Minimal spectral representations of infinitely divisible and max–infinitely divisible processes

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1. Spectral representations

2. Uniqueness under a “new” notion of minimality

3. Max-stable case: Connections to the “old” notion of minimality

4. Examples

5. Final comments

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Spectral Representations
Max-id processes

Definition

\( \{X_t, t \in T\} \) is max-id if for all \( n \),

\[
\{X_t, t \in T\} \overset{d}{=} \left\{ \max_{i=1,\cdots,n} X_t^{(i,n)}, \ t \in T \right\},
\]

for some iid \( \{X_t^{(i,n)}, t \in T\}, \ i = 1, \cdots, n. \)

- Ch. 5, Resnick (1987): Max-id laws in \( \mathbb{R}^n \) have the form:

\[
F(x) = \exp\{ -\mu(-\infty, x]^{c} \}, \ x \in \mathbb{R}^n,
\]

for some \( \sigma \)-finite measure \( \mu \) on \( \mathbb{R}^n \) – Balkema and Resnick [1977], Gerritse [1986] and Vatan [1985].

- Balkema et al. [1993]: spectral representations for max-id processes.

Note: WLOG we will suppose

\[
\text{essinf}(X_t) = 0, \ \text{for all} \ t \in T.
\]
The Poisson calculus for max-id processes

- Let \((E, \mathcal{E}, \mu)\) be a \(\sigma\)-finite measure space.
- Let \(\Pi_\mu = \{U_i, \ i \in \mathbb{N}\}\) be a Poisson random measure on \((E, \mathcal{E})\) with intensity \(\mu\).

**Definition**

\(\mathcal{L}^\vee(E, \mathcal{E}, \mu)\) is the set of non-negative measurable functions \(f : E \rightarrow \mathbb{R}_+\), such that

\[\mu\{f > a\} < \infty, \ \text{for all} \ a > 0.\]

Define

\[I^\vee(f) \equiv \int_E \check{f} d\Pi_\mu := \sup_{U \in \Pi_\mu} f(U).\]

**Note:** For \(f \in \mathcal{L}^\vee(E, \mu)\), we have \(\text{essinf}(I^\vee(f)) = 0\) and

\[P(I^\vee(f) \leq x) = P(\Pi_\mu \cap \{f > x\}) = 0) = e^{-\mu\{f > x\}}, \ (x > 0).\]
Spectral representations

For \( f_t \in \mathcal{L}^\vee (E, \mu) \), \( t \in T \) it is easy to see that \( X_t := \int_E f_t \, d\Pi_\mu, \ t \in T \) is a max-id process with fidi

\[
P\{X_{t_i} \leq x_i, \ i = 1, \ldots, k\} = \exp \left\{ -\mu \left( \bigcup_{i=1}^k \{ f_{t_i} > x_i \} \right) \right\}, \ (x_i \geq 0)
\]

Conversely:

**Definition**

\( \{X_t, \ t \in T\} \) satisfies **Condition S**, if exists a countable \( T_0 \subset T \), s.t.

\[
\forall t, \ X_t = \plim X_{t_n}, \text{ for some } t_n \in T_0.
\]

**Theorem (Balkema et al. [1993] & Kabluchko and S. [2012])**

If \( \{X_t, \ t \in T\} \) is max-id satisfies **Condition S** and \( \text{essinf}(X_t) = 0 \), then there exist \( \{f_t, \ t \in T\} \subset \mathcal{L}^\vee (\mathbb{R}, \text{Leb}) \)

\[
\{X_t, \ t \in T\} \overset{d}{=} \{I^\vee (f_t), \ t \in T\}.
\]

**Note:** \( \{f_t, \ t \in T\} \) above is called a spectral representation of \( X \).
Examples

- (mixed moving maxima)

\[ X_t = \bigvee_i F_i(t - U_i), \quad t \in \mathbb{R}^2. \]

where the PPP \( \Pi_\mu = \{(U_i, F_i(\cdot))\} \) has intensity
\( \mu(du, dF) = duP(dF) \), where \( P \) is the law of a random field
\( F = \{F(t)\}_{t \in \mathbb{R}^2}. \)

Think of \( U_i \) as storm locations and \( F_i(\cdot) \) as storm profiles.

- (Penrose type processes) \( \xi = \{\xi(t)\}_{t \in \mathbb{R}} \) process with stationary increments. Then,

\[ X_t := \min_i |U_i + \xi_i(t)|, \quad t \in \mathbb{R}, \]

is stationary min-id, where \( (U_i, \xi_i(\cdot)) \) is a PPP with intensity
\( dudP_\xi. \)

- See Kabluchko and S. [2012] for more examples: max-stable processes, Poisson lines, \( \cup \)-id random sets.
Pictures of max-id random fields
Minimality
A “new” minimality concept

Definition

The spec rep \( \{ f_t \}_{t \in T} \subset \mathcal{L}^\vee(E, \mathcal{E}, \mu) \) of \( X \) is **minimal** if:

(i) \( \sigma\{f_t, t \in T\} = \mathcal{E} \) (mod \( \mu \))

(ii) \( \text{supp}\{f_t, t \in T\} = E \) (mod \( \mu \)) i.e. there is no \( A \in \mathcal{E} \) with \( \mu(A) > 0 \) s.t. \( f_t = 0 \) on \( A \).

Theorem (Kabluchko and S. [2012])

*Under Condition S the max-id process \( X \) has a minimal spec rep on the space \( \mathcal{L}^\vee(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu) \), for some \( \sigma \)-finite Borel measure \( \mu \).*

Notes:

1. The proof is not innovative – book-keeping + use of prior results of Balkema et al. [1993], Vatan [1985], and Kuratowski’s Thm.
2. The definition is the “right one” because of the following **uniqueness** result.
Uniqueness

Theorem (Kabluchko and S. [2012])

If \( \{ f_t \}_{t \in T} \subset \mathcal{L}^\vee(E, \mathcal{E}, \mu) \) and \( \{ g_t \}_{t \in T} \subset \mathcal{L}^\vee(F, \mathcal{F}, \nu) \) are two minimal reps of \( X \), then there exists a measure space isomorphism \( \Phi : (E, \mu) \to (F, \nu) \), s.t.

\[
\forall t, \ f_t = g_t \circ \Phi, \ \text{mod} \ \mu.
\]

Moreover, \( \Phi \) is mod \( \mu \) unique.

Notes:

1. Similar results are well-known in the stable and max–stable cases under a somewhat different minimality concept. They imply important structural results through connections with non-singular flows and ergodic theory: Hardin Jr. [1982], Rosiński [1995], Samorodnitsky [2005], Roy and Samorodnitsky [2008], Roy [2010], de Haan and Pickands III [1986], Kabluchko [2009], Wang and Stoev [2010].

2. How to use this uniqueness result?
Stationry max-id processes and measure-preserving flows

If $X$ is stationary,

$$\{f_t\}_{t \in \mathbb{R}} \quad \text{and} \quad \{g_t\}_{t \in \mathbb{R}} := \{f_{t-\tau}\}_{t \in \mathbb{R}},$$

are both minimal spec reps and then, for any $\tau \in \mathbb{R}$:

$$g_t \circ \Phi_\tau \equiv f_{t-\tau} \circ \Phi_\tau = f_t \pmod{\mu}$$

Uniqueness yields the flow property:

$$\Phi_{t+s} = \Phi_t \circ \Phi_s \pmod{\mu}$$

and using “standard” techniques (Mackey [1962]) one can get a measurable version of the flow $\{\phi_t\}_{t \in \mathbb{R}}$, defined everywhere.

Notes:

1. Now $\phi_t : E \to E$ is measure-preserving! Not just non-singular...

2. Hence the spec rep has the flow representation:

$$f_t = f_0 \circ \phi_t \pmod{\mu}$$
The precise statement

Theorem (Kabluchko and S. [2012])

Let $X = \{X_t, \ t \in \mathbb{R}^d\}$ be continuous in probability, stationary max-id random field. Then

$$X \overset{d}{=} \left\{ \int_{E} f_0 \circ \phi_t d\Pi \mu \right\}_{t \in \mathbb{R}^d},$$

where $f_0 \in \mathcal{L}^\vee(E, \mathcal{E}, \mu)$ and $\{\phi_t\}_{t \in \mathbb{R}^d}$ is a measurable, measure preserving action on a $\sigma$-finite Borel space $(E, \mathcal{E}, \mu)$.

Notes:

1. Recall that for convenience, we are assuming throughout:

$$\text{essinf}(X_t) = 0, \ t \in \mathbb{R}^d.$$ 

The result trivially extends to other max-id processes.

2. The spec rep $\{f_0 \circ \phi_t\}_{t \in \mathbb{R}^d}$ may be chosen to be minimal.
Max-stable case
Max-stable case

Let $E = (0, \infty) \times F$ and $\Pi_\mu = \{(\epsilon_i, V_i)\}_{i \in \mathbb{N}}$ be a PPP with intensity

$$\mu(dx, dv) = dx \nu(dv).$$

Define $f_t(x, v) := x^{-1}g_t(v)$, where $g_t \in L^1_+(F, \nu)$. Then

$$X_t := \int_E f_t \, d\Pi_\mu \equiv \bigvee_{i \in \mathbb{N}} \frac{g_t(V_i)}{\epsilon_i}, \quad t \in T$$

is a max-stable process.

Notes:

1. $X_t$ is well-defined because $g_t \in L^1_+(\nu)$ implies $f_t \in \mathcal{L}^\vee(E, \mu)$.

   Indeed, for all $a > 0$

   $$\mu\{f_t > a\} = \int_F \int_0^\infty \mathbb{I}(g_t(v) > ax) \, dx \nu(dv) = a^{-1} \int_F g_t(v) \nu(dv) < \infty.$$  

2. Max-stability follows from thinning.
More precisely, the fidi of $X$ are:

$$P(X_{t_i} \leq x_i, \ i = 1, \cdots, k) = \exp\{-\mu(\bigcup_i \{f_{t_i} > x_i\})\}$$

$$= \exp\left\{-\int_F \left( \int_0^\infty \max_{i=1,\cdots,k} \mathbb{I}(g_{t_i}(v) > x_i x) \, dx \right) \nu(dv) \right\}$$

$$= \exp\left\{-\int_F \left( \max_{i=1,\cdots,k} g_{t_i}/x_i \right) d\nu \right\}.$$
Non-singular flows in the max-stable case

Recall

Fact

For a stationary max-stable 1-Fréchet process, we have

$$X \overset{d}{=} \left\{ \int_{F} \left( g_0 \circ \varphi_t(v) \frac{d\nu \circ \varphi_t}{d\nu}(v) \right) M_1(dv) \right\}_{t \in \mathbb{R}}$$

where $M_1$ is 1-Fréchet sup-measure on $(F, \mathcal{F})$ with control measure $\nu$, $g_0 \in L^1_+(\nu)$ and $\{\varphi_t\}$ is a non-singular flow on $F$.

Notes:

1. The flow $\varphi_t : F \to F$ is non-singular if $\nu \circ \varphi_t \sim \nu$ and hence the above Radon-Nikodym derivative $d\nu \circ \varphi_t/d\nu$ makes sense.

2. Question: $X$ is max-id, so what is its measure-preserving flow representation in terms of a PPP?!
From max-stable to max-id

- Let $g_t \in L_+^1(F, \nu)$ and $M_1$ be a 1-Fréchet sup-measure.
- Define the max-stable process
  \[ X_t = \int_F g_t(v)M_1(dv), \quad t \in T. \]
- What is the PPP spec rep of $X = \{X_t\}_{t \in T}$ as a max-id process?
From max-stable to max-id

- Let $g_t \in L^1_+(F, \nu)$ and $M_1$ be a 1-Fréchet sup-measure.
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  \[ X_t = \int_F g_t(v) M_1(dv), \quad t \in T. \]

  What is the PPP spec rep of $X = \{X_t\}_{t \in T}$ as a max-id process?

Let $E := (0, \infty) \times F$ and $\Pi_\mu = \{(\epsilon_i, V_i)\}_{i \in \mathbb{N}}$ be a PPP with intensity
  \[ \mu(dx, dv) = dx \nu(dv) \]

- Note that $M_1(B) := \bigvee_i \epsilon_i^{-1} I_B(V_i)$ is an independently scattered 1–Fréchet sup-measure.

- Thus,
  \[ \int_{(0, \infty) \times F} x^{-1} g_t(v) \Pi_\mu(dx, dv) = \int_F g_t(v) M_1(dv). \]

- This gives the natural max-id spec rep of a max-stable process.
Dorothy Maharam’s construction

If \( \varphi_t : F \to F \) is a non-singular flow on \( (F, \nu) \), then

\[
\phi_t(x, \nu) := \left( \frac{d\nu \circ \varphi_t}{d\nu}(\nu)^{-1}x, \varphi_t(\nu) \right)
\]

is a measure-preserving flow on

\[
((0, \infty) \times F, dx\nu(d\nu)).
\]
If \( \phi_t : F \rightarrow F \) is a non-singular flow on \((F, \nu)\), then

\[
\phi_t(x, v) := \left( \frac{d\nu \circ \phi_t}{d\nu}(v)^{-1} x, \phi_t(v) \right)
\]

is a measure-preserving flow on

\[\left( (0, \infty) \times F, dx \nu(dv) \right)\].

Now, as above, define \( f_t(x, v) := x^{-1} g_t(v) \) and note that

\[
f_t(x, v) = f_0 \circ \phi_t(x, v) = x^{-1} \frac{d\nu \circ \phi_t}{d\nu}(v) g_t \circ \phi_t(v)
\]

Since \( \bigvee_i e_i^{-1} \Pi_B(V_i) = M_1(B) \), recall that

\[
\int_{(0, \infty) \times F} x^{-1} g_t(v) \Pi_\mu(dx, dv) = \int_F g_t(v) M_1(dv),
\]

The max-id spec rep is generated by a measure-preserving flow!
Old and new minimality

Let $X = \{X_t\}_{t \in T}$ be max-stable with spec rep

$$\{X_t\}_{t \in T} \overset{d}{=} \left\{ \int_F \bigvee g_t(v) M_1(dv) \right\}_{t \in T}, \quad (g_t \in L_+^1(F, \mathcal{F}_\nu)).$$

Recall that $\{g_t\}_{t \in T}$ is minimal if:

1. (ratio $\sigma$-alg) $\rho\{g_t, \ t \in T\} := \sigma\{g_t/g_s, \ t, s \in T\} \sim \mathcal{F}$ (mod $\nu$).
2. (full support) $\text{supp}\{g_t, \ t \in T\} = F$ (mod $\nu$).

What is the connection b/w new and old minimality?

Lemma ("Old" $\Rightarrow$ "New")

If $\{g_t\}_{t \in T} \subset L_+^1(F, \mathcal{F}_\nu)$ is minimal then $\{f_t\}_{t \in T} \subset L_+^\wedge(E, \mu)$ is minimal.

Notes:

1. "Old" $\Rightarrow$ "New" is great news! Because all "old" results on max-stable proc can be reproduced with the "new" tools.
2. Proof is a nice exercise.
3. Open problem: I don't know if "New" $\Rightarrow$ "Old".
Old and new minimality

Let \( X = \{X_t\}_{t \in T} \) be max-stable with spec rep

\[
\{X_t\}_{t \in T} \overset{d}{=} \left\{ \int_F g_t(v) M_1(dv) \right\}_{t \in T}, \quad (g_t \in L^1_+ (F, \mathcal{F} \nu)).
\]

Recall that \( \{g_t\}_{t \in T} \) is minimal if:

1. (ratio \( \sigma \)-alg) \( \rho\{g_t, \ t \in T\} := \sigma\{g_t/g_s, \ t, s \in T\} \sim F \ (\text{mod } \nu). \)
2. (full support) \( \text{supp}\{g_t, \ t \in T\} = F \ (\text{mod } \nu). \)

What is the connection b/w new and old minimality?
Old and new minimality

Let \( X = \{X_t\}_{t \in T} \) be max-stable with spec rep
\[
\{X_t\}_{t \in T} \overset{d}{=} \left\{ \int_T g_t(v) M_1(dv) \right\}_{t \in T}, \quad (g_t \in L^1_+(F, \mathcal{F}_\nu)).
\]

Recall that \( \{g_t\}_{t \in T} \) is minimal if:
1. (ratio \( \sigma \)-alg) \( \rho\{g_t, \ t \in T\} := \sigma\{g_t/g_s, \ t, s \in T\} \sim \mathcal{F} \ (\text{mod } \nu)\).
2. (full support) \( \text{supp}\{g_t, \ t \in T\} = F \ (\text{mod } \nu)\).

What is the connection b/w new and old minimality?

**Lemma ("Old" \( \Rightarrow \) "New")**

If \( \{g_t\}_{t \in T} \subset L^1_+(F, \nu) \) is minimal then \( \{f_t\}_{t \in T} \subset L^\vee(E, \mu) \) is minimal.

**Notes:**
1. "Old" \( \Rightarrow \) "New" is great news! Because all "old" results on max-stable proc can be reproduced with the "new" tools.
2. Proof is a nice exercise.
3. **Open problem:** I don’t know if "New" \( \Rightarrow \) "Old".
You may wonder...

What about the sum infinitely divisible case?
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What about the sum infinitely divisible case?

- Define the class $\mathcal{L}^+(E, \mu) \ni f : \int_E 1 \land |f|^2 d\mu < \infty$.
- Let $\Pi_\mu = \{ U_i, \ i \in \mathbb{N} \}$ be a PPP on $(E, \mu)$ and define
  \[ I^+(f) := \text{plim}_{\epsilon \downarrow 0} \left( \sum_{U \in \Pi} f(U) \mathbb{I}(|f(U)| > \epsilon) - \int_E f \mathbb{I}(\epsilon < |f| \leq 1) d\mu \right). \]

**Theorem (Kabluchko and S. [2012])**

*Under Condition S, any sum-id process $X$ has a minimal spec rep*

\[ \{ X_t \}_{t \in T} \overset{d}{=} \{ I^+(f_t) + c_t \}_{t \in T}, \]

*for some constants $c_t$, over a Borel $\sigma$-finite $(E, \mu)$.

**Notes:** In close parallel with the max-id case:

- minimal spec reps are unique.
- stationary processes correspond to measure-preserving flows.
Main messages and contributions

- Provide PPP-based stochastic integral representations for both sum and max-id processes.
- Unifing and simple notion of a minimal spec rep was developed.
- Stationary sum and max-id processes can be associated with measure-preserving flows.
- Tools for classification and ergodic theory decompositions!
- Clarified/cleaned-up a bit the theory on spec rep’s of continuous-time sum-id processes.


A random set $A$ is $\cup$-id if for all $n \in \mathbb{N}$

$$A \overset{d}{=} A_{1,n} \cup \cdots \cup A_{n,n},$$

for some iid $A_{i,n}, \; i = 1, \cdots, n$.

**Note:** $X_t := I_A(t)$ is a max-id process.

**Theorem**

If $A$ is a shift-invariant $\cup$-id random set in $\mathbb{R}^d$ that is continuous in probability. Then exists a $\sigma$-finite Borel space $(E, \mu)$ with a PPP $\Pi_\mu$ and a measure-preserving action $\varphi_t : E \to E$, such that

$$A \overset{d}{=} \{ t \in \mathbb{R}^d : \Pi_\mu \cap \varphi_t(A_0) \neq \emptyset \},$$

for some non-random $A_0$ with $\mu(A_0) < \infty$. 