By the property of $\bar{v}$ there exists a profile $\hat{v} = (\hat{v}_1, \hat{v}_2) \in A^2$ such that $f^4(\bar{v}, \bar{v}) = f^4(v, \hat{v})$. We construct $\hat{v}$ as follows: (1) $\hat{v}_{ij} = v_{ij}$ (ii) $\hat{v}_{ij'} = v_{ij'}$ for all $j' \notin L(v) \cup W(v)$, $j' \neq j$ (iii) $\hat{v}_{ij'} = 1 - v_{ij'}$ for all $j' \in L(v) \cup W(v)$, $j' \neq j$ for $i \in \{1, 2\}$. Therefore, $(v, \hat{v})$ is contructed by permutations of component values in the profile $(\bar{v}, \bar{v})$.

By our previous claim and consistency, we have $f^4(\bar{v}, \bar{v}) = f^4(v, \hat{v}) = f^4(f(v), f(v), \hat{v})$. Now, we construct a profile $\tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in A^2$ such that $f^4(v, \hat{v}) = f^4(\tilde{v}, v)$. We construct $\tilde{v}$

[8] Recall that component anonymity only states that the aggregator is invariant *only* over the alternative for which the opinions are permuted.

as follows: (1) $\tilde{v}_{ij} = f_j(v)$ (ii) $\tilde{v}_{ij'} = v_{ij'}$ for all $j' \notin L(v) \cup W(v)$, $j' \neq j$ (iii) $\tilde{v}_{ij'} = 1 - v_{ij'}$ for all $j' \in L(v) \cup W(v)$, $j' \neq j$ for $i \in \{1, 2\}$. Therefore, $(\tilde{v}, v)$ is contructed by permutations of component values in the profile $(f(v), f(v), \hat{v})$.

By our previous claim, we have $f(v, \hat{v}) = f(\tilde{v}, v)$. Also, note that $f_j(\tilde{v}) = f_j(v)$. By consistency, component anonymity and component unanimity, we have $f_j^4(\bar{v}, \bar{v}) = f_j^4(v, \hat{v}) = f_j^3(f(v), \hat{v}) = f^4(f(v), f(v), \hat{v}) = f_j^4(\tilde{v}, v) = f_j^3(f_j(v), v) = f_j^2(v)$.

Therefore, our claim is true and f^2 is a Bipartite Rule where an alternative $j \in X$ is in the favoured set $F \subset X$ if $f_j(0, 1) = 1$ and it is in the non-favoured set F^C if $f_j(0, 1) = 0$.

We show that if f^2 is a Bipartite Rule then f^k is a Bipartite Rule, $k \in \{1, \ldots, N\}$. To see this, take any profile $v \in A^K$. Then we have $f(v) = f^2(\ldots f^2(f^2(v_1, v_2), v_3), \ldots, v_K)$. Since f^2 is separable, we can focus our attention to any arbitrary alternative j. Suppose $j \in L(v)$. Then by component unanimity $f_j(v) = 0$. Similarly $f_j(v) = 1$ if $W(v)$. Suppose $j \notin L(v) \cup W(v)$. Let

$$f^2(v_{1j}, v_{2j}) = z^1.$$

$$f^2(f^2(v_{1j}, v_{2j}), v_{3j}) = z^2.$$

$$\vdots$$

$$f^2(...(f^2(v_{1j}, v_{2j}), ...v_{kj})) = z^{k-1}.$$

In view of the nature of f^2 there must exist q such that $z^q \in \{0, 1\}$ such that $z^{q'} = z^q$ for all $q' \geq q$. Therefore, f^K is a Bipartite Rule with $j \in X$ in the favoured set F if $f_j(0, 1) = 1$ or j in the non-favoured set F^C if $f_j(0, 1) = 0$. This completes the proof. ■

Theorem 2 implies that the result of the previous model holds in this setting but with a stronger set of axioms. These aggregators are also similar to Unanimity Rules described in Bervoets and Merlin (2012).

3.2 Indepedence of axioms

We show the independence of the axioms below.

Component unanimity: Constant Rules satisfy all axioms except component unanimity.

Component anonymity: L-min aggregators satisfy all axioms except component anonymity.

Consistency: The following aggregator satisfies all the axioms except consistency. An aggregator f^P is a *Parity aggregator* if for any profile $v \in A^n$, $N \in \mathbf{N}$: (i) $f_j^P(v) = \min_{i \in N}(v_{ij})$ if N is odd and (ii) $f_j^P(v) = \max_{i \in N}(v_{ij})$ if N is even. This aggregator satisfies the other component unanimity and component anonymity but is not consistent. We show that it violates consistency. Note that the aggregator is separable so it is sufficient to show its violation of consistency for some arbitrary alternative j. Suppose $v_j = (0, 1, 1)$ is the vector of opinions of voters 1, 2 and 3 for an alternative j. By definition we have $f_j^P(v_j) = \min(0, 1, 1) = 0$.

Consider the partition $I = \{\{1,2\},\{3\}\}$. By applying the aggregator to the subgroups we have $f_j^P(f_j^P(0,1),1) = f^P(1,1) = 1$. Therefore, $f_j^P(v_j) \neq f_j^P(f_j^P(v_{\{1,2\}}), f_j^P(v_3))$.

None of the axioms can be weaked to give separability of the aggregator. To see this note that the L-min aggregator satisfies consistency, component unaninimity and anonymity but not component anonymity. Moreover, the L-min aggregator is not separable. Therefore, component anonymity plays a vital role in characterizing separable aggregators.

4 Conclusion

This paper examines the structure of consistent, multidimensional, multilevel aggregators in two distinct models. We characterize a class of separable rules called component-wise α-median rules and generalize the one-dimensional results of Fung and Fu (1975). These can also be seen as component-wise α-median aggregators. If the set of evaluations is finite, separability is no longer guaranteed. In addition to consistency, stronger notions of unanimity and anonymity are required to characterize a class of separable rules called Bipartite Rules.

References

Austen-Smith, D. and J. S. Banks (2000): *Positive Political Theory I: Collective Preference*, University of Michigan Press.

——— (2005): *Positive Political Theory II: Strategy and Structure*, University of Michigan Press.

Barberà, S., H. Sonnenschein, and L. Zhou (1991): "Voting by Committees," *Econometrica: Journal of the Econometric Society*, 595–609.

Bervoets, S. and V. Merlin (2012): "Gerrymander-proof Representative Democracies," *International Journal of Game Theory*, 41(3), 473–488.

Chambers, C. P. (2008): "Consistent Representative Democracy," *Games and Economic Behavior*, 62(2), 348–363.

——— (2009): "An Axiomatic Theory of Political Representation," *Journal of Economic Theory*, 144(1), 375–389.

Fung, L. W. and K. S. Fu (1975): "An Axiomatic Approach to Rational Decision Making in a Fuzzy Environment," *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, 227–256.

Goodin, R. E. and C. List (2006): "A Conditional Defense of Plurality Rule: Generalizing May's Theorem in a Restricted Informational Environment," *American Journal of Political Science*, 50(4), 940–949.

Kasher, A. and A. Rubinstein (1997): "On the Question "Who is a J?": A Social Choice Approach," *Logique et Analyse*, 160, 385–395.

Katz, R. S. (1998): "Malapportionment and Gerrymandering in Other Countries and Alternative Electoral Systems," *Contributions in Political Science*, 384, 245–260.

Macé, A. (2013): "Voting with Evaluations: When Should We Sum? What Should We Sum?" Mimeo.

Nermuth, M. (1994): "Two-Stage Discrete Aggregation: The Ostrogorski Paradox and Related Phenomena," *Social Choice and Welfare*, 9, 99–116.

Perote-Peña, J. (2005): "Gerrymandering-proof Social Welfare Functions," *Mimeo.*

Plott, C. R. (1973): "Path Independence, Rationality, and Social Choice," *Econometrica*, 41, 1075–1091.

Samuels, D. and R. Snyder (2001): "The Value of a Vote: Malapportionment in Comparative Perspective," *British Journal of Political Science*, 31, 651–671.