ERGODIC THEORY OF DIFFUSIONS AND CONTROLLED DIFFUSIONS

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WARNING

The slides are not complete either in the statements of the claims or their proofs.

Many details will be filled in during the lectures, but not all. Mostly, I shall give only sketches of proofs.

The lectures are rated '**A**', i.e., 'for mature audience only'. There is no sex or violence, but working knowledge of Brownian motion and stochastic calculus will be needed.

All random processes described in these lectures are fictional and any similarity with the behavior of any person or persons living or dead is purely coincidental.

General reference: **THE RED BOOK***

**Ergodic Control of Diffusion Processes*, by A. Arapostathis, V. S. Borkar, M. K. Ghosh, Cambridge Uni. Press, Cambridge, UK, 2012.

OVERVIEW OF DIFFUSION THEORY

Consider the d-dimensional diffusion

$$X(\cdot) = [X_1(\cdot), \cdots, X_d(\cdot)]^T$$

satisfying the s.d.e.

$$X(t) = X_0 + \int_0^t m(X(s))ds + \int_0^t \sigma(X(s))dW(s).$$

- $m : \mathcal{R}^d \mapsto \mathcal{R}^d, \sigma : \mathcal{R}^d \mapsto \mathcal{R}^{d\times m}$ are Lipschitz, and,

- W is an m-dimensional standard Brownian motion independent of X_0.

Solution concepts:

1. *Strong solution:* Given an m-dimensional Brownian motion W and an $\mathcal{R}^d$-valued random variable X_0 independent of W on a probability space, construct X satisfying the above on this probability space.

2. *Weak solution:* Find a probability space on which there exist W, X_0, X, with X_0, W independent and prescribed in law as above, and X satisfying the above s.d.e.

Uniqueness notions:

- For strong solutions, uniqueness $\Longleftrightarrow$ (if X, X' two solutions, then $X = X'$ a.s.)

- For weak solutions, uniqueness $\Longleftrightarrow$ (if X, X' two solutions, then X, X' agree in law.)

Ito formula

Define the '*extended generator*' $\mathcal{L}$ as:

$$\mathcal{L}f := \langle \nabla f, m \rangle + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^T \nabla^2 f\right).$$

Then for $t > s \geq 0$,

$$\begin{aligned} f(X(t)) &= f(X(s)) + \int_s^t \mathcal{L}f(X(y))dy \\ &+ \int_0^t \langle \nabla f(X(y), \sigma(X(y))dW(y) \rangle. \end{aligned}$$

Ito-Krylov formula: As above for $f \in W^{2,p}_{\mathrm{loc}}(\mathcal{R}^d), p \geq d$, when σ is *uniformly non-degenerate*, i.e.,

$$\lambda_{\mathrm{min}}(\sigma\sigma^T) \geq \delta \text{ for some } \delta > 0.$$

Martingale formulation:

Weak solution $\Longleftrightarrow$ for all $f \in C^2(\mathcal{R}^d)$,

$$f(X(t)) - \int_0^t \mathcal{L}f(X(s))ds, \ t \geq 0,$$

is a local martingale w.r.t. $\mathcal{F}_t :=$ natural filtration* of X.

('$\Longrightarrow$' immediate from the Ito formula, converse from 'martingale representation theorem'.)

*'right-continuous completion'

A brief historical perspective

1. **Kolmogorov:** Using

$$
\begin{aligned}
& E[(X(t+\Delta)-X(t))I\{\|X(t+\Delta)-X(t)\| \leq \epsilon\}|X(t)] \\
\approx\ & m(X(t))\Delta, \quad \text{and,} \\
& E[(X(t+\Delta)-X(t))(X(t+\Delta)-X(t))^T \times \\
& \qquad I\{\|X(t+\Delta)-X(t)\| \leq \epsilon\}|X(t)] \\
\approx\ & \sigma(X(t))\sigma(X(t))^T\Delta,
\end{aligned}
$$

(plus a few technical conditions), can derive the evolution equations for the transition density.

$$p(s,x;t,y)dy \approx P(X(t) \approx y | X(s) = x), t > s.$$

In forward time t:

$$\frac{\partial p}{\partial t} = \mathcal{L}^* p$$

(Kolmogorov forward equation / Fokker-Planck or 'master' equation) and in backward time s:

$$\frac{\partial p}{\partial s} + \mathcal{L} p = 0$$

(Kolmogorov backward equations), with the initial, resp., terminal, condition

$$\lim_{t-s \to 0} p(s,x;t,y) = \delta_x(y).$$

Limitation: Needed PDE theoretic results (then unavailable) in order to go anywhere starting with this. This prompted the next development.

2. **Semigroup approach (Feller, Dynkin):**
Define $T_t : L_\infty(\mathcal{R}^d) \mapsto L_\infty(\mathcal{R}^d)$ by

$$T_t f(x) := E\left[f(X(t)) | X(0) = x\right].$$

Then for $s, t \geq 0$,

$$T_t \circ T_s = T_s \circ T_t = T_{t+s}, \; T_0 = I.$$

That is, $T_t, t \geq 0$, is a semigroup $\Longrightarrow$ Hille-Yosida theory of semigroups applies. A 'clean' theory possible if T_t maps $C_b(\mathcal{R}^d)$ to itself. Then X is called a *Feller process* (*strong Feller* if it maps $L_\infty(\mathcal{R}^d)$ to $C_b(\mathcal{R}^d)$).

In semigroup theory, a key role is played by the generator $\mathcal{G}$ given by

$$\lim_{\delta \downarrow 0} \left\| \frac{T_\delta f - f}{\delta} - \mathcal{G} f \right\| = 0$$

for f in a dense 'domain' $\mathcal{D}(\mathcal{G})$, with the associated evolution equation (cf. Komogorov equations)

$$\frac{\partial T_t f}{\partial t} = T_t \mathcal{G} f = \mathcal{G} T_t f.$$

Limitation: $\mathcal{D}(\mathcal{G})$ can be difficult to get a handle on.

3. **Parallel development Ito (foreshadowed by W. Doeblin):** Diffusion as a stochastic process given by the s.d.e.

4. **Martingale approach (Stroock-Varadhan):** Works with the extended generator $\mathcal{L}$ which coincides with $\mathcal{G}$ on its domain, but is easier to work with. Specificaion only in terms of 'law'.

Local existence/uniqueness + 'non-explosion' $\Longleftrightarrow$ global existence/uniqueness.

Sufficient conditions for non-explosion: linear growth, Khasminskii criterion, stochastic Liapunov functions. **We assume non-explosion.**

1. m measurable, bounded, σ continuous, non-degenerate: existence/uniqueness of weak solutions (Stroock-Varadhan).
   For degenerate case with m and σ continuous, only existence, no uniqueness (Hartman example).

2. σ Lipschitz, non-degenerate, m locally bounded, measurable with linear growth: existence/uniqueness of strong solutions
(Zvonkin, Veretennikov)

3. m, σ locally bounded, measurable with linear growth, σ non-degenerate: existence through smooth approximations (Krylov), no uniqueness guarantee (examples by Nadirashvili, Safanov)

Theory extends to time-dependent coefficients, but caution is required: non-degeneracy is not sufficient for the existence of densities (example by Fabes, Kenig). Leads to two parameter semigroup $T_{s,t}, t > s \geq 0$, satisfying

$$T_{t,t} = I,\ T_{s,t} \circ T_{u,s} = T_{u,t} \text{ for } 0 \leq u \leq s \leq t.$$

Degenerate case: selection of Markov family through Krylov selection or viscosity solutions.

The PDE connection

Assume σ non-degenerate, Lipschitz, m measurable with linear growth.

1. Consider the elliptic equation

$$\begin{aligned} \mathcal{L}u(x) - \alpha u(x) &= -f(x) \ \ \forall\ x \in D, \\ u(x) &= h(x) \ \ \forall\ x \in \partial D, \end{aligned}$$

   for f, h continuous,$\alpha > 0$, D a bounded open set with ‘exterior cone’ condition.

Unique solution $u \in W^{2,p}_{\text{loc}}(D) \cap C(\bar{D}), p \geq 2$, given by

$$u(x) = E\left[\int_0^\tau e^{-\alpha t} f(X(t))dt + e^{-\alpha\tau} h(X(\tau))|X(0) = x\right]$$

where $\tau := \inf\{t \geq 0 : X(t) = \partial D\}$.

2. Consider the parabolic equation

$$\begin{aligned} 0 \;=\; & \frac{\partial}{\partial t}u(x,t) + \mathcal{L}u(x,t) - \alpha u(x,t) + f(x,t) \\ & \forall\; x \in D,\; t \in (0,T), \\ & u(x,t) = h(x)\;\forall\; x \in \partial D, t \in (0,T), \\ & u(x,T) = g(x)\;\forall x \in D, \end{aligned}$$

for f, h, g continuous, $T, \alpha > 0$.

Unique solution

$u \in W^{2,1,p}_{\text{loc}}(D \times (0,T)) \cap C(\bar{D} \times [0,T)) \cap C(D \times [0,T]),$
$p \geq 2,$

given by

$$u(x,s) = \\ E[\int_s^{\tau \wedge T} e^{-\alpha t} f(X(t))dt + e^{-\alpha \tau \wedge T} \times \\ (g(X(T))I\{\tau > T\} + h(X(\tau))I\{\tau < T\})|X(s) = x]$$

where $\tau := \text{inf}\{t \geq s : X(t) = \partial D\}$.

Ergodic theory of Markov processes

Ergodic theorem

Let $S_t, t \geq 0$, be a semigroup of measure-preserving transformations on a probability space $(\Omega, \mathcal{F}, P)$, i.e., $\forall t$,

$$P(S_t^{-1}(A)) := P(\{\omega \in \Omega : S_t(\omega) \in A\}) = P(A) \ \forall \ A \in \mathcal{F}.$$

Define the *invariant σ-field* $\mathcal{I} :=$ the P-completion of

$$\{A \in \mathcal{F} : S_t^{-1}(A) = A \ \forall t\}.$$

Ergodic theorem:

$$\lim_{T \uparrow \infty} \frac{1}{T} \int_0^T f(S_t(\omega))dt = E[f|\mathcal{I}] \text{ a.s.}$$

$\forall \ f \in L_1(\Omega, \mathcal{F}, P)$.

$\{S_t\}$ is *ergodic* if $\mathcal{I}$ is trivial, i.e., $A \in \mathcal{I} \Longrightarrow P(A) = 0$ or 1. Then for f as above,

$$\lim_{T \uparrow \infty} \frac{1}{T} \int_0^T f(S_t(\omega))dt = E[f] \text{ a.s.}$$

Here P is an $invariant$ (probability) measure for $\{S_t\}$.

Let Ω be Polish with $\mathcal{F} :=$ its Borel σ-field completed w.r.t. P.

Define $\mathcal{M} :=$ the set of invariant probability measures of $\{S_t\}$.

- $\mathcal{M}$ is closed convex,

- Any limit point of $\mu_T := \frac{1}{T}\int_0^T \nu \circ S_t^{-1} dt, \nu \in \mathcal{P}(\Omega)$, in $\mathcal{P}(\Omega)$ as $T \uparrow \infty$ is in $\mathcal{M}$ (nonempty if $\{\mu_t\}$ is tight).

- Extreme points of $\mathcal{M}$ are ergodic and are mutually singular.

- Every $\eta \in \mathcal{M}$ is a barycenter of ergodic measures.

ERGODIC THEORY OF MARKOV PROCESSES

Take $\Omega := D([0,\infty);\mathcal{R}^d)$ with $\mathcal{F} :=$ the Borel σ-field completed w.r.t. P. For $t \in \mathcal{R}$, let $\theta_t : \Omega \mapsto \Omega$ denote the shift operator: $\theta_t(\omega(\cdot)) = \omega(t + \cdot)$.

$\xi \in \mathcal{P}(\Omega)$ is *stationary* if $\theta_t, t \in \mathcal{R}$, is measure-preserving on $(\Omega, \mathcal{F}, \xi)$.

Assumption: Transition kernel $p(dy|x,t)$ is continuous in x (Feller property).

The set of all stationary measures compatible with the transition kernel $p(dy|\cdot,\cdot)$ is closed convex.

Let $\mathcal{J}$ denote the set of its extreme points and $\mathcal{E}$ the set of ergodic measures.

CLAIM: $\xi \in \mathcal{J} \Longrightarrow \xi \in \mathcal{E}$, i.e., it is ergodic.

Sketch of Proof: Consider $t \in (-\infty, \infty)$. If the claim is false, there exist mutually singular $\xi_1, \xi_2 \in \mathcal{E}$ and $0 < a < 1$ such that $\xi = a\xi_1 + (1-a)\xi_2$. Then for some $A \in \mathcal{F}$,

$$\Lambda_1 := \frac{d\xi_1}{d\xi} = \frac{I_A}{a},\ \Lambda_2 := \frac{d\xi_2}{d\xi} = \frac{I_{A^c}}{1-a}.$$

Let $\xi_i(t)$ denote the restrictions of ξ_i to $\mathcal{F}_t :=$ the natural filtration, and

$$\Lambda_i(t) = \frac{d\xi_i(t)}{d\xi(t)} = E[\Lambda_i|\mathcal{F}_t]$$

the corresponding R-N derivatives. Then $\Lambda_i(t) \to \Lambda_i$ a.s. and by stationarity, $\Lambda_i(t) = \Lambda_i$ a.s.

It follows that

$$\frac{\Lambda_1(t+s)}{\Lambda_1(t)} = 1 \text{ a.s. } A.$$

Write

$$\begin{aligned} \xi_{t+s}(d\omega, d\omega') &= \xi_t(d\omega)\nu_{t,s}(d\omega'|\omega), \\ \xi^i_{t+s}(d\omega, d\omega') &= \xi^i_t(d\omega)\nu^i_{t,s}(d\omega'|\omega), \;\; i = 1, 2. \end{aligned}$$

Then

$$\frac{d\xi^i_{t+s}}{d\xi_{t+s}}(\omega,\omega') = \frac{d\xi^i_t}{d\xi_t}(\omega)\frac{d\nu^i_{t,s}}{d\nu_{t,s}}(\omega'|\omega)\ a.s.$$

That is, for $i = 1, 2$,

$$\Lambda^i_{t+s}(\omega,\omega') = \Lambda^i_t(\omega)\frac{d\nu^i_{t,s}}{d\nu_{t,s}}(\omega'|\omega).$$

Hence for $i = 1, 2$, almost surely,

$$\frac{d\nu^i_{t,s}}{d\nu_{t,s}} = 1 \Longrightarrow \nu^i_{t,s} = \nu_{t,s}.$$

That is, the regular conditional law of the canonical process X_{t+s} given $\mathcal{F}_t$ is $P(s, X_t, dy)$ a.s. Hence $\xi_i \in \mathcal{J}$, a contradiction to extreme point property of ξ. Thus ξ must be ergodic. $\square$

Let $\mu \in \mathcal{P}(\mathcal{R}^d)$. A set $A \in \mathcal{B}(\mathcal{R}^d)$ is μ-*invariant* if $p(A|x,t) = 1$ for μ-a.s. $x \in A$ and at $t \geq 0$. Then $\mathcal{I}_\mu :=$ the set of μ-invariant sets is a σ-field.

Let $\xi \in \mathcal{J}$ and μ its one dimensional marginal.

Lemma: If $B \in \mathcal{I}_\mu$, then $I_B(\omega(0)) = I_B(\omega(t))$ a.s.

Proof

$$
\begin{aligned}
& \xi(I_B(\omega(0)) \neq I_B(\omega(t))) \\
= \ & \int_B \mu(dx) P(t, x, B^c) + \int_{B^c} \mu(dx) P(t, x, B) \\
= \ & 0.
\end{aligned}
$$

$\square$

Thus $I_B(\omega(0))$ is $\mathcal{I}_\mu$ measurable.

Theorem: Let $\xi \in \mathcal{J}$. If $C \in \mathcal{F}$ is ξ-invariant, then $I_C(\omega(\cdot)) = I_B(\omega(t))$ a.s. for some $B \in \mathcal{B}(\mathcal{R}^d)$.

Sketch of Proof Let $X :=$ the canonical process. Then

$$f(X_t) := E_{X_t}[I_C(X)] \to I_C \text{ a.s.}$$

By stationarity, $f(X_t)$ must be of the form $I_B(X(t))$ a.s. (cf. Lemma above). $\square$

Thus ergodic decomposition of path space $\Longleftrightarrow$ decomposition of state space (*Doeblin decomposition*).

ERGODIC THEORY OF

NON-DEGENERATE DIFFUSIONS

Say that a diffusion is *positive recurrent* if for some bounded open $D \subset \mathcal{R}^d, x \in D^c$ and $\tau_D := \min\{t \geq 0 : X(t) \in D\}$, $E_x[\tau_D] < \infty$.

For this definition to make sense, we need:

1. If $E_x[\tau_D] < \infty$ for given D, x as above, it is true for all x.

2. If $E_x[\tau_D] < \infty$ for some D as above, it is true for all such D.

Proof of the first claim (sketch):

Let $R > 0$ be such that $\{x\} \cup D \subset B_R := \{y : \|y\| < R\}$, $\tau_R := \min\{t \geq 0 : X(t) \notin B_R\}$. Then

$$\psi_R(y) := E_y[\tau_D \wedge \tau_R] \geq 0$$

is the unique solution in $W^{2,p}_{loc}(B_R \backslash \bar{D}) \cap C(\bar{B}_R \backslash D)$ to

$$\begin{aligned} \mathcal{L}\psi &= -1 \quad \text{in} \quad B_R \backslash \bar{D}, \\ \psi &= 0 \quad \text{on} \quad \partial D \cup \partial B_R. \end{aligned}$$

But

$$\psi_R(x) = E_x[\tau_D \wedge \tau_R] \uparrow E_x[\tau_D] \quad \text{as} \quad R \uparrow \infty.$$

By Harnack's inequality and elliptic regularity,

$$\psi_R \uparrow \psi \in W^{2,p}_{loc}(\bar{D}^c) \cap C(D^c)$$

uniformly on compacts, where $\psi \geq 0$ solves

$$\begin{aligned} \mathcal{L}\psi &= -1 \quad \text{in} \quad D^c, \\ \psi &= 0 \quad \text{on} \quad \partial D. \end{aligned}$$

Hence one has:

$$E_y[\tau_D] = \psi(y) < \infty.$$

$\square$

We also have: for C compact in $\bar{D}^c$,

$$\max_{y \in C} E_y[\tau_D] < \infty.$$

Proof of the second claim (sketch):

Let $G \subset \bar{D}^c$ be a bounded open set and τ_G its first hitting time. Pick $R > r > 0$ such that $\bar{D} \cup \bar{G} \subset B_r$. Define

$$\begin{aligned}
\zeta_0 &:= \min\{t > \tau_0 : X(t) \in \partial B_r\},, \\
\sigma_k &:= \min\{t > \zeta_k : X(t) \in \partial B_R\}, \\
\zeta_{k+1} &:= \min\{t > \sigma_k : X(t) \in \partial B_r\},
\end{aligned}$$

for $k \geq 0$. Then $\{\sigma_k, \zeta_k\}$ are finite a.s.

Claim: $p_0 := \sup_{x \in \partial B_r} P_x(\tau_G > \sigma_1) < 1$.

Proof of claim:

$\varphi(x) := P_x(\tau_G > \sigma_1)$ is the unique solution to

$$\begin{aligned} \mathcal{L}\varphi &= 0 \quad \text{on} \quad B_R \backslash \bar{G}, \\ \varphi &= 0 \quad \text{on} \quad \partial G, \\ \varphi &= 1 \quad \text{on} \quad \partial B_r. \end{aligned}$$

By strong maximum principle, φ cannot have a maximum in $B_R \backslash \bar{G}$, hence the claim.

$$
\begin{aligned}
E_x[\tau_G] &\leq E_x[\tau_0] + \sum_{k=1}^{\infty} E_x[\zeta_k I\{\zeta_{k-1} < \tau_G < \zeta_k\}] \\
&= E_x[\tau_0] + \sum_{k=1}^{\infty} \sum_{m=1}^{k} E_x[(\zeta_m - \zeta_{m-1}) \times \\
&\qquad\qquad\qquad\qquad I\{\zeta_{k-1} < \tau_G < \zeta_k\}] \\
&= E_x[\tau_0] + \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} E_x[(\zeta_m - \zeta_{m-1}) \times \\
&\qquad\qquad\qquad\qquad I\{\zeta_{k-1} < \tau_G < \zeta_k\}] \\
&= E_x[\tau_0] + \sum_{m=1}^{\infty} E_x[(\zeta_m - \zeta_{m-1}) I\{\tau_G > \zeta_{m-1}\}] \\
&\leq E_x[\tau_0] + \sum_{m=1}^{\infty} p_0^{m-1} \sup_{y\in\partial B_r} E_y[\zeta_1] \\
&\leq E_x[\tau_0] + \frac{\sup_{y\in\partial B_r} E_y[\zeta_1]}{1-p_0} \\
&< \infty. \qquad \square
\end{aligned}
$$

Theorem: If X is positive recurrent, it has a unique invariant distribution which has a strictly positive density.

Sketch of proof: From PDE theory, $p(dy|x,t) = \varphi(y|x,t)dy$ for some $\varphi(\cdot|\cdot,\cdot) > 0$. If μ is an invariant probability measure,

$$\mu(dy) = \int \mu(dx)\varphi(y|x,t)dy,$$

implying that μ has a strictly positive density. Also, if μ, μ' are two invariant probability measures, they are mutually absolutely continuous w.r.t. the Lebesgue measure, hence w.r.t. each other. Thus they must be identical.

Existence:

Define empirical measures $\nu_t \in \mathcal{P}(\mathcal{R}^d), t > 0$, by:

$$\int f d\nu_t := \frac{1}{t}\int_0^t f(X(s))ds, \quad f \in C_b(\mathcal{R}^d).$$

Claim: Almost surely, any limit point of ν_t in $\mathcal{P}(\mathcal{R}^d)$ as $t \uparrow \infty$ is an invariant probability measure.

Proof of the claim:

For $f \in$ a countable convergence determining class of compactly supported C^2 functions,

$$
\begin{aligned}
\frac{\int_0^t \mathcal{L}f(X(s))ds}{t} &= \frac{f(X(t)) - f(X(0))}{t} - \\
&\quad \frac{1}{t}\int_0^t \langle \nabla f(X(s)), \sigma(X(s))dW(s)\rangle \\
&\to 0 \text{ a.s.}
\end{aligned}
$$

by the strong law of large numbers for square-integrable martingales.

Thus outside a P-null set,

$$\int \mathcal{L} f d\nu = 0$$

for f as above, for any limit point ν of ν_t as $t \uparrow \infty$. The claim follows by Echeverria's theorem (to be proved later). $\square$

Thus it suffices to exhibit one such ν.

Khasminskii construction:

Ler $B_r, B_R, \{\sigma_k, \zeta_k\}$ be as before. Then $Y_n := X(\zeta_n), n \geq 0$, is a ∂B_r-valued Markov chain.

Compact state space $\Longrightarrow$ at least one invariant probability measure

PDE thory $\Longrightarrow$ transition kernel mutually absolutely continuous w.r.t. surface measure of $\partial B_r \Longrightarrow$
unique invariant probability $\eta \in \mathcal{P}(\partial B_r)$.

Define $\mu \in \mathcal{P}(\mathcal{R}^d)$ by

$$\int f d\mu := \frac{\int_{\partial B_r} E_x[\int_0^{\zeta_1} f(X(s))ds]\eta(dx)}{\int_{\partial B_r} E_x[\zeta_1]\eta(dx)}.$$

Then

$$\frac{\int_0^{\zeta_k} f(X(s))ds}{\zeta_k} = \frac{\sum_{m=1}^k \int_{\zeta_{m-1}}^{\zeta_m} f(X(s))ds}{\sum_{m=1}^k (\zeta_m - \zeta_{m-1})}$$
$$\overset{k\uparrow 0}{\rightarrow} \int f d\mu \text{ a.s.}$$

This completes the existence proof.

Digression: Pseudo-atom construction

Let $\{X_n, n \geq 0\}$ be a discrete time φ-irreducible Markov chain on a Polish space S with transition kernel $p(dy|x)$, satisfying the *minorization* condition:

There exists a Borel set $B \subset S$ satisfying $\varphi(B) > 0$, $\delta > 0$ and $\nu \in \mathcal{P}(S)$ with $\nu(B) = 1$, such that

$$p(A|x) \geq \delta\nu(A)I_B(x) \ \forall \ \text{Borel} \ \ A.$$

Let $S^* := S \times \{0,1\}$. For Borel $A \subset S$, let
$A_0 := A \times \{0\}$, $A_1 := A \times \{1\}$.

Construct an S^*-valued process $(\hat{X}_n, i_n), n \geq 0$ (called the *split chain*) as follows:

1.

$$
\begin{aligned}
P((\hat{X}_0, i_0) \in A_0) &= (1-\delta) P(X_0 \in A \cap B) \\
&\quad + P(X_0 \in A \cap B^c), \\
P((\hat{X}_0, i_0) \in A_1) &= \delta P(X_0 \in A \cap B).
\end{aligned}
$$

2. If $\widehat{X}_n = x \in B, i_n = 0$, then $\widehat{X}_{n+1} = y$ according to the probability

$$\frac{1}{1-\delta}(p(dy|x) - \delta\nu(dy)).$$

Moreover, if $y \in B$, $i_{n+1} = 1$ with probability δ. Otherwise $i_{n+1} = 0$.

3. If $\widehat{X}_n = x \in B$ and $i_n = 1$, then $\widehat{X}_{n+1} = y$ according to probability $\nu(dy)$ and $i_{n+1} = 0$ or 1 with probability $1-\delta$, δ resp.

4. If $\widehat{X}_n \notin B$ and $i_n = 0$, then $\widehat{X}_{n+1} = y$ according to $p(dy|x)$ and if $y \in B$, $i_{n+1} = 1$ with probability δ. Otherwise $i_{n+1} = 0$.

5. $B^c \times \{1\}$ is never visited.

Theorem: $\widehat{X}, X$ agree in law.

$B \times \{1\}$ acts like an atom and is called a *pseudo-atom*.

Theorem: Let μ = the (unique) invariant probability measure for X. Then

$$E[f(X(t))] \to \int f d\mu \ \forall \ f \in C_b(\mathcal{R}^d).$$

Sketch of proof:

Let Y denote the stationary solution corresponding to $p(dy|\cdot)$. Consider the Markov Chains $\{X(n)\}, \{Y(n)\}$. Let $(\hat{X}_n, i_n), (\hat{Y}_n, j_n), n \geq 0$, denote the corresponding split chains. Define the *coupling time at the pseudo-atom*

$$\tau := \min\{n \geq 0 : ((\hat{X}_n, i_n), (\hat{Y}_n, j_n)) \in (B \times \{1\}) \times (B \times \{1\})\}.$$

Can show $\tau < \infty$ a.s. Couple the two split chains at τ. Then for $f \in C_b(\mathcal{R}^d)$,

$$
\begin{aligned}
& |E[f(X(t))] - E[f(Y(t))]| \\
= \ & |E[(f(\hat{X}(t)) - f(\hat{Y}(t)))I\{\tau > t\}]| \\
\leq \ & KP(\tau > t) \to 0 \ \text{ as } \ t \uparrow \infty.
\end{aligned}
$$

The claim follows. $\square$

Stochastic Liapunov theory

Stochastic Liapunov condition: Suppose there exists a C^2 function $V : \mathcal{R}^d \mapsto \mathcal{R}$ such that

- $\lim_{\|x\|\uparrow\infty} V(x) = \infty$, and,

- there exist $\epsilon, C > 0$ and a bounded set $B \subset \mathcal{R}^d$ such that

$$\mathcal{L}V(x) \leq -\epsilon + C I_B.$$

Theorem Under above condition, X is positive recurrent.

Proof: Let τ denote the first hitting time of $A :=$ a bounded open set containing B. Then by Dynkin formula and Fatou's lemma,

$$\begin{aligned}
\inf_{y \in A} V(y) - V(x) &\leq E_x[V(X(\tau))] - V(x) \\
&\leq E_x\left[\int_0^\tau \mathcal{L}V(X(s))ds\right] \\
&\leq -\epsilon E_x[\tau].
\end{aligned}$$

The claim follows. $\square$

Converse: Let $k : [0, \infty) \mapsto [0, 1]$ be a continuous onto increasing function and $\beta := \int k(\|x\|) d\mu(x) \in (0, 1)$. Consider the *Poisson equation*

$$\mathcal{L}\psi(x) + k(\|x\|) - \beta = 0.$$

If this has a solution ψ, then for $B := \{x : k(\|x\|) - \beta \leq \frac{1-\beta}{2}\}$ and $\tau :=$ the first hitting time of B,

$$\psi(x) \geq E_x[\int_0^\tau (k(\|X(t)\|) - \beta) dt + \psi(X(\tau))].$$

It follows that

$$\lim_{\|x\| \uparrow \infty} \psi(x) = \infty.$$

Also, for $C := \max_{y \in B} |k(\|y\|) - \beta|$ and $\epsilon := \frac{1-\beta}{2}$,

$$\mathcal{L}\psi \leq -\epsilon + C I_B.$$

Thus converse holds.

For existence of ψ, use the *vanishing discount* argument:

For $\alpha \in (0,1)$, consider

$$\mathcal{L}\psi_\alpha(x) + k(\|x\|) - \alpha\psi_\alpha(x) = 0.$$

This has a unique bounded solution

$$\psi_\alpha(x) := E_x\left[\int_0^\infty e^{-\alpha t} k(\|X(t)\|) dt\right].$$

Let $\bar{\psi}_\alpha(\cdot) := \psi_\alpha(\cdot) - \psi_\alpha(0)$. Then

$$\mathcal{L}\bar{\psi}_\alpha(x) + k(\|x\|) - \alpha\bar{\psi}_\alpha(x) - \alpha\psi_\alpha(0) = 0. \tag{1}$$

But for $\tau :=$ the 'coupling time at the pseudo-atom',

$$\begin{aligned} |\bar{\psi}(x)| &= |E[\int_0^\infty e^{-\alpha t}(f(X_x(t)) - f(X_0(t)))dt]| \\ &= |E[\int_0^\infty e^{-\alpha t}(f(\hat{X}_x(t)) - f(\hat{X}_0(t)))dt]| \\ &= |E[\int_0^\tau e^{-\alpha t}(f(\hat{X}_x(t)) - f(\hat{X}_0(t)))dt]| \\ &\le KE[\tau] < \infty. \end{aligned}$$

Then by elliptic regularity, $\bar{\psi}_\alpha \to \psi$ in an appropriate sense along a subsequence as $\alpha \downarrow 0$. Letting $\alpha \downarrow 0$ in (1) along this subsequence, we get the Poisson equation for ψ.

Variants: *Geometric ergodicity*

$$\mathcal{L}V \leq -\gamma V + CI_B.$$

Can show $E\left[e^{a\tau_B}\right] < \infty$ for some $a > 0$, where τ_B is the first hitting time of a bounded open neighborhood of B. Thus $P(\tau > t) \leq Ke^{-at}$ and therefore $E[f(X(t))] \to \int f d\mu$ at an exponential rate.

h-norm ergodicity

$$\mathcal{L}V \leq -\gamma h + CI_B,$$

where $\lim_{\|x\|\uparrow\infty} h(x) = \infty$. Then $E[f(X(t))] \to \int f d\mu$ for all $f \in C(\mathcal{R}^d)$ that are $O(h)$.

CONTROLLED DIFFUSIONS

Controlled diffusion:

$$X(t) = X_0 + \int_0^t m(X(s), u(s))ds + \int_0^t \sigma(X(s))dW(s),$$

where for a compact metric ‘action space’ U,

- the map $(x, u) \mapsto m(x, u) : \mathcal{R}^d \times U \mapsto \mathcal{R}^d$ is continuous in x, u and Lipschitz in x uniformly in u, and,

- $u(\cdot)$ is a measurable U-valued *control* process that is *non-anticipative*: for $t > s \geq 0$, $W(t) - W(s)$ is independent of right-continuous completion of $\sigma(W(y), u(y), y \leq s)$. (†)

Say that $u(\cdot)$ is:

- *admissible* if $(\dagger)$ holds,

- *feedback* if it is adapted to the natural flitration of X,

- *Markov* if $u(t) = v(X(t), t)\ \forall t$ for some measurable v,

- *stationary Makov* if $u(t) = v(X(t)), t \geq 0$.

Relaxed control: Replace U by $\mathcal{P}(U)$ and consider $\mathcal{P}(U)$-valued control. (‘Young measures’)

‘Chattering lemma’: This is a legitimate relaxation.

By abuse of terminology, we continue to use notation $m(X(t), u(t))$ instead of $\int m(X(t), y)u(t, dy)$. ‘Control’ will always taken to be relaxed.

The original framework then coresponds to $u(t) = \delta_{\tilde{u}(t)}$. We call this a precise control.

Similarly define precise stationary Markov control etc.

Ergodic occupation measures:

Under a stationary Markov control v, X is a time-homogeneus Markov process.

Say v is a *stable* stationary Markov control (SSM) if X has an invariant distribution η. (Unique if nondegenerate.)

Define the *ergodic occupation measure* $\mu^v \in \mathcal{P}(\mathcal{R}^d \times U)$ by

$$\mu^v(dxdu) = \eta(dx)v(x, du).$$

Let $\mathcal{G} :=$ the set of ergodic occupation measures.

Define the *controlled extended generator* $\mathcal{L}$ by:

for $f \in C^2(\mathcal{R}^d)$,

$$\mathcal{L}f(x,u) := \langle m(x,u), \nabla f \rangle + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^T(x)\nabla^2 f(x)\right).$$

Theorem: $\mathcal{G} = \{\mu : \int \mathcal{L}f d\mu = 0 \ \forall\ f \in C_b^2(\mathcal{R}^d)\}$.

(To be proved later.)

Corollary: $\mathcal{G}$ is closed convex.

Sketch of proof: $\int \mathcal{L} f d\mu = 0$ holds under weak convergence $\Longrightarrow$ closed.

Suppose $\int \mathcal{L} f d\mu_i = 0$ with

$$\mu_i(dx, du) = \eta_i(dx) v_i(du|x), i = 1, 2.$$

Let $\mu = a\mu_1 + (1-a)\mu_2$, $a \in (0,1)$. Then

$$\mu(dx, du) = \eta(dx) v(du|x)$$

where $\eta = a\eta_1 + (1-a)\eta_2$, and,

$$v(du|x) = a\frac{d\eta_1}{d\eta}(x) v_1(du|x) + (1-a)\frac{d\eta_2}{d\eta}(x) v_2(du|x),$$

satisfies $\int \mathcal{L} f d\mu = 0 \Longrightarrow$ convex. $\square$

Define *empirical measures* $\nu_t \in \mathcal{P}(\mathcal{R}^d \times U), t > 0$, by

$$\int f d\nu_t := \frac{1}{t}\int_0^t \int f(X(s), y)u(s, dy)ds, \ \ f \in C_b(\mathcal{R}^d \times U).$$

Theorem: $\nu_t \in \mathcal{P}((\mathcal{R}^d \cup \{\infty\}) \times U) \stackrel{t\uparrow\infty}{\to} \{a\delta_\infty + (1-a)\mu :$ $\mu \in \mathcal{G}, a \in [0,1]\}$. If $\{\nu_t, t > 0\}$ is tight, then $\nu_t \to \mathcal{G}$.

Proof For $f \in$ a countable convergence determining set in $C_0^2(\mathcal{R}^d \times U)$, a.s.,

$$\begin{aligned} 0 \ \stackrel{t\uparrow\infty}{\leftarrow} & \ \frac{\int_0^t \mathcal{L}f(X(s), u(s))ds}{t} \\ = & \ \frac{f(X(t)) - f(X(0))}{t} - \frac{\int_0^t \langle \nabla f(X(s)), \sigma(X(s))dW(s)\rangle}{t}. \end{aligned}$$

It follows that $\int \mathcal{L}f d\nu_t \to 0$ a.s., implying the claim. $\square$

Ergodic control problem: For $\bar{k} \in C(\mathcal{R}^d \times \mathcal{P}(U)) \geq 0$, $k(x,u) := \int k(x,y)u(dy)$, minimize

$$\limsup_{T \uparrow \infty} \frac{1}{T} \int_0^T k(X(t), u(t))dt$$

(a.s. version) or

$$\limsup_{T \uparrow \infty} \frac{1}{T} \int_0^T E[k(X(t), u(t))]dt$$

(average version).

In the non-degenerate case, under SSM v with ergodic occupation measure μ, this equals $\int \bar{k} d\mu$ a.s.

1. (*Near-monotone case*) $\liminf_{\|x\|\uparrow\infty} \min_u k(x,u) > \beta$ where $\beta := \inf_{\mu\in\mathcal{G}} \int \bar{k} d\mu$.

2. (*Stable case*) $\mathcal{G}$ compact.

Theorem: Under either condition, an optimal SSM exists and under any admissible u,

$$\liminf_{T\uparrow\infty} \frac{1}{T}\int_0^T k(X(t),u(t))dt \geq \min_{\mathcal{G}} \int \bar{k} d\mu \quad \text{a.s.}$$

Assume non-degeneracy

$\Longrightarrow$ each SSM v has a unique stationary distribution and unique ergodic occupation measure.

Theorem: Extreme points of $\mathcal{G}$ correspond to precise controls.

Corollary Under above conditions, an optimal precise SSM exists.

Sketch of proof: Let $\mu(dx,du) = \eta(dx)v(du|x)$ be an extreme point of $\mathcal{G}$. Suppose there exist:

- a bounded (w.l.o.g.) set A of measure > 0,

- $\gamma : \mathcal{R}^d \mapsto (0,1)$ with $\gamma(x) \wedge (1-\gamma(x)) \geq \epsilon > 0$ on A,

- SSMs $v_i(du|x), i = 1,2$, such that $v_1(du|x) = v_2(du|x)$ on A^c, $v_1(du|x) \neq v_2(du|x)$ a.e. on A, and

$$v(du|x) = \gamma(x)v_1(du|x) + (1-\gamma(x))v_2(du|x).$$

Need: $\delta \in (0,1)$, $\widehat{v}(du|x)$ such that

$$\mu_v = \delta \mu_{v_1} + (1-\delta)\mu_{\widehat{v}} \ .$$

That is,

$$\begin{aligned} v(du|x) &= \delta \frac{d\eta_{v_1}}{d\eta_v}(x) v_1(du|x) + (1-\delta)\frac{d\eta_{\widehat{v}}}{d\eta_v}(x)\widehat{v}(du|x) \\ &= \frac{\delta\varphi_{v_1}(x)v_1(du|x) + (1-\delta)\varphi_{\widehat{v}}(x)\widehat{v}(\cdot|x)}{\delta\varphi_{v_1}(x) + (1-\delta)\varphi_{\widehat{v}}(x)} \ . \end{aligned}$$

Fact: $0 < \delta_1 \leq \varphi_u(x) \leq \delta_2 < \infty \ \forall x \in A$.

Let SSM $u \in \mathcal{U} :=$ { the SSM that agree with v on A^c}.

Define

$$\delta \;=\; \frac{\delta_1 \epsilon}{\delta_1 \epsilon + \delta_2 (1-\epsilon)},$$

$$w(\cdot|x) \;:=\; v(\cdot|x) + \frac{\delta \varphi_{v_1}(x)}{(1-\delta)\varphi_u(x)} (v(\cdot|x) - v_1(\cdot|x)), \; u \in \mathcal{U}.$$

Can show: The map $\eta_u \mapsto \eta_w$ has a fixed point $\widehat{v} \Longrightarrow$

$$\mu_v = \delta \mu_{v_1} + (1-\delta)\mu_{\widehat{v}} \; .$$

The claim follows. $\square$

Uniform stability

Assume: All stationary Markov controls are stable.

Let $\mathcal{U}$ be a set of SSM, h an inf-compact function on domain dependent on the context, and :

$\mathcal{M}(\mathcal{U}) :=$ { ergodic occupation measures $\mu_v, v \in \mathcal{U}$},

$\mathcal{H}(\mathcal{U}) :=$ { invariant probability measures $\eta_v, v \in \mathcal{U}$},

$\tau(D) :=$ the first hitting time of an open ball D.

Then the following statements are equivalent:

1. For some open ball G and some $x \notin D$,

$$\sup_{v \in \mathcal{U}} E_x^v \left[\int_0^{\tau(D)} h(X_t) dt \right] < \infty.$$

2. For all open balls D and compact $\Gamma \subset \mathcal{R}^d$,

$$\sup_{v \in \mathcal{U}} \sup_{x \in \Gamma} E_x^v \left[\int_0^{\tau(D)} h(X_t) dt \right] < \infty.$$

3. $\sup_{\mu \in \mathcal{M}(\mathcal{U})} \int h d\mu < \infty$.

4. There exist non-negative inf-compact $\mathcal{V} \in C^2(\mathcal{R}^d)$, $k > 0$ such that

$$\mathcal{L}\mathcal{V}(x,u) \leq k - h(x,u) \ \forall u \in U.$$

5. For any compact $\Gamma \subset \mathcal{R}^d$ and $t_0 > 0$, the set of mean empirical measures $\nu_t, \ t \geq t_0, \ v \in \mathcal{U}, \ x \in \Gamma$, is tight.

6. $\mathcal{H}(\mathcal{U})$ is tight.

7. $\mathcal{M}(\mathcal{U})$ is tight.

8. $\mathcal{M}(\mathcal{U})$ is compact.

9. For some open ball D and $x \notin \bar{D}$, $\{\tau(D),\ v \in \mathcal{U}\}$ is uniformly integrable.

10. For all open balls D and compact $\Gamma \subset \mathcal{R}^d$, $\{\tau(D),\ x \in \Gamma, v \in \mathcal{U}\}$ is uniformly integrable.

(Sketch)^2 of Proof:

1. Equivalence of $(1), (2), (3)$:

   $(2) \implies (1)$ free, $(1) \implies (2)$ by Harnack.

   $(2) \implies (3) \implies (1)$ by Khasminskii.

2. $(3) \implies (4) \implies (1)$:

   $(3) \implies (4)$ via the 'HJB equation'.

   $(4) \implies (1)$ by Dynkin's formula.

3. (4) $\Longrightarrow$ (5):

For $R >> 0$,

$$
\begin{aligned}
0 &< \left(\inf_{\|x\|>R} h(x) \right) E\left[\int_0^t I\{\|X(s)\| > R\} ds \right] \\
&\leq E\left[\int_0^t h(X(s)) ds \right] \\
&\leq kt + \mathcal{V}(x).
\end{aligned}
$$

Dividing by t and letting $t \uparrow \infty$, the claim follows.

4. (5) $\Longrightarrow$ (6) $\Longleftrightarrow$ (7) $\Longleftrightarrow$ (8) $\Longrightarrow$ (3):

Since limit points of tight measures are tight, the first claim follows. The equivalences are easy to prove. The last claim follows by an explicit construction of a suitable h.

5. (10) $\Longrightarrow$ (9) obvious.

6. For (9) $\Longrightarrow$ (6), let $v_n \to v_\infty$ in $\mathcal{U}$. Then the corresponding processes converge in law. Using Skorohod construction, we may consider the convergence to be a.s. Then the return times in Khasminskii construction converge a.s., and by uniform integrability, so do the expectations in the Khasminskii representation. Thus stationary distributions depend continuously on v. Continuous image of a compact set is compact. Therefore the set of invariant distributions for $v \in \mathcal{U}$ is compact.

7. For (6) $\Longrightarrow$ (10), let $D_1 = D$ in the Khasminskii construction. By a p.d.e. argument,

$$(x, v) \mapsto E_x \left[\int_0^{\tau(D)} I_{B_R}(X(s)) ds \right]$$

is continuous. By tightness of invariant distributions and the Khasminskii representation,

$$\sup_{\mathcal{U}} \sup_{x \in \partial D_2} E_x \left[\int_0^{\tau(D_1)} I_{B_R^c}(X(s)) ds \right] \overset{R \uparrow \infty}{\to} 0.$$

Thus $(x, v) \mapsto E_x[\tau(D_1)]$ is continuous. The claim follows. $\square$

DEGENERATE PROBLEMS

Main Result:

$\int \mathcal{L} f d\pi = 0 \; \forall f \in C_0^2(\mathcal{R}^d) \Longrightarrow \pi$ is the marginal of a stationary solution $(X(\cdot), u(\cdot))$,

i.e., an ergodic occupation measure.

Corollary: Ergodic control problem $\Longleftrightarrow$

Minimize $\int \bar{k} d\mu$: $\int \mathcal{L} f d\mu = 0 \; \forall f \in C_0^2(\mathcal{R}^d)$

$\Longrightarrow$ existence of optimal controls under near-monotonicity / stability hypotheses.

(Sketch)^N of proof of 'Main Result':

Define

- $\mathcal{L}_n : \mathcal{D}(\mathcal{L}_n) :=$ Range$(I - \frac{1}{n}\mathcal{L}) \mapsto C_b(\mathcal{R}^d \times U)$ by:

  $\mathcal{L}_n g := n[(I - \frac{1}{n}\mathcal{L})^{-1} - I]g \ \forall \ g \in \mathcal{D}(\mathcal{L}_n)$,

- $M := \{F \in C_b(\mathcal{R}^d \times \mathcal{R}^d \times U)$:

  $F(x, y, u) = \sum_{i=1}^m f_i(x) g_i(y, u) + f(y, u)$,

  $f_i \in C_b(\mathcal{R}^d), f \in C_b(\mathcal{R}^d \times U), g_i \in \mathcal{D}(\mathcal{L}_n)\}$,

- for $F \in M$,

$$\Lambda F := \int \left[\sum_{i=1}^{m} f_i(x)[(I - \frac{1}{n}\mathcal{L})^{-1} g_i](x) + f_i(x,u) \right] \pi(dx, du).$$

Can check:

- for $f_n := (I - \frac{1}{n}\mathcal{L})^{-1} f$,

  $\|f_n - f\| \to 0, \ \mathcal{L}_n f_n = \mathcal{L} f, \ \int \mathcal{L}_n f_n d\pi = 0.$

- Λ is well-defined: if F has two different representations, they lead to the same ΛF,

- $|\Lambda F| \leq \|F\|$, $\Lambda \mathbf{1} = 1, \Lambda F \geq 0$ for $F \geq 0$,

- $F(x, y, u) = h(x) \Longrightarrow \Lambda F = \int h d\pi_1$, where $\pi_1(\cdot) := \pi(\cdot, U)$,

- $F(x, y, u) = f(y, u) \Longrightarrow \Lambda F = \int f d\pi$.

$\Longrightarrow$

$\Lambda F = \int F d\nu$ for some $\nu \in \mathcal{P}(\mathcal{R}^d \times \mathcal{R}^d \times U) = \pi_1(dx)\eta(dy, du|x)$.

Furthermore, $\int \eta(A \times B|x)\pi_1(dx) = \pi(A \times B)$.

$\Longrightarrow$ can construct $\mathcal{R}^d \times U$-valued stationary Markov chain $\{(Y_k, Z_k)\}$.

Let $\zeta^n(t), t \geq 0$, be a Poisson process with rate n.

Set $X^n(t) := Y_{\zeta^n(t)}, U^n(t) := W_{Z^n(t)}, t \geq 0$.

$(X^n(\cdot), U^n(\cdot))$ is a stationary solution of the martingale problem corresponding to $\mathcal{L}_n$ with marginal π.

Final step: Let $n \uparrow \infty$, let $(X(\cdot), U(\cdot))$ be a limit point in law.

Fact: $(X(\cdot), U(\cdot))$ the desired stationary solution.

$\Longrightarrow$ the set of such laws is a closed convex set.

The extreme points are *ergodic*.

However: for a fixed initial law, the situation is different!

Define an equivalence class, called the *marginal class*, by:

$$(X(\cdot),U(\cdot)) \approx (X'(\cdot),U'(\cdot)) \Longleftrightarrow (X(t),U(t)),(X'(t),U'(t))$$

agree in law for a.e. t.

Claim: for fixed initial law, extremal marginal classes are singletons which are Markov, albeit possibly time-inhomogeneous.

Sketch of proof:

Use the fact that extreme points of the closed convex set of probability measures on a product space with a given marginal are the measures for which the regular conditional law on the other space is a.s. Dirac.

Suppose Markov property fails at t, then the law of $(X([0,t]), X(t))$ is a mixture of laws of $\{(\delta_{f_{X(t)}(\cdot)}, X(t))\}$.

$\Longrightarrow$ the marginal class of $(X(\cdot), U(\cdot))$ is not extremal.

Corollary: Existence of optimal Markov **OR** ergodic control.

Open issue: both together?

Another open issue: characterize all limit ergodic occupation measures attainable with a given initial law (a la L. C. M. Kallenberg's work on controlled Markov chains)

SINGULAR PERTURBATIONS

Consider the two time-scale system

(non-degenerate)

$$\begin{aligned} dZ_t^\epsilon &= h(Z_t^\epsilon, X_t^\epsilon, U_t)dt + \gamma(Z_t^\epsilon)dB_t, \\ dX_t^\epsilon &= \frac{1}{\epsilon} b(Z_t^\epsilon, X_t^\epsilon, U_t)dt + \frac{1}{\sqrt{\epsilon}}\sigma(Z_t^\epsilon, X_t^\epsilon)dW_t, \end{aligned}$$

as $\epsilon \downarrow 0$. Assume relaxed control $\Longrightarrow$

$$h(z,x,u) = \textstyle\int h'(z,x,y)u(dy),\ b(z,x,u) = \int b'(z,x,y)du(y).$$

Intuition: The fast time-scale sees the slow time-scale as *quasi-static* and the slow time-scale sees the fast time-scale as *quasi-equilibrated*.

This motivates:

- the *associated system*

$$d\tilde{X}_\tau = \bar{b}(z, \tilde{X}_\tau, \tilde{U}_\tau)d\tau + \sigma(z, \tilde{X}_\tau)d\tilde{W}_\tau,$$

- the *averaged system*

$$dZ_t = \bar{h}(Z_t, \mu_t)dt + \gamma(Z_t)d\bar{B}_t,$$

where

$$\bar{h}(z, \nu) := \int h'(z, x, u)\nu(dx, du).$$

Define

$$\widehat{\mathcal{L}}^{\epsilon} f(z,x,u) =$$
$$\langle \nabla^z f(z,x), h'(z,x,u)\rangle + \frac{1}{2}\mathrm{tr}(\gamma(z)\gamma^T(z)\nabla^2 f)(z,x)$$
$$+\frac{1}{\epsilon}\Big(\langle \nabla^x f(z,x), b'(z,x,u)\rangle + \frac{1}{2}\mathrm{tr}(\sigma(z)\sigma^T(z)\nabla^2 f)(z,x)\Big),$$

$$\mathcal{L}_z f(x) =$$
$$\Big(\langle \nabla^x f(z,x), b'(z,x,u)\rangle + \frac{1}{2}\mathrm{tr}(\sigma(z)\sigma^T(z)\nabla^2 f)(z,x)\Big),$$

$$\widetilde{\mathcal{L}}^{\nu} f(z) =$$
$$\langle \nabla f(z), \bar{h}(z,\nu)\rangle + \frac{1}{2}\mathrm{tr}(\gamma(z)\gamma(z)^T\nabla^2 f(z)),$$

and ergodic occupation measures:

$$\mathcal{G}^\epsilon \ := \ \{\mu \in \mathcal{P}(\mathcal{R}^d \times \mathcal{R}^m \times U) : \int \widehat{\mathcal{L}}^\epsilon f d\mu = 0 \ \forall \ f \in C_b^2\},$$

$$\mathcal{G}_z \ := \ \{\mu \in \mathcal{P}(\mathcal{R}^m \times U) : \int \mathcal{L}_z f d\mu = 0 \ \forall \ f \in C_b^2\},$$

$$\begin{aligned} \mathcal{G} \ := \ & \{\mu(dz, dxdu) = \eta(dz)\nu(dxdu|z) \in \mathcal{P}(\mathcal{R}^d \times \mathcal{R}^m \times U) : \\ & \int \tilde{\mathcal{L}}^{\nu(\cdot \cdot|z)} f(z) d\eta(dz) = 0 \ \forall \ f \in C_b^2\}, \end{aligned}$$

corresponding to the full system, the associated system and the averaged system, resp.

Let ergodic occupation measures $\mu^\epsilon(dz, dx, du)$ for the overall system

$\rightarrow \mu(dz, dx, du) = \eta(dz)\nu(dxdu|z)$

along a subsequence as $\epsilon \downarrow 0$.

Take $f(z, x) = f_1(z)f_2(x)$ in

$$\epsilon \int \widehat{\mathcal{L}}^\epsilon f d\mu^\epsilon = 0$$

and let $\epsilon \downarrow 0$ along the subsequence to obtain

$$\int f_1(z) \int \mathcal{L}_z f_2(x, u)\nu(dxdu|z)d\eta(dz) = 0$$

$\Longrightarrow$ for η-a.s. z, $\nu(dxdu|z) \in \mathcal{G}_z$.

Now take $f(z,x) = g(z)$ in

$$\int \widehat{\mathcal{L}}^{\epsilon} f d\mu^{\epsilon} = 0$$

and let $\epsilon \downarrow 0$ along the subsequence to obtain

$$\int \widetilde{\mathcal{L}}^{\nu(\cdot\cdot|z)} f(z) d\eta(dz) = 0$$

$\Longrightarrow \mu \in \mathcal{G}$.

Converse: Under technical conditions, every $\mu \in \mathcal{G}$ is attainable as such a limit.

$\Longrightarrow$ control problem for the (lower dimensional) averaged system well approximates the original control problem.

Small noise limit for stationary distributions

Consider

$$dX_t = b(X_t)dt + \epsilon\sigma(X_t)dW_t.$$

Here, b, σ are smooth and bounded with bounded derivatives, σ non-degenerate.

Let $a(\cdot) := \sigma(\cdot)\sigma(\cdot)^T$.

Assume:

- for some $\alpha > 0, \beta \in (0, 1]$,

$$\limsup_{\|x\|\uparrow\infty} \left[\alpha \sup_x \lambda_{\max}(a(x)^T) + \|x\|^{1-\beta} b(x)^T \left(\frac{x}{\|x\|}\right)\right] < 0.$$

- $b(0) = 0$ and is the globally asymptotically stable equilibrium of

$$\dot{x}(t) = b(x(t)).$$

Let $\eta^\epsilon(dx) = \varphi^\epsilon(x)dx$ denote the stationary distribution. Then it is easy to show that

$$\eta^\epsilon \to \delta_0 \text{ in } \mathcal{P}(\mathcal{R}^d).$$

Furthermore, for

$$\mathcal{L}_\epsilon(\cdot) := \langle b, \nabla(\cdot)\rangle + \frac{1}{2}\mathrm{tr}\left(a\nabla^2(\cdot)\right),$$

we have

$$\int \mathcal{L}_\epsilon f d\eta = 0 \ \forall\ f \in C_b^2(\mathcal{R}^d)$$

$$\implies \mathcal{L}_\epsilon^* \varphi^\epsilon = 0.$$

For $\phi^\epsilon := -\epsilon^2 \log(\varphi^\epsilon)$,

$$\frac{\epsilon^2}{2}\mathrm{tr}(a\nabla^2\phi^\epsilon) + \min_u \left[(\tilde{b}^\epsilon - u)^T \nabla\phi^\epsilon + \frac{1}{2}u^T a^{-1} u\right] - \epsilon^2 c^\epsilon,$$

where,

$$\begin{aligned} \tilde{b}_i^\epsilon &:= -b_i + \epsilon^2 \sum_j \frac{\partial}{\partial x_j} a_{ij}, \\ c^\epsilon &:= \frac{\epsilon^2}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} a_{ij} - \sum_i \frac{\partial}{\partial x_i} b_i. \end{aligned}$$

This is the HJB equation for an ergodic control problem for

$$d\tilde{X}_t = (\tilde{b}^\epsilon(\tilde{X}_t) - u_t)dt + \epsilon\sigma(\tilde{X}_t)dW_t,$$

with cost

$$\limsup_{T\uparrow\infty} \frac{1}{T}\int_0^T E[\frac{1}{2}u_t a(\tilde{X}_t)^{-1}u_t - \epsilon^2 c(\tilde{X}_t)]dt.$$

Fact: $\phi^\epsilon - \phi^\epsilon(0)$ is relatively compact in $C(\mathcal{R}^d)$.

Let $\epsilon \downarrow 0 \Longrightarrow$ a limit point ϕ satisfies (in viscosity sense)

$$\min_u \left[(b-u)^T\nabla\phi + \frac{1}{2}u^T a^{-1} u\right] = 0.$$

This is HJB equation for

$$\dot{y}(t) = -b(y(t)) - u(t), \quad x(0) = x,$$

with cost

$$\frac{1}{2}\int_0^\infty u(t)a(y(t))^{-1}u(t)dt,$$

to be minimized over all $y(\cdot)$ as above satisfying $y(t) \to 0$ as $t \uparrow \infty$. $\phi(x)$ is the corresponding minimum cost.

Thus ϕ is the 'rate function' for concentration of η^ϵ near zero.

More generally, $\dot{x}(t) = b(x(t))$ may have multiple equilibria $\{x_j\}$

$\Longrightarrow$ need to fall back upon the Freidlin-Wentzell device to obtain:

$$\phi(x) = \inf \min_j \left(\int_0^\infty u(t)a(y(t))^{-1}u(t)dt + \phi(x_j) \right),$$

where the infimum is over all trajectories of the above controlled o.d.e. that tend to one of the equilibria as $t \uparrow \infty$.