**Lectures on Probability and Stochastic Processes III**
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# Martingale Problems

by

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This short write-up can be considered as *background material* for the short course on Martingale Problems. It is intended to serve a twofold purpose -

- The uninitiated can familiarise themselves with some of the Definitions and results by going through the write-up before the Lectures.
- It can serve as a ready reference during the lectures.

We also indicate texts/references at the end to which the reader can refer for more details. The list of references is not intended to be complete. This is done with the understanding that those familiar with these concepts will not need them, while for those who are looking at any of the following material for the first time, the given reference(s) can serve as a starting point.

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# 1 Metric Spaces

Throughout the course, our state space will be a complete, separable metric space. We give below the relevant definitions. A more general account can be found *e.g.* in [Rud66] and [Rud76].

**Definitions**

1. Let $E$ be an arbitrary set. Let $d : E \times E \to [0, \infty)$ be a function satisfying

   (a) $d(x, y) = 0$ if and only if $x = y$

   (b) $d(x, y) = d(y, x)$ for all $x, y \in E$

   (c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in E$ (triangle inequality)

   $(E, d)$ is called a metric space with metric $d$.

2. A sequence $\{x_n\} \subset E$ converges to $x \in E$ if $d(x_n, x) \to 0$ as $n \to \infty$.

3. A sequence $\{x_n\} \subset E$ is said to be a Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

4. A metric space $(E, d)$ is complete if every Cauchy sequence in $E$ converges.

5. A subset $F \subset E$ is dense in $E$ if for every $\epsilon > 0$ and any $x \in E$, there exists a $y \in F$ with $d(x, y) < \epsilon$.

6. A metric space $(E, d)$ is separable if it contains a countable dense subset.

7. A subset $F \subset E$ is closed if

$$\{x_n\}_{n\geq 1} \subset F \text{ and } \lim_{n\to\infty} d(x_n, x) = 0 \text{ implies } x \in F.$$

8. A subset $O \subset E$ is open if $O^c$ is closed.

9. A subset $K \subset E$ is compact if and only if every sequence $\{x_n\} \in K$ has a convergent subsequence $\{x_{n_j}\}$ such that $\lim_{j\to\infty} x_{n_j} \in K$.

# 2 General Theory of Processes

A more general account can be found in [IW81].

Let $(E, d)$ be a complete, separable metric space. Let $\mathcal{E}$ denote the Borel $\sigma$-field on $E$, *i.e.* the smallest $\sigma$-field containing all open sets in $E$.

Let $(X_t)_{\{t\geq 0\}}$ be a $E$-valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

DEFINITION **2.1** *The process $X$ is measurable if it is* jointly measurable *in $t$ and $\omega$. i.e. $X : [0, \infty) \times \Omega \to E$ is $\mathcal{E}/\mathcal{B}([0, \infty)) \otimes \mathcal{F}$.*

Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration or an increasing family of sub-$\sigma$-fields of $\mathcal{F}$. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is called a filtered probability space.

DEFINITION **2.2** *A process $X$ is $(\mathcal{F}_t)_{t \geq 0}$ - adapted if $X_t$ is $\mathcal{F}_t$ measurable for each $t \geq 0$.*

DEFINITION **2.3** *A process $X$ is $(\mathcal{F}_t)_{t \geq 0}$ - progressively measurable if the restriction of the process $X$ to $[0, t] \times \Omega$ is $\mathcal{E}/\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ measurable for each $t \geq 0$.*

The law of the process is completely determined by its finite dimensional distributions.

DEFINITION **2.4** *Two processes $X$ and $Y$ (possibly defined on different probability spaces) are said to be versions of each other if they have the same finite dimensional distributions.*

DEFINITION **2.5** *If $X$ and $Y$ are defined on the same probability space,*

- *$Y$ is said to be a modification of $X$ if $\mathbb{P}\{X_t = Y_t\} = 1$ for all $t \geq 0$.*
- *$X$ and $Y$ are said to be indistinguishable if $\mathbb{P}\{X_t = Y_t$ for all $t \geq 0\} = 1$. i.e. $\exists N \in \mathcal{F}$, with $\mathbb{P}(N) = 0$, $N$ being independent of $t$, such that for $\omega \notin N$, $X_t = Y_t$ for all $t \geq 0$.*

## 2.1 Martingales

We will be using several results and ideas from Martingale Theory throughout the lecture series. Here we give the basic definitions. We start with the definition of conditional expectation.

DEFINITION **2.6** *Let $X$ be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}|X| < \infty$. Let $\mathcal{G}$ be a sub $\sigma$-algebra of $\mathcal{F}$. Then there exists an almost surely unique $\mathcal{G}$ measurable random variable $Y$ satisfying*

$$\int_F X d\mathbb{P} = \int_F Y d\mathbb{P} \ \ \forall \ F \in \mathcal{G}.$$

*$Y$ is called the Conditional Expectation of $X$ given $\mathcal{G}$ and is denoted by $\mathbb{E}[X|\mathcal{G}]$.*

For properties and more details see [Wil91].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space.

Definition **2.7** *A real valued process $M$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale if*

1. *$M$ is $(\mathcal{F}_t)_{t\geq 0}$ - adapted*
2. *$\mathbb{E}|M_t| < \infty$ for all $t \geq 0$*
3. *$\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ a.s. for all $0 \leq s \leq t$.*

Definition **2.8** *A real valued process $M$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a submartingale (supermartingale) it satisfies conditions 1 and 2 of Definition 2.7 and the equality in condition 3 of 2.7 is replaced by $\geq$ ($\leq$ respectively).*

We state a the following result.

Theorem **2.1** *Let $M$ be a submartingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$. Then there exists an $\Omega^0 \in \mathcal{F}$ with $\mathbb{P}(\Omega^0) = 1$, and such that for every $\omega \in \Omega^0$*

$$M_{t+} = \lim_{\substack{s\downarrow t \\ s\in\mathbb{Q}}} M_s, \ \ \&\ M_{t-} = \lim_{\substack{s\uparrow t \\ s\in\mathbb{Q}}} M_s$$

*exist for all $t \geq 0$ and $t > 0$ repectively.*

*Further, (defining $M_{t+}(\omega)$ suitably for $\omega \notin \Omega^0$ the process,) $(M_{t+})$ is r.c.l.l. and is also a submartingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$.*

*If, in addition, $M$ is a martingale, then $(M_{t+})$ is a r.c.l.l. modification of $(M_t)$.*

For a proof see *e.g.* [KS91, Proposition I.3.14].

Definition **2.9** *A martingale is square integrable if $\mathbb{E}M_t^2 < \infty$ for all $t \geq 0$.*

Theorem **2.2** *Let $M$ be a square integrable martingale. Then $M^2$ is a submartingale.*

We will also use the following result during the lectures.

THEOREM **2.3** *Let $M$ be a r.c.l.l. square integrable martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$. Then there exists a r.c.l.l. $(\mathcal{F}_t)_{t\geq 0}$ - martingale $(N_t)$ and a $(\mathcal{F}_t)_{t\geq 0}$ - adapted, increasing process $(A_t)$ such that*

$$M_t^2 = N_t + A_t.$$

*If the increasing process $(A_t)$ is continuous with $A_0 \equiv 0$, then the above decomposition is also unique. In such a case $A$ is called the (predictable) quadratic variation process of $M$ and is denoted by $\langle M, M\rangle$.*

There is a similar concept of cross quadratic variation of martingales $M^1$ and $M^2$, denoted by $\langle M^1, M^2\rangle$ and has the property that $M_t^1 M_t^2 - \langle M^1, M^2\rangle_t$ is a martingale.

If $M^1$ and $M^2$ are independent, then $\langle M^1, M^2\rangle \equiv 0$.

(Note: The above two definitions given are not the most general, but will suffice for the purposes of the lectures.)

## 2.2 Some special processes

### 2.2.1 Brownian Motion

A $\mathbb{R}$-valued process $W$ is a Standard Brownian motion if

1. $W_0 \equiv 0, W_t \sim N(0, t)$ for all $t > 0$. (Gaussian)
2. $\mathcal{L}(W_t - W_s) = \mathcal{L}(W_{t-s})$ for all $0 \leq s < t$. (Stationary increments)
3. $(W_t - W_s)$ is independent of $\mathcal{F}_s^W$ for all $0 \leq s < t$. (Independent increments)
4. $(W_t)$ is continuous almost surely.

Using the independent increment property of Brownian motion and the fact that $\mathbb{E}(W_t) = 0$ one can show that $W_t$ and $W_t^2 - t$ are martingales.

A $\mathbb{R}^d$-valued process $W = (W^1, \ldots, W^d)$ is a d-dimensional Standard Brownian motion if $W^1, \ldots, W^d$ are independent (1-dimensional) standard Brownian motions.

As before, it is easy to check that $W_t^i$ and $W_t^i W_t^j - \delta_{ij} t$ are martingales.

If, $W_0 \equiv x \in \mathbb{R}^d$ in the above definition, $W$ is called a Brownian motion starting at $x$.

### 2.2.2 Poisson Process

A $\mathbb{R}$-valued process $N$ is a Poisson Process with intensity $\lambda$ if

1. $N_0 \equiv 0, N_t \sim$ Poisson$(\lambda t)$ for all $t > 0$.
2. $\mathcal{L}(N_t - N_s) = \mathcal{L}(N_{t-s})$ for all $0 \leq s < t$. (Stationary increments)
3. $(N_t - N_s)$ is independent of $\mathcal{F}_s^N$ for all $0 \leq s < t$. (Independent increments)

Definition **2.10** *Let $N_t$ be a Poisson Process with intensity $\lambda$. Define*

$$\tilde{N}_t = N_t - \lambda t \quad \text{for all } t \geq 0.$$

*$\tilde{N}_t$ is called the compensated Poisson process.*

Using the independent increment property of Poisson processes and the fact that $\mathbb{E}(N_t) = \lambda t$ one can show that $\tilde{N}_t$ and $\tilde{N}_t^2 - \lambda t$ are martingales.

# 3 Stochastic Calculus

Let $B$ be a standard Brownian motion. We will use the stochastic integral $\int f dB$ along with some of its properties during the lectures. However, we skip the definition here and direct the reader to [KS91]. We will however state some the properties of the integral.

1. The integral $X_t = \int_0^t f_s dB_s$ is defined for all $(\mathcal{F}_t^B)_{t \geq 0}$ - adapted processes $f$ satisfying
$$\mathbb{E} \int_0^t f_s^2 d\langle B, B \rangle_s = \mathbb{E} \int_0^t f_s^2 ds < \infty.$$
2. $X_t$ is a continuous, square-integrable martingale with $\mathbb{E}X_t^2 = \int_0^t f_s^2 ds$.
3. For any two processes $f$ and $g$ as above, the cross-quadratic variation between the stochastic integrals $\int f dB$ and $\int g dB$ is given by
$$\left\langle \int f dB, \int g dB \right\rangle_t = \int_0^t f_s g_s ds.$$

More generally, if $M$ is a square-integrable martingale, then the stochastic integral $\int f dM$, defined for a suitable class of processes, is a square integrable martingale. Further for any two martingales $M$ and $N$ and processes $f$ and $g$ for which the stochastic integrals are defined

$$\left\langle \int f dM, \int g dN \right\rangle_t = \int_0^t f_s g_s d\langle M, N\rangle_s.$$

## 3.1 Ito's formula

We end this write-up by giving the change of variable formula or Ito's formula for continuous semi-martingales.

Definition **3.11** *A continuous $(\mathcal{F}_t)_{t\geq 0}$ - adapted process $X$ is a $(\mathcal{F}_t)_{t\geq 0}$ - semimartingale if it can be written as*

$$X_t = X_0 + M_t + (A_t^1 - A_t^2)$$

*where $M$ is a $(\mathcal{F}_t)_{t\geq 0}$ - martingale, and $A^i, (i = 1, 2)$ are $(\mathcal{F}_t)_{t\geq 0}$ - adapted increasing processes.*

Theorem **3.4** *Let $X$ be a continuous semimartingale as above and let $f \in C_b^2(\mathbb{R})$. Then*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M, M\rangle_s \\ &= f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)d(A^1 - A^2)_s \\ &\quad + \frac{1}{2}\int_0^t f''(X_s)d\langle M, M\rangle_s \end{aligned} \tag{3.1}$$

Note that the theorem 3.4 implies that $f(X_t)$ is also a semimartingale.

## 3.2 Stochastic Differential Equations

Equation (3.1) is also frequently written in differential form (where the differentials are to be interpreted as (stochastic) integrals) as

$$\begin{aligned} df(X_t) &= f'(X_t)dX_s + \frac{1}{2}f''(X_t)d\langle M, M\rangle_t \\ &= f'(X_t)dM_t + f'(X_t)d(A^1 - A^2)_t + \frac{1}{2}f''(X_t)d\langle M, M\rangle_t. \end{aligned}$$

More generally, an equation of the type

$$dZ_t = a(Z_t)dt + b(Z_t)dB_t$$

is called a Stochastic Differential Equation (SDE) driven by the Brownian motion $B$. This is once again to be interpreted as a Stochastic integral equation. In particular, a process $Z$ is a solution if it satisfies

$$Z_t = Z_0 + \int_0^t a(Z_s)ds + \int b(Z_s)dB_s \quad \text{a.s.}$$

For more on this, see [IW81], [KS91].

# References


[IW81] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes.* North-Holland, Amsterdam, 1981.

[KS91] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1991.

[Rud66] Walter Rudin. *Real and Complex Analysis.* Tata McGraw-Hill Publishing Company Ltd., Bombay-New Delhi, 1966.

[Rud76] Walter Rudin. *Principles of Mathematical Analysis.* McGraw-Hill Book Company, Singapore, third edition, 1976.

[Wil91] David Williams. *Probability with Martingales.* Cambridge University Press, Cambridge, 1991.