Martingale Problems

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Outline

Definition

Definition 1.1

An operator A is said to satisfy the positive maximum principle if for $f \in D(A)$ and $x_0 \in E$,

$$f(x_0) = \sup_{y \in E} f(y) \geq 0$$

then $Af(x_0) \leq 0$.

Lemma 1

Suppose that A is an operator on $C_b(E)$ and that for every $x \in E$, there exists a solution (X_t^x) for the $D([0, \infty), E)$ martingale problem for (A, δ_x). Then A satisfies the positive maximum principle.

Proof

Proof.

- Suppose $f \in D(A)$ attains its maximum at $x \in E$.
- Then
$$E[f(X_t^x)] - f(x) = \mathbb{E}\left[\int_0^t Af(X_s^x)ds\right]$$
- Thus
$$\frac{1}{t}\int_0^t \mathbb{E}\left[Af(X_s^x)\right] ds \leq 0.$$
- Since $s \mapsto \mathbb{E}\left[Af(X_s^x)\right]$ is right continuous,
$$Af(x) = \lim_{t \to 0} \frac{1}{t}\int_0^t \mathbb{E}\left[Af(X_s^x)\right] ds \leq 0$$

□

Invariant measures

Theorem 1

Suppose A is an operator on $C_b(E)$ such that $D(A)$ is an algebra that separates points in E, $\mathbf{1} \in D(A)$, and satisfying

- *the positive maximum principle*
- *Separability condition*

Suppose $\mu \in \mathcal{P}(E)$ is such that

$$\int_E Af d\mu = 0 \qquad \forall\, f \in D(A).$$

Then there exists a progressively measurable, stationary solution $(X_t)_{t\geq 0}$ to the martingale problem for (A, μ).

Evolution Equation

Recall the evolution equation

$$\int_E f d\nu_t = \int_0^t \left(\int_E Af d\nu_s \right) ds + \int_E f d\nu_0 \quad \forall f \in D(A)$$

A collection $\{\nu_t\}_{t \geq 0} \subset \mathcal{M}(E)$ which satisfies the above evolution equation, and such that

$$t \mapsto \nu_t(B) \text{ is measurable } \forall B \in \mathcal{E}$$

is a solution.
Also, recall $\nu_t = \nu$ for all t implies $\int Af d\nu = 0$.

Mimicking one dimensional distributions

Theorem 2

Suppose A is an operator on $C_b(E)$ such that $D(A)$ is an algebra that separates points in E, $\mathbf{1} \in D(A)$, and satisfying

- *the positive maximum principle*
- *Separability condition*

If $\{\mu_t : 0 \leq t \leq T\} \subset \mathcal{P}(E)$ is a solution of

$$\langle f, \mu_t \rangle = \langle f, \mu_0 \rangle + \int_0^t \langle Af, \mu_r \rangle dr, \qquad t \leq T,\ f \in D(A).$$

Then there exists a progressively measurable solution $(X_t)_{t<T}$ to the martingale problem for (A, μ_0), with $\mathcal{L}(X_t) = \mu_t$ for all $t < T$.

Space-time process

Corollary 2

In addition, if the martingale problem for (A, μ_0) is well-posed, then the evolution equation admits a unique solution.

Proof of Theorem 2.

- Let $g \in C_0^1([0, \infty)), f \in D(A), t \leq T$. Then integration by parts argument implies

$$g(t)\langle f, \mu_t\rangle = g(0)\langle f, \mu_0\rangle + \int_0^t \{g'(r)\langle f, \mu_r\rangle + g(r)\langle Af, \mu_r\rangle\} \, dr.$$

- Let $E_1 = [0, \infty) \times E$ and $\Gamma(s) = T(1 - e^{-s})$, $\nu_s = \delta_{\Gamma(s)} \otimes \mu_{\Gamma(s)}$.

Introducing Auxillary Jumps

- Define
  $D(A_1) =$ linear span $\{g \otimes f : g \in C_0^1([0,\infty)), f \in D(A)\}$,
  $$A_1(g \otimes f)(s,x) = e^{-s}\left(\frac{\partial g}{\partial s}(s)f(x) + g(s)Af(x)\right).$$
- Then
  $$\langle f_1, \nu_s\rangle = \langle f_1, \nu_0\rangle + \int_0^s \langle A_1 f_1, \nu_r\rangle dr, \qquad 0 \le s < \infty, f_1 \in D(A_1).$$
- Define an operator B on $C_b(E_1 \times \{-1,1\})$:
  $$D(B) = \text{ linear span } \{f_1 \otimes h : f_1 \in D(A_1), h \in C(\{-1,1\})\}$$
  $$B(f_1 \otimes h)(s,x,n) = h(n)(A_1 f_1)(s,x) + \left\{h(-n)\int_{E_1} f_1(s,y)\nu_0(dsdy) - h(n)f_1(s,x)\right\}.$$

- Let $\nu \in \mathcal{P}(E_1 \times \{-1,1\})$ be defined by

$$\nu = \left(\int_0^\infty e^{-s}\nu_s ds\right) \otimes \left(\frac{1}{2}\delta_{\{1\}} + \frac{1}{2}\delta_{\{-1\}}\right).$$

- Then

$$\int_{E_1\times\{-1,1\}} (BF)d\nu = 0 \qquad \forall F \in D(B).$$

- $\exists$ a progressively measurable stationary solution $(\tilde{\alpha}_s, \tilde{X}_s, \tilde{N}_s)$ to the martingale problem for (B,ν).
- W.l.o.g., assume that the stationary process is defined for $-\infty < s < \infty$
- $(\tilde{\alpha}_s, \tilde{N}_s)$ admit an r.c.l.l. modification, say (α_s, N_s).
- Compact embedding $\hat{E}, \mathcal{A}, \mathcal{A}_1, \mathcal{B}$ and measure $\hat{\nu}$:
- Get (α_s, Z_s, N_s), an r.c.l.l. solution to the martingale problem for $(\mathcal{B}, \hat{\nu})$ and $(\alpha_s, Z_s, N_s)_{-\infty<s<\infty}$ is a stationary process.

Solution of the Martingale Problem for $\mathcal{A}_1$

- Define

$$\tau_{-1} = \sup\{t < 0 : N_t \neq N_0\},$$

$$\tau_1 = \inf\{t > 0 : N_t \neq N_0\},$$

$$\tau_2 = \inf\{t > \tau_1 : N_t \neq N_{\tau_1}\}.$$

- Let $\hat{\alpha}_s = \alpha_{s+\tau_1}, \hat{Z}_s = Z_{s+\tau_1}, \hat{\mathcal{F}}_s = \mathcal{F}_{s+\tau_1}$.
- Define $\mathbb{Q}^t$ on $(\Omega, \hat{\mathcal{F}}_t)$ by

$$d\mathbb{Q}^t = \frac{1}{\tau_1 - \tau_{-1}} \mathbb{I}_{\{N_{t+\tau_1} \neq N_{\tau_1}\}} d\mathbb{P}.$$

- Then $\exists$ a measure $\mathbb{Q}$ on Ω such that
  - its restriction to $\hat{\mathcal{F}}_t$ is $\mathbb{Q}^t$
  - $(\hat{\alpha}_s, \hat{Z}_s)$ is a solution to the $(\mathcal{A}_1, \hat{\nu}_0)$ martingale problem with respect to $(\hat{\mathcal{F}}_t)$ on $(\Omega, \mathcal{F}, \mathbb{Q})$.

Solution of the Martingale Problem for $\mathcal{A}$

- Further for every $F \in C_b([0,\infty) \times \hat{E})$,

$$\mathbb{E}\int_0^\infty e^{-s}F(\hat{\alpha}_s, \hat{Z}_s)ds = \int_{[0,\infty)\times\hat{E}} Fd\hat{\nu}$$
$$= \int_0^\infty \int e^{-s}F(\alpha, z)\hat{\nu}_s(d\alpha dz)ds$$
$$= \int_0^\infty \int e^{-s}F(\alpha, z)\delta_{\Gamma(s)}(d\alpha)\mu_{\Gamma(s)} \circ \hat{g}^{-1}(dz)ds.$$

- Then $\hat{\alpha}_s = \Gamma(s)$.
- Define $\mu_t^* = \mu_t \circ \hat{g}^{-1}$ and $Z_t^* = \hat{Z}_{\Gamma^{-1}(t)}$, for $0 \leq t < T$.
- Then Z^* is an r.c.l.l. solution to the martingale problem for $(\mathcal{A}, \mu_0^*)$.
- Changing variables

$$\mathbb{E}\int_0^T F(t, Z_t^*)dt = \int_0^T F(t,z)\mu_t^*(dz)dt$$

Solution of Martingale Problem

- Clearly (μ_t^*) satisfies

$$\langle G, \mu_t^* \rangle = \langle G, \mu_0^* \rangle + \int_0^t \langle \mathcal{A}G, \mu_r^* \rangle dr \quad \forall G \in D(\mathcal{A}).$$

- $D(\mathcal{A})$ being convergence determining, this implies $t \mapsto \mu_t^*$ is weakly continuous.
- Further

$$\mathcal{L}(Z_t^*) = \mu_t^* \quad \text{for all } t, 0 \leq t < T.$$

- In particular, $\mathbb{Q}(Z_t^* \in \hat{g}(E)) = 1$ for all t.
- Define $X_t^* = \hat{g}^{-1}(Z_t^*)$.
- Then $\mathcal{L}(X_t^*) = \mu_t$ and X_t^* is a progressively measurable solution to the martingale problem for (A, μ_0).

The Hille-Yosida Theorem

- Let X be a Markov process with associated semigroup $(T_t)_{t\geq 0}$.
- Let L be its generator.
- X is a solution of the martingale problem for L.

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Hille-Yosida Theorem

A linear operator L on $C_b(E)$ is the generator of a strongly continuous contraction semigroup on $C_b(E)$ if and only if

1. *$D(L)$ is dense in $C_b(E)$*
2. *L is dissipative i.e. $\|\lambda f - Af\| \geq \lambda \|f\|$ for all $f \in D(A), \lambda > 0$.*
3. *$Range(\lambda - L) = C_b(E)$ for all $\lambda > 0$.*

Evolution equation - Uniqueness

Theorem 3

If $\{\nu_t\}, \{\mu_t\}$ both solve the evolution equation

$$\int f d\rho_t = \int f d\rho_0 + \int_0^t \int_E Lf d\rho_s ds$$

with $\mu_0 = \nu_0$, then $\mu_t = \nu_t$ for all $t \geq 0$.

Proof. Multiplying by $\lambda e^{-\lambda t}$, integrating w.r.t. t, rearranging terms and using Fubini's theorem, we get

$$\int f d\rho_0 = \int_0^\infty \int_E e^{-\lambda s}(\lambda f - Lf) d\rho_s ds$$

$$\int_0^\infty \int_E e^{-\lambda s}(\lambda f - Lf) d\nu_s = \int_0^\infty \int_E e^{-\lambda s}(\lambda f - Lf) d\mu_s$$

Uniqueness (contd.)

- Since Range $(\lambda - L) = C_b(E)$, we get equality of the measures

$$\int_0^\infty e^{-\lambda s}\nu_s ds = \int_0^\infty e^{-\lambda s}\mu_s ds$$

- These are uniquely determined by the measure ρ_0
- Uniqueness of Laplace transform implies $\nu_t = \mu_t$. □

Corollary 3

The martingale problem for L is well-posed.

Let (Y_t) be any other solution with $\mathcal{L}(Y_0) = \mathcal{L}(X_0)$.
Let $\nu_t = \mathcal{L}(X_t), \mu_t = \mathcal{L}(Y_t)$. We get, from the above theorem $\nu_t = \mu_t$ for all t. □

Outline

1. Evolution Equations
   - Maximum Principle
   - From Evolution Equation to Martingale Problem
   - Evolution equation and Generator
2. Charachterization of Markovian Generators
   - Hille-Yosida type theorem for Markov processes
   - Proof of Sufficiency
   - Proof of Necessity
3. Extensions
   - Controlled Martingale Problem
   - Discontinuous Coefficients

Dissipativity

- Recall that a linear operator A is dissipative if

$$\|\lambda f - Af\| \geq \lambda \|f\| \quad \text{for all } f \in D(A), \lambda > 0$$

- Dissipativity $\Rightarrow$ $(\lambda - A)$ is one-to-one
- A satisfies the positive maximum principle $\Rightarrow$ A is dissipative
  - Suppose $f \in D(A)$ attains its maximum at $x \in E$
  - let x_0 be such that $f(x_0) = \|f\|$.
  - Then $Af(x_0) \leq 0$.
  - $\|\lambda f - Af\| \geq \lambda f(x_0) - Af(x_0) \geq \lambda f(x_0) = \lambda \|f\|$

Charachterization of Generators of Markov Processes

Let $\mathcal{P} = \Big\{ \mu \in \mathbb{P}(E) : \exists \eta \in \mathbb{P}(E)$ such that for all $f \in D(A)$,

$$\int (\lambda - A) f d\mu = \lambda \int f d\eta \Big\}$$

Theorem 4

- *Suppose that A is a linear operator with $D(A)$ being an algebra and $\mathbf{1} \in D(A)$ and $A\mathbf{1} = \mathbf{0}$.*
- *Suppose that the separability condition is satisfied*

Then the martingale problem for A is well-posed if and only if

1. *$D(A)$ is measure determining*
2. *A satisfies the positive maximum principle*
3. *Range $(\lambda - A)$ is measure determining over $\mathcal{P}$*

Existence

Proof.

- Consider the compact embedding of the martingale problem into $\hat{E}, \mathcal{A}$
- $\mathcal{A}$ satisfies the positive maximum principle, since A does
- $\therefore \mathcal{A}$ is dissipative
- $(\lambda - \mathcal{A})$ is one-to-one on its range.
- $(\lambda - \mathcal{A})^{-1}$ is well defined on Range $(\lambda - \mathcal{A})$.
- Dissipitivaty $\implies$ that $(\lambda - \mathcal{A})^{-1}$ is a contraction
- $A\mathbf{1} = \mathbf{0} \implies$ that $(\lambda - \mathcal{A})^{-1}$ is a positive operator.
- $(I - \lambda^{-1}\mathcal{A})^{-1}$ is a positive contraction operator
- $\therefore \forall z, (I - \lambda^{-1}\mathcal{A})^{-1} g(z)$ defines a positive linear functional on Range $(\lambda - \mathcal{A})$.

Existence (contd.)

- Extend to $C(\hat{E})$ by Hahn-Banch Theorem.
- Riesz Representation Theorem $\implies \exists$ some transition function μ_λ

$$(I - \lambda^{-1}A)^{-1}g(z) = \int g(w)\mu_\lambda(z, dw)$$

- Since $\mathbb{I}_{\hat{g}(E)} \in D(\mathcal{A})$, we get $\mu_\lambda(z, \hat{g}(E)) = 1$ for all $z \in \hat{g}(E)$.
- Get $\hat{g}(E)$ valued Markov processes Z^n corresponding to μ_n
- Note: $X^n = \hat{g}^{-1}Z^n$ is a solution of the martingale problem for $A^n = n\left[(I - n^{-1}A)^{-1} - I\right]$
- $\{Z^n\}$ has some weak subsequential limit Z
- $X = \hat{g}^{-1}Z$ is a progressively measurable solution of the martingale problem for A. □

Uniqueness

- Let X and Y be two solutions of the martingale problem for A with $\mathcal{L}(X_0) = \mathcal{L}(Y_0) = \rho_0$.
- Then $\mu_t = \mathcal{L}(X_t)$ and $\nu_t = \mathcal{L}(Y_t)$ both satisfy the corresponding forward equation
- As in Theorem 3, but now using the fact that Range$(\lambda - A)$ is measure determining we get uniqueness of solution to the evolution equation.
- This implies that the $\mu_t = \nu_t$ for all $t \geq 0$.
- Hence the martingale problem for A is well-posed.

$\square$

Domain of A

- Let μ, ν be such that $\int f d\mu = \int f d\nu \quad \forall f \in D(A)$
- Let X, Y be solutions of the (A, μ) and (A, ν) martingale problems respectively
- Let $\mu_t = \mathcal{L}(X_t)$ and $\nu_t = \mathcal{L}(Y_t)$
- Then

$$\int f d\mu_t = \int f d\mu + \int_0^t \left(\int A f d\mu_s \right)$$
$$\int f d\nu_t = \int f d\nu + \int_0^t \left(\int A f d\nu_s \right)$$

- Well-posedness $\Rightarrow$ uniqueness for evolution equation
- $\therefore \mu_t = \nu_t$ for all t
- Hence $\mu = \nu$
- Thus $D(A)$ is measure determining

Positive maximum principle

- By hypothesis there exists a solution of the martingale problem for (A, δ_x) for each $x \in E$.
- Hence there exists a r.c.l.l. solution for the martingale problem for $(\mathcal{A}, \delta_z)$ for each $z \in \hat{E}$.
- Hence $\mathcal{A}$ satisfies the positive maximum principle
- It follows that A satisfies the positive maximum principle
  - For if $f \in D(A)$ attains a positive maximum at $x \in E$, $u = f \circ \hat{g}^{-1}$ attains its positive maximum at $\hat{g}(x) \in \hat{E}$
  - Then $Af(x) = \mathcal{A}u(\hat{g}(x)) \leq 0$.

□

Range($\lambda - A$)

- Let $\mu, \nu \in \mathcal{P}$ with

$$\int (\lambda - A) f d\mu = \int (\lambda - A) f d\nu \quad \forall f \in D(A)$$

- By definition of $\mathcal{P}$, $\exists \eta \in \mathbb{P}(E)$ such that the common value above is $= \lambda \int f d\eta$
- Let
  - $E_0 = E \times \{-1, 1\}$
  - $D(B^\eta) = \{f_1 f_2 : f_1 \in D(A), f_2 \in C(\{-1, 1\})$
  - $B^\eta f_1 f_2(x, n) = f_2(n) A f_1(x) + \lambda \left(f_2(-n) \int f_1 d\eta - f_1(x) f_2(n) \right)$
- The martingale problem for B^η is well-posed

Range($\lambda - A$) (Contd.)

- Let

$$\mu^0 = \mu \otimes \left(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1} \right); \qquad \nu^0 = \nu \otimes \left(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1} \right)$$

- Further

$$\int B^\eta f d\mu = \int B^\eta f d\nu = 0 \quad \forall f \in D(B^\eta)$$

- This implies that μ and ν are invariant measures for the Markov process charachterised via the martingale problem for B^η
- Uniqueness stationary distribution!!
- $\mu = \nu$

Outline

Stochastic Control

- State space - E
- Control space - $U, V = \mathcal{P}(U)$
- $A : D(A) \subset C_b(E) \to M(E \times U)$

Definition 3.1

An $E \times U$ - valued process $(X_t, u_t)_{t\geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a solution to the controlled martingale problem for (A, μ) with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if

1. $\mathcal{L}(X_0) = \mu$,
2. *For $f \in D(A)$,*

$$f(X_t) - \int_0^t Af(X_s, u_s)ds,$$

is a $(\mathcal{F}_t)$- martingale.

Relaxed Control

Definition 3.2

An $E \times V$ - valued process $(X_t, \pi_t)_{t \geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a solution to the relaxed controlled martingale problem for (A, μ) with respect to a filtration $(\mathcal{F}_t)_{t \geq 0}$ if

1. $\mathcal{L}(X(0)) = \mu$,
2. *for* $f \in D(A)$,

$$f(X_t) - \int_0^t \int_U Af(X_s, u)\pi_s(du)ds$$

is a $(\mathcal{F}_t)$-martingale.

Cost criteria

Let k be a running cost function

Ergodic cost

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E} \int_U [k(X_s, u)\pi_s(du)]\, ds$$

Discounted Cost

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha s} \int_U k(X_s, u)\pi_s(du) ds\right], \alpha > 0$$

Finite Horizon Cost

$$\mathbb{E}\left[\int_0^T \int_U k(X_s, u)\pi_s(du) ds\right], T > 0$$

Minimize cost over a class of controls

Find the optimal control

Occupation measures

- Let (X, π) be a solution of the relaxed control martingale problem for A.

Ergodic occupation measure If (X, π) is stationary with $\mathcal{L}(X_t, \pi_t) = \mu$. Then the associated cost is $\int k d\mu$.
μ is called the ergodic occupation measure

Discounted occupation measure

$$\int f d\mu = \alpha \mathbb{E}\left[\int_0^\infty \int_U e^{-\alpha t} f(X_t, u)\pi_t(du)dt\right]$$

Finite time occupation measure

$$\int f d\mu = T^{-1}\mathbb{E}\left[\int_0^T \int_U f(X_t, u)\pi_t(du)dt\right]$$

Conditions on A

1. Separability Condition is satisfied
2. $D(A)$ is an algebra that separates points in E and contains constant functions. Furthermore, $A\mathbf{1} = 0$
3. For each $u \in U$, $A^u f \equiv Af(\cdot, u)$ satisfies the positive maximum principle

Theorem 5

Suppose $\mu \in \mathcal{P}(E \times U)$ satisfy

$$\int Afd\mu = 0 \quad \forall f \in D(A).$$

Then there exists a stationary solution (X, π) of the relaxed controlled martingale problem for A such that $\mu = \mathcal{L}(X_t, \pi_t)$ for all $t \geq 0$.

More results

Theorem 6

Suppose $\mu \in \mathcal{P}(E \times U)$ satisfy

$$\int Af d\mu = \alpha \left(\int f d\mu - \int f d\nu_0 \right) \quad \forall f \in D(A).$$

Then there exists a solution (X, π) of the relaxed controlled martingale problem for A such that μ is the discounted occupation measure for this process.

- Similar result holds finite time occupation measures
- Finding the optimal control is now equivalent to optimising over a class of occupation measures

Martingale Problem with discontinuous Af

- $A : C_b(E) \to B(E)$
- Separability condition & Positive maximum principle
- There exists a complete separable metric space U, an operator $\hat{A} : D(A) \to C_b(E \times U)$ and a transition function η from $(E, \mathcal{E})$ into $(U, \mathcal{B}(U))$ such that

$$(Af)(x) = \int_U \hat{A}f(x,u)\eta(x,du).$$

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$$(Af)(x) = \int_U \hat{A}f(x,u)\eta(x,du).$$

- Results carry over to this set-up

An example

Consider the following well-posed SDE

$$dX_t = a(X_t)dt + b(X_t)dW_t$$

where

- a, b are measurable functions & W is a Standard Brownian Motion
- Then $(t, X_t, \mathcal{Y}_t)$ is a Markov process.
- X is then the unique solution of the martingale problem for A where $Af(x) = a(x)f'(x) + \frac{1}{2}b^2(x)f''(x)$
- Let $U = \mathbb{R}^2$, $\eta(x, du) = \delta_{a(x)} \otimes \delta_{b(x)}$
- $\hat{A}(x, (u_1, u_2)) = u_1 f'(x) + \frac{1}{2}u_2^2 f''(x)$

THANK YOU