Martingale Problems

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Modulus of continuity

Definition 1.1

The modulus of continuity w is defined by

$$w(x, \delta, T) = \inf_{\{t_i\}} \max_i \sup_{s,t \in [t_{i-1}, t_i)} d(x_t, x_s)$$

where $x \in D([0, \infty), E), \delta > 0, T < \infty$, and $\{t_i\}$ ranges over all partitions

$$0 = t_0 < t_1 < \ldots < t_{n-1} \leq T < t_n; \min_{1 \leq i \leq n} (t_i - t_{i-1}) > \delta, n \geq 1$$

Compact Sets of $D([0,\infty),E)$

Theorem 1

Let (E,d) be a complete and separable metric space. Then $\bar{F}$ is compact in $D([0,\infty),E)$ if and only if the following two conditions hold.

1. *For every rational $t \geq 0$, there exists a compact set $\Gamma_t \subset E$ such that $x_t \in \Gamma_t$ for all $x \in F$.*
2. *For each $T > 0$,*

$$\lim_{\delta \to 0} \sup_{x \in F} w(x,\delta,T) = 0.$$

Relative Compactness of Processes

Theorem 2

Let $\{X^n\}$ be a sequence of processes with sample paths in $D([0,\infty), E)$. Then $\{X^n\}$ is relatively compact if and only if the following two conditions hold.

1. *For every $\eta > 0$ and rational $t \geq 0$, there exists a compact set $\Gamma_{\eta,t} \subset E$ such that*

$$\liminf_{n\to\infty} \mathbb{P}\left\{X_t^n \in \Gamma_{\eta,t}\right\} \geq 1 - \eta.$$

2. *For every $\eta > 0$ and $T > 0$, there exists a $\delta > 0$ such that*

$$\limsup_{n\to\infty} \mathbb{P}\left\{w(X^n, \delta, T) \geq \eta\right\} \leq \eta.$$

A Conditional Modulus of Continuity

Theorem 3

Let Y^n be a sequence of $D([0,\infty),\mathbb{R})$-valued processes. Suppose

- *for every $\epsilon > 0$ and $t \geq 0$, $\exists$ a compact set $K_{\epsilon,t} \subset E$ such that*

$$\mathbb{P}\{Y_t^n \in K_{\epsilon,t}\} \geq 1 - \epsilon \quad \forall\ n.$$

- *for each $T > 0$, $\exists$ a family $\{\gamma_n(\delta) : 0 < \delta < 1, n \geq 1\}$ of nonnegative random variables and $\beta > 0$ satisfying*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{E}[\gamma_n(\delta)] = 0$$

$$\mathbb{E}\left[\left|Y_{t+u}^n - Y_t^n\right|^\beta |\mathcal{F}_t^{Y^n}\right] \leq \mathbb{E}\left[\gamma_n(\delta)|\mathcal{F}_t^{Y^n}\right]$$

for $0 \leq t \leq T, 0 \leq u \leq \delta$.

Then $\{Y^n\}$ is relatively compact.

Theorem 4

Suppose that $D(A) \subset C_b(E)$ is an algebra. Let A_n be operators on $B(E)$, $n = 1, 2, \ldots$, and X^n be solutions to the $D([0,\infty), E)$ - martingale problem for A_n.
Suppose that for every $f \in D(A)$, there exists $f_n \in D(A_n)$ satisfying

$$\limsup_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |f_n(X_t^n) - f(X_t^n)|\right] = 0$$

$$\limsup_{n\to\infty} \mathbb{E}\left[\|A_n f_n \circ X^n\|_{p,T}\right] < \infty \text{ for some } p \in (1,\infty].$$

Then $\{f \circ X^n\}_{n\geq 1}$ is relatively compact for each $f \in D(A)$.
More generally, $\{(g_1, g_2, \ldots, g_k) \circ X^n\}_{n\geq 1}$ is relatively compact in $D([0,\infty), \mathbb{R}^k)$ for all $g_1, g_2, \ldots, g_k \in D(A), 1 \leq k \leq \infty$.

Proof

Proof.

- Note $f \in D(A)$ implies f^2 also belongs to $D(A)$.
- Choose f_n's and h_n's in $D(A_n)$ for f and f^2 respectively and for some p and p'.
- Note

$$
\begin{aligned}
(f(X^n_{t+u}) - f(X^n_t))^2 &= f^2(X^n_{t+u}) - f^2(X^n_t) \\
&\quad - 2f(X^n_t)\left[f(X^n_{t+u}) - f(X^n_t)\right], \\
f^2(X^n_{t+u}) - f^2(X^n_t) &= \left(f^2(X^n_{t+u}) - h_n(X^n_{t+u})\right) \\
&\quad - \left(f^2(X^n_t) - h_n(X^n_t)\right) + \left(h_n(X^n_{t+u}) - h_n(X^n_t)\right), \\
f(X^n_{t+u}) - f(X^n_t) &= \left(f(X^n_{t+u}) - f_n(X^n_{t+u})\right) \\
&\quad - \left(f(X^n_t) - f_n(X^n_t)\right) + \left(f_n(X^n_{t+u}) - f_n(X^n_t)\right)
\end{aligned}
$$

Proof (Contd.)

- For $0 < u < \delta$,

$$\mathbb{E}\left[\left(f(X^n_{t+u}) - f(X^n_t)\right)^2 |\mathcal{G}^n_t\right] \leq \mathbb{E}\left[\gamma_n(\delta)|\mathcal{G}^n_t\right]$$

where

$$\begin{aligned}\gamma_n(\delta) = 2 &\sup_{s\in[0,T+1]} |f^2(X^n_s) - h_n(X^n_s)| \\ &+ 4\|f\| \sup_{s\in[0,T+1]} |f(X^n_s) - f_n(X^n_s)| \\ &+ \delta^{1/q'}\|A_n h_n\|_{p',T+1} + 2\|f\|\delta^{1/q}\|A_n f_n\|_{p,T+1}.\end{aligned}$$

- From the hypothesis

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{E}[\gamma_n(\delta)] = 0$$

Proof (Contd.)

- Relative compactness of $f(X^n)$ follows
- For $f_1, f_2, \ldots, f_k \in D(A)$,
  define $\gamma_n^j(\delta)$ for each f_j as above
- Set $\gamma_n(\delta) = \sum_{j=1}^k \gamma_n^j(\delta)$.
- Result follows for all $k < \infty$
- Relative compactness for $k = \infty$ follows

□

Compactification of state space

- A be an operator on $C_b(E)$.
- Let $\{g_k\} \subset D(A)$ be as in the separability condition *i.e.*

$$\{(f, Af) : f \in D(A)\} \subset \text{ bp-closure } (\{(g_k, Ag_k) : k \geq 1\}$$

- Let $\|g_k\| = a_k$ and $\hat{E} = \prod_{k=1}^{\infty}[-a_k, a_k]$
- Define $\hat{g} : E \to \hat{E}$ by

$$\hat{g}(x) = (g_1(x), ..., g_k(x), ...)$$

- $\hat{g}$ is continuous & one-to-one if $\{g_k\}$ separate points
- $\hat{g}(E)$ is Borel & $\hat{g}^{-1} : \hat{g}(E) \to E$ is measurable
- Extend $\hat{g}^{-1}$ to $\hat{E}$ by setting $\hat{g}^{-1}(z) = e$ for $z \notin \hat{g}(E)$.

Embedding

- Let $\mathcal{U}$ be the algebra generated by
$$\{u_k \in C(\hat{E}) : u_k((z_1, ..., z_k, ...)) = z_k\}.$$
- Define operator $\mathcal{A}$ with domain $\mathcal{U}$ as follows.
$$\mathcal{A}(cu_{i_1}u_{i_2}...u_{i_k})(z) = \begin{cases} cAg_{i_1}g_{i_2}...g_{i_k}(x) & \text{if } z = \hat{g}(x) \\ 0 & \text{otherwise.} \end{cases}$$
- Note
$$u_k(\hat{g}(x)) = g_k(x) \quad \text{and} \quad \mathcal{A}u_k(\hat{g}(x)) = Ag_k(x).$$

Lemma 1

Let X be a solution of the martingale problem for A. Let $Z_t = \hat{g}(X_t)$ for all t. Then Z is a solution to the martingale problem for $\mathcal{A}$.

Proof of Lemma 1

Proof:

- Z is a $\hat{g}(E)$ valued process.
- Let $u \in \mathcal{U}$ with $u(\hat{g}(x)) = g(x)$.
- For $0 \leq t_1 < t_2 < \cdots < t_m < t < r$, $H_1, \ldots, H_m \in B(\hat{E})$,

$$\mathbb{E}\left[\left(u(Z_r) - u(Z_t) - \int_t^r \mathcal{A}u(Z_s)ds\right)\prod_{i=1}^m H_i(Z_{t_i})\right]$$
$$= \mathbb{E}\left[\left(g(X_r) - g(X_t) - \int_t^r Ag(X_s)ds\right)\prod_{i=1}^m h_i(X_{t_i})\right]$$
$$= 0$$

  where $h_i = H_i \circ \hat{g}$.
- Thus Z is a solution of the martingale problem for $\mathcal{A}$.

Equivalence

Lemma 2

If Z is a solution of the martingale problem for $\mathcal{A}$ with

$$\mathbb{P}(Z_t \in \hat{g}(E)) = 1 \quad \forall \quad t \geq 0 \tag{1}$$

then $X_t = \hat{g}^{-1}(Z_t)$ defines a solution to the martingale problem for A.

Lemma 3

If the martingale problem for A is well - posed, then there exists a unique solution Z to the martingale problem for $\mathcal{A}$ satisfying (1).

Progressive measurability

Corollary 4

If $D(A)$ is an algebra that separates points and if separability condition holds, then well - posedness in the class of progressively measurable processes implies well - posedness in the class of all measurable processes.

Proof.

- Let X be a measurable solution.
- Let $Z_t = \hat{g}(X_t)$. Z is a solution of the martingale problem for $\mathcal{A}$ with $Z_t \in \hat{g}(E)$.
- Since $\hat{E}$ is compact, Z has a r.c.l.l. modification $\hat{Z}$. $\hat{Z}_t \in \hat{g}(E)$ a.s.
- $Y_t = \hat{g}^{-1}(\hat{Z}_t)$ is a progressively measurable modification of X.

Convergence of f.d.d.'s to those of a Markov Process

Theorem 5

Let $D(A)$ - algebra, separating points in E, vanishing nowhere. Assume separability condition. Suppose that the martingale problem for A is well-posed.
Let X^n, X be progressively measurable solutions to the martingale problems for A^n, A respectively. Suppose that $X_0^n \Rightarrow X_0$. Further, suppose that $\{X_t^n : n \geq 1\}$ is tight for all $t \geq 0$.
If for all $f \in D(A)$ there exist $f_n \in D(A^n)$ such that

$$\|f_n - f\| \to 0 \text{ as } n \to \infty; \quad \sup_n \|A^n f_n\| < \infty$$

$$\sup_{x \in K} |A^n f_n(x) - Af(x)| \to 0 \text{ as } n \to \infty, \quad \forall \text{ compact } K.$$

then the finite dimensional distributions of the proceess X_n converge to those of X.

Proof

Proof.

- Let $\{g_k\}_{k\geq 1}$ be as in separability condition.
- Let $Z^n = \hat{g}(X^n), \quad Z = \hat{g}(X)$.
- Then Z is a solution of the martingale problem for $\mathcal{A}$.
- Using Theorem 4, we get

$$Z^n = \hat{g}(X^n) = (g_1(X^n), \ldots, g_k(X^n), \ldots)$$

  is tight in $D([0,\infty), \hat{E})$.
- Let $Z^{n_k} \Rightarrow \tilde{Z}$.
- Fix $u \in \mathcal{U}, H_i \in C(\hat{E})$.
- Define $f = u \circ \hat{g}, \ h_i = H_i \circ \hat{g}$

Solution of martingale problem for $\mathcal{A}$

$$
\begin{aligned}
&\mathbb{E}\left[\left(u(\tilde{Z}_r) - u(\tilde{Z}_t) - \int_t^r \mathcal{A}u(\tilde{Z}_s)ds\right)\prod_{i=1}^m H_i(\tilde{Z}_{t_i})\right] \\
&= \lim_{k\to\infty} \mathbb{E}\left[\left(u(Z_r^{n_k}) - u(Z_t^{n_k}) - \int_t^r \mathcal{A}u(Z_s^{n_k})ds\right)\prod_{i=1}^m H_i(\tilde{Z}_{t_i}^{n_k})\right] \\
&= \lim_{k\to\infty} \mathbb{E}\left[\left(f(X_r^{n_k}) - f(X_t^{n_k}) - \int_t^r Af(X_s^{n_k})ds\right)\prod_{i=1}^m h_i(X_{t_i}^{n_k})\right] \\
&= \lim_{k\to\infty} \mathbb{E}\Big[\Big((f - f_{n_k})(X_r^{n_k}) - (f - f_{n_k})(X_t^{n_k}) \\
&\qquad - \int_t^r (Af - A^{n_k} f_{n_k})(X_s^{n_k})ds\Big)\prod_{i=1}^m h_i(X_{t_i}^{n_k})\Big] \\
&\leq \lim_{k\to\infty} \left(2\|f_{n_k} - f\|\prod_{i=1}^m \|h_i\| + \mathbb{E}\int_t^r |(Af - A^{n_k} f_{n_k})(X_s^{n_k})|\,ds \prod_{i=1}^m h_i(X_{t_i}^{n_k})\right)
\end{aligned}
$$

Proof (Contd.)

- The first term on the RHS above tends to zero by hypothesis. (*viz.,*. $\|f_{n_k} - f\| \to 0$)
- Moreover

$$\mathbb{E}\left|(Af - A^{n_k} f_{n_k})(X_s^{n_k})\right|$$
$$\le \sup_{x \in K} |Af(x) - A^{n_k} f_{n_k}(x)| + (\|Af\| + \|A^{n_k} f_{n_k}\|)\,\mathbb{P}\left(X_s^{n_k} \in K^c\right)$$

which can be made arbitrarily small for large enough n (using tightness of $\{X_s^n\}$) for every s

- DCT implies RHS above tends to zero
- $\tilde{Z}$ is a solution of the martingale problem for $\mathcal{A}$.

Proof (Contd.)

- Let $u \in \mathcal{U}$. Then $u^2 \in \mathcal{U}$.
- Let $M_t^u = u(\tilde{Z}_t) - \int_0^t \mathcal{A}u(\tilde{Z}_s)ds$.
- $\langle M^u, M^u \rangle_t = \int_0^t (\mathcal{A}u^2 - 2u\mathcal{A}u)(\tilde{Z}_s)ds$ is continuous
- Hence $(M_t^u)_{t\geq 0}$ and $(u(\tilde{Z}_t))_{t\geq 0}$ are continuous in probability
- $(u(\tilde{Z}_t))_{t\geq 0}$ has no fixed points of discontinuity for all $u \in \mathcal{U}$.
- $(\tilde{Z}_t)_{t\geq 0}$ cannot have any fixed points of discontinuity
- For every t, $\tilde{Z}_t^{n_k} \Rightarrow \tilde{Z}_t$
- $\{X_t^n : n \geq 1\}$ is tight in E for every t
- $\{Z_t^n : n \geq 1\}$ is tight in $\hat{g}(E)$ for every t
- Hence

$$\mathbb{P}\left\{\tilde{Z}_t \in \hat{g}(E)\right\} = 1 \quad \forall\ t \geq 0. \qquad (2)$$

Proof (Contd.)

- Recall $Z = \hat{g}(X)$
- Then Z as well $\tilde{Z}$ are solutions of the martingale problem for $(\mathcal{A}, \mu \circ \hat{g}^{-1})$ satisfying (2) and hence have the same law
- Hence

$$\hat{g}(X^n) \Rightarrow \hat{g}(X)$$

- We also have for all $t_1, \ldots, t_j$ and for all j,

$$(\hat{g}(X^n_{t_1}), \ldots, \hat{g}(X^n_{t_j})) \Rightarrow (\hat{g}(X_{t_1}), \ldots, \hat{g}(X_{t_j}))$$

- Finally, since $\{X^n_t : n \geq 1\}$ is tight and $D(A)$ is a measure determining class (an algebra that separates points) we have

$$(X^n_{t_1}, \ldots, X^n_{t_j}) \Rightarrow (X_{t_1}, \ldots, X_{t_j}).$$

Weak Convergence to a Markov Process

Theorem 6

In addition to the hypotheses of Theorem 5, assume that $D(A)$ strongly separates points in E, (i.e. $f(x_n) \to f(x) \forall f \in D(A)$ implies $x_n \to x$). Then $X^n \Rightarrow X$ (as processes in $D([0,\infty), E)$).

Proof.

- Since $D(A)$ strongly separates points in E, we get that $\hat{g}^{-1} : \hat{g}(E) \mapsto E$ is continuous
- As before we get

$$\hat{g}(X^n) \Rightarrow \hat{g}(X)$$

- This now implies the result

□

Outline

Stationary Distribution of a Markov Process

Definition 2.1

$\mu \in \mathcal{P}(E)$ is a stationary distribution or an invariant measure for the Markov process determined by A, if the solution X of the martingale problem for (A, μ) is a stationary process, i.e., if $\mathbb{P}\{X_{t+s_1} \in \Gamma_1, \ldots, X_{t+s_k} \in \Gamma_k\}$ is independent of $t \geq 0$ for all $0 \leq s_1 < s_2 < \ldots < s_k$, $\Gamma_1, \Gamma_2, \ldots, \Gamma_k \in \mathcal{E}$ and for all $k \geq 1$.

- In particular, $\mathbb{P}\{X_t \in \Gamma\} = \mu(\Gamma)$ for all t
- For the transition probability P

$$\mu(\Gamma) = \int_E P(t, x, \Gamma)\mu(dx) \quad \forall\, t > 0, \Gamma \in \mathcal{E}$$

Markov Processes

- For the associated semigroup T_t

$$\int_E f d\mu = \int_E T_t f d\mu \quad \forall\ f \in B(E), t > 0$$

- For Generator L

$$\int_E (Lf) d\mu = 0, \qquad \forall\ f \in D(L)$$

Markov Processes

- For the associated semigroup T_t

$$\int_E f d\mu = \int_E T_t f d\mu \quad \forall\, f \in B(E), t > 0$$

- For Generator L

$$\int_E (Lf) d\mu = 0, \qquad \forall\, f \in D(L)$$

- Can generator L be replaced by operator A for which X is a unique solution of its martingale problem?

Existence of a stationary solution

Theorem 7

Let $D(A)$ be an algebra that separates points and vanishes nowhere. Suppose A satisfies the separability condition and that for all $\nu \in \mathcal{P}(E)$, there exists a solution to the $D([0,\infty), E)$ -martingale problem for (A,ν). Suppose that $\mu \in \mathcal{P}(E)$ satisfies

$$\int_E Af d\mu = 0 \quad \forall\, f \in D(A).$$

Then on some probability space, there exists a filtration $(\mathcal{G}_t)_{t\geq 0}$ and a $(\mathcal{G}_t)_{t\geq 0}$- progressively measurable process X such that X is a stationary process and that X is a solution of the martingale problem for (A,μ) w.r.t. $(\mathcal{G}_t)_{t\geq 0}$.

Yosida Approximations

Proof.

- Since the martingale problem for (A, δ_x) admits a solution for all $x \in E$, A satisfies

$$\|(\lambda - A)f\| \geq \lambda \|f\| \quad \forall f \in D(A), \lambda > 0$$

- Hence $(I - n^{-1}A) = n^{-1}(n - A)$ is one to one
- For $n \geq 1$ define A_n on $\mathcal{R}(I - n^{-1}A)$ by

$$A_n f = n[(I - n^{-1}A)^{-1} - I]f$$

- For $f \in D(A)$, define $f_n := (I - n^{-1}A)f$
- Then $A_n f_n = Af$ & $\|f_n - f\| \to 0$
- For $g = (I - n^{-1}A)f$, $f \in D(A)$,

$$\int_E A_n g d\mu = \int_E Af d\mu = 0.$$

Construction of a suitable transition probability function

- Construct a probability measure ν on $E \times E$ and transition function $\eta : E \times \mathcal{B}(E) \to [0,1]$ satisfying

$$\nu(E \times B) = \nu(B \times E) = \mu(B)$$

$$\nu(B_1 \times B_2) = \int_{B_1} \eta(x, B_2)\mu(dx)$$

$$\int_E g(y)\eta(x, dy) = (I - n^{-1}A)^{-1}g(x) \quad \mu - \text{ a.s.}$$

- Then

$$A_n f = n[(I - n^{-1}A)^{-1} - I]f = n \int_E (f(y) - f(x))\eta(x, dy)$$

- Moreover

$$\int_E \eta(x, B)\mu(dx) = \nu(E \times B) = \mu(B)$$

Stationary solutions for A_n

- Let $Y_0, Y_1, \ldots, Y_k, \ldots$ be an E-valued Markov chain with initial distribution μ and transition function η.
- Then $\{Y_k : k \geq 0\}$ is a stationary sequence
- Let V be an independent Poisson process with parameter n
- Define $X_t^n = Y_{V_t}$
- Then X^n is a stationary Markov (Jump) Process with initial distribution μ, and a solution of the martingale problem for A_n
- Since $\mathcal{L}(X_t^n) = \mu$, $\{X_t^n : n \geq 1\}$ is tight for all t
- Hence as in Theorem 5 we get (via subsequential limits on $\hat{g}(E)$) a progressively measurable solution of the martingale problem for A
- Further X is stationary

A criterion for Invariant measures

Theorem 8

Let $D(A)$ be an algebra that separates points and vanishes nowhere. Suppose A satisfies the separability condition. Suppose that the $D([0,\infty), E)$- martingale problem for (A, δ_x) is well posed for all $x \in E$. Let $(T_t)_{t\geq 0}$ be the semigroup associated with the martingale problem for A.
Further suppose that every progressively measurable solution to the martingale problem for (A, μ) admits an r.c.l.l. modification.
If $\mu \in \mathcal{P}(E)$ satisfies

$$\int_E Af d\mu = 0 \quad \forall f \in D(A)$$

then μ is an invariant measure for the semigroup $(T_t)_{t\geq 0}$.

Outline

The Kolmogorov's forward equation

- Suppose $f(X_t) - \int_0^t Af(X_s)ds$ is a martingale for all $f \in D(A)$
- Taking expectations and denoting $\mathcal{L}(X_t) = \nu_t$, we get

$$\int_E f d\nu_t = \int_0^t \left(\int_E Af d\nu_s \right) ds + \int_E f d\nu_0 \quad \forall f \in D(A) \qquad (3)$$

  Equation (3) is called the Kolmogorov's Forward equation or the Fokker-Planck equation

- When $\nu_t = \mu$ the above equation reduces to

$$\int_E Af d\mu = 0 \quad \forall f \in D(A)$$

A perturbed evolution equation

- Let $\lambda \in C_b(E)$.
- Let $\mathcal{M}(E)$ be the set of all positive finite measures on $(E, \mathcal{E})$.
- Consider

$$\int_E f d\nu_t = \int_E f d\nu_0 + \int_0^t \left(\int_E (Af - \lambda(\cdot) f) d\nu_s \right) ds, \quad f \in D(A) \tag{4}$$

- A collection $\{\nu_t\}_{t \geq 0} \subset \mathcal{M}(E)$ is a solution of (3) (or (4)) if
  1. $\{\nu_t\}_{t \geq 0}$ satisfies (3) (or (4))
  2. and

$$t \mapsto \nu_t(B) \text{ is measurable } \forall B \in \mathcal{E}$$

Feynman-Kac

- If $(X_t)_{t\geq 0}$ is a solution of the $D([0,\infty),E)$-martingale problem for (A,ν_0) then (using integration by parts)

$$f(X_t)\exp\left\{-\int_0^t \lambda(X_s)ds\right\} - \int_0^t \left[\exp\left\{-\int_0^s \lambda(X_u)du\right\} \left(Af(X_s) - \lambda(X_s)f(X_s)\right)\right]ds$$

  is a martingale.
- Define

$$\nu_t(B) = \mathbb{E}\left(\mathbb{I}_B(X_t)\exp\left\{-\int_0^t \lambda(X_s)ds\right\}\right).$$

- $(\nu_t)_{t\geq 0}$ is a solution of the perturbed evolution equation
- Uniqueness?

$\lambda \geq 0$

- Let $\{\mu_t\}_{t \geq 0}$ be a solution of (4) and define $\mu'_t = \mu_t e^{\alpha t}$
- Then for $f \in D(A)$

$$\int_E f d\mu'_t = \int_E f d\mu'_0 + \int_0^t \left(\int_E (Af - \lambda(\cdot)f + \alpha f)\, d\mu'_s \right) ds$$

- Conversely if $\{\mu'_t\}_{t \geq 0} \subset \mathcal{M}(E)$ satisfies the above equation then $\mu_t = \mu'_t e^{-\alpha t}$ satisfies (4).
- So without loss of generality we consider $\lambda \geq 0$
- Operator $A - \lambda$
- Killing Intensity $\lambda(x)$

Theorem

Theorem 9

Suppose that for all $x \in \mathcal{P}(E)$, there exists a solution to the $D([0,\infty), E)$-martingale problem for (A, δ_x). Further suppose that every progressively measurable solution admits a r.c.l.l. modification.
If $\{\nu_t\} \subset \mathcal{P}(E)$ and $\{\mu_t\} \subset \mathcal{P}(E)$ are solutions of (3) with $\nu_0 = \mu_0$, then $\nu_t = \mu_t$ for all $t \geq 0$.

Proof. Let $E_0 = E \times \{-1, 1\}$, $\beta > 0, \nu_0 \in \mathcal{P}(E)$.
Let $D(B) \subset C_b(E_0)$ be the linear span of

$$\{f_1 f_2 : f_1 \in D(A), f_2 \in C(\{-1, 1\})\}.$$

$$Bf_1 f_2(x, v) = f_2(v)Af_1(x) + \beta(f_2(-v)\int_E f_1 d\nu_0 - f_1(x)f_2(v)).$$

Proof (Contd.)

- $D(B)$ is an algebra that separates points in E_0
- B satisfies the separability condition.
- Since B is a jump perturbation of A we know that the martingale problem for $(B, \delta_{(x,v)})$, admits a r.c.l.l. solution for every $(x, v) \in E_0$.
- Let $\mu \in \mathcal{P}(E^0) = \mu_1 \times \delta_v$, where $\mu_1 \in \mathcal{P}(E)$, $v \in \{-1, 1\}$.
- Let (Y, V) be a progressively measurable solution to the martingale problem for B with $V(0) = v$.
- Let $Z_t = \hat{g}(Y_t)$

Proof (Contd.)

- Define

$$D(\mathcal{B}) = \{uf_2 : u \in \mathcal{U}, f_2 \in C(\{-1,1\})\}$$

$$\mathcal{B}uf_2(z,v) = f_2(v)\mathcal{A}u(z) + \beta(f_2(-v)\int_{\hat{E}} ud\tilde{\nu}_0 - f_2(v)u(z))$$

where

$$\tilde{\nu}_0 = \nu_0(\hat{g}^{-1}(\Gamma \cap \hat{g}(E))).$$

- (Z,V) is a solution of the martingale problem for $\mathcal{B}$.
- Let $(\hat{Z},\hat{V})$ denote its r.c.l.l. modification
- Define

$$D(\mathcal{C}) = \{uh : u \in \mathcal{U}, h \in C_b(\mathbb{Z}^+)\}$$

$$\mathcal{C}uh(z,n) = h(n)\mathcal{A}u(z) + \beta(h(n+1)\int_{\hat{E}} ud\tilde{\nu}_0 - h(n)u(z))$$

Proof (Contd.)

- Define

$$\tau_0 \equiv 0, \quad \tau_k \equiv \inf\{t > \tau_{k-1} : V_t = (-1)^k v\} \quad ; \quad k \geq 1$$
$$N_t = k \quad \text{if } \tau_k \leq t < \tau_{k+1}.$$

- $(\hat{Z}, N)$ is a r.c.l.l. solution of the martingale problem for $\mathcal{C}$.
- The one dimensional distributions of $(\hat{Z}_t, N_t)_{t\geq 0}$ are uniquely determined by $Y_0, \beta, \tilde{\nu}_0$ and $(T_s)_{s\geq 0}$, the semigroup corresponding to the martingale problem for A.
- In turn, the one dimensional distributions of V and of Y are uniquely determined by $Y_0, (T_s)_{s\geq 0}, \ \beta$ and $\tilde{\nu}_0$.
- The martingale problem for B is well - posed.

Proof (Contd.)

- Let $(\nu_t)_{t\geq 0}$ be a solution of (3).
- Define

$$\nu = (\beta \int_0^\infty e^{-\beta t} \nu_t dt) \times (\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}).$$

- $\int_{E_0} Bf d\nu = 0$ for all $f \in D(B)$.
- Hence ν is an invariant measure for the Markov process characterised by B.
- If $\{\gamma_t\}$ denotes the one dimensional distributions for the solution of the $D([0,\infty), E_0)$-martingale problem for $(B, \nu_0 \times \delta_1)$ and (Y, V) is any stationary solution to the $D([0,\infty), E_0)$-martingale problem for B, then

$$P(Y_s, V_s \in \Gamma) = \lim_{t\to\infty} t^{-1} \int_0^t \gamma_u(\Gamma) du \ \forall\ s.$$

Proof (Contd.)

- Stationary distribution for B is unique.
- Thus if $(\mu_t)_{t\geq 0} \subset \mathcal{P}(E)$ is another solution, μ defined by

$$\mu = (\beta \int_0^\infty e^{-\beta t} \mu_t dt) \times (\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1})$$

  is a stationary distribution
- Uniqueness implies

$$\int_0^\infty e^{-\beta t} \nu_t dt = \int_0^\infty e^{-\beta t} \mu_t dt.$$

- Since $\beta > 0$ was arbitrary, we get $\nu_t = \mu_t \ \forall\ t \geq 0$.

□

Cemetery

- Assume $\mathbf{1} \in D(A)$ with $A\mathbf{1} = \mathbf{0}$.
- Choose a point $\Delta \notin E$ and let $E^{\Delta} = E \cup \{\Delta\}$
- Define a metric d' on E^{Δ} by

$$d'(\Delta, \Delta) = 0, \; d'(\Delta, x) = d'(x, \Delta) = 1,$$

$$d'(x, y) = d(x, y) \wedge 1 \text{ for } x, y \in E$$

- Define

$$\tilde{\lambda}(\Delta) = 0, \tilde{\lambda}(x) = \lambda(x) \quad \forall \; x \in E$$
$$D(A^{\Delta}) = \{f \in C_b(E^{\Delta}) : f|_E \in D(A)\}$$
$$A^{\Delta} f(\Delta) = 0; \quad A^{\Delta} f(x) = Af(x) \quad \forall \; x \in E, f \in D(A^{\Delta})$$
$$Cf(x) = \tilde{\lambda}(x)(f(\Delta) - f(x)) \quad \forall f \in C_b(E^{\Delta}), x \in E^{\Delta}$$

The operator $A^\Delta + C$

- $D(A)$ algebra $\implies$ $D(A^\Delta)$ is an algebra
- $D(A)$ separates points $\implies$ $D(A^\Delta)$ is separates points
- Separability condition for A $\implies$ the same for A^Δ
- Well-posedness of the martingale problem for A implies Well-posedness of the martingale problem for A^Δ
- C is a Jump Operator with all jumps going to Δ
- Existence of a solution to the $D([0,\infty), E)$ - martingale problem for (A, δ_x) for all x $\implies$ existence of a solution to the $D([0,\infty), E^\Delta)$- martingale problem for $(A^\Delta + C, \delta_y)$ for all y
- Solution X of the martingale problem for $A^\Delta + C$:
  - X_t evolves as a solution of A until it is killed
  - At the time of Death it jumps to cemetery Δ and stays there
  - If $X_t = y$, the process is killed (independntly) at time t with intensity $\lambda(y)$.

Well-posedness of $A^\Delta + C$

Theorem 10

The martingale problem for $A^\Delta + C$ is well-posed.

Proof.

- Let $(T_t^\Delta)_{t\geq 0}$ be the semigroup associated with the well-posed martingale problem for A^Δ.
- Let X be a measurable solution to the martingale problem for $A^\Delta + C$.
- Since $\mathbb{I}_E \in D(A^\Delta + C)$ and $(A^\Delta + C)\mathbb{I}_E = -\tilde{\lambda}\mathbb{I}_E = -\tilde{\lambda}$, we get

$$M_t = \mathbb{I}_E(X_t) + \int_0^t \tilde{\lambda}(X_s)ds$$

is a martingale.

Proof (Contd.)

- $\tilde{\lambda} \geq 0 \implies \mathbb{I}_E(X_t)$ is a supermartingale.
- Get an r.c.l.l. modification (N_t) of $\mathbb{I}_E(X_t)$
- Let
$$\tau = \inf\{t > 0 : N_t = 0\}.$$
- Then $N_u = 0$ for $u \geq \tau$ a.s.
- Thus $\mathbb{I}_E(X_t) = \mathbb{I}_{\{\tau > t\}}$ a.s.
- Integration by parts implies
$$\mathbb{I}_{\{\tau > t\}} \exp\left\{\int_0^t \tilde{\lambda}(X_s)ds\right\}$$
is a martingale.

Proof (Contd.)

- Consider the compact embedding into $\hat{E}$
- $\mathcal{C}u(z) = \mathcal{A}u(z) + \hat{\lambda}(z)(u(\hat{g}(\Delta)) - u(z))$.
- Then $Z_t = \hat{g}(X_t)$ is a solution of the martingale problem for $\mathcal{C}$.
- Let $\hat{Z}$ be the r.c.l.l. modification of Z
- We get
$$\widetilde{M}_t = \mathbb{I}_{\{\tau > t\}} \exp\left\{\int_0^t \hat{\lambda}(\hat{Z}_s)ds\right\}$$
is a non-negative mean one martingale.
- Fix $T > 0$. Define $\mathbb{Q}$ on $D([0,\infty), \hat{E})$ by
$$\mathbb{Q}(\hat{\theta}_{t_1} \in \Gamma_1, ..., \hat{\theta}_{t_m} \in \Gamma_m) = \mathbb{E}^{\mathbb{P}}\left[\prod_{i=1}^m \mathbb{I}_{\Gamma_i}(\hat{Z}_{t_i})\widetilde{M}_t\right]$$
for all $0 \leq t_1 < \ldots < t_m \leq T, \Gamma_1, \ldots, \Gamma_m$.

Proof (Contd.)

The following are martingales

- $f(X_t) - \int_0^t (Af(X_s) - \lambda(X_s)f(X_s)ds$
- $f(X_t)\exp\{\int_0^t \tilde{\lambda}(X_s)ds\} - \int_0^t Af(X_s)\exp\{\int_0^s \tilde{\lambda}(X_u)du\}ds$
- $f(X_t)\mathbb{I}_{\{\tau>t\}}\exp\{\int_0^t \lambda(X_s)ds\} - \int_0^t Af(X_s)\mathbb{I}_{\{\tau>s\}}\exp\{\int_0^s \lambda(X_u)du\}ds$
- $u(\hat{Z}_t)\widetilde{M}_t - \int_0^t \mathcal{A}u(\hat{Z}_s)\widetilde{M}_s ds$

Hence, for $0 \leq t_1 < \ldots < t_{m+1} \leq T, h_1, \ldots, h_m \in C(\hat{E})$,

$$\mathbb{E}^{\mathbb{Q}}\left[\left(u(\hat{\theta}_{t_{m+1}}) - u(\hat{\theta}_{t_m}) - \int_{t_m}^{t_{m+1}} \mathcal{A}u(\hat{\theta}_s)ds\right)\prod_{k=1}^m h_k(\hat{\theta}_{t_k})\right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[\left(u(\hat{Z}_{t_{m+1}})\widetilde{M}_{t_m} - u(\hat{Z}_{t_m})\widetilde{M}_{t_{m+1}} - \int_{t_m}^{t_{m+1}} \mathcal{A}u(\hat{Z}_s)\widetilde{M}_s ds\right)\prod_{k=1}^m h_k(\hat{Z}_{t_k})\right]$$

$$= 0.$$

Proof (Contd.)

- Under Q, $\hat{\theta}$ is a solution of the martingale problem for $\mathcal{A}$ with

$$\mathbb{Q}(\hat{\theta}_t \in \hat{g}(E^{\Delta})) = 1 \quad \forall\, t.$$

- $X'_t := \hat{g}^{-1}(\hat{\theta}_t)$ is a solution of the martingale problem for A^{Δ}.
- For $u \in B(\hat{E})$

$$\mathbb{E}^{\mathbb{Q}}[u(\hat{\theta}_t)] = \mathbb{E}^{\mathbb{Q}}[u \circ \hat{g}(X'_t)] = \mathbb{E}^{\mathbb{Q}}[[T^{\Delta}_t(u \circ \hat{g})](X'_0)] = \mathbb{E}^{\mathbb{P}}[[T^{\Delta}_t(u \circ \hat{g})](X_0)]$$

- $\mathbb{E}^{\mathbb{P}}[u(\hat{Z}_t)\exp\{\int_0^t \hat{\lambda}(\hat{Z}_r)dr\}\mathbb{I}_{\{\tau > t\}}] = \mathbb{E}^{\mathbb{P}}[T^{\Delta}_t(u \circ \hat{g})(X_0)]$ for all $0 \leq t \leq T$.
- Arguing exactly similarly for the process $X_{s+\cdot}$ we get

$$\mathbb{E}^{\mathbb{P}}[\mathbb{I}_F u(\hat{Z}_t)\exp\{\int_s^t \hat{\lambda}(\hat{Z}_r)dr\}\mathbb{I}_{\{\tau > t\}}] = \mathbb{E}^{\mathbb{P}}[\mathbb{I}_F[T^{\Delta}_{t-s}(u \circ \hat{g})](X_s)]$$

for all $s \leq t \leq s + T, F \in \mathcal{F}_s^{\hat{Z}}$

Proof (Contd.)

Thus for all $s \leq t \leq s+T$

$$\mathbb{E}^{\mathbb{P}}[u(\hat{Z}_t)\exp\{\int_s^t \hat{\lambda}(\hat{Z}_r)dr\}\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_s^{\hat{Z}}] = T_{t-s}^{\Delta}(u\circ\hat{g})(X_s) \text{ a.s.}$$

Choosing $u = f\circ\hat{g}^{-1}$

$$\mathbb{E}^{\mathbb{P}}[f(X_t)] - \mathbb{E}^{\mathbb{P}}[T_t^{\Delta}f(X_0)] = \int_0^t \mathbb{E}^{\mathbb{P}}[CT_{t-s}^{\Delta}f(X_s)]ds.$$

$$\mathbb{E}^{\mathbb{P}}[f(X_t)] = \mathbb{E}^{\mathbb{P}}[T_t^{\Delta}f(X_0)] + \int_0^t \mathbb{E}^{\mathbb{P}}[T_s^{\Delta}CT_{t-s}^{\Delta}f(X_0)]ds$$
$$+ \int_0^t\int_0^s \mathbb{E}^{\mathbb{P}}[CT_{s-r}^{\Delta}CT_{t-s}^{\Delta})f(X_r)]drds$$

and so on.

Proof (Contd.)

Thus the distribution of X_t is completely determined determined by $C, (T_s^\Delta)_{s\geq 0}$ and X_0.
Hence we have uniqueness of one-dimensional marginals. □

Theorem 11

If $\{\mu_t\}_{t\geq 0} \subset \mathcal{M}(E)$ and $\{\nu_t\}_{t\geq 0} \subset \mathcal{M}(E)$ are solutions of (4) with $\mu_0 = \nu_0$, then $\mu_t = \nu_t$ for all $t \geq 0$.

Proof. Theorem 9 implies uniqueness of solution to

$$\int_{E^\Delta} f d\gamma_t = \int_{E^\Delta} f d\tilde{\nu}_0 + \int_0^t \Big(\int_{E^\Delta} (A^\Delta + C) f d\gamma_s\Big) ds.$$

Since $1 \in D(A)$ with $A1 = 0$, we get
$\nu_t(E) = \nu_0(E) - \int_0^t \int_E \lambda d\nu_s ds \leq 1.$
Set $\tilde{\nu}_t(U) = \nu_t(U \cap E) + (1 - \nu_t(E))\mathbb{I}_U(\Delta)$
Then $\tilde{\nu}_t$ is a solution to (4). □