Lecture Notes

on

Directed trees and the limit

by

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Directed Trees

1.1 Drainage Network Model

Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$ and $\mathcal{F}$ the σ algebra generated by finite dimensional cylinder sets. Fix $0 < p < 1$. On $(\Omega, \mathcal{F})$ we assign a product probability measure P_p which is defined by its marginals as

$$P_p\{\omega : \omega(u) = 1\} = 1 - P_p\{\omega : \omega(u) = 0\} = p, \text{ for } u \in \mathbb{Z}^d.$$

Let $\{U_{u,v} : u, v \in \mathbb{Z}^d,\ v(d) = u(d) + 1\}$ be i.i.d. uniform $(0,1]$ random variables on some probability space $(\Xi, \mathcal{S}, \mu)$. Here and subsequently we express the co-ordinates of a vector u as $u = (u(1), \ldots, u(d))$.

Consider the product space $(\Omega \times \Xi, \mathcal{F} \times \mathcal{S}, \mathbb{P} := P_p \times \mu)$. For $(\omega, \xi) \in \Omega \times \Xi$ let $\mathcal{V}(= \mathcal{V}(\omega, \xi))$ be the random vertex set defined by

$$\mathcal{V}(\omega, \xi) = \{u \in \mathbb{Z}^d : \omega(u) = 1\}.$$

Note that if $u \in \mathcal{V}(\omega, \xi)$ for some $\xi \in \Xi$ then $u \in \mathcal{V}(\omega, \xi')$ for all $\xi' \in \Xi$ and thus we say that a vertex u is open in a configuration ω if $u \in \mathcal{V}(\omega, \xi)$ for some $\xi \in \Xi$.

For $u \in \mathbb{Z}^d$ let

$$N_u = N_u(\omega, \xi) = \Big\{v \in \mathcal{V}(\omega, \xi) : v(d) = u(d) + 1 \text{ and } \sum_{i=1}^{d} |v(i) - u(i)| = \min\big\{\sum_{i=1}^{d} |w(i) - u(i)| : w \in \mathcal{V}(\omega, \xi), w(d) = u(d) + 1\big\}\Big\}.$$

Note that N_u is non-empty almost surely and that N_u is defined for all u, irrespective of it being open or closed. For $u \in \mathbb{Z}^d$ let

$$M(u) \in N_u(\omega, \xi) \text{ be such that } U_{u,M(u)}(\xi) = \min\{U_{u,v}(\xi) : v \in N_u(\omega, \xi)\}. \quad (1.1)$$

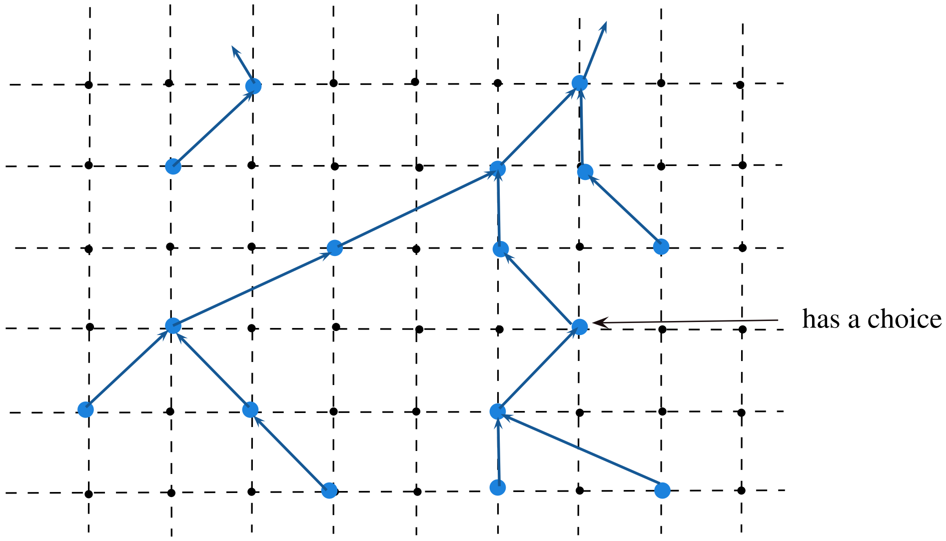

Figure 1.1: The drainage network model

For each $u \in \mathbb{Z}^d$, $M(u)$ is open, almost surely unique and $M(u)(d) = u(d) + 1$. On $\mathcal{V}(\omega, \xi)$ we assign the edge set $\mathcal{E} = \mathcal{E}(\omega, \xi) := \{\langle u, M(u) \rangle : u \in \mathcal{V}(\omega, \xi)\}$.

Consider that graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of the vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. From any vertex $u \in \mathcal{V}(\omega, \xi)$, there is exactly one edge *going up* from u; thus the graph $\mathcal{G}$ contains no loops almost surely. Hence, the graph $\mathcal{G}$ consists of only trees. Then we have

Thoerem 1.1. *For $d = 2$ and $d = 3$, $\mathcal{G}$ consists of one single tree $\mathbb{P}$–almost surely; while for $d \geq 4$, $\mathcal{G}$ is a forest consisting of infinitely many disjoint trees $\mathbb{P}$–almost surely.*

Regarding the geometric structure of the graph $\mathcal{G}$ we have

Thoerem 1.2. *For any $d \geq 2$, the graph $\mathcal{G}$ contains no bi-infinite path $\mathbb{P}$–almost surely.*

1.1.1 Proof of Theorem 1.1

For $\mathbf{u} \in \mathbb{Z}^d$, let us define $M^0(\mathbf{u}) = \mathbf{u}$ and for $n \geq 1$, $M^n(\mathbf{u}) = M(M^{n-1}(\mathbf{u}))$. For any $\mathbf{u} \in \mathbb{Z}^d$, set $\mathbf{u}(1; d-1)$ as the $(d-1)$-dimensional vector having the first $(d-1)$ co-ordinates of $\mathbf{u}$, while $\mathbf{u}(d)$ will represent the d^{th} co-ordinate of $\mathbf{u}$.

For $\mathbf{u} \in \mathbb{Z}^d$, define $X_{\mathbf{u}}(n) = M^n(\mathbf{u})(1; d-1)$ for $n \geq 0$. Now, observe that for $\mathbf{u} \in \mathbb{Z}^d$, $X_{\mathbf{u}}(n)$ has the same distribution as $\mathbf{u}(1; d-1) + \sum_{i=1}^n I_i$, where $I_1, I_2, \ldots$ are i.i.d. copies of $X_{\mathbf{0}}(1)(1; d-1)$ and $\mathbf{0} = (0, 0, \ldots, 0)$ is the origin. Hence $\{X_{\mathbf{u}}(n) : n \geq 0\}$

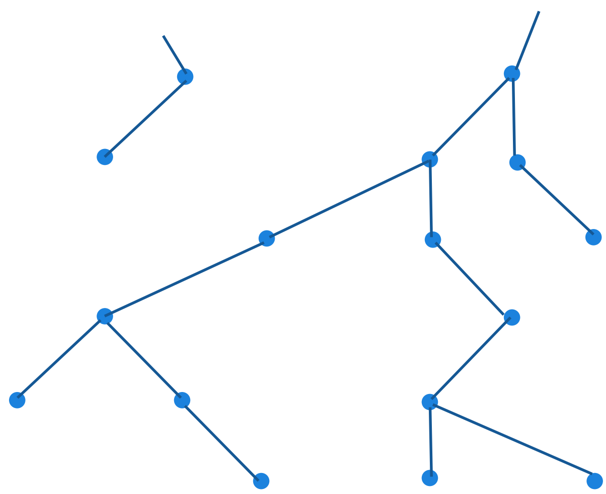

Figure 1.2: The graph of drainage network

is a symmetric random walk starting at $(\mathbf{u}(1), \mathbf{u}(2), \dots, \mathbf{u}(d-1))$, with i.i.d. steps, each step size having distribution $X_{\mathbf{0}}(1)(1; d-1)$. We will refer the step size random variable $I^{(d-1)}$.

For $k \geq 0$ let $\Delta_k := \{\mathrm{v} \in \mathbb{Z}^{d-1} : ||\mathrm{v}||_1 \leq k\}$ denote the $(d-1)$-dimensional diamond of radius k and let $\delta\Delta_k := \{\mathrm{v} \in \mathbb{Z}^{d-1} : ||\mathrm{v}||_1 = k\}$ denote its boundary. The distribution of the step size random variable $I^{(d-1)}$ is given by

$$\mathbb{P}(I^{(d-1)} = \mathrm{v}) = p_{\mathrm{v}} = \begin{cases} p & \text{if } \mathrm{v} = \mathrm{o} \\ \frac{(1-p)^{|\Delta_{k-1}|}(1-(1-p)^{|\delta\Delta_k|})}{|\delta\Delta_k|} & \text{for } \mathrm{v} \in \delta\Delta_k,\ k \geq 1 \end{cases} \tag{1.2}$$

where $\mathrm{o} = (0, 0, \dots, 0)$ is the origin in the $\mathbb{Z}^{d-1}$.

However, for $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$, the corresponding walks (or paths) from $\mathbf{u}(1; d-1)$ and $\mathbf{v}(1; d-1)$ are not independent. Let $u, v \in \mathbb{Z}^{d-1}$, set $\mathbf{u} = (u, 0)$ and $\mathbf{v} = (v, 0)$. Though the random walks $X_{\mathbf{u}}$ and $X_{\mathbf{v}}$ are not independent, they are jointly Markov, i.e., $\{(X_{\mathbf{u}}(n), X_{\mathbf{v}}(n)) : n \geq 1\}$ is a Markov chain taking values in $\mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1}$. Furthermore, setting $Z_n (= Z_n(u, v)) := X_{\mathbf{u}}(n) - X_{\mathbf{v}}(n)$, we observe that $\{Z_n : n \geq 0\}$ is a time-homogeneous Markov chain with state space $\mathcal{S} \subseteq \mathbb{Z}^{d-1}$. This follows on the Markov property of the process $\{(X_{\mathbf{u}}(n), X_{\mathbf{v}}(n)) : n \geq 0\}$ and the spatial invariance of the model.

The connectedness or otherwise of the graph $\mathcal{G}$, is equivalent to whether or not the walks $X_{\mathbf{u}}$ and $X_{\mathbf{v}}$ hit for every pair of point $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$. From the description of the model, it is clear that it is enough to consider points $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}(d) = \mathbf{v}(d) = 0$. Thus we need to show that for points $u, v \in \mathbb{Z}^{d-1}$, the Markov chain Z_n is absorbed at the origin. Again, by spatial invariance of the model, we can take u as the origin in $\mathbb{Z}^{d-1}$. For $d = 2$ and 3 we show that Z_n gets absorbed at the origin with probability 1; while for $d \geq 4$, Z_n is a transient Markov chain and hence has a positive probability of not being absorbed.

In this connection observe that instead of the above Z_n if we had considered a modified Markov chain $\tilde{Z}_n$ where the origin o is no longer an absorbing state, but from the origin we move in one step to some fixed vertex $\mathrm{u} \neq \mathrm{o}$ with probability 1 and the other transition probabilities are kept unchanged, then to show that the original process Z_n is absorbed at o almost surely, it suffices to show that the modified Markov process $\tilde{Z}_n$ is recurrent. A more formal argument for this would require Z_n and $\tilde{Z}_n$ to be coupled together until they hit the origin, which occurs almost surely if the modified process is recurrent. For the case $d = 3$, we will show that $\tilde{Z}_n$ is recurrent. The proof is divided into three subsections according as $d = 2$, $d = 3$ and $d \geq 4$.

Case d = 2

Fix $i < j$ and observe that $X_{(i,0)}(n) \leq X_{(j,0)}(n)$ for every $n \geq 1$. Thus the Markov chain $Z_n := X_{(j,0)}(n) - X_{(i,0)}(n)$ with $Z_0 = j - i$ has as its state space the set of all non-negative integers. Since the marginal distributions of the increments of both $X_{(i,0)}(n)$ and $X_{(j,0)}(n)$ is given by the distribution $I^{(1)}$ (see (1.2)) and hence identical having finite means, $\{Z_n : n \geq 0\}$ is a non-negative martingale. Hence, by the martingale convergence theorem (see Theorem 35.4, Billingsley [1979] pg. 416), Z_n converges almost surely as $n \to \infty$. Since $\{Z_n : n \geq 0\}$ is also a time-homogeneous Markov chain with 0 as the only absorbing state, we must have, $Z_n \to 0$ as $n \to \infty$ with probability 1. Since this is true for all $i < j$, we have the result for $d = 2$.

Case d = 3

Throughout this subsection the letters $\mathbf{u}$, $\mathbf{v}$ in bold font denote vectors in $\mathbb{Z}^3$, u, v in roman font denote vectors in $\mathbb{Z}^2$ and u, v in slanted font denote integers. Fix two vectors $\mathbf{u} := (\mathrm{u}, 0)$ and $\mathbf{v} := (\mathrm{v}, 0)$ in $\mathbb{Z}^2 \times \{0\}$ and let $\tilde{Z}_n (= \tilde{Z}_n(\mathrm{u}, \mathrm{v}))$ be the time-homogeneous Markov chain with state space $\mathbb{Z}^2$ as defined at the beginning of this section. We shall exhibit, by a Lyapunov function technique, that this Markov chain $\tilde{Z}_n$ is recurrent, thereby showing that Z_n is absorbed at the origin with probability 1.

Consider the function $f : \mathbb{R}^2 \to [0, \infty)$ defined by $f(\mathrm{x}) := \sqrt{\log(1 + \|\mathrm{x}\|_2^2)}$ where $\|.\|_2$ is the standard L_2 norm (Euclidean distance). Since $f(\mathrm{x}) \to \infty$ as $\|\mathrm{x}\|_2 \to \infty$, by Foster's criterion (see Asmussen [1987] Proposition 5.3 of Chapter I, pg. 18) the following lemma implies that $\tilde{Z}_n$ is recurrent.

Lemma 1.1. *For all $n \geq 0$, there exists $n_0 \geq 0$ such that, for all $\|\mathrm{x}\|_2 \geq n_0$, we have*

$$\mathbb{E}\big(f(\tilde{Z}_{n+1}) - f(\tilde{Z}_n) | \tilde{Z}_n = \mathrm{x}\big) < 0.$$

Proof : Let $g : [0, \infty) \to [0, \infty)$ be defined as $g(x) := \sqrt{\log(1 + x)}$. Clearly $g(x) \geq 0$ for all $x \geq 0$ and $g(x) \to \infty$ as $x \to \infty$. Also, for $x, y \geq 0$, the Taylor series expansion yields

$$g(x) - g(y) \leq (x - y)g^{(1)}(y) + \frac{(x - y)^2}{2} g^{(2)}(y) + \frac{(x - y)^3}{6} g^{(3)}(y), \tag{1.3}$$

which holds because the fourth derivative

$$g^{(4)}(s) = -\frac{3}{(1 + s)^4 g(s)} - \frac{11}{4(1 + s)^4 (g(s))^3} - \frac{18}{8(1 + s)^4 (g(s))^5} - \frac{15}{16(1 + s)^4 (g(s))^7} < 0$$

for $s > 0$. The first three derivatives of g are

$$\begin{aligned}
g^{(1)}(s) &= \frac{1}{2(1 + s)g(s)} \\
g^{(2)}(s) &= -\frac{1}{2(1 + s)^2 g(s)} - \frac{1}{4(1 + s)^2 (g(s))^3} \\
g^{(3)}(s) &= \frac{1}{(1 + u)^3 g(s)} + \frac{3}{4(1 + s)^3 (g(s))^3} + \frac{3}{8(1 + s)^3 (g(s))^5}.
\end{aligned}$$

Note that, for all s large,

$$g^{(3)}(s) \leq \frac{3}{(1 + s)^3 g(s)}.$$

Assuming for the moment that (we will prove this shortly) for some $\alpha > 0$

$$\mathbb{E}\Big(\|\tilde{Z}_{n+1}\|_2^2 - \|\tilde{Z}_n\|_2^2 \big| \tilde{Z}_n = \mathrm{x}\Big) = \alpha + o(\|\mathrm{x}\|_2^{-2}) \tag{1.4}$$

$$\mathbb{E}\Big(\big(\|\tilde{Z}_{n+1}\|_2^2 - \|\tilde{Z}_n\|_2^2\big)^2 \big| \tilde{Z}_n = \mathrm{x}\Big) \geq 2\alpha \|\mathrm{x}\|_2^2 \tag{1.5}$$

$$\mathbb{E}\Big(\big(\|\tilde{Z}_{n+1}\|_2^2 - \|\tilde{Z}_n\|_2^2\big)^3 \big| \tilde{Z}_n = \mathrm{x}\Big) = O(\|\mathrm{x}\|_2^2) \tag{1.6}$$

as $\|\mathrm{x}\|_2 \to \infty$; and using the above estimates and expression for derivatives we have, for all $\beta := \|\mathrm{x}\|_2^2$ large and for some non-negative constants C_1 and C_2,

$$\begin{aligned}
&\mathbb{E}\big(f(\tilde{Z}_{n+1}) - f(\tilde{Z}_n) \mid \tilde{Z}_n = \mathrm{x}\big) \\
&\leq \frac{\alpha + C_1/\beta}{2(1+\beta)\sqrt{\log(1+\beta)}} - \frac{2\alpha\beta}{4(1+\beta)^2\sqrt{\log(1+\beta)}} \\
&- \frac{2\alpha\beta}{8(1+\beta)^2\sqrt{\big(\log(1+\beta)\big)^3}} + \frac{3C_2\beta}{(1+\beta)^3\sqrt{\log(1+\beta)}} \\
&= \frac{1}{8(1+\beta)^2\sqrt{\log(1+\beta)}}\Big[4\alpha + 4C_1 + 4C_1/\beta + 24C_2\beta/(1+\beta) \\
&\qquad - 2\alpha\beta/\log(1+\beta)\Big].
\end{aligned}$$

The term inside the square braces tends to $-\infty$ as $\beta \to \infty$; therefore, for all sufficiently large β, the term is negative. Thus to complete the proof of the lemma we need to show (1.4), (1.5) and (1.6).

Since $\mathbb{P}(||I^{(2)}||_1 > k)$ has an super exponential decay (in k), all moments of $I^{(2)}$ exist. For any $k \geq 1$ and $i, j \geq 0$, define

$$\begin{aligned}
m_i &:= \sum_{\mathrm{u}:=(u_1,u_2)\in\mathbb{Z}^2} u_1^i\, p_{\mathrm{u}} \text{ and } m_{i,j} := \sum_{\mathrm{u}:=(u_1,u_2)\in\mathbb{Z}^2} u_1^i u_2^j\, p_{\mathrm{u}} \\
m_i(k) &:= \sum_{\mathrm{u}:=(u_1,u_2)\in D_k} u_1^i\, p_{\mathrm{u}} \text{ and } m_{i,j}(k) := \sum_{\mathrm{u}:=(u_1,u_2)\in D_k} u_1^i u_2^j\, p_{\mathrm{u}}
\end{aligned}$$

where $p_{\mathrm{u}} = \mathbb{P}(I^{(2)} = \mathrm{u})$. Since $(-u_1, -u_2), (-u_1, u_2)$ and $(u_1, -u_2)$ are in D_k whenever $(u_1, u_2) \in D_k$ with $\mathbb{P}(I^{(2)} = (-u_1, -u_2)) = \mathbb{P}(I^{(2)} = (-u_1, u_2)) = \mathbb{P}(I^{(2)} = (u_1, -u_2)) = \mathbb{P}(I^{(2)} = (u_1, u_2))$, it is clear that, for every $k \geq 1$ we have

$$m_i = m_i(k) = 0 \text{ for all odd } i, \text{ and } m_{i,j} = m_{i,j}(k) = 0 \text{ whenever either } i \text{ or } j \text{ is odd.} \tag{1.7}$$

By the exponential decay of the tail, we have, when both i and j are even, $m_i(k) \to m_i$ and $m_{i,j}(k) \to m_{i,j}$ as $k \to \infty$. Moreover, $k^2(m_2 - m_2(k)) \leq k^2 \sum_{u:(u_1,u_2)\notin D_k} u_1^2 p_u \leq \sum_{j=k+1}^{\infty} j^4(1-p)^{1+2j(j-1)} \to 0$ as $k \to \infty$ since the sum $\sum_{j=1}^{\infty} j^4(1-p)^{1+2j(j-1)} < \infty$. A similar result holds for $m_0(k)$ and so we have

$$m_2(k) = m_2 + o(k^{-2}) \text{ and } m_0(k) = m_0 + o(k^{-2}) \text{ as } k \to \infty. \tag{1.8}$$

Now we proceed to compute the expectations.

$$\begin{aligned}
&\mathbb{E}\left(\|\tilde{Z}_{n+1}\|_2^2 - \|\tilde{Z}_n\|_2^2 \big| \tilde{Z}_n = \mathrm{x}\right) \\
&= \sum_{\mathrm{a},\mathrm{b}\in\mathbb{Z}^2} (\|\mathrm{x}+\mathrm{a}-\mathrm{b}\|_2^2 - \|\mathrm{x}\|_2^2)\mathbb{P}\{X_{\mathrm{x}}(1) = \mathrm{x}+\mathrm{a}, X_{\mathrm{o}}(1) = \mathrm{b}\big|X_{\mathrm{o}}(0) = \mathrm{o}, X_{\mathrm{x}}(0) = \mathrm{x}\},
\end{aligned} \tag{1.9}$$

where we have used the spatial invariance of the model.

To calculate the above sum we let $k := \|\mathrm{x}\|_2/4$. Note for $\mathrm{a}, \mathrm{b} \in D_k$ we have $\mathbb{P}\{X_{\mathrm{x}}(1) = \mathrm{x}+\mathrm{a}, X_{\mathrm{o}}(1) = \mathrm{b}\big|X_{\mathrm{o}}(0) = \mathrm{o}, X_{\mathrm{x}}(0) = \mathrm{x}\} = p_{\mathrm{a}}p_{\mathrm{b}}$, thus, using (1.7) and (1.8),

$$\begin{aligned}
T_1(1) &:= \sum_{\mathrm{a},\mathrm{b}\in D_k} (\|\mathrm{x}+\mathrm{a}-\mathrm{b}\|_2^2 - \|\mathrm{x}\|_2^2)\mathbb{P}\{X_{\mathrm{x}}(1) = \mathrm{x}+\mathrm{a}, X_{\mathrm{o}}(1) = \mathrm{b}\big|X_{\mathrm{o}}(0) = \mathrm{o}, X_{\mathrm{x}}(0) = \mathrm{x}\} \\
&= \sum_{\mathrm{a},\mathrm{b}\in D_k} \left[(a_1-b_1)^2 + 2x_1(a_1-b_1) + (a_2-b_2)^2 + 2x_2(a_2-b_2)\right] p_{\mathrm{a}}p_{\mathrm{b}} \\
&= 4m_2(k)m_0(k) = 4m_2 + o(k^{-2}) \text{ as } k \to \infty.
\end{aligned} \tag{1.10}$$

Also, if $\mathrm{b} \notin D_k$ then, taking $\|\mathrm{b}\|_1 = k + l$ for some $l \geq 1$, the occurrence of the event $\{X_{\mathrm{o}}(1) = \mathrm{b}\}$ requires that all the vertices in the diamond D_{k+l-1} be closed and that at least one vertex of δD_{k+l} be open — an event which occurs with probability $(1-p)^{1+2(k+l-1)(k+l)} - (1-p)^{1+2(k+l)(k+l+1)}$. Moreover, if $\{X_{\mathrm{o}}(1) = \mathrm{b}\}$ occurs then $X_{\mathrm{x}}(1)$ must lie in the smallest diamond centred at x which contains the vertex b, thus $\|X_{\mathrm{x}}(1) - X_{\mathrm{o}}(1)\|_2 \leq \|X_{\mathrm{x}}(1)\|_1 + \|X_{\mathrm{o}}(1)\|_1 \leq (\|\mathrm{x}\|_1 + \|\mathrm{b}\|_1) + \|\mathrm{b}\|_1 = 6k + 2l$. Now noting that there are $4(k+l)$ vertices on δD_{k+l} and that an argument similar to the above may be given when $\mathrm{a} \notin D_k$, we have

$$\begin{aligned}
&T_2(1) \\
&:= \sum_{\mathrm{a}\notin D_k \text{ or } \mathrm{b}\notin D_k} (\|\mathrm{x}+\mathrm{a}-\mathrm{b}\|_2^2 - \|\mathrm{x}\|_2^2)\mathbb{P}\{X_{\mathrm{x}}(1) = \mathrm{x}+\mathrm{a}, X_{\mathrm{o}}(1) = \mathrm{b}\big|X_{\mathrm{o}}(0) = \mathrm{o}, X_{\mathrm{x}}(0) = \mathrm{x}\} \\
&\leq 2\sum_{l\geq 1} 4(k+l)\left((6k+2l)^2 + (4k)^2\right)(1-p)^{1+2(k+l-1)(k+l)}\left[1-(1-p)^{4(k+l)}\right] \\
&= o(k^{-2}) \text{ as } k \to \infty.
\end{aligned} \tag{1.11}$$

This establishes (1.4) with $\alpha = 4m_2$.

Similar calculations establish (1.6) and completes the proof of Lemma 1.1. ∎

Case $\mathbf{d} \geq 4$

For notational simplicity we present the proof only for $d = 4$. Throughout this subsection the letters $\mathbf{u}$, $\mathbf{v}$ in bold font denote vectors in $\mathbb{Z}^4$, u, v in roman font denote vectors in $\mathbb{Z}^3$ and u, v in slanted font denote integers. We first show that on $\mathbb{Z}^4$ the graph $\mathcal{G}$ admits two distinct trees with positive probability, i.e.,

$$\mathbb{P}\{\mathcal{G} \text{ is disconnected}\} > 0. \tag{1.12}$$

For $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^4$, we have already noted that the random walks $X_{\mathbf{u}}$ and $X_{\mathbf{v}}$ are not independent and so, to obtain our theorem, we cannot use the fact that with positive probability two independent random walks on $\mathbb{Z}^3$ do not intersect. Nonetheless, if $\mathbf{u}$ and $\mathbf{v}$ are sufficiently far apart their dependence on each other is weak. In the remainder of this section we formalize this notion of weak dependence by coupling two independent random walks and the processes $\{X_{\mathbf{u}}(n), X_{\mathbf{v}}(n) : n \geq 0\}$ and obtain the desired result.

For $\mathbf{v} = (\mathrm{v}, 0)$, given $\epsilon > 0$ define the event

$$\begin{aligned} A_{n,\epsilon}(\mathbf{v}) := \{X_{\mathbf{v}}(n^4) \in X_{\mathbf{0}}(n^4) + (\Delta_{n^{2(1+\epsilon)}} \setminus \Delta_{n^{2(1-\epsilon)}}), \\ X_{\mathbf{v}}(i) \neq X_{\mathbf{0}}(i) \text{ for all } i = 1, \dots, n^4\}, \end{aligned} \tag{1.13}$$

where $\mathbf{0} := (0, 0, 0, 0)$.

Lemma 1.2. *For $0 < \epsilon < 1/3$ there exist constants $C, \beta > 0$ and $n_0 \geq 1$ such that, for all $n \geq n_0$,*

$$\inf_{\mathrm{v} \in \Delta_{n^{1+\epsilon}} \setminus \Delta_{n^{1-\epsilon}}} \mathbb{P}(A_{n,\epsilon}(\mathbf{v})) \geq 1 - Cn^{-\beta}.$$

Assuming the above lemma we proceed to complete the proof of (1.12). We shall return to the proof of the lemma later.

For $i \geq 1$ and $n \geq n_0$, let $\tau_i (= \tau_i(n)) := 1 + n^4 + (n^4)^2 + \cdots + (n^4)^{2^{i-1}}$ and take $\tau_0 = 1$. For fixed $\mathbf{v}$, we define

$$B_0 = B_0(\mathbf{v}) := \{X_{\mathbf{v}}(1) \in X_{\mathbf{0}}(1) + (\Delta_{n^{1+\epsilon}} \setminus \Delta_{n^{1-\epsilon}})\},$$

and having defined $B_0, \dots, B_{i-1}$ we define

$$\begin{aligned} B_i = B_i(\mathbf{v}) := \{X_{\mathbf{v}}(\tau_i) \in X_{\mathbf{0}}(\tau_i) + (\Delta_{n^{2^i(1+\epsilon)}} \setminus \Delta_{n^{2^i(1-\epsilon)}}) \text{ and} \\ X_{\mathbf{v}}(j) \neq X_{\mathbf{0}}(j) \text{ for all } \tau_{i-1} + 1 \leq j \leq \tau_i\}. \end{aligned}$$

Clearly,

$$\begin{aligned}
&\mathbb{P}\{M^j(\mathbf{v}) \neq M^j(\mathbf{0}) \text{ for all } j \geq 1\}\\
&= \mathbb{P}\{X_{\mathbf{v}}(j) \neq X_{\mathbf{0}}(j) \text{ for all } j \geq 1\} \geq \mathbb{P}(\cap_{i=0}^{\infty} B_i)\\
&= \lim_{i\to\infty} \mathbb{P}(\cap_{j=0}^{i} B_j) = \lim_{i\to\infty} \prod_{l=1}^{i} \mathbb{P}(B_l | \cap_{j=0}^{l-1} B_j)\mathbb{P}(B_0).
\end{aligned} \tag{1.14}$$

Since $\mathbb{P}(B_0) > 0$, from (1.14) we have that $\mathbb{P}(M^j(\mathbf{v}) \neq M^j(\mathbf{0})$ for all $j \geq 1) > 0$ if $\sum_{l=1}^{\infty} 1 - \mathbb{P}(B_l | \cap_{j=0}^{l-1} B_j) < \infty$.

For fixed $l \geq 1$, let $\mathrm{u}_1 := X_{\mathbf{0}}(\tau_l)$ and $\mathrm{v}_1 := X_{\mathbf{v}}(\tau_l)$. Set $\mathbf{u}_1 = (\mathrm{u}_1, 0)$ and $\mathbf{v}_1 = (\mathrm{v}_1, 0)$. For $(\omega, \xi) \in B_l(\mathbf{v})$, since $\mathrm{v}_1 \in \mathrm{u}_1 + (\Delta_{n^{2^l(1+\epsilon)}} \setminus \Delta_{n^{2^l(1-\epsilon)}})$. Since $\{(X_{\mathbf{0}}(n), X_{\mathbf{v}}(n)) : n \geq 0\}$ is a Markov process, we have

$$\begin{aligned}
&\mathbb{P}\big(B_{l+1} | \cap_{j=0}^{l} B_j\big) = \mathbb{P}\Big\{X_{\mathbf{v}}(\tau_{l+1}) \in X_{\mathbf{0}}(\tau_{l+1}) + (\Delta_{n^{2^i(1+\epsilon)}} \setminus \Delta_{n^{2^i(1-\epsilon)}})\\
&\qquad \text{and } X_{\mathbf{v}}(j) \neq X_{\mathbf{0}}(j) \text{ for all } \tau_l + 1 \leq j \leq \tau_{l+1} \Big| \cap_{j=0}^{l} B_j\Big\}\\
&= \mathbb{P}\Big\{X_{\mathbf{v}}(\tau_{l+1}) \in X_{\mathbf{0}}(\tau_l) + (\Delta_{n^{2^i(1+\epsilon)}} \setminus \Delta_{n^{2^i(1-\epsilon)}}) \text{ and } X_{\mathbf{v}}(j) \neq X_{\mathbf{0}}(j)\\
&\qquad \text{for all } \tau_l + 1 \leq j \leq \tau_{l+1} \Big| X_{\mathbf{v}}(\tau_l) \in X_{\mathbf{0}}(\tau_l) + (\Delta_{n^{2^l(1+\epsilon)}} \setminus \Delta_{n^{2^l(1-\epsilon)}})\Big\}\\
&= \mathbb{P}\Big\{X_{\mathbf{v}_1}\big((n^4)^{2^l}\big) \in X_{\mathbf{u}_1}\big((n^4)^{2^l}\big) + (\Delta_{n^{2^{l+1}(1+\epsilon)}} \setminus \Delta_{n^{2^{l+1}(1-\epsilon)}}) \text{ and } X_{\mathbf{v}_1}(j) \neq X_{\mathbf{u}_1}(j)\\
&\qquad \text{for all } 1 \leq j \leq (n^4)^{2^l} \Big| X_{\mathbf{v}_1}(0) \in X_{\mathbf{u}_1}(0) + (\Delta_{n^{2^l(1+\epsilon)}} \setminus \Delta_{n^{2^l(1-\epsilon)}})\Big\}\\
&= \mathbb{P}\Big\{X_{\mathbf{v}_1}\big((n^4)^{2^l}\big) \in X_{\mathbf{0}}\big((n^4)^{2^l}\big) + (\Delta_{n^{2^{l+1}(1+\epsilon)}} \setminus \Delta_{n^{2^{l+1}(1-\epsilon)}}) \text{ and } X_{\mathbf{v}_1}(j) \neq X_{\mathbf{0}}(j)\\
&\qquad \text{for all } 1 \leq j \leq (n^4)^{2^l} \Big| X_{\mathbf{v}_1}(0) \in X_{\mathbf{0}}(0) + (\Delta_{n^{2^l(1+\epsilon)}} \setminus \Delta_{n^{2^l(1-\epsilon)}})\Big\}\\
&\geq \inf_{\mathrm{v}_2 \in (\Delta_{n^{2^l(1+\epsilon)}} \setminus \Delta_{n^{2^l(1-\epsilon)}})} \mathbb{P}\big(A_{n^{2^l},\epsilon}(\mathrm{v}_2, 0)\big) \geq 1 - C(n^{2^l})^{-\beta}.
\end{aligned} \tag{1.15}$$

Thus $\sum_{l=1}^{\infty} \big(1 - \mathbb{P}(B_l | \cap_{j=0}^{l-1} B_j)\big) \leq C \sum_{l=1}^{\infty} (n^{2^l})^{-\beta} < \infty$; thereby completing the proof of (1.12).

To prove Lemma 1.2, we have to compare the trees $\{M^n(\mathbf{0})\}$ and $\{M^n(\mathbf{v})\}$ and independent "random walks" $\{\mathbf{0} + (\sum_{i=1}^{n} I_1^{(3)}(i), n)\}$ and $\{\mathbf{v} + (\sum_{i=1}^{n} I_2^{(3)}(i), n)\}$ where $\{I_1^{(3)}(i) : i \geq 1\}$ and $\{I_2^{(3)}(i) : i \geq 1\}$ are independent collections of i.i.d. copies of the random variable $I^{(3)}$ given in (1.2).

We now describe a method to couple the trees and the independent random walks. Before embarking on the formal details of the coupling procedure we present the main idea.

From a vertex $\mathbf{0}$ we construct the 'path' $\{\mathbf{0} + (\sum_{i=1}^n I_1^{(3)}(i), n)\}$. Now consider the vertex $\mathbf{v}$ with $\mathbf{v} = (v_1, v_2, v_3, 0)$. In case the diamond $D := \{\mathrm{u} \in \mathbb{Z}^3 : ||\mathrm{u}||_1 \leq ||I_1^{(3)}(1)||_1\}$ is disjoint from the diamond $D' := \{\mathrm{u} \in \mathbb{Z}^3 : ||\mathrm{u} - (v_1, v_2, v_3)||_1 \leq ||I_2^{(3)}(i)||_1\}$ then we take $M^1(\mathbf{v}) = \{\mathbf{v} + (I_1^{(3)}(i), 1)\}$. While if the two diamonds are not disjoint, then we have to define $M^1(\mathbf{v})$ taking into account the configuration inside the diamond D. Similarly, we may obtain $M^2(\mathbf{v})$ by considering the diamonds $\{\mathrm{u} \in \mathbb{Z}^3 : ||\mathrm{u} - I_1^{(3)}(i)||_1 \leq ||I_2^{(3)}(i)||_1\}$ and $\{\mathrm{u} \in \mathbb{Z}^3 : ||\mathrm{u} - M^1(\mathbf{v})||_1 \leq ||I_2^{(3)}(2)||_1\}$. Note that if, for each $i = 1, \dots, n$ the two diamonds involved at the ith stage are disjoint, then the growth of the tree $\{(M^i(\mathbf{0}), M^i(\mathbf{v})) : 0 \leq i \leq n\}$ is stochastically equivalent to that of the pair of independent 'random walks' $(\mathbf{0} + (\sum_{i=1}^n I_1^{(3)}(i), n), \mathbf{v} + (\sum_{i=1}^n I_2^{(3)}(i), n))$.

We start with two vertices $\mathbf{u} := (\mathrm{u}, 0)$ and $\mathbf{v} := (\mathrm{v}, 0)$ in $\mathbb{Z}^4$ with $\mathrm{u}, \mathrm{v} \in \mathbb{Z}^3$. Let $\{U_1^{\mathrm{u}}(\mathrm{z}) : \mathrm{z} \in \mathbb{Z}^3\}, \{U_2^{\mathrm{u}}(\mathrm{z}) : \mathrm{z} \in \mathbb{Z}^3\}$ and $\{U_1^{\mathrm{v}}(\mathrm{z}) : \mathrm{z} \in \mathbb{Z}^3\}, \{U_2^{\mathrm{v}}(\mathrm{z}) : \mathrm{z} \in \mathbb{Z}^3\}$ be four independent collections of i.i.d. random variables, each of these random variables being uniformly distributed on $[0, 1]$.

Let k_{u} and l_{v} be defined as

$$k_{\mathrm{u}} := \min\{k : U_1^{\mathrm{u}}(\mathrm{z}) < p \text{ for some } \mathrm{z} \in (\mathrm{u} + \Delta_k)\}$$
$$l_{\mathrm{v}} := \min\{l : U_1^{\mathrm{v}}(\mathrm{z}) < p \text{ for some } \mathrm{z} \in (\mathrm{v} + \Delta_l)\}.$$

Now define m_{v} as

$$m_{\mathrm{v}} := \min\{m : \text{ either } U_1^{\mathrm{v}}(\mathrm{z}) < p \text{ for some } \mathrm{z} \in (\mathrm{v} + \Delta_m) \setminus (\mathrm{u} + \Delta_{k_{\mathrm{u}}}) \text{ or } U_1^{\mathrm{u}}(\mathrm{z}) < p \text{ for some } \mathrm{z} \in (\mathrm{v} + \Delta_m) \cap (\mathrm{u} + \Delta_{k_{\mathrm{u}}})\}.$$

Also, define the sets

$$N_{\mathrm{u}} := \{\mathrm{z} \in (\mathrm{u} + \Delta_{k_{\mathrm{u}}}) : U_1^{\mathrm{u}}(\mathrm{z}) < p\}$$
$$N_{\mathrm{v}}^1 := \{\mathrm{z} \in (\mathrm{v} + \Delta_{l_{\mathrm{v}}}) : U_1^{\mathrm{v}}(\mathrm{z}) < p\}$$
$$N_{\mathrm{v}}^2 := \{\mathrm{z} \in (\mathrm{v} + \Delta_{m_{\mathrm{v}}}) \setminus (\mathrm{u} + \Delta_{k_{\mathrm{u}}}) : U_1^{\mathrm{v}}(\mathrm{z}) < p\} \cup \{\mathrm{z} \in (\mathrm{v} + \Delta_{m_{\mathrm{v}}}) \cap (\mathrm{u} + \Delta_{k_{\mathrm{u}}}) : U_1^{\mathrm{u}}(\mathrm{z}) < p\}.$$

We pick

(a) $\phi(\mathrm{u}) \in N_{\mathrm{u}}$ such that $U_2^{\mathrm{u}}(\phi(\mathrm{u})) = \min\{U_2^{\mathrm{u}}(\mathrm{z}) : \mathrm{z} \in N_{\mathrm{u}}\}$;

(b) $\zeta(\mathrm{v}) \in N_{\mathrm{v}}^1$ such that $U_2^{\mathrm{v}}(\zeta(\mathrm{v})) = \min\{U_2^{\mathrm{v}}(\mathrm{z}) : \mathrm{z} \in N_{\mathrm{v}}^1\}$;

(c) $\psi(\mathrm{v}) \in N^2_{\mathrm{v}}$ such that $U^{\mathrm{v}}_2(\psi(\mathrm{v})) = \min\{U^{\mathrm{v}}_2(\mathrm{z}) : \mathrm{z} \in N^2_{\mathrm{v}}\}$.

Taking $\phi^0(\mathrm{u}) = \mathrm{u}$, $\phi^n(\mathrm{u}) = \phi(\phi^{n-1}(\mathrm{u}))$, and similarly for $\zeta^n(\mathrm{v})$ and $\psi^n(\mathrm{v})$, we note that the distribution of $\Big\{\big((\phi^n(\mathrm{u}), n), (\zeta^n(\mathrm{v}), n)\big) : n \geq 0\Big\}$ is the same as that of $\Big\{\big((\mathrm{u} + \sum_{i=1}^n I^{(3)}_1(i), n),\ (\mathrm{v} + \sum_{i=1}^n I^{(3)}_2(i), n)\big) \colon n \geq 0\Big\}$, i.e. two independent "random walks" one starting from $(\mathrm{u}, 0)$ and the other starting from $(\mathrm{v}, 0)$. Also the distribution of $\{(M^n(\mathrm{u}, 0), M^n(\mathrm{v}, 0)) : n \geq 0\}$ and that of $\{((\phi^n(\mathrm{u}), n), (\psi^n(\mathrm{v}), n)) : n \geq 0\}$ are identical. Thus, the procedure described above may be used to construct the trees from $(\mathrm{u}, 0)$ and $(\mathrm{v}, 0)$.

Now observe that $\{(\phi^n(\mathrm{u}), n)\}$ describes both the random walk and the tree starting from $(\mathrm{u}, 0)$. Also if $\Delta_{k_{\mathrm{u}}} \cap \Delta_{m_{\mathrm{v}}} = \emptyset$, then $m_{\mathrm{v}} = l_{\mathrm{v}}$ and, more importantly, $\zeta(\mathrm{v}) = \psi(\mathrm{v})$. Hence the 'random walk' and the tree from $(\mathrm{u}, 0)$ are coupled and so are the 'random walk' and the tree from $(\mathrm{v}, 0)$. In particular, this happens when both $k_{\mathrm{u}} < [||\mathrm{u} - \mathrm{v}||_1/2]$ and $m_{\mathrm{v}} < [||\mathrm{u} - \mathrm{v}||_1/2]$. Let $k_0 = ||\mathrm{u} - \mathrm{v}||_1/2$. From the above discussion we have

$$\begin{aligned}
&\mathbb{P}\big\{\zeta(\mathrm{v}) \neq \psi(\mathrm{v})\big\} \\
&\leq \mathbb{P}\Big[\big\{(U^{\mathrm{u}}_1(\mathrm{z})) > p \text{ for all } \mathrm{z} \in (\mathrm{u} + \Delta_{k_0})\big\} \bigcup \big\{(U^{\mathrm{v}}_1(\mathrm{z})) > p \text{ for all } \mathrm{z} \in (\mathrm{v} + \Delta_{k_0})\big\}\Big] \\
&\leq 2\mathbb{P}\big\{(U^{\mathrm{u}}_1(\mathrm{z})) > p \text{ for all } \mathrm{z} \in (\mathrm{u} + \Delta_{k_0})\big\} \leq 2(1-p)^{\#\Delta_{k_0}}.
\end{aligned}$$

Since $(1/2)k^3 \leq \#\Delta_k \leq 2k^3$, the above inequality gives

$$\mathbb{P}\big\{\zeta(\mathrm{v}) = \psi(\mathrm{v})\big\} \geq 1 - C_1 \exp(-C_2||\mathrm{u} - \mathrm{v}||^3_1) \tag{1.16}$$

for constants $C_1 = 2$ and $C_2 = (1/2)|\log(1-p)|$.

With the above estimate at hand, we look at the process $\{(\phi^n(\mathrm{u}), \zeta^n(\mathrm{v})) : n \geq 0\}$. Without loss of generality we take $\mathrm{u} = \mathrm{o}$. For $\epsilon > 0$ and constant $K > 0$ (to be specified later) define

$$\begin{aligned}
B_{n,\epsilon}(\mathrm{v}) := \{&\zeta^{n^4}(\mathrm{v}) \in \phi^{n^4}(\mathrm{o}) + (\Delta_{n^{2(1+\epsilon)}} \setminus \Delta_{n^{2(1-\epsilon)}}), \\
&||\zeta^i(\mathrm{v}) - \phi^i(\mathrm{o})||_1 \geq K \log n \text{ for all } i = 1, \ldots, n^4\}.
\end{aligned} \tag{1.17}$$

This event is an independent random walk version of the event $A_{n,\epsilon}(\mathrm{v}, 0)$ defined in (1.13), except that here we require that the two random walks come no closer than $K \log n$ at any stage.

We will show that there exists $\alpha > 0$ such that

$$\sup_{\mathrm{v} \in (\Delta_{n^{(1+\epsilon)}} \setminus \Delta_{n^{(1-\epsilon)}})} \mathbb{P}\big((B_{n,\epsilon}(\mathrm{v}))^c\big) < C_3 n^{-\alpha} \tag{1.18}$$

for some constant $C_3 > 0$.

Since $(B_{n,\epsilon}(\mathrm{v}))^c \subseteq E_{n,\epsilon}(\mathrm{v}) \cup F_{n,\epsilon}(\mathrm{v}) \cup G_{n,\epsilon}(\mathrm{v})$ where

$$\begin{aligned}
E_{n,\epsilon}(\mathrm{v}) &:= \big\{ ||\zeta^i(\mathrm{v}) - \phi^i(\mathrm{o})||_1 \leq K \log n \text{ for some } i = 1, \ldots, n^4 \big\}, \\
F_{n,\epsilon}(\mathrm{v}) &:= \big\{ \zeta^{n^4}(\mathrm{v}) \notin \phi^{n^4}(\mathrm{o}) + \Delta_{n^{2(1+\epsilon)}} \big\}, \\
G_{n,\epsilon}(\mathrm{v}) &:= \big\{ \zeta^{n^4}(\mathrm{v}) \in \phi^{n^4}(\mathrm{o}) + \Delta_{n^{2(1-\epsilon)}} \big\},
\end{aligned}$$

to prove (1.18) it suffices to show

Lemma 1.3. *There exist $\alpha > 0$ and constants $C_4, C_5, C_6 > 0$ such that for all n sufficiently large we have*

(*a*) $\sup_{\mathrm{v} \in (\Delta_{n^{(1+\epsilon)}} \setminus \Delta_{n^{(1-\epsilon)}})} \mathbb{P}(E_{n,\epsilon}(\mathrm{v})) < C_4 n^{-\alpha}$,

(*b*) $\sup_{\mathrm{v} \in (\Delta_{n^{(1+\epsilon)}} \setminus \Delta_{n^{(1-\epsilon)}})} \mathbb{P}(F_{n,\epsilon}(\mathrm{v})) < C_5 n^{-\alpha}$,

(*c*) $\sup_{\mathrm{v} \in (\Delta_{n^{(1+\epsilon)}} \setminus \Delta_{n^{(1-\epsilon)}})} \mathbb{P}(G_{n,\epsilon}(\mathrm{v})) < C_6 n^{-\alpha}$.

The proof of this lemma is straightforward using the estimates from random walk and central limit theorem (see the paper version for the proof).

Proof of Lemma 1.2 : Let $\mathbf{v} := (\mathrm{v}, 0) \in \mathbb{Z}^4$. Observe that $A_{n,\epsilon}(\mathbf{v}) \supseteq B_{n,\epsilon}(\mathrm{v}) \cap \{M^i(\mathbf{0}) = \sum_{j=1}^i X_j,\ M^i(\mathbf{v}) = \mathrm{v} + \sum_{j=1}^i Y_j \text{ for all } 1 \leq i \leq n^4\}$. Hence

$$\begin{aligned}
&\mathbb{P}\big(A_{n,\epsilon}(\mathbf{v})\big) \\
&\geq \mathbb{P}\Big[B_{n,\epsilon}(\mathrm{v}) \cap \big\{X_{\mathbf{0}}(i) = \sum_{j=1}^{i} I_1^{(3)}(j),\ X_{\mathbf{v}}(i) = \mathrm{v} + \sum_{j=1}^{i} I_2^{(3)}(j) \text{ for } 1 \leq i \leq n^4\big\}\Big] \\
&= \mathbb{P}\Big[B_{n,\epsilon}(\mathrm{v}) \cap \big\{X_{\mathbf{0}}(i) = \sum_{j=1}^{i} I_1^{(3)}(j),\ X_{\mathbf{v}}(i) = \mathrm{v} + \sum_{j=1}^{i} I_2^{(3)}(j) \text{ for } 1 \leq i \leq n^4 - 1\big\}\Big] \\
&\times \mathbb{P}\Big[X_{\mathbf{0}}(n^4) = \sum_{j=1}^{n^4} I_1^{(3)}(j),\ X_{\mathbf{v}}(n^4) = \mathrm{v} + \sum_{j=1}^{n^4} I_2^{(3)}(j) \ \Big|\ B_{n,\epsilon}(\mathrm{v}) \cap \Big\{X_{\mathbf{0}}(i) = \sum_{j=1}^{i} I_1^{(3)}(j), \\
&\qquad X_{\mathbf{v}}(i) = \mathrm{v} + \sum_{j=1}^{i} I_2^{(3)}(j) \text{ for } 1 \leq i \leq n^4 - 1\Big\}\Big] \\
&\geq \mathbb{P}\Big[B_{n,\epsilon}(\mathrm{v}) \cap \big\{X_{\mathbf{0}}(i) = \sum_{j=1}^{i} I_1^{(3)}(j),\ X_{\mathbf{v}}(i) = \mathrm{v} + \sum_{j=1}^{i} I_2^{(3)}(j) \text{ for } 1 \leq i \leq n^4 - 1\big\}\Big] \\
&\qquad \times \Big(1 - C_1 \exp\big(-C_2 (K \log n)^3\big)\Big),
\end{aligned}$$

where the last inequality follows from (1.16) after noting that given $B_{n,\epsilon}(\text{v})$, $X_{\mathbf{0}}(i) = \sum_{j=1}^{i} I_1^{(3)}(j)$ and $X_{\mathbf{v}}(i) = \text{v} + \sum_{j=1}^{i} I_2^{(3)}(j)$ hold for all $1 \leq i \leq n^4 - 1$, we have $||X_{\mathbf{0}}(n^4 - 1) - X_{\mathbf{v}}(n^4 - 1)||_1 \geq K \log n$. The above argument may be used iteratively for $i = 1, \dots, n^4 - 1$ and together with (1.18) we have

$$\begin{aligned}
\mathbb{P}\big(A_{n,\epsilon}(\mathbf{v})\big) &\geq \Big(1 - C_1 \exp(-C_2(K\log n)^3)\Big)^{n^4} \mathbb{P}\big(B_{n,\epsilon}(\text{v})\big) \\
&\geq \Big(1 - C_1 n^4 \exp(-C_2 K^3 \log n)\Big)\Big(1 - C_3 n^{-\alpha}\Big) \\
&\geq \Big(1 - C_1 n^4 n^{-C_2 K^3}\Big)\Big(1 - C_3 n^{-\alpha}\Big) \\
&= \Big(1 - C_1 n^{-C_2 K^3 + 4}\Big)\Big(1 - C_3 n^{-\alpha}\Big).
\end{aligned}$$

Taking K such that $C_2 K^3 > 4$ (i.e. $K^3 > 8|\log(1-p)|^{-1}$) we have

$$\mathbb{P}\big(A_{n,\epsilon}(\mathbf{v})\big) \geq 1 - C_1 n^{-C_2K^3+4} - C_3 n^{-\alpha} \geq 1 - Cn^{-\beta},$$

for some constant $C > 0$ and $\beta := \min\{\alpha, C_2K^3 - 4\} > 0$. This completes the proof of Lemma 1.2. ∎

Finally to complete the theorem we need to show that $\mathcal{G}$ admits infinitely many trees almost surely. For $k \geq 2$, define $D^k(n,\epsilon) := \{(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k) : \mathbf{u}_i \in \mathbb{Z}^4$ such that $n^{1-\epsilon} \leq ||M^0(\mathbf{u}_i) - M^0(\mathbf{u}_j)||_1 \leq n^{1+\epsilon}$ for all $i \neq j\}$. Define the event $A(n,\epsilon,\mathbf{u}_1,\mathbf{u}_2,\dots,\mathbf{u}_k) := \{n^{2(1-\epsilon)} \leq ||M^{n^4}(\mathbf{u}_i) - M^{n^4}(\mathbf{u}_j)||_1 \leq n^{2(1+\epsilon)}$ and $M^t(\mathbf{u}_i) \neq M^t(\mathbf{u}_j)$ for all $t = 1, \dots, n^4$ and for all $i \neq j\}$. Using Lemma 1.2, we can easily show, for $0 < \epsilon < 1/3$ and for all large n

$$\inf\Big\{\mathbb{P}(A(n,\epsilon,\mathbf{u}_1,\mathbf{u}_2,\dots,\mathbf{u}_k) : (\mathbf{u}_1,\mathbf{u}_2,\dots,\mathbf{u}_k) \in D^k(n,\epsilon)\Big\} \geq 1 - \frac{C_k}{n^\beta} \tag{1.19}$$

where C_k is a constant independent of n (depending on k) and β is as in Lemma 1.2. We may now imitate the method following the statement of Lemma 1.2 to obtain

$$\mathbb{P}\Big\{M^t(\mathbf{u}_i) \neq M^t(\mathbf{u}_j) \text{ for all } t \geq 1 \text{ and for } 1 \leq i \neq j \leq k\Big\} > 0.$$

Thus, by translation invariance and ergodicity, we have that for all $k \geq 2$

$$\mathbb{P}\Big\{\mathcal{G} \text{ contains at least } k \text{ trees}\Big\} = 1.$$

This shows that $\mathcal{G}$ contains infinitely many trees almost surely.

1.1.2 Geometry of the graph $\mathcal{G}$

We now prove Theorem 1.2 for $d = 2$; with minor modifications the same argument carries through for any dimensions.

For $t \in \mathbb{Z}$ consider the set $O_t := \mathcal{G} \cap \{y = t\}$, the set of open vertices on the line $\{y = t\}$. For $x \in O_t$ and $n \geq 0$ let $C_t^n(x) := \{y \in O_{t-n} : M^n(y) = x\}$ be the set of the nth order children of the vertex $x \in O_t$. Now consider the set of vertices in O_t which have nth order children, i.e., $M_t^{(n)} := \{x \in O_t : C_t^n(x) \neq \emptyset\}$. Clearly, $M_t^{(n)} \subseteq M_t^{(m)}$ for $n > m$ and so $R_t := \lim_{n\to\infty} M_t^{(n)} = \cap_{n\geq 0} M_t^{(n)}$ is well defined. Moreover, this is the set of vertices in O_t which have bi-infinite paths. We want to show that $\mathbb{P}(R_t = \emptyset) = 1$ for all $t \in \mathbb{Z}$. Since $\{R_t : t \in \mathbb{Z}\}$ is stationary, it suffices to show that $\mathbb{P}(R_0 = \emptyset) = 1$.

Suppose that $\mathbb{P}(|R_0| = 0) < 1$. Then, we claim:

$$\mathbb{P}(|R_0| = 0) + \mathbb{P}(|R_0| = 1) < 1.$$

Suppose $\mathbb{P}(|R_0| = 0) + \mathbb{P}(|R_0| = 1) = 1$. Fix $N > 0$ and for any $n > 0$,

$$\begin{aligned}
&\mathbb{P}(R_0 \cap [-N, N] = \emptyset) = \mathbb{P}(R_n \cap [-N, N] = \emptyset) \\
&= \mathbb{P}(R_0 = \emptyset) + \mathbb{P}(R_n \cap [-N, N] = \emptyset \,|\, |R_0| = 1)\mathbb{P}(|R_0| = 1).
\end{aligned}$$

We observe that given $|R_0| = 1$, R_n is given by a symmetric random walk (with finite moments) starting at R_0. Letting $n \to \infty$ and noting that the random walk will escape any finite set, we conclude that

$$\lim_{n\to\infty} \mathbb{P}(R_n \cap [-N, N] = \emptyset \,|\, |R_0| = 1) = 1.$$

Thus, we have $\mathbb{P}(R_0 \cap [-N, N] = \emptyset) = 1$ for any $N > 0$. Letting $N \to \infty$, we have $\mathbb{P}(R_0 = \emptyset) = 1$ which is a contradiction. So, $\mathbb{P}(|R_0| \geq 2) > 0$.

Definition 1.1. A vertex $x \in R_t$ is called a *branching point* if

$$|(C_t^1(x) \cap R_{t+1})| \geq 2,$$

i.e, x has at least two distinct infinite branches of progeny.

Note that this notion of 'branching point' is similar to that of 'encounter point' of Burton and Keane [1989].

We denote the set of branching points at level t by B_t. Since, $\mathbb{P}(|R_0| \geq 2) > 0$, we have $\mathbb{P}(|R_{-1}| \geq 2) > 0$. Fix N so large that

$$\mathbb{P}(A(N)) := \mathbb{P}(|R_{-1} \cap [-N, N]| \geq 2) > 0.$$

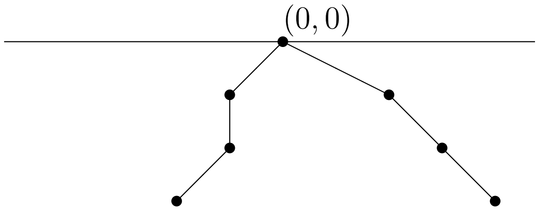

Figure 1.3: Branching point

The event says that there are at least two distinct vertices in $[-N, N] \times \{-1\}$ which has infinite set of progeny. and depends only on points $\{(i, j) : j \leq -1\}$. Consider, now

$$B(N) := \{(0,0) \text{ open and no other points in } [-2N-2, 2N+2] \times \{0\} \text{ is open}\}.$$

$\mathbb{P}(B(N)) > 0$. Further, $A(N)$ and $B(N)$ are independent. Then,

$$\mathbb{P}(0 \in B_0) \geq \mathbb{P}(A(N) \cap B(N)) > 0. \tag{1.20}$$

Further, by spatial invariance of the model, $\mathbb{P}(x \in B_0)$ is independent of x.

Now, we define $r_0(n) := |(R_0 \cap ([-n, n] \times \{0\})|$ and $r_1(n) := |R_1 \cap ([-n, n] \times \{1\})|$. We arrange the points of $R_1 \cap ([-n, n] \times \{1\})$ as $u_1, \ldots, u_{r_1(n)}$, in an increasing order of the x-co-ordinates . By our construction of $\mathcal{G}$, neither u_2 nor $u_{r_1(n)-1}$ nor any of the vertices between them can be connected to a vertex in R_0 which lies outside $[-n, n] \times \{0\}$. Thus, each of the vertices $u_2, u_3, \ldots, u_{r_1(n)-1}$ will have at least one ancestor in the set $R_0 \cap ([-n, n] \times \{0\})$. Moreover, each of the branching points in $u_2, \ldots, u_{r_1(n)-1}$, has at least two distinct ancestors in the set $R_0 \cap ([-n, n] \times \{0\})$. Thus, if $r_1^{(2)}(n)$ is the number of branching points in $[-n, n] \times \{1\}$, we must have have

$$r_0(n) \geq (r_1(n) - 2 - r_1^{(2)}(n)) + 2(r_1^{(2)}(n) - 2) = r_1(n) + r_1^{(2)}(n) - 6. \tag{1.21}$$

But, by stationarity we have $\mathbb{E}(r_1(n)) = \mathbb{E}(r_0(n))$ for all $n \geq 1$. Thus, for n sufficiently large, from (1.20) we have

$$0 = \mathbb{E}(r_0(n) - r_1(n)) \geq \mathbb{E}(r_1^{(2)}(n)) - 6 = (2n+1)\mathbb{P}(0 \in B_0) - 6 > 0.$$

This contradiction establishes Theorem 1.2.

1.2 Directed Spanning Forest

We consider a homogeneous Poisson point process $\mathcal{N}$ on $\mathbb{R}^2$ with intensity 1. In the original version, Coupier and Tran, [2011] considered *Directed Spanning Forest* in the direction along the x-axis; to be consistent, here, we will consider along y-axis.

For each point $x \in \mathcal{N}$, let $M(x)$ be the unique point of $\mathcal{N}$ which is closest in the L_2 norm in the upper half plane at x. The point $M(x)$ will be called the mother vertex of x. We join the edges between every point x and its mother $M(x)$. This defines a graph with vertex set $V = \mathcal{N}$ and edge set $E = \{\langle x, M(x)\rangle : x \in \mathcal{N}\}$. For $x \in \mathcal{N}$, define γ_x as the path starting at x. Clearly this path can be written as an union of line segments, each finite in length, i.e., $\gamma_x = \bigcup_{i=0}^{\infty}[x_i, M(x_i)]$ where $x_0 = x$ and for $i \geq 1$, $x_i = M(x_{i-1})$ and is identified as a subset of $\mathbb{R}^2$.

It is easy to observe that this graph does not have a loop. From any vertex, $x \in \mathcal{N}$, it continues infinitely in the positive direction of y-axis. Furthermore, two paths γ_x and γ_y, starting from two points $x, y \in \mathcal{N}$, either coincides with each other at some point $z \in \mathcal{N}$ and continues together from that point onwards or they do not cross and are disjoint subsets of $\mathbb{R}^2$. Therefore, the graph is composed of disjoint infinite trees.

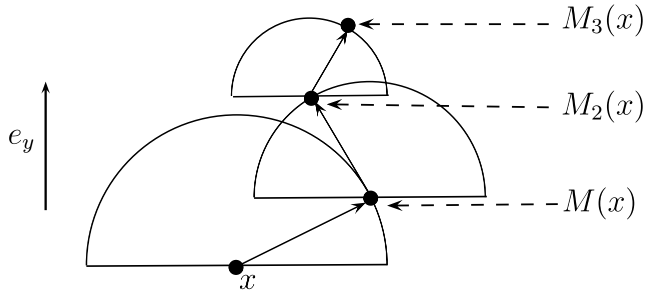

Figure 1.4: Construction of DSF from a point of $x \in \mathcal{N}$.

Thoerem 1.3. *The DSF constructed on the homogeneous PPP $\mathcal{N}$ is almost surely a tree.*

A similar model in discrete version has been considered by RSS, where they prove in dimension 2 or 3, the discrete version is almost surely a tree, and in dimension 4 onwards this is a forest. Furthermore, when suitably scaled, the two dimensional model converge to the Brownian web.

The proof is inspired by the literature from Percolation theory. The strategy here is to show that if the probability that there are at least two infinite trees is positive, then there are *special points* with positive density. This will allow us to use Burton-Keane type argument from percolation theory giving comparison between volume and surface area.

Let us define, η as the number of disjoint infinite paths of the DSF. As we have noted the DSF is identified as a subset of $\mathbb{R}^2$, the number of disjoint infinite paths η is the number of components (topological) of this set. The statement of the theorem is equivalent to showing $\mathbb{P}(\eta = 1) = 1$. Contrary to that we assume

$$\mathbb{P}(\eta \geq 2) > 0.$$

Note here, that this event $\{\eta \geq 2\}$ is translation invariant and the ergodicity of the PPP will actually imply that the above prob is either 0 or 1, and here 1 under the assumption. However, we do not require this.

1.2.1 Main Lemma

For $m_1, m_2 \geq 1$, let $C_{m_1,m_2} = [-m_1, m_1) \times [-m_2, m_2)$. Let F_{m_1,m_2} be the following event: there exists a path γ_x in the DSF with $x \in C_{m_1,m_2} \cap \mathcal{N}$ which does not meet any other path γ_z for all $z \in \{(u,v) \in \mathbb{R}^2 : v < m_2\} \setminus C_{m_1,m_2}$. More precisely,

$$F_{m_1,m_2} = \Big\{\text{there exists } x \in \mathcal{N} \cap C_{m_1,m_2} \text{ such that} \\ \text{for all } z \in \mathcal{N} \cap \{(u,v) \in \mathbb{R}^2 : v < m_2\} \setminus C_{m_1,m_2}, \gamma_x \cap \gamma_z = \emptyset\Big\}.$$

Lemma 1.4. *If $\mathbb{P}(\eta \geq 2) > 0$, then there exists $m_1, m_2 \geq 1$ such that $\mathbb{P}(F_{m_1,m_2}) > 0$.*

First we prove the theorem assuming lemma. Fix $L \geq 1$. Now consider the lattice

$$\mathbb{Z}_{L,m_1,m_2} = \{z = (2m_1 k, 2m_2 l) : -L \leq k, l \leq L\}$$

having $(2L+1)^2$ points and the rectangular region

$$\Re_{L,m_1,m_2} = \bigcup_{z \in \mathbb{Z}_{L,m_1,m_2}} \big(z + C_{m_1,m_2}\big)$$

where $\big(z + C_{m_1,m_2}\big)$ is obtained by translating the cell C_{m_1,m_2} to $z \in \mathbb{Z}_{L,m_1,m_2}$. Define $F^z_{m_1,m_2}$ as the event F_{m_1,m_2} translated to the cell $z + C_{m_1,m_2}$.

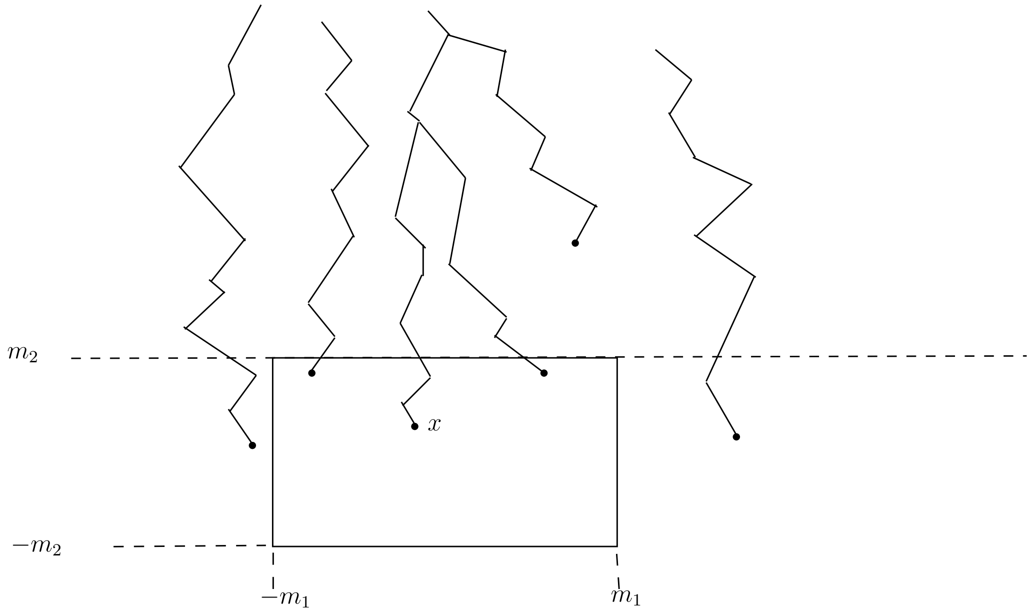

Figure 1.5: The event F_{m_1,m_2}.

Clearly, by translation invariance of PPP,

$$\mathbb{P}(F^z_{m_1,m_2}) = \mathbb{P}(F_{m_1,m_2}) \quad \text{for all } z \in \mathbb{Z}_{L,m_1,m_2}.$$

Clearly, if both $F^z_{m_1,m_2}$ and $F^{z'}_{m_1,m_2}$ occurs for $z, z' \in \mathbb{Z}_{L,m_1,m_2}, z \neq z'$, then there are at least two distinct paths (infinite) in the DSF, one starting in the cell $\big(z + C_{m_1,m_2}\big)$ and the other starting in the cell $\big(z' + C_{m_1,m_2}\big)$. Therefore, the number of events F_{m_1,m_2} occurring simultaneously, will be dominated by the number of distinct (infinite) paths starting from inside $\Re_{L,m_1,m_2}$ which is, in turn, dominated by the number edges of the DSF going out of the rectangle $\Re_{L,m_1,m_2}$.

Define, η_{L,m_1,m_2} as the number of edges of the DSF going out of the rectangle $\Re_{L,m_1,m_2}$. Thus, by the above observation,

$$\eta_{L,m_1,m_2} \geq \Sigma_{z\in\mathbb{Z}_{L,m_1,m_2}} 1_{F^z_{m_1,m_2}}. \tag{1.22}$$

Hence, by taking expectation, for m_1, m_2 and $L \geq 1$,

$$\begin{aligned}\mathbb{E}\big(\eta_{L,m_1,m_2}\big) &\geq \mathbb{E}\big(\Sigma_{z\in z_{L,m_1,m_2}} 1_{F^z_{m_1,m_2}}\big)\\ &= \Sigma_{z\in z_{L,m_1,m_2}} \mathbb{P}(F^z_{m_1,m_2}) = (2L+1)^2\mathbb{P}(F_{m_1,m_2}).\end{aligned} \tag{1.23}$$

Next, we show that, the number of edges going out of the rectangle $\Re_{L,m_1,m_2}$ cannot grow faster than $L^{3/2}$, in the expected sense.

Lemma 1.5. *For all $m_1, m_2 \geq 1$, there exists a constant $C(= C(m_1, m_2))$ such that for all L, large enough, we have*

$$\mathbb{E}\big(\eta_{L,m_1,m_2}\big) \leq CL^{3/2}. \tag{1.24}$$

Clearly, from (1.22) and (1.24), we have a contradiction, if $\mathbb{P}(F_{m_1,m_2}) > 0$, which proves the result.

Proof of Lemma 1.5: We decompose the set of edges, going out of $\Re_{L,m_1,m_2}$, into two sets; one whose length are more than $\sqrt{L}$ and the other whose length are no more than $\sqrt{L}$. We show that the expected number of edges in both sets are of the order of $L^{3/2}$. It should be noted that the edges going out of $\Re_{L,m_1,m_2}$, essentially depend on the perimeter of the rectangle.

We define $\eta^{<}_{L,m_1,m_2}$, $\eta^{>}_{L,m_1,m_2}$ as the number of edges going out of $\Re_{L,m_1,m_2}$ which are shorter than $\sqrt{L}$ and larger than $\sqrt{L}$, respectively.

For the edges which are shorter than $\sqrt{L}$ and going out of $\Re_{L,m_1,m_2}$ must start in a strip of width $\sqrt{L}$ as shown in the Figure 1.6. Thus, the number of edges of length shorter than $\sqrt{L}$ in $\eta^{<}_{L,m_1,m_2}$ is dominated by the number of points in the strip, clearly.

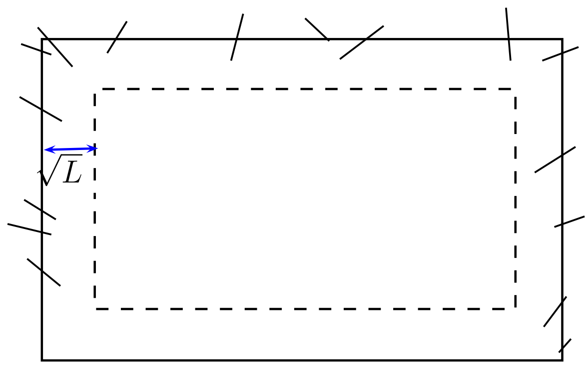

Figure 1.6: Edges in $\eta^{<}_{L,m_1,m_2}$ starting in a strip of width $\sqrt{L}$.

Clearly the expected number of points in the strip is given by the area, which is dominated by $C_1L^{3/2}$, for some constant $C_1(= C_1(m, M))$. Hence, we have

$$\mathbb{E}(\eta^{<}_{L,m_1,m_2}) \leq C_1(m, M)L^{3/2}. \tag{1.25}$$

For the number of edges which are larger than $\sqrt{L}$, we note that each such edge will be associated with a Poisson point $x \in \mathcal{N} \cap \Re_{L,m_1,m_2}$ for which the semi-circular region of radius $\sqrt{L}$ centered at x will contain no Poisson points.

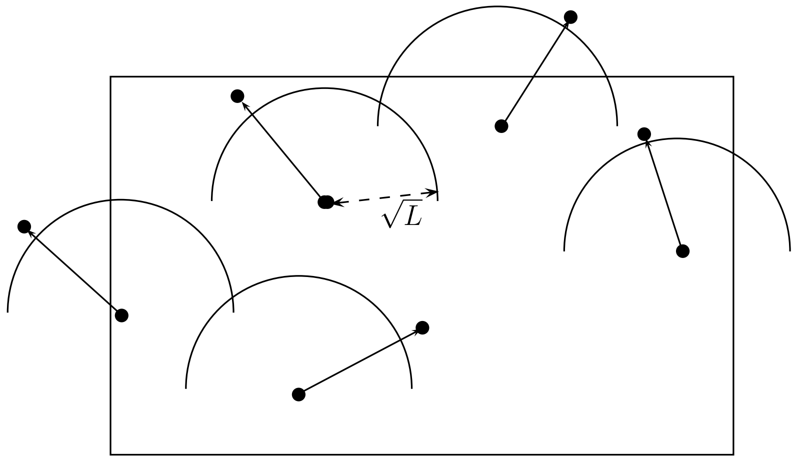

Figure 1.7: Poisson points with semi circular region of radius $\sqrt{L}$ containing no Poisson points.

Let Z be the number of such points of Poisson process in $\Re_{L,m_1,m_2}$ such that for each a semi-circular region of radius $\sqrt{L}$ contains no other Poisson points. Thus, we have

$$\eta^{>}_{L,m_1,m_2} \leq Z. \tag{1.26}$$

It is easy to compute the expectation of Z. Indeed,

$$\mathbb{E}(Z) \leq C_2(m, M) L^2 \exp(-\pi L/2) \tag{1.27}$$

for suitable choice of $C_2(m, M)$. Combining we have the desired result of Lemma 2. ∎

1.2.2 Proof of Main Lemma

Next, we prove the main lemma. This divided into two parts. First, we show that if $\mathbb{P}(\eta \geq 2) > 0$, then there exist $m_1, m_2 \geq 1$ such that the probability that box C_{m_1,m_2} will contain three disjoint infinite paths starting points is positive. This will imply that the path in the middle is sandwiched on both sides by two paths which are disjoint from it and all the paths starting from the box. The main lemma will then follow by manipulating the realization of the Poisson points to create a shield so that the paths starting from below the bottom line of the box cannot coalesce with the path in the middle.

For $m_1, m_2 \geq 1$, define T_{m_1,m_2} be the top edge of the box C_{m_1,m_2}, i.e.,

$$T_{m_1,m_2} = \{(x, y) : y = m_2, -m_1 \leq x \leq m_1\}.$$

Now, for $\delta > 0$ and $m_1, m_2 \geq 1$, we define the event

$$A^{\delta}_{m_1,m_2} = \Big\{\text{there exist } u, v \in \mathcal{N} \cap C_{m_1,m_2} \text{such that } \gamma_u \cap \gamma_v = \emptyset$$
$$(\gamma_u \cup \gamma_v) \cap \partial(C_{m_1,m_2}) \subset T_{m_1,m_2}, \mathcal{N} \cap C^{\delta,T}_{m_1,m_2} = \emptyset\Big\}$$

where $C^{\delta,T}_{m_1,m_2} = (-m_1, m_1] \times (m_2 - \delta, m_2]$. This event says that there exist two Poisson points in C_{m_1,m_2} from which two infinite disjoint paths emerge which touch only the top edge of the box C_{m_1,m_2} and a strip of width δ at the top edge does not contain any Poisson points.

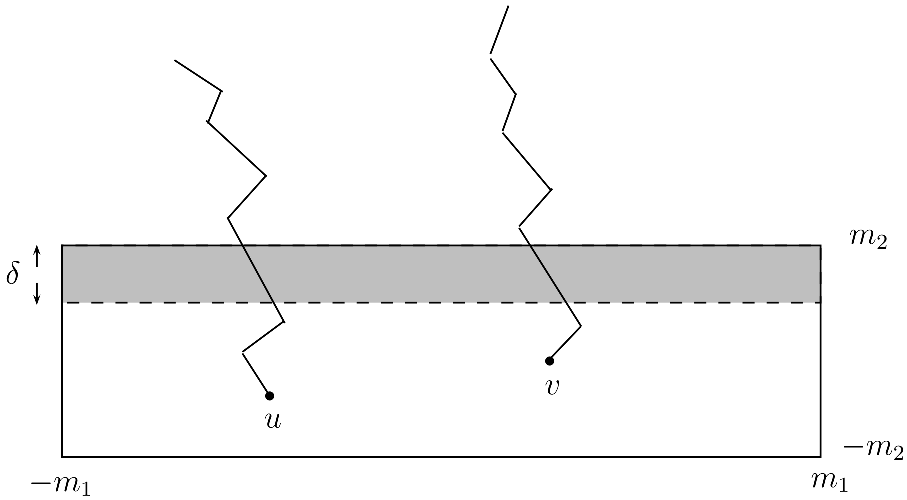

Figure 1.8: The event $A^{\delta}_{m_1,m_2}$. The shaded region δ strip contain no Poisson points.

Lemma 1.6. *If* $\mathbb{P}(\eta \geq 2) > 0$, *then there exists* $1 \leq m_2 \leq m_1$ *and* $\delta > 0$ *such that*

$$\mathbb{P}\big(A^{\delta}_{m_1,m_2}\big) > 0.$$

Proof : We have

$$0 < \mathbb{P}(\eta \geq 2) = \mathbb{P}\Big[\bigcup_{m=1}^{\infty}\big\{\text{there exist } u, v \in C_{m,m} \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset\big\}\Big]$$
$$\leq \sum_{m=1}^{\infty} \mathbb{P}\big\{\text{there exist } u, v \in C_{m,m} \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset\big\}.$$

Since the sum is strictly positive, we can choose $m \geq 1$ such that

$$\mathbb{P}\big\{\text{there exist } u, v \in C_{m,m} \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset\big\} > 0.$$

The path between two lines of $= m$ and $y = -m$ is a compact set in $\mathbb{R}^2$. Now, we have

$$\begin{aligned}
0 &< \mathbb{P}\big\{\text{there exist } u, v \in C_{n,m} \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset\big\} \\
&= \mathbb{P}\Big[\bigcup_{n=m+1}^{\infty} \big\{\text{there exist } u, v \in C_{n,m} \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset, \\
&\qquad (\gamma_u \cup \gamma_v) \cap \partial(C_{n,m}) \subseteq T_{n,m}\big\}\Big] \\
&\leq \sum_{n=m+1}^{\infty} \mathbb{P}\big\{\text{there exist } u, v \in C_{n,m} \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset, \\
&\qquad (\gamma_u \cup \gamma_v) \cap \partial(C_{n,m}) \subseteq T_{n,m}\big\}.
\end{aligned}$$

So, we can choose $n \geq m + 1$, such that

$$\begin{aligned}
&\mathbb{P}\big\{\text{there exist } u, v \in C_{n,m} \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset, \\
&\qquad (\gamma_u \cup \gamma_v) \cap \partial(C_{n,m}) \subseteq T_{n,m}\big\} > 0.
\end{aligned}$$

Again, in any given box, there are almost surely only finitely many points of the process. Hence the distance of them from a fixed given line is almost surely positive. Thus, taking $\delta_k = \frac{1}{k}$, we have

$$\begin{aligned}
0 &< \mathbb{P}\big\{\text{there exist } u, v \in C_{n,m} \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset, \\
&\qquad (\gamma_u \cup \gamma_v) \cap \partial(C_{n,m}) \subseteq T_{n,m}\big\} \\
&= \mathbb{P}\Big[\bigcup_{k=1}^{\infty} \big\{\text{there exist } u, v \in C_{n,m} \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset, \\
&\qquad (\gamma_u \cup \gamma_v) \cap \partial(C_{n,m}) \subseteq T_{n,m}, \mathcal{N} \cap C_{n,m}^{\delta_k, T} = \emptyset\big\}\Big] \\
&\leq \sum_{k=1}^{\infty} \mathbb{P}(A_{n,m}^{\delta_k}).
\end{aligned}$$

So, we choose δ_k such that

$$\mathbb{P}(A_{n,m}^{\delta_k}) > 0.$$

■

For $R \geq n$, define the event

$$\begin{aligned}
A_{n,m}^{\delta,R} = \big\{&\text{there exist } u, v \in C_{n,m} \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset, (\gamma_u \cup \gamma_v) \cap \partial(C_{n,m}) \subseteq T_{n,m}, \\
&\mathcal{N} \cap C_{n,m}^{\delta_k, T} = \emptyset, |\mathcal{N} \cap C_{n,m,R}^{\delta,R}| \geq 1, \ |\mathcal{N} \cap C_{n,m,R}^{\delta,L}| \geq 1\big\}
\end{aligned}$$

where $C_{n,m,R}^{\delta,R} = [n, R) \times [m - \delta, m)$ and $C_{n,m,R}^{\delta,L} = [-R, -n) \times [m - \delta, m)$ and $|B|$ denote the cardinality of the set B. This event says that there exists at least one Poisson point in the strips $[n, R) \times [m - \delta, m)$ and $[-R, -n) \times [m - \delta, m)$ along with the occurrence of $A^{\delta}_{m,M}$.

Lemma 1.7. *If* $\mathbb{P}(\eta \geq 2) > 0$*, then there exists* $1 \leq m < n < R$ *and* $\delta > 0$ *such that* $\mathbb{P}(A^{\delta,R}_{n,m}) > 0$.

The proof follows similarly as in Lemma 1.6, using the fact that the infinite strips $(-\infty, -n) \times [m - \delta, m)$ and $[n, \infty) \times [m - \delta, m)$ must contain at least one Poisson point almost surely.

The most important observation is that the event $A^{\delta,R}_{n,m}$ depend only on the configuration of the Poisson process in $\mathbb{R} \times [m, \infty) \cup C_{R,m}$.

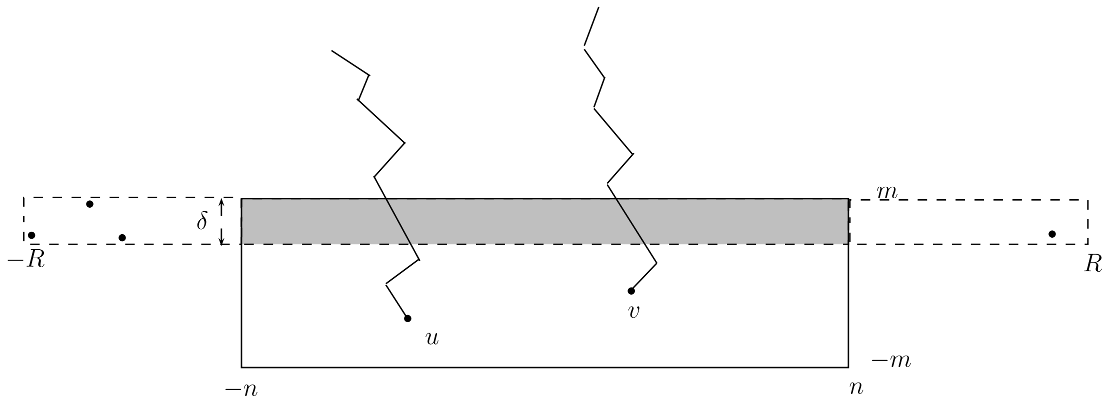

Figure 1.9: The event $A^{\delta,R}_{n,m}$.

Now, we show that at least two of these events will occur simultaneously with positive probability. Define, for $i \geq 0$,

$$
\begin{aligned}
A^{\delta,R,i}_{n,m} = \big\{&\text{there exist } u, v \in \big((2iR, 0) + C_{n,m}\big) \cap \mathcal{N} \text{ such that } \gamma_u \cap \gamma_v = \emptyset, \\
&(\gamma_u \cup \gamma_v) \cap \partial\big((2iR, 0) + C_{n,m}\big) \subseteq \big((2iR, 0) + T_{n,m}\big), \mathcal{N} \cap \big((2iR, 0) + C^{\delta_k, T}_{n,m}\big) = \emptyset, \\
&\big|\mathcal{N} \cap \big((2iR, 0) + C^{\delta,R}_{n,m,R})\big)\big| \geq 1,\ \big|\mathcal{N} \cap \big((2iR, 0) + C^{\delta,L}_{n,m,R})\big)\big| \geq 1\big\}.
\end{aligned}
$$

In other words, $A^{\delta,R,i}_{n,m}$ is the event $A^{\delta,R}_{n,m}$ translated to the box $\big((2iR, 0) + C_{n,m}\big)$.

Lemma 1.8. *If* $\mathbb{P}(\eta \geq 2) > 0$*, then there exists* $1 \leq m < n < R$*,* $\delta > 0$ *and* $i \geq 1$ *such that*

$$\mathbb{P}(A^{\delta,R,0}_{n,m} \cap A^{\delta,R,i}_{n,m}) = \mathbb{P}(A^{\delta,R}_{n,m} \cap A^{\delta,R,i}_{n,m}) > 0.$$

Proof : Choose m, n, R and δ as in Lemma 1.7. Clearly, by translation invariance of the Poisson process, we have, for any $i \geq 1$,

$$\mathbb{P}(A_{n,m}^{\delta,R}) = \mathbb{P}(A_{n,m}^{\delta,R,0}) = \mathbb{P}(A_{n,m}^{\delta,R,i}).$$

If $\mathbb{P}(A_{n,m}^{\delta,R,i_1} \cap A_{n,m}^{\delta,R,i_2}) = 0$ for all pairs $i_1, i_2 \geq 0$, we have

$$1 \geq \mathbb{P}\Big(\bigcup_{i=0}^{\infty} A_{n,m}^{\delta,R,i}\Big) = \sum_{i=0}^{\infty} \mathbb{P}\big(A_{n,m}^{\delta,R,i}\big) = \infty.$$

This proves the Lemma, by translation invariance. ∎

Now we are in a position to prove the main lemma.

Proof of main lemma : Choose m, n, R, δ and i as in Lemma 1.8 so that $\mathbb{P}(A_{n,m}^{\delta,R} \cap A_{n,m}^{\delta,R,i}) > 0$. On the event $A_{n,m}^{\delta,R,0} \cap A_{n,m}^{\delta,R,i}$, we have two pairs of points $(u_1, v_1) \in C_{m,m}$ and $(u_2, v_2) \in \big((2iR, 0) + C_{m,m}\big)$ such that $\gamma_{u_1} \cap \gamma_{v_1} = \emptyset$ and $\gamma_{u_2} \cap \gamma_{v_2} = \emptyset$. Since the paths are non-crossing, we must have at least 3 paths with positive probability.

Furthermore, the event $A_{n,m}^{\delta,R} \cap A_{n,m}^{\delta,R,i}$ depend only on the configuration of Poisson points on the region

$$D = \big(\mathbb{R} \times (m, \infty)\big) \bigcup C_{R,m} \bigcup \big((2iR, 0) + C_{R,m}\big).$$

We now manipulate the realization of the Poisson point pair on D^c to create a shield so that no path from bottom can coalesce with the middle path, constructed above.

We consider three isosceles triangles with base on $y = m$ and the inclined sides being parallel to each other, in such a way that the region $C_{R,m} \cup \big((2iR, 0) + C_{R,m}\big)$ is contained within the inner triangle and the distances between the corresponding inclined sides being 2δ and $\delta/2$ respectively between the inner and the middle triangle and between the middle triangle and the outer triangle.

Now, we consider two trapezoidal regions, joined at the top between the middle triangle and the outer triangle. For each such region, we select a set of finitely many disjoint squares $\{B_1^1, B_2^1, \ldots, B_{N_1}^1\}$ and $\{B_1^2, B_2^2, \ldots, B_{N_2}^2\}$. Each of these squares has side length $\delta/16$ are chosen so that bottom line of square B_i^1 lies above top line of B_{i+1}^1 for $i = 1, \ldots, N_1 - 1$ and bottom line of square B_i^2 lies above top line of B_{i+1}^2 for $i = 1, \ldots, N_2 - 1$. Furthermore, we select the squares so that the centres of squares B_i^1 and B_{i+1}^1 and B_i^2 and B_{i+1}^2 are at most $\delta/4$ distance away and the centres of B_1^1 and B_1^2 are also at most $\delta/4$ distance away and the centres of $B_{N_1}^1$ and $B_{N_2}^2$ are at most $\delta/4$ distance away from the $y = m$ line (See Figure 1.10)

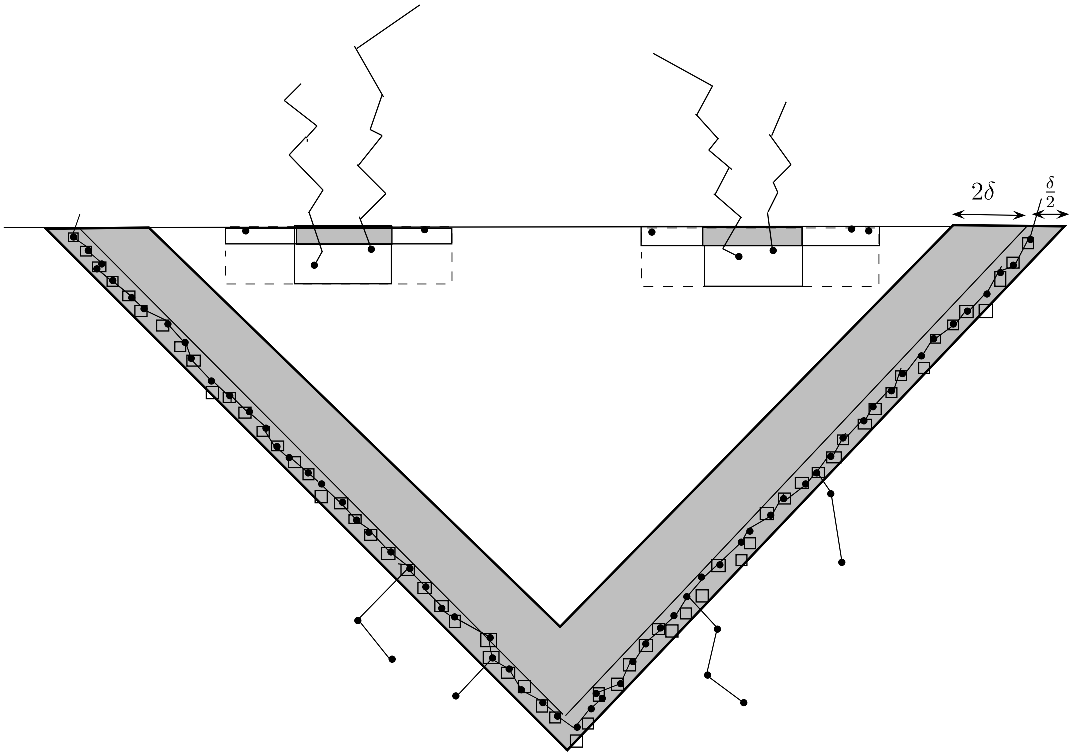

Figure 1.10: Poisson points with semi circular region of radius $\sqrt{L}$ containing no Poisson points.

Now, consider the event

$$E = \big\{\text{Each of squares } B_i^1, i = 1, \ldots, N_1 \text{ and } B_i^2, i = 1, \ldots, N_2 \text{ has at least one Poisson point and there are no Poisson points in the remaining trapezoidal region between the outer triangle and the inner triangle}\big\}.$$

Clearly, $\mathbb{P}(E) > 0$ and the event depend upon the configuration of the Poisson process in D^c. Thus, we have

$$\mathbb{P}\big((A_{R,m}^{\delta,R} \cap A_{R,m}^{\delta,R,i}) \cap E\big) > 0.$$

Choosing m_1 and m_2 suitably, we now have

$$F_{m_1,m_2} \subseteq (A_{R,m}^{\delta,R} \cap A_{R,m}^{\delta,R,i}) \cap E,$$

which proves the lemma. ∎

1.3 Infinite Oriented Cluster

We consider the oriented lattice in $\mathbb{Z}^2_{\text{even}} = \{(m,n) \in \mathbb{Z}^2 : m+n \text{ is even}\}$ and oriented edges from (m,n) to $(m+1,n+1)$ and to $(m-1,n+1)$. The oriented edges from u to v is denoted by $[u,v\rangle$. As usual, each edge is open with probability p and closed with probability $1-p$, independently of all other edges. We denote by $\mathbb{P}_p$, the corresponding product measure and by $\mathbb{E}_p$, the corresponding expectation with respect to $\mathbb{P}_p$.

For $u, v \in \mathbb{Z}^2_{\text{even}}$, we say that v can be reached from u, if there is a finite sequence of vertices and edges, $v_0 = u, e_1, v_1, e_2, \ldots, v_{m-1}, e_m, v_m = v$ such that $e_i = [v_{i-1}, v_i\rangle$ is open for $1 \le i \le m$. We denote this by $\{u \longmapsto v\}$. If there is no such sequence, then we say that v cannot be reached from u, and we denote this by $\{u \not\longmapsto v\}$. For $(x,y) \in \mathbb{Z}^2_{\text{even}}$, we denote oriented percolation cluster at (x,y), by

$$C_{(x,y)} = \big\{(z,w) \in \mathbb{Z}^2_{\text{even}} : (x,y) \longmapsto (z,w)\big\}.$$

Let $\Omega_{(x,y)} = \big\{|C_{(x,y)}| = \infty\big\}$ and $\theta(p) = \mathbb{P}(\Omega_{(0,0)})$. Define

$$\overrightarrow{p_c} = \sup\{p : \theta(p) = 0\}.$$

It is well known that [Durrett 1985]

$$0 < \overrightarrow{p_c} < 1.$$

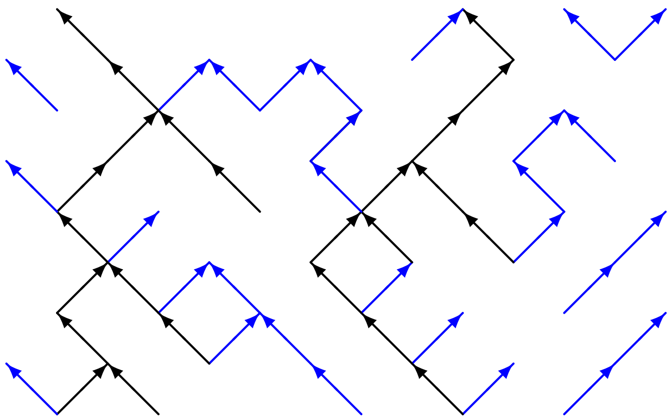

Figure 1.11: Configuration of oriented percolation

By definition

$$\theta(p) \begin{cases} = 0 & \text{for } p < \overrightarrow{p_c} \\ > 0 & \text{for } p > \overrightarrow{p_c}. \end{cases}$$

Furthermore, Bezuideninout and Grimmett [1990] has shown that $\theta(p) = 0$ if and only if $p \leq \overrightarrow{p_c}$.

Now, we say $(x, y) \in \mathbb{Z}^2_{\text{even}}$, is a percolation point if $|C_{(x,y)}| = \infty$. If $p \leq \overrightarrow{p_c}$, there are no percolation points, however if $p > \overrightarrow{p_c}$, there are infinitely many percolation points. Let

$$\mathcal{K} = \{(x, y) \in \mathbb{Z}^2_{\text{even}} : |C_{(x,y)}| = \infty\}.$$

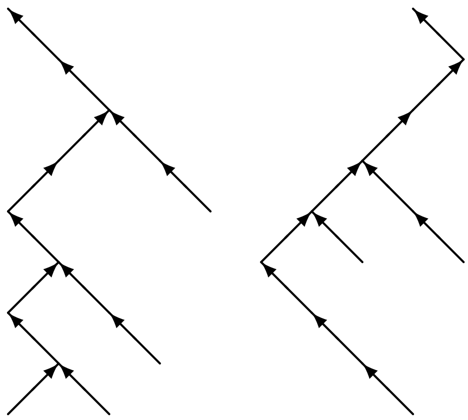

Figure 1.12: Graph consisting of rightmost infinite paths

Since from any $(x, y) \in \mathbb{Z}^2_{\text{even}}$, the rightmost $(u, v) \in \mathbb{Z}^2_{\text{even}}$, with $v = y + n$, which is reachable from (x, y) must have the property that $u \leq x + n$, we can define a rightmost infinite path $\gamma_{(x,y)}$ from $(x, y) \in \mathcal{K}$. More precisely, for $(x, y) \in \mathcal{K}$, the rightmost infinite path $\gamma_{(x,y)}$ is an infinite sequence of vertices and open edges, $(x, y) = v_0, e_1, v_1, e_2, \ldots, e_n v_n, \ldots$ with $v_n = (x_n, y_n)$ and $e_n = [v_{n-1}, v_n\rangle$ such that, for each $n \geq 1$,

$$\{k \geq 1 : (x, y) \longmapsto (x_n + k, y_n) \in \mathcal{K}\} = \emptyset.$$

Similarly we define a leftmost infinite path as $l_{(x,y)}$ for $(x, y) \in \mathcal{K}$. Thus, the infinite oriented cluster of $C_{(x,y)}$ will be contained inside the random cone generated by $\gamma_{(x,y)}$ and $l_{(x,y)}$.

For any (finite or infinite) path Γ, let $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices in Γ and the set of edges in Γ respectively. Let G be the random graph consisting of edges in $\gamma_{(x,y)}$ for $(x,y) \in \mathcal{K}$. In other words, the vertex set of $G, V(G) = \mathcal{K}$ and $E(G) = \{(x,y) : [x,y\rangle \in E(\Gamma_u), u \in \mathcal{K}\}$. Clearly, there are no loops in G, hence it is a forest.

For any infinite oriented path Γ and $v \in \Gamma$ (note Γ may not be the rightmost infinite path, it is just one fixed path), we define

$$b_r(v,\Gamma) = \big\{u \in \mathbb{Z}^2_{\text{even}} \setminus V(\Gamma) : u \text{ lies to the right of } \Gamma \text{ and } v \longmapsto u \text{ uses no edges of } \Gamma\big\},$$
$$b_\ell(v,\Gamma) = \big\{u \in \mathbb{Z}^2_{\text{even}} \setminus V(\Gamma) : u \text{ lies to the left of } \Gamma \text{ and } v \longmapsto u \text{ uses no edges of } \Gamma\big\}.$$

These are called right and left bud of Γ planted at v. (See figure 1.13). Given two vertices u, v of Γ such that $u \longmapsto v$ in Γ, i.e., there exists a finite sequence $v_0 = u, e_1, v_1, \ldots, e_n, v_n = v$ such that $v_i \in V(\Gamma)$ and $e_i \in E(\Gamma)$, we define by $\Gamma[u,v]$ as the piece between u and v.

Further, let

$$C_r(\Gamma[u,v)) = \bigcup_{v' \in V(\Gamma[u,v] \setminus v)} b_r(v',\Gamma)$$
$$C_l(\Gamma[u,v)) = \bigcup_{v' \in V(\Gamma[u,v] \setminus v)} b_r(v',\Gamma).$$

Clearly, if Γ is rightmost infinite path, then the right buds of Γ are finite, hence $C_r(\Gamma([u,v)))$ is also finite. Similarly left buds are finite if Γ is leftmost infinite path.

For each $u \in \mathbb{Z}^2_{\text{even}}$, two edges are adjacent to u, which are above, namely if $u = (u_1, v_1)$, then the edges (u_1+1, v_1+1), $(u_1-1, v+1)$. These will be called upper edges of u. There are two edges which are lower edges to u, namely (u_1+1, v_1-1) and (u_1-1, v_1-1). Since G consists of oriented paths which contain no loops, each vertex $u = (u_1, v_1)$ of G must be adjacent to only one upper edge. We say that the vertex at the upper edge at u is called the *mother* $M(u)$ of u. On the other hand, at most two vertices of G, can have the same mother u, in other words, $D(u) = \{v \in GV(G) : M(v) = u\}$. If $D(u)$ has two vertices, then they are called *sisters* and they are of the form (u_1-1, v_1-1) and $(u+1, v_1-1)$. The vertex (u_1-1, v_1-1) will be called older sister and (u_1+1, u_1-1) is called the younger sister.

Define, $M^0(u) = u$ and for $n \geq 1$, $M^n(u) = M(M^{u-1}(u))$, the nth ancestor of u. Define, $D^n(u_a) = \{v \in \mathcal{K} : M^n(v) = u\}$ for $n \geq 0$ and $D(u) = \bigcup_{n\geq 0} D^n(u)$. The set $D^n(u)$ is the nth generation descendants of u and $D(u)$ is the set of all descendants of u.

We say that two vertices $u,v \in \mathcal{K}$ are connected if they have common ancestor, i.e., there exists $m, n \geq 0$ such that $M_m(u) = M_n(v)$. This defines an equivalence relation and equivalence classes are connected trees.

Thoerem 1.4. *For $p \in (\overrightarrow{p_c}, 1)$,*

(a) G has a unique connected component

(b) D for all u, $D(u)$ is finite

(c) For each $u \in V(G)$, there is an ancestor with a younger sister almost surely.

1.3.1 Kuczek's Construction

For $A \subset (-\infty, \infty)$, we denote a random subset by $\xi_n^A = \{x :$ there exists $x' \in A$ such that $(x', 0) \rightarrow (x, n)\}$ for $n > 0$. The right most point at level n is defined by $r_n = \sup \xi_n(-\infty, \infty]$ where $\sup \emptyset = -\infty$. Thus, r_n is rightmost point on the n^{th} level which is reachable from some point in $(-\infty, 0] \times \{0\}$.

It is known that

$$\lim_{n \to \infty} \frac{r_n}{n} = \inf \Big\{ \frac{\mathbb{E}_p(r_n)}{n} : n \geq 1 \Big\} = \alpha(p) \text{ a. s. and in } L^1.$$

Furthermore, it is known

$$\alpha(p) = \begin{cases} -\infty & \text{if } p < \overrightarrow{p_c} \\ 0 & \text{if} p = \overrightarrow{p_c} \\ > 0 & \text{if} p > \overrightarrow{p_c}. \end{cases}$$

Now, we define

$$\xi_0^1 = \xi_0^{\{0\}}$$

and for all $n \geq 0$,

$$\xi_{n+1}^1 = \begin{cases} \{x : (y, n) \rightarrow (x, n+1) \text{ for some } y \in \xi_n^1\} & \text{if the set is non-empty} \\ (n+1) & \text{otherwise} \end{cases}$$

and finally set $r_n^1 = \sup \xi_n^1$.

It is easy to observe that on the event, $\Omega_{0,0} = \{(0,0) \in \mathcal{K}\}$,

$$r_n^1 = r_n \geq \gamma_{(0,0)}(n), \text{ where } \gamma_{(0,0)}(n) \in \mathbb{Z} \text{ satisfies } (\gamma_{(0,0)}(n), n) \in \gamma_{(0,0)}.$$

Let $T_0 = 0$ and $T_m = \inf\{n \geq T_{m-1} + 1 : (r_n^1, n) \in \mathcal{K}\}$ for $m \geq 1$. Define $\tau_0 = 0$ and $\tau_1 = T_1$ and $\tau_2 = T_2 - T_1, \ldots, \tau_m = T_m - T_{m-1}$ where $\tau_i = 0$ if $T_{i-1} = T_i = \infty$. The collection of points $\{(r_{T_m}^1, T_m) : m \geq 0\}$ are called the *break points* for path from $(0,0)$. Observe that, on $\Omega_{(0,0)}$, we must have $(r_{T_m}^1, T_m) \in \gamma_{(0,0)}$ for all $m \geq 1$. Also, define $X_0 = 0$, and $X_1 = r_{T_1}^1, X_2 = r_{T_2}^1 - r_{T_1}^1, \ldots, X_m = r_{T_m}^1 - r_{T_{m-1}}^1$ where $X_i = 0$ if $T_{i-1} = T_i = \infty$.

Proposition 1.1. *(***Kuczek***) For $p \in (\overrightarrow{p_c}, 1)$, on $\Omega_{(0,0)}$, $\{(X_m, \tau_m) : m \geq 1\}$ are independently and identically distributed with all moments and $\frac{\gamma_{(0,0)}(n) - \alpha(p)n}{\sqrt{n\sigma^2}}$ converges to $N(0,1)$ in distribution as $n \to \infty$ for some $\sigma > 0$.*

On $\Omega_{(0,0)}$, the rightmost infinite path $\gamma_{(0,0)}$ is well defined and all break points $(r_{\tau_m}^1, T_m) : m \geq 1$ are well defined. By Proposition 1.1, on the event $\Omega_{(0,0)}$, we define an integer-valued random walk $\xi_{(0,0)} = \{\xi_{(0,0)}(t) : t \geq 0\}$ as follows: $\xi_{(0,0)}(0) = 0$ and $\xi_{(0,0)}(t) = \sum_{i=1}^{N(t)} X_i$ for $t > 0$, where $N(t)$ is the largest integer m such that $T_m \leq t$. In the same way, we define the random walk $\xi_{(x,y)} = \{\xi_{(x,y)}(t) : t \geq y\}$ on the event $\Omega_{(x,y)}$, $(x,y) \in \mathbb{Z}^2_{\text{even}}$.

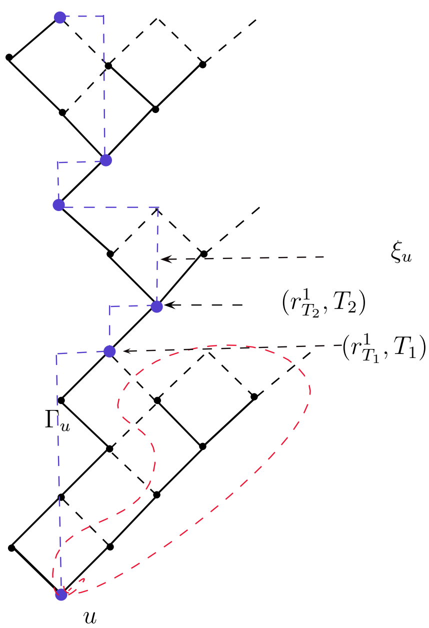

Figure 1.13: This figure represent right bud at u on γ_u and note that the break points form random walk ξ_u.

Proposition 1.2. *For any $p \in (\overrightarrow{p_c}, 1)$ and any pair $u_1, u_2 \in \mathbb{Z}^2_{even}$, conditioned on the event $\Omega_{u_1} \cap \Omega_{u_2}$, the following statements are equivalent.*

(a) ξ_{u_1} meets ξ_{u_2}, i.e., for some $t_0 \geq \min(u_1(2), u_2(2))$, $\xi_{u_1}(t_0) = \xi_{u_2}(t_0)$.

(b) The rightmost infinite paths γ_{u_1} and γ_{u_2} meet.

Proof : Clearly, (a) implies (b). So, enough to show (b) implies (a).

Let Γ_1 and Γ_2 be two realizations of γ_{u_1} and γ_{u_2} respectively. Suppose that Γ_1 and Γ_2 meet at $u_{1,2} \in \mathcal{K}$. So, for $i, j \geq 0$, $u_{1,2} = M^i(u_1) = M^j(u_2)$. We assume that $M^{i-1}(u_1)(1) < M^{j-1}(u_2)(1)$ where $u(1)$ denote the first co-ordinate of the point $u \in \mathbb{Z}^2_{\text{even}}$. Note that $u_{1,2}$ is a break point for u_1, as all the branches of Γ_1 are bounded on the right by Γ_2 and consequently no point on the right of $u_{1,2}$ is reachable from u_1. However, $u_{1,2}$ need not be a break point for u_2 (See figure 1.13). We need to show, that there is a point $v_{1,2} \in \Gamma_1 \cap \Gamma_2$ such that $v_{1,2}$ is a common jump point of both ξ_{u_1} and ξ_{u_2}.

Let $\{v_m = (x_m, y_m) : m \geq 0\}$ be the jump points of ξ_{u_2}, ordered by their y-co-ordinate. By definition, each v_m is also a break point of Γ_2. If $u_{1,2}$ is a jump point of ξ_{u_2}, then we have nothing to show. If not, there is a k st. $u_{1,2} \in \Gamma_2([v_{k-1}, v_k))$. By definition of break point, y_k the second co-ordinate of v_k, is greater than y for $(x, y) \in C_r(\Gamma_2(u_{1,2}, v_k))$. Since $C_r(\Gamma_1(u_{1,2}, v_k)) = C_r(\Gamma_2(u_{1,2}, v_k)) \subseteq C_r(\Gamma_2(u_{k-1}, v_k))$, we have that v_k is a break point, hence a jump point of ξ_{u_1}. Hence, the proposition follows. ∎

We define $R_{\alpha(p)}$ as the line (in $\mathbb{R}^2$) defined by the equation $y = x/\alpha(p)$. Conditioned on $\Omega_{(0,0)}$, we have the following property of $\gamma_{(0,0)}$.

Proposition 1.3. *Suppose $p \in (\overrightarrow{p_c}, 1)$. Then conditioned on the event $\Omega_{(0,0)}$, almost surely, the rightmost infinite path $\gamma_{(0,0)}$ crosses the line $R_{\alpha(p)}$ infinitely many times.*

The proof of this a direct consequence of the Proposition 1.1 by Kuczek. Since $\frac{\gamma_{(0,0)}(n)-\alpha(p)n}{\sqrt{n\sigma^2}}$ converges to a normal distribution, both the events $\liminf_{n\to\infty}\{\gamma_{(0,0)}(n) \geq \alpha(p)n\}$ and $\liminf_{n\to\infty}\{\gamma_{(0,0)}(n) \leq \alpha(p)n\}$ has probability 0.

1.3.2 Proof of Theorem 1.4

We introduce some more notations. For $u \in \mathbb{Z}^2_{\text{even}}$, we define $\wedge_u$ as 45°-degree cone starting at u. Thus, V_u consists of all vertices v such that there is an oriented path (not considering open or closed edges) from u to v. Thus, $C_u \subseteq \wedge_u$. Similarly, we define Λ_u

as the inverted 45° cone at u. Then, Λ_u is the set of all vertices v such that there is an oriented path from v to u.

For any $n \geq 0$ and $u = (x, y)$, let

$$\Lambda_n(u) = \{v = (x', y') \in \Lambda_u : y - y' \leq n\}$$

For any finite set $A \subseteq \mathbb{Z}^2_{\text{even}}$, contained in a horizontal line, we define

$$\Lambda_A = \big\{v \in \mathbb{Z}^2_{\text{even}} : \text{ there is a oriented path from }, v \text{ to some } u \in A\big\} = \bigcup_{u \in A} \Lambda_u$$

and $\Lambda_A(n) = \{v = (x', y') \in \Lambda_A : y - y' \leq n\}$ for $n \geq 0$, where y is the second co-ordinate of some point in A. Given $u \in \mathbb{Z}^2_{\text{even}}$ and $v \in \Lambda_u$, we say that there is an anti-oriented open path from u to v, if $v \mapsto u$. For $u \in \mathbb{Z}^2_{\text{even}}$, we define

$$C_u^{\text{anti}} := \{v \in \mathbb{Z}^2_{\text{even}} : v \mapsto u\},$$

as anti-oriented cluster of u. clearly it is a random subset of Λ_u. On the event $\{|C_u^{\text{anti}}| = \infty\}$, we define l_u^{anti} for the leftmost anti-oriented infinite open path from u. When $p > \overrightarrow{p_c}$, we have

$$\mathbb{P}_p(|C_u^{\text{anti}}| = |C_u| = \infty) = \theta(p)^2 > 0 \text{ for } u \in \mathbb{Z}^2_{\text{even}}. \tag{1.28}$$

This follows from the fact that $\{|C_u| = \infty\}$ depends only on edges $\wedge_u$ while $\{|C_u^{\text{anti}}| = \infty\}$ depends only of the edges in Λ_u, hence they are independent. Now, reversing the orientation of each of the edges in Λ_u, we see that $\{|C_u^{\text{anti}}| = \infty\}$ has the same probability as that $\{|C_u| = \infty\}$. The vertices u, satisfying $\{|C_u^{\text{anti}}| = \infty, |C_u| = \infty\}$ are called bi-directional percolation points. Let $\widetilde{\mathcal{K}}$ denote the set of bi-directional percolation points.

Proof of Theorem : It suffices to prove the result for any two vertices $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$, with $y_1 = y_2$, conditioned on the event $\Omega_{u_1} \cap \Omega_{u_2}$, since it is always possible to find two vertices which are ancestors of u_1 and u_2, on the same horizontal line. Furthermore, by translation invariance, it is enough to consider $u_2 = (0, 0)$ and $u_1 = (-n_0, 0)$ for some $n_0 \geq 1$.

Let Γ be a realization of $\gamma_{(0,0)}$. By Proposition 1.3, Γ crosses the line $R_{\alpha(p)}$ infinitely many times. For some vertex $v \in V(\Gamma)$, let $e = [u, v)$ be the lower edge of v in Γ. We call v, a crossing point if $[u, v\rangle \cap R_{\alpha(p)} \neq \emptyset$.

Given a realization of Γ, we define a sequence of independent events $E(K, \Gamma)$, for $k \geq 1$. Fix $\epsilon_0 > 0$ such that $\Phi(-\epsilon_0) > \frac{1}{3}$. By Proposition 1.1, we choose $N_0 \geq 1$ such that

$$\mathbb{P}_p\big(\gamma_{(0,0)}(n^2) - \alpha(p)n^2 < -n\sigma\epsilon_0 | \Omega_{(0,0)}\big) \geq \frac{1}{3} \tag{1.29}$$

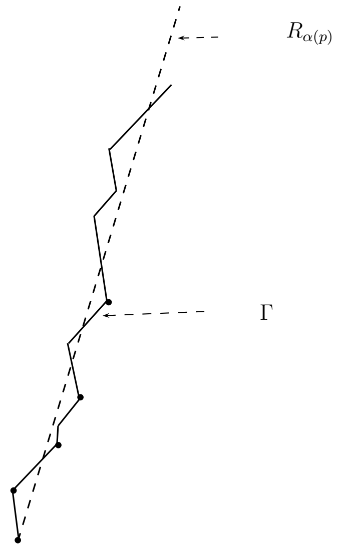

Figure 1.14: Crossing points.

for all $n \geq N_0$.

Set $v_0(\Gamma) = (x_0, y_0) = (0, 0)$. We choose the first crossing point $v_1(\Gamma) = (x_1, y_1)$ which satisfy the following condition

$$y_1 > \max\{n_0/(\epsilon_0\sigma), N_0\}^2.$$

Having chosen $v_1(\Gamma), \ldots, v_{k-1}(\Gamma)$, we choose $v_k(\Gamma) = (x_k, y_k)$, the first crossing point of Γ after $v_{k-1}(\Gamma)$, satisfying the following condition

$$y_k - y_{k-1} > \max\{2y_{k-1}/(\epsilon_0\sigma), N_0\}^2. \tag{1.30}$$

Now, we define $E(1, \Gamma)$ as

$$E(1, \Gamma) = \big\{\text{there is an anti-oriented open path from } v_1(\Gamma) \text{ to the half line } (-\infty, n_0) \times \{0\}\big\}.$$

Iteratively, for $k \geq 2$, we define $E(k, \Gamma)$ as

$$E(k, \Gamma) = \big\{\text{there is an anti-oriented open path from } v_k(\Gamma) \text{ to the half line } (-\infty, x_{k-1} - 2y_{k-1}] \times \{y_{k-1}\} \text{ and to the line} (-\infty, \infty) \times \{0\}\big\}.$$

Define $\text{Area}(K, \Gamma) = \Lambda_{v_k(\Gamma)}(y_k - y_{k-1}) \cup \Lambda_{A_k}(y_{k-1})$ where

$$A_1 = \{u = (x, y) \in \Lambda_{v_1(\Gamma)}(y_1) : x \leq -n_0, y = 0\}$$

and, for $k \geq 2$,

$$A_k = \left\{u = (x, y) \in \Lambda_{v_k(\Gamma)}(y_k - y_{k-1}) : x \leq x_{k-1} - 2y_{k-1}, y = y_{k-1}\right\}.$$

See figure 1.15 to observe that $\{\text{Area}(K, \Gamma) : k \geq 1\}$ are edge disjoint. Thus, for a fixed given path Γ, the events $\{E(K, \Gamma) : k \geq 1\}$ are independent.

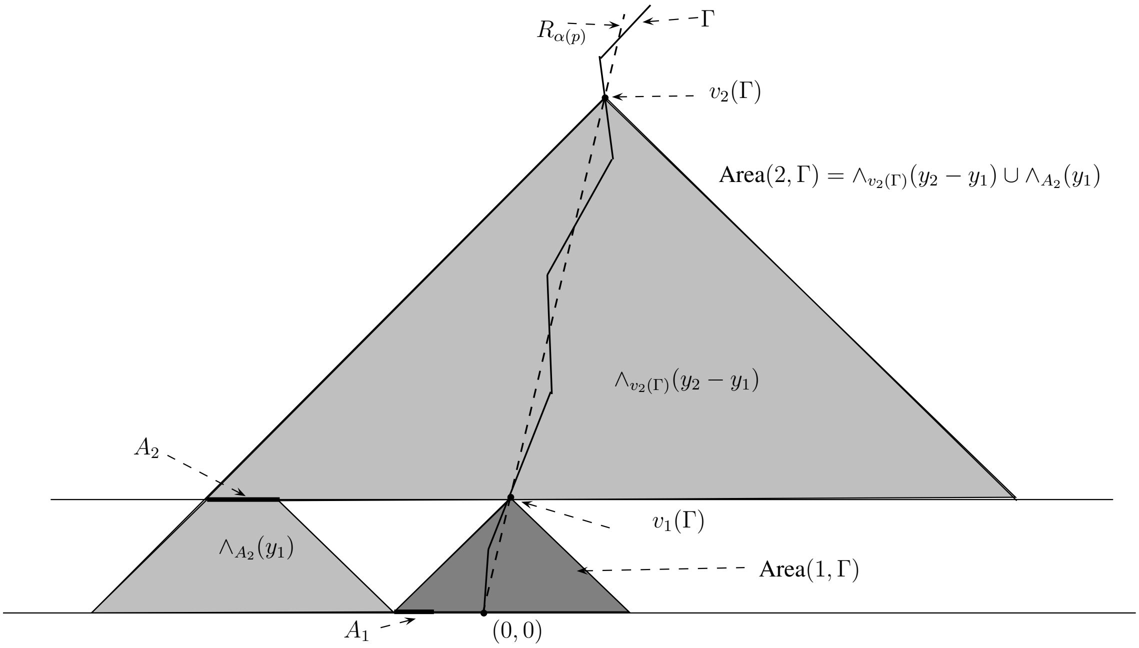

Figure 1.15: This figure represents $v_1(\Gamma), v_2(\Gamma)$, Area$(1, \Gamma)$(darker shaded region) and Area$(2, \Gamma)$ (lighter shaded region). Note that Area$(1, \Gamma)$ and Area$(2, \Gamma)$ are edge-disjoint areas.

Now, using (1.28) and (1.29), used on leftmost anti-oriented path from $v_k(= v_k(\Gamma))$, we have

$$\begin{aligned} \mathbb{P}_p(E(k, \Gamma)) &\geq \mathbb{P}\big(|C^{\text{anti}}_{v_k}| = \infty, l^{\text{anti}}_{v_k} \cap A_k \neq \emptyset\big) \\ &= \mathbb{P}_p\big(l^{\text{anti}}_{v_k} \cap A_k \neq \emptyset \big| |C^{\text{anti}}_{v_k}| = \infty\big)\mathbb{P}\big(|C^{\text{anti}}_{v_k}| = \infty\big) \geq \frac{1}{3}\theta(p) > 0. \qquad (1.31) \end{aligned}$$

We note here that in the above calculations Γ was used only to find the vertices $v_k(\Gamma)$. On the event $\{\gamma_{(0,0)} = \Gamma\}$, let $E^*(1, \Gamma)$ be the event that there is an anti-oriented open from $\Gamma(0, v_1(\Gamma))$ to A_1 and let $E^*(K, \Gamma)$ be the event that there is anti-oriented open path $\Gamma(v_{k-1}(\Gamma), v_k(\Gamma))$ to A_k and then to $(-\infty, +\infty) \times \{0\}$.

On the event $\{\gamma_{(0,0)} = \Gamma\}$, the event $E^*(K, \Gamma)$ only depends on the edges of Area $(\mathcal{K}, \Gamma)$ which are lying to the left of Γ, since the edges of Γ are open.

Now, we define two events,

$$F(\Gamma) = \{\Gamma \text{ is open}\} \text{ and } G(\Gamma) = \{v \not\mapsto \infty \text{ in } R(\Gamma) \text{ for each } v \in \Gamma\}$$

where $R(\Gamma)$ is the set of edges lying to the right of Γ. Clearly, we have

$$\{\gamma_{(0,0)} = \Gamma\} = F(\Gamma) \cap G(\Gamma).$$

Now, $G(\Gamma)$ depends only on the edges which are to the right of Γ. Thus, the events $E^*(k, \Gamma)$ and $G(\Gamma)$, conditioned on $F(\Gamma)$, are independent for each $k \geq 1$. Now, we observe that

$$E^*(k, \Gamma) \cap F(\Gamma) = E(k, \Gamma) \cap F(\Gamma).$$

Thus,

$$\mathbb{P}_p(E^*(k, \Gamma) | \gamma_{(0,0)} = \Gamma) = \mathbb{P}_p(E^*(k, \Gamma) | F(\Gamma)) = \mathbb{P}_p(E(k, \Gamma) | F(\Gamma)). \tag{1.32}$$

Now, for any $k \geq 1$, the event $F_k(\Gamma)$ be the event all edges in $\Gamma(v_{k-1}(\Gamma), v_k(\Gamma))$ are open. Clearly, $F_k(\Gamma)$ is increasing. By FKG inequality,

$$\mathbb{P}_p(E(k, \Gamma) | F(\Gamma)) = \mathbb{P}_p(E(k, \Gamma) | F_k(\Gamma)) \geq \mathbb{P}_p(E(k, \Gamma)). \tag{1.33}$$

On the other hand, $E^*(k, \Gamma)$ depends only on edges in Area(k, Γ) which are lying to the left of Γ. Since Area(k, Γ) are edge disjoint for different k, we have $E^*(k, \Gamma)$ are independent for $k \geq 1$. Furthermore, from (1.30), (1.31), (1.32) and (1.33), we have

$$\mathbb{P}_p(E^*(k, \Gamma) | \gamma_{(0,0)}) \geq \theta(p)/3.$$

for $k \geq 1$. Thus, by second Borel-Cantelli lemma, $E^*(k, \Gamma)$ must occur infinitely often. This proves that for some $u = (u', 0)$ with $u' \leq -n_0$, there is a open path from u to $v_k(\Gamma)$ for some $k \geq 1$. Thus, by the definition of rightmost path, γ_{u_1} and γ_{u_2} must meet.

Proof of Theorem (b): For any $u \in \mathcal{K}$, by definition of $D(u, G)$, it is clear that $D(u, G) \subseteq C_u^{\text{anti}}$. So, if $u \in \mathcal{K} \setminus \widetilde{\mathcal{K}}$, i.e., u is a percolation point but not a bi-directional percolation point, then $|C_u^{\text{anti}}| < \infty$, hence $|D(u, G)| < \infty$. Thus, it suffices to prove the result only for $u \in \widetilde{\mathcal{K}}$. By translation invariance, it is enough to show

$$|D(u, G)| < \infty \text{ when } (0, 0) \in \widetilde{\mathcal{K}}.$$

Let $l^{\text{anti}}_{(0,0)}$ be the leftmost anti-oriented infinite open path from $(0,0)$ and L^{anti} be a possible realization of $l^{\text{anti}}_{(0,0)}$, which crosses $R_{\alpha(p)}$ infinitely many times. We want to show that, $|D((0,0),G)| < \infty$ on the event $\{l^{\text{anti}}_{(0,0)} = L^{\text{anti}}\}$.

By part (a), we know that, on the event $\{l^{\text{anti}}_{(0,0)} = L^{\text{anti}}\}$, with probability 1, for all $n \geq 1$, there exists some point $v_n(L^{\text{anti}})$ in L^{anti}, from which there is an oriented open path to $[n,\infty) \times \{0\}$. On the other hand, by ergodic theorem, with probability 1, there exists infinitely many $m > 0$ such that $(m,0) \in \widetilde{\mathcal{K}}$. Thus, on the event $\{l^{\text{anti}}_{(0,0)} = L^{\text{anti}}\}$, with probability 1, there exists $m_0 > 0$ and $v' \in L^{\text{anti}}$ such that $(m_0,0) \in \widetilde{\mathcal{K}}$ and there is an oriented path from v' to $v'' \in [m_0,\infty) \times \{0\}$. Let us denote this path by $\pi = \pi(v',v'')$ (see figure 1.16).

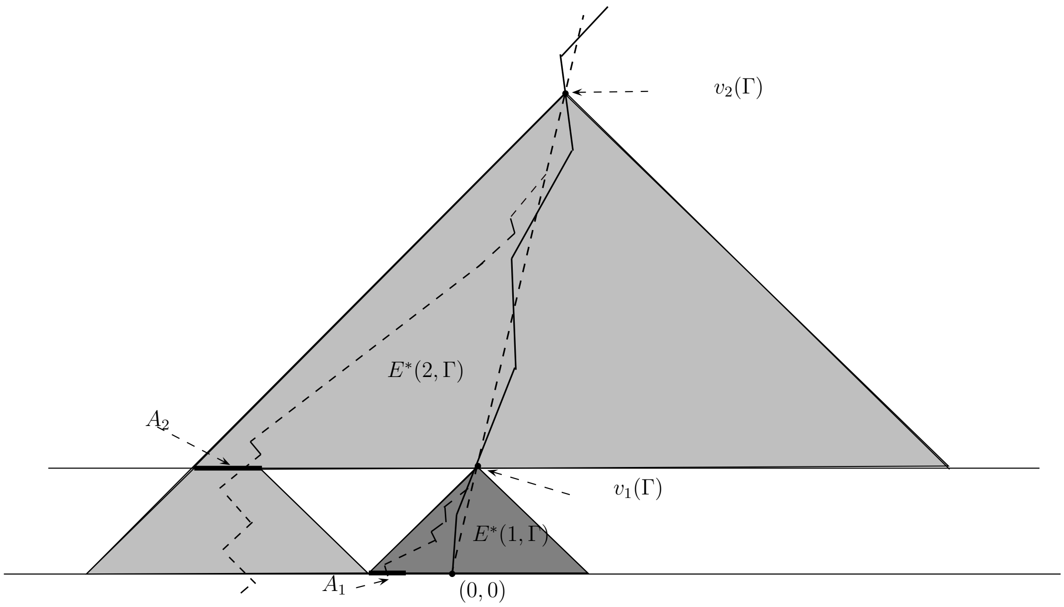

Figure 1.16: On the event $\{\gamma_{(0,0)} = \Gamma\}$, the event $E^*(k,\Gamma)$ depends only on the edges of $\text{Area}(k,\Gamma)$ which are to the left of Γ.

Define

$$\Lambda_{(0,0)}(L^{\text{anti}},\pi) = \{u \in \Lambda_{(0,0)} \setminus L^{\text{anti}} : u \text{ lies to the right of } L^{\text{anti}} \text{ and above } \pi\}.$$

and

$$\begin{aligned} C_l(L^{\text{anti}}(v,(0,0))) = \{ & w \notin L^{\text{anti}} : w \text{ lies to the left of } L^{\text{anti}} \text{ and for some} \\ & z \in L^{\text{anti}}(v',(0,0)), z \neq v' \text{ and } w \mapsto z \text{ uses no edges of } L^{\text{anti}}\}. \end{aligned}$$

We claim

$$D((0,0),G) \subseteq C_l(L^{\text{anti}}(v',(0,0))) \bigcup L^{\text{anti}}(v,(0,0)) \bigcup \Lambda_{(0,0)}(L^{\text{anti}},\pi). \tag{1.34}$$

Indeed, if $u \in C^{\text{anti}}_{(0,0)}$ and u does not belong to the right hand side of (1.34), then it is easy to find a path from u to $(m_0, 0)$, implying that the rightmost infinite open path from u does not lead to $(0,0)$. Hence $u \notin D((0,0),G)$.

By definition of leftmost anti oriented infinite path, we have

$$|C_\ell(L^{\text{anti}}(v',(0,0)))| < \infty.$$

Clearly, $|L^{\text{anti}}(v',(0,0))| < \infty$ and $|\Lambda_{(0,0)}(L^{\text{anti}},\pi)| < \infty$. This proves the result.

Proof of Theroem (c) : This is easy. For any $(x,y) \in \mathcal{K}$, by ergodicity, there exists $(x',y) \in \mathcal{K}$ where $x' > x$. Since $\gamma_{(x,y)}$ and $\gamma_{(x',y)}$ meet almost surely at $v \in K_1$ (say), v must have two daughters. ∎

Brownian Web

2.1 Introduction

We begin with a metric space $(\bar{\mathbb{R}}^2, \rho)$ where $\bar{\mathbb{R}}^2$ is the compactification of $\mathbb{R}^2$ under the metric ρ, with

$$\rho\big((x_1, t_1), x_2, t_2)\big) = \Big| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \Big| \vee \big|\tanh(t_1) - \tanh(t_2)\big|. \tag{2.1}$$

The space $\bar{\mathbb{R}}^2$ can be viewed as the image of the compact space $[-\infty, \infty] \times [-\infty, \infty]$ under the mapping

$$(x, t) \to (\Phi(x, t), \psi(t))$$

where

$$\Phi(x, t) = \frac{\tanh(x)}{1 + |t|} \text{ and } \psi(t) = \tanh(t). \tag{2.2}$$

For $t_0 \in [-\infty, \infty]$, let $C(t_0)$ denote the set of all functions f from $[t_0, \infty]$ to $[-\infty, -\infty]$ such that $\Phi(f(t), t)$ is continuous. Now, define

$$\Pi = \bigcup_{t_0 \in [-\infty, \infty]} C[t_0] \times \{t_0\}.$$

Now $(f, t_0) \in \Pi$, represents a path in $\bar{\mathbb{R}}^2$ with starting point at $(f(t_0), t_0) \in \bar{\mathbb{R}}^2$. The space Π is given the following metric

$$d((f_1, t_1), (f_2, t_2)) = \big(\sup_{t \in [-\infty, \infty]} \big|\Phi(\widehat{f_1}(t), t) - \Phi(\widehat{f_2}(t), t)\big|\big) \vee |\psi(t_1) - \psi(t_2)| \tag{2.3}$$

where, for $i = 1, 2$,

$$\widehat{f_1}(t) = \begin{cases} f_i(t) & \text{if } t \geq t_i \\ f_i(t_i) & \text{if } t < t_i. \end{cases}$$

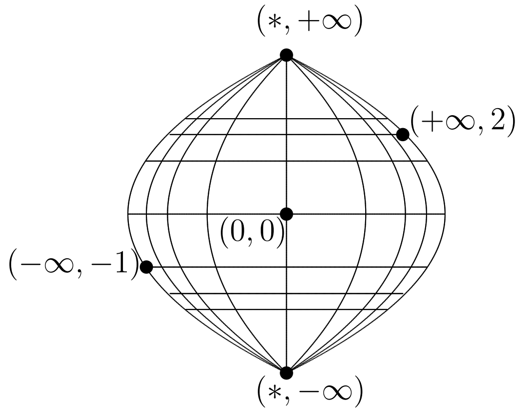

Figure 2.1: The compactification $\bar{\mathbb{R}}^2$ of $\mathbb{R}^2$.

We observe now that $(f_n, t_n) \in \Pi$ converges to $(f, t) \in \Pi$ if both the terms in (2.3) converges to 0. This implies $\psi(t_n) \to \psi(t)$ i.e., $t_n \to t$. (in the usual distance). For the first term, observe that when $t \to \pm\infty$, $\psi(x, t) \to 0$. Thus the convergence of the first term to 0 is equivalent to that the supremum taken over $t \in [-M, M], M \geq 0$ converges to 0, i.e.,

$$\sup_{t\in[-M,M]} \left| \frac{\tanh(\widehat{f_n}(t)) - \tanh(\widehat{f}(t))}{1 + |t|} \right| \to 0$$

for all M. This, in turn, is equivalent to $\sup\limits_{t\in[-M,M]} \left| \widehat{f_n}(t) - f(t) \right| \to 0$ as $n \to \infty$ for all $M > 0$. So, convergence in this topology is equivalent to the convergence of starting points of the paths as well as uniform convergence of the paths to the limiting path on compact intervals.

Proposition 2.1. *The space* Π *is complete and separable under the metric* d.

Next, we define $\mathcal{H}$ as space of all compact subsets of (Π, d), equipped with the induced Hausdorff metric $d_{\mathcal{H}}$ i.e.,

$$d_{\mathcal{H}}(K_1, K_2) = \sup_{g_1\in K_1}\inf_{g_2\in K_2} d(g_1, g_2) \vee \sup_{g_2\in K_2}\inf_{g_1\in K_1} d(g_1, g_2).$$

Proposition 2.2. *The space* $\mathcal{H}$ *is complete and separable under the metric* $d_{\mathcal{H}}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where we have a family of i.i.d. Brownian motion $\{B_i : i \geq 1\}$ is defined. Now, suppose $\mathcal{D} = \{x_j, t_j : j \geq 1\}$ be any countable dense set in

$(\bar{\mathbb{R}}^2, \rho)$. Define, for $(x_j, t_j) \in \mathcal{D}$, and $t \geq t_j$

$$W_j(t) = x_j + B_j(t - b_j), \quad \text{for} \quad j \geq 1.$$

Thus, W_j's are independent Brownian motions starting at x_j at time b_j. Note, $(W_j, t_j) \in \Pi$, for each $j \geq 1$. Now, we define coalescing Brownian motion paths from this family $\{W_j, j \geq 1\}$ as follows:

When two paths meet for the first time, they coalesce into a single path, which is the path of the Brownian motion having lower index. More mathematically, define

$$\widetilde{W}_1(t) = W_1(t) \quad \text{for} \quad t \geq t_1.$$

Having defined $\widetilde{W}_1, \widetilde{W}_2, \ldots, \widetilde{W}_k(t)$, we define $\widetilde{W_{k+1}}$ as follows: let $\tau_{k+1}(\tau_{k+1} \geq t_{k+1})$ be the first time $W_{k+1}(t)$ hit one of $\widetilde{W}_1, \ldots, \widetilde{W}_k(t)$ and let j be the index, $j \in \{1, \ldots, k\}$ of path which is hit by W_{k+1}. Then, define

$$\widetilde{W}_{k+1}(t) = \begin{cases} W_{k+1}(t) & \text{for } t \leq \tau_{k+1} \\ \widetilde{W}_j(t) & \text{for } t > \tau_{k+1}. \end{cases}$$

We define $\mathcal{W}_k = \mathcal{W}_k(\mathcal{D}) = \{\widetilde{W}_j : 1 \leq j \leq k\}$ and $\mathcal{W} = \mathcal{W}(\mathcal{D}) = \bigcup_{k \geq 1} W_k(\mathcal{D})$.

Proposition 2.3. *The subset $\overline{\mathcal{W}}$ is compact almost surely in (Π, d) where $\overline{\mathcal{W}}$ is the closure of $\mathcal{W}$ in (Π, d).*

It is therefore the case that $\overline{\mathcal{W}} \in \mathcal{H}$. Note for any $k \geq 1$, $\mathcal{W}_k \in \mathcal{H}$ since it is a finite set.

Remark. It can also be shown that all the paths in $\overline{\mathcal{W}}$ are Hölder continuous with exponent α, for any $\alpha < \frac{1}{2}$.

Thoerem 2.1. *The $\mathcal{H}$-valued random variable $\overline{\mathcal{W}}$ satisfies the following properties:*

(a) For any $(x, t) \in \mathbb{R}^2$, almost surely there is a unique element $W_{x,t} \in \overline{\mathcal{W}}$, starting at the point (x, t).

(b) For any $n \geq 1$ and $(x_{x_1,t_1}), (x_2, t_2), \ldots, (x_n, t_n)$, the joint distribution of $(W_{x_1,t_1}, \ldots, W_{x_n,t_n})$ is the same as that of coalescing Brownian motions with unit diffusion constants starting at points $(x_1, t_1), (x_2, t_2), \ldots, (x_n, t_n)$.

(c) For any countable dense subset $\mathcal{D}$ of $\mathbb{R}^2$, almost surely $\overline{\mathcal{W}} = \overline{\bigcup_{(x_n,t_n)\in\mathcal{D}} \{W_{x_n,t_n}\}}$.

Remark. It is furthermore known that the Brownian web is characterized by the three properties (a), (b) and (c) above. In other words, for any $(\mathcal{H}, d_{\mathcal{H}})$-valued random variable satisfying the properties (a), (b) and (c), must be the same in distribution as that of $\overline{\mathcal{W}}$.

2.2 Path counting

For $t > 0$ and t_0, $a, b \in \mathbb{R}$ with $a < b$, let $\eta(t_0; t; a, b)$ be the number of distinct points in $\mathbb{R} \times \{t_0 + t\}$ that are touched by paths $\overline{\mathcal{W}}$ which also touch some point in $[a, b] \times \{t_0\}$. Also, define $\hat{\eta} = (t_0; t; a, b) = \eta(t_0; t; a, b) - 1$.

Proposition 2.4. *The distribution of $\hat{\eta}$ is same as that of the number of points in $[a, b] \times \{t_0 + t\}$ which are touched by some paths in $\overline{\mathcal{W}}$ which also touch $\mathbb{R} \times \{t_0\}$.*

The proof uses a duality argument (see paper by Fontes et. al. [arxive version]).

Proposition 2.5. *Let $\hat{\eta} = (t_0; t; a, b)$ be the counting random variable for $\overline{\mathcal{W}}$. Then,*

$$\mathbb{P}(\hat{\eta} \geq k) \leq \big[\theta(b - a, t)\big]^k$$

for $k \geq 1$, where $\theta(b - a, t)$ is the probability that two independent Brownian motions starting at a distance $b - a$ apart have not met before time t.

In fact, it is known that

$$\mathbb{P}(\hat{\eta} \geq k) \leq \mathbb{P}(\hat{\eta} \geq k - 1)\mathbb{P}(\hat{\eta} \geq 1) \leq \big[\mathbb{P}(\hat{\eta} \geq 1)\big]^k = \big[\theta(b - a, t)\big]^k$$

where the first inequality follows by using FKG inequality.

Thoerem 2.2. *Let $\mathcal{W}'$ be $\mathcal{H}$-valued random variable. Suppose that conditions a), b) hold. Further*

c′) suppose that $\hat{\eta}_{\mathcal{W}'}(t_0; t; a, b) \overset{d}{=} \hat{\eta}_{\overline{\mathcal{W}}}(t_0; t; a, b)$ *for all* $t > 0$, $t_0, a, b \in \mathbb{R}$, $a < b$.

Then $\mathcal{W}'$ has the same distribution as the $\overline{\mathcal{W}}$.

There is further minimality property of the Brownian web. In fact, the condition (c′) can be replaced by c$''$) for all $t > 0$, $t_0, a, b \in \mathbb{R}$, $a < b$, $\eta_{\mathcal{W}'}$ is stochastically dominated by $\eta_{\overline{\mathcal{W}}}$, i.e., for all $k \geq 1$,

$$\mathbb{P}(\eta_{\mathcal{W}'} \geq k) \leq \mathbb{P}(\eta_{\overline{\mathcal{W}}} \geq k).$$

It has further been proved that (c$''$) can be replaced by

c$'''$) for all $t > 0$, $t_0, a, b \in \mathbb{R}$, $a < b$,

$$\mathbb{E}(\eta_{\mathcal{W}'}) \leq \mathbb{E}(\eta_{\overline{\mathcal{W}}}).$$

It is also known that

$$\mathbb{E}(\eta_{\overline{\mathcal{W}}}) = 1 + \frac{(b-a)}{\sqrt{\pi t}}.$$

2.3 Convergence Criteria

Let K be a $\mathcal{H}$-valued random variable. For $t > 0$, and $t_0, a, b \in \mathbb{R}$, $a < b$, we define

$$\begin{aligned} K_{t_0,t}[a,b] = \big\{ x \in [a,b] : \text{ there exists } y \in \mathbb{R} \text{ and a path in } K \\ \text{which touches both } (x, t_0) \text{ and } (y, t_0 + t) \big\}. \end{aligned}$$

Set

$$l_{t_0,t}[a,b] = \inf\{K_{t_0,t}[a,b]\} \text{ and } r_{t_0,t}[a,b] = \sup\{K_{t_0,t}[a,b]\}.$$

We now define

$$\begin{aligned} N_{t_0,t}[a,b] &= \big\{ y \in \mathbb{R} : \text{ there exists } x \in [a,b] \text{ and a path in } K \\ &\qquad \text{which touches } (x, t_0) \text{ and } (y, t_0 + t) \big\} \\ N^-_{t_0,t}[a,b] &= \big\{ y \in \mathbb{R} : \text{ there exists } x \in [a,b] \text{ and a path in } K \\ &\qquad \text{which touches } (l_{t_0,t}[a,b], t_0) \text{ and } (y, t_0 + t) \big\} \\ N^+_{t_0,t}[a,b] &= \big\{ y \in \mathbb{R} : \text{ there exists } x \in [a,b] \text{ and a path in } K \\ &\qquad \text{which touches } (r_{t_0,t}[a,b], t_0) \text{ and } (y, t_0 + t) \big\}. \end{aligned}$$

Observe that

$$\big| N_{t_0,t}[a,b] \big| = \eta(t_0; t; a, b).$$

Suppose $\{\chi_m\}$ be a sequence of $\mathcal{H}$-valued random variables with distribution μ_m. We define two conditions (B1$'$) and (B2$'$) as follows:

(B1′) For $\beta > 0$,

$$\lim_{\epsilon\downarrow 0}\limsup_{m\to\infty}\sup_{t>\beta}\sup_{(t_0,a)\in\mathbb{R}^2}\mu_m\big(\big|N_{t_0,t}[a-\epsilon,a+\epsilon]\big|>1\big)=0.$$

(B2′) For $\beta > 0$,

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\limsup_{m\to\infty}\sup_{t>\beta}\sup_{(t_0,a)\in\mathbb{R}^2}\mu_m\big(N_{t_0,t}[a-\epsilon,a+\epsilon]\neq N^+_{t_0,t}[a-\epsilon,a+\epsilon]\bigcup N^-_{t_0,t}[a-\epsilon,a+\epsilon]\big)=0.$$

We define the criteria for convergence of finite dimensional distribution. Let $\mathcal{D}$ be a *countable dense* subset of $\mathbb{R}^2$.

(I1) For $y \in \mathbb{R}^2$, there exists $\theta_n^y \in \chi_n$ such that for any deterministic collection $y_1, y_2, \ldots, y_m \in \mathcal{D}$, the distribution of $(\theta_n^{y_1}, \ldots, \theta_n^{y_m})$ converges to the distribution of coalescing Brownian motions starting at $(y_1, \ldots, y_m)$.

Thoerem 2.3. Convergence Theorem : *Suppose $\{\mu_m\}$ is tight. If conditions (I1) and (B1′) and (B2′) hold, then $\{\chi_m\}$ converges in distribution to the Brownian web $\overline{\mathcal{W}}$.*

Suppose $\{\chi_m\}$ have only non-crossing paths, the conditions (B1′) and (B2′) are implied by

(B1) For all $t > 0$,

$$\lim_{\epsilon\downarrow 0}\limsup_{m\to\infty}\sup_{t>\beta}\sup_{(t_0,a)\in\mathbb{R}^2}\mu_m(\eta(t_0;t;a,a+\epsilon)\geq 2)=0.$$

(B2) For all $t > 0$,

$$\lim_{\epsilon\downarrow 0}\limsup_{m\to\infty}\sup_{t>\beta}\sup_{(t_0,a)\in\mathbb{R}^2}\quad\mu_m(\eta(t_0;t;a,a+\epsilon)\geq 3)=0.$$

If further $\{\chi_m\}$ is spatially stationary, one can take $(t_0, a) = (0, 0)$. as

$$\sup_{(a,t_0)\in\mathbb{R}^2}\quad\mu_m(\eta(t_0;t;a,a+\epsilon)\geq 2)=\mu_m(\eta(0;t;0,\epsilon)\geq 2)$$

and

$$\sup_{(a,t_0)\in\mathbb{R}^2}\quad\mu_m(\eta(t_0;t;a,a+\epsilon)\geq 3)=\mu_m(\eta(0_0;t;0,\epsilon)\geq 3).$$

There is also an alternative convergence criteria developed Newman et. al. [2005], which replaces conditions (B1) and (B2) by a density bound using the dual counting variable $\hat{\eta}$ and the minimality of the Brownian web. This is more robust than the previous criteria and more useful in non-overlapping cases.

Thoerem 2.4. *Let $(\chi_m)_{m\geq 1}$ be a sequence of $\mathcal{H}$-valued random variables having μ_m as the distribution of chi. Suppose that $\{\mu_m\}$ is tight and the condition (I1) holds. Then $\mathcal{X}_n$ converges in distribution to the standard Brownian web $\mathcal{W}$:*

(E) *If $\mathcal{X}$ is any subsequential weak limit of $(\mathcal{X}_n)_{n\geq 1}$, then for all $t > 0$ and $t_0, a, b \in \mathbb{R}$ with $a < b$, we have $\mathbb{E}[\hat{\eta}_{\mathcal{X}}(t_0, t; a, b)] \leq \mathbb{E}[\hat{\eta}_{\mathcal{W}}(t_0, t; a, b)] = \frac{b-a}{\sqrt{\pi t}}$.*

To verify condition (E) in Theorem 2.4 for $\mathcal{X}_n$, by following the strategy : we show that for any $\delta > 0$ and for any subsequential limit, the paths (in the subsequential limit) starting below $y = t_0 \in \mathbb{R}$ with have locally finitely many intersection with the line $y = t_0 + \delta$. Furthermore, the distribution of the paths in the subsequential limit after $t_0 + \delta$ is the same as coalescing Brownian motions starting at those points. Under the assumption of condition (I1), the second part is standard and holds generally. For a proof see Newman et. al. [2005].

In what follows, for any $s < t$ and $K \in \mathcal{H}$, we will denote

$$\begin{aligned} K_s &:= \{\pi \in K : \sigma_\pi = s\}, & K_{s^-} &:= \{\pi \in K : \sigma_\pi \leq s\}, \\ K(s) &:= \{\pi(s) : \pi \in K\}, & K^t &:= \{\pi^t : \pi \in K\}, \end{aligned} \tag{2.4}$$

where π^t denotes the path obtained from π by restricting π to the time interval $[t, \infty)$. We will also denote $K^t_{s^-} := (K_{s^-})^t$.

It was further proved that condition (E) can be replaced by condition

(E') For any $t_0 \in \mathbb{R}$, if $\mathcal{Z}$ is a subsequential weak limit of $(\mathcal{X}_{n,t_0^-})_{n\in\mathbb{N}}$, where $\mathcal{X}_{n,t_0^-} := (\mathcal{X}_n)_{t_0^-}$, then for all $t > 0$ and $a < b$, we have

$$\mathbb{E}[\hat{\eta}_{\mathcal{Z}}(t_0, t; a, b)] \leq \mathbb{E}[\hat{\eta}_{\mathcal{W}}(t_0, t; a, b)] = \frac{b-a}{\sqrt{\pi t}}. \tag{2.5}$$

Condition (E') simplifies (E) by effectively singling out the subset of paths in $\mathcal{X}$ starting before or at time t_0, which are the only relevant paths for verifying condition (E).

Assume that for a subsequence n_k, we have $\mathcal{X}_{n_k,t_0^-}$ converges weakly to $\mathcal{Z}$. To verify (E'), it suffices to prove (2.5), which will follow from the next two lemmas.

Lemma 2.1. *Let $\mathcal{Z}$ be the weak limit of $\mathcal{X}_{n_k,t_0^-}$. Then for any $\delta > 0$, $\mathcal{Z}(t_0 + \delta)$ is a.s. a locally finite subset of $\mathbb{R}$.*

Lemma 2.2. *Let $\mathcal{Z}$ be the weak limit of $\mathcal{X}_{n_k,t_0^-}$. Then for any $\delta > 0$, $\mathcal{Z}^{t_0+\delta}$ has the same distribution as that of $\mathcal{W}_{t_0+\delta,\mathcal{Z}} := \{\pi \in \mathcal{W} : \sigma_\pi = t_0 + \delta, \pi(t_0 + \delta) \in \mathcal{Z}(t_0 + \delta)\}$, where $\mathcal{W}$ is a standard Brownian web independent of $\mathcal{Z}$.*

Lemma 2.2 implies that

$$\mathbb{E}[\hat{\eta}_{\mathcal{Z}}(t_0, t; a, b)] = \mathbb{E}\big[\hat{\eta}_{\mathcal{Z}^{t_0+\delta}}(t_0+\delta, t-\delta; a, b)\big] \leq \mathbb{E}[\hat{\eta}_{\mathcal{W}}(t_0+\delta, t-\delta; a, b)] = \frac{b-a}{\sqrt{\pi(t-\delta)}},$$

from which (2.5) follows by letting $\delta \downarrow 0$.

2.3.1 Tightness condition

Let $\Lambda_{L,T} = [-L, L] \times [-T, T] \subseteq \mathbb{R}^2$. For $C(\chi_0 t_0) \in \mathbb{R}^2$ and $u, t > 0$, let $R(\chi_0, t_0, u, t)$ denote the rectangle $[x_0 - u, x_0 + u] \times [t_0, t_0 + t] \subseteq \mathbb{R}^2$. Define $A_{t,u}(x_0, t_0)$ to be event that K contains a path touching both $R(x_0, t_0, \frac{u}{2}, t)$ and (at a later time) either the left boundary or the right boundary of a bigger triangle $R(x_0, t_0, u, 2t)$.

(T1) $\widetilde{g}(t, u, L_1, T) = \frac{1}{t} \sup_{\limsup (x_0,t_0)\in\Lambda_{L,T}} \sup \quad \mu_m(A_{t_u}(x_0, t_0))$

Proposition 2.6. *If for every $u, L, T > 0$, $\lim_{t\downarrow 0} \widetilde{g}(t, u; L, T) = 0$, then $\{\mu_m\}$ is tight.*

Thus, the general convergence theorem can be modified as

Corollary 2.4.1. *Suppose $\{\chi_m\}$ is a sequence of $\mathcal{H}$-valued random variables having distribution μ_m which satisfy the conditions (I1), (T1), (B1′) and (B2′), then $\{\chi_m\}$ converges in distribution to the Brownian web $\overline{\mathcal{W}}$.*

Thoerem 2.5. *Suppose $\{\chi_m\}$ is a sequence of $\mathcal{H}$-valued random variables, whose paths are non-crossing. Suppose, further, for any countable dense set $\mathcal{D}$.*

I1′ For each $y \in \mathcal{D}$, there is a path $\theta_m^y \in \chi_m$ such that θ_m^y converges in distribution to a Brownian motion Z^y, starting at y.

Then, the family $\{\chi_m\}$ is tight.

Thus, for a family having non-crossing paths, the convergence theorem simplifies to

Corollary 2.5.1. *Suppose $\{\chi_m\}$ is a sequence of $\mathcal{H}$-valued random variables whose paths are non-crossing. Suppose that the conditions (I1), (B1) and (B2) are satisfied. Then, χ_m converges in distribution to the Brownian web $\overline{\mathcal{W}}$.*

Corollary 2.5.2. *Suppose $\{\chi_m\}$ is a sequence of $\mathcal{H}$-valued random variables whose paths are non-crossing. Suppose that the condition (I1) and (E) is satisfied. Then, χ_m converges in distribution to the Brownian web $\overline{\mathcal{W}}$.*

2.4 Convergence of coalescing random walks

Consider the nearest neighbour discrete random walk starting from every vertex $(x, t) \in \mathbb{Z}^2_{\text{even}}$. If the two walks meet, they coalesce from that point onwards. Let Y denote the collection of all discrete time coalescing random walks in $\mathbb{Z}$, starting from every vertex of $\mathbb{Z}^2_{\text{even}}$. In other words, suppose $X^{\mathbf{z}}$ be the random walk path starting at $\mathbf{z} = (x, t) \in \mathbb{Z}^2_{\text{even}}$. Then,

$$Y = \{X^{\mathbf{z}} : \mathbf{z} \in \mathbb{Z}^2_{\text{even}}\}.$$

Fix $\delta > 0$ and scale the set of random walks using the usual diffusive scaling, i.e., scale the space by δ and time by δ^2. We denote by Y_δ, the scaled collection of paths, i.e.,

$$Y_\delta = \{(\delta x_1, \delta^2 x_2) \in \mathbb{R}^2 : (x_1, x_2) \in Y\}.$$

The closure of Y_δ is almost surely compact in (Π, d).

Remark. If general random walks (not necessarily nearest neighbour walks, and hence paths could cross each, without coalescing) are considered, then Y_δ is almost surely compact in (Π, d) of the increment random variable of the random walk has finite absolute first moment.

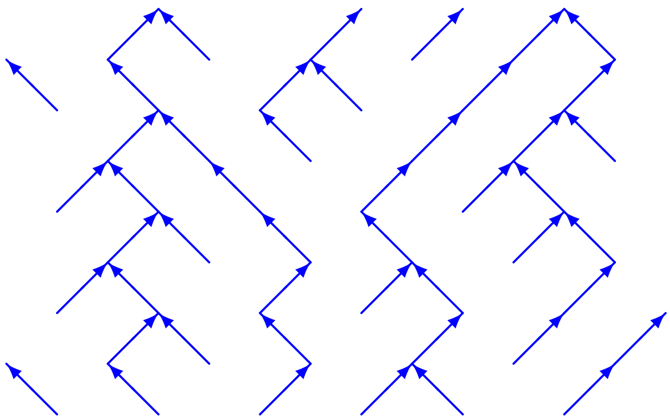

Figure 2.2: Nearest neighbour independent random walk system

Remark. The closure of Y_δ is obtained by adding paths of the form (f, t), where $t \in \delta^2\mathbb{Z} \cup \{-\infty, \infty\}$ and $f = +\infty$ or $f = -\infty$.

Thoerem 2.6. *For any sequence $\delta_m \downarrow 0$, $\overline{Y}_{\delta_m}$ converges in distribution to $\overline{\mathcal{W}}$.*

Proof : It is enough to verify (I1), (B1) and (B2). The condition (I1) follows the Donsker's invariance principle. For (B1) and (B2), we have the following proposition.

Let $X_1, X_2, \ldots, X_n$ be n independent simple symmetric coalescing random walks on $\mathbb{Z}$, starting on points $l_1 < l_2 < \cdots < l_n$ respectively. We can assume that l_i's are all even. Let $T \geq 1$ and $\eta := \big(\eta(X_1, \ldots, X_n)\big) = |\{X_1(T), X_2(T), \ldots, X_n(T)\}|$, as the number of random walk remaining at time T.

Proposition 2.7. *For the coalescing random walks, we have, for any $k \geq 1$,*

$$\mathbb{P}(\eta \geq k+1) \leq \mathbb{P}(\eta \geq k)\mathbb{P}(\eta \geq 2) \leq \big[\mathbb{P}(\eta \geq 2)\big]^k. \tag{2.6}$$

Proof : Fix m such that $k \leq m \leq n$ and consider a simple random walks Y_m starting at l_m. Suppose $\xi_1, \xi_2, \ldots, \xi_{m-1}$ are non-crossing paths having coalescing properties such that $\eta(\xi_1, \xi_2, \ldots, \xi_{n-1}) = k-1$ starting at $l_1, l_2, \ldots, l_{m-1}$ respectively. Suppose the collection of such paths be denoted by Π_1. Furthermore ξ_n be another path of simple symmetric random walk starting at l_n. Denote by Π_2 the collection of such paths. Note that both the collections are finite. Then, we have

$$\begin{aligned}
&\mathbb{P}(\eta \geq k+1) \\
&= \sum_{m=k}^{n-1} \sum_{(\xi_1,\xi_2,\ldots,\xi_{m-1})\in\Pi_1, \xi_n\in\Pi_2} \mathbb{P}\big\{(X_1, X_2, \ldots, X_{m-1}) = (\xi_1, \xi_2, \ldots, \xi_{m-1}), \\
&\qquad X_n = \xi_n, \xi_{m-1}(t) < Y_m(t) < \xi_n(t) \text{ for } 0 \leq t \leq T\big\} \\
&= \sum_{m=k}^{n-1} \sum_{(\xi_1,\xi_2,\ldots,\xi_{m-1})\in\Pi_1, \xi_n\in\Pi_2} \mathbb{P}\big\{\xi_{m-1}(t) < Y_m(t) < \xi_n(t) \text{ for } 0 \leq t \leq T\big| \\
&\qquad (X_1, X_2, \ldots, X_{m-1}) = (\xi_1, \xi_2, \ldots, \xi_{m-1}), X_n = \xi_n\big\} \\
&\qquad \times \mathbb{P}\big\{(X_1, X_2, \ldots, X_{m-1}) = (\xi_1, \xi_2, \ldots, \xi_{m-1}), X_n = \xi_n\big\} \\
&= \sum_{m=k}^{n-1} \sum_{(\xi_1,\xi_2,\ldots,\xi_{m-1})\in\Pi_1, \xi_n\in\Pi_2} \mathbb{P}\big\{\xi_{m-1}(t) < Y_m(t) < \xi_n(t) \text{ for } 0 \leq t \leq T\big\} \\
&\qquad \times \mathbb{P}\big\{(X_1, X_2, \ldots, X_{m-1}) = (\xi_1, \xi_2, \ldots, \xi_{m-1})\big\}\mathbb{P}\big\{X_n = \xi_n\big\}.
\end{aligned}$$

Consider the events, $A = \{Y_m(t) > \xi_M(t) : 0 \leq t \leq T\}$ and $B = \{Y_m(t) < \xi_m(t) : 0 \leq t \leq T\}$. Suppose the increments of Y_m are given by the random variables $\{I_i : i \geq 0\}$ where I_i's are i.i.d. and taking values in $\{-1, +1\}$ with probability $1/2$ each. Consider the sample space $\{-1, +1\}^T$ and the product probability measure with $1/2$ probability for -1 and $+1$. Clearly the event

$$A = \{l_m + \sum_{j=1}^{t} I_i > \xi_{m-1}(t) : 0 \leq t \leq T\}$$

is increasing. Similarly, the event

$$B = \{l_m + \sum_{j=1}^{t} I_i < \xi_n(t) : 0 \leq t \leq T\}$$

is decreasing. Thus, by FKG inequality, $P(A \cap B) \leq P(A)P(B)$.

Now summing over all possible values of ξ_n, we have

$$\begin{aligned}&\sum_{\xi_n \in \Pi_2} \mathbb{P}\{X_n = \xi_n\}\mathbb{P}\{l_m + \sum_{j=1}^{t} I_i < \xi_n(t) : 0 \leq t \leq T\} \\ &\leq \mathbb{P}\big(Y_m(t) < X_n(t) \text{ for } 0 \leq t \leq T\big) \leq \mathbb{P}\big(\eta(X_1, \dots, X_n) \geq 2\big).\end{aligned}$$

Therefore, summing over all possible $\xi_1, \xi_2, \dots, \xi_{m-1}$, we have the result. ■

We now complete the proof of Theorem 2.6. We scale it back to $\delta = 1$, and observe that $\mathbb{P}\{\mu_m(0; t; 0, \epsilon) \geq 2\}$ is dominated by two random walks starting at $(0, 0)$ and $(n, 0)$ do not coalesce by time t/δ_m^2 where $n = 2\left(\left[\frac{t}{2\delta_m}\right] + 1\right)$. As $\delta_m \to 0$, by Donsker's invariance principle, this probability converges to

$$\begin{aligned}\theta(\epsilon, t) = \mathbb{P}\big(&\text{Two independent brownian motions starting at distance } \epsilon \\ &\text{do not coalesce before time } t\big) \leq \frac{C\epsilon}{\sqrt{t}}\end{aligned}$$

for some $C > 0$. This completes the proof.

Remark. The Proposition 2.5 follows from Proposition 2.7 and Theorem 2.6.

Remark. If the random walk increments satisfy the condition of finiteness of fifth absolute moment, then, diffusively scaled random walks paths converge to the Brownian web.

2.5 Convergence of two-dimensional drainage network

For $\mathbf{z} = (z_1, z_2) \in \mathbb{Z}^2$, we define the path $X^{\mathbf{z}} = \{X^{\mathbf{z}}(s) : s \geq z_2\}$ in $\mathbb{R}$ as the linearly interpolated line composed of all edges $\{[\mathbf{z}_i, \mathbf{z}_{i+1}] : i \geq 0\}$ where $\mathbf{z}_0 = \mathbf{z}$ and $M(\mathbf{z}_i) = \mathbf{z}_{i+1}$

is the mother of $\mathbf{z}_i$. Clearly $X^{\mathbf{z}}$ is a continuous path starting at (z_1, z_2). We let

$$\chi = \{X^{\mathbf{z}} : \mathbf{z} \in \mathbb{Z}^2\}$$

which we also call the drainage network. We consider the diffusive scaling of χ, i.e., for $\delta > 0$,

$$\chi_\delta = \{(\delta x_1, \delta^2 x_2) \in \mathbb{R}^2 : (x_1, x_2) \in \chi\}.$$

Again, it can be shown that $\overline{\chi}_\delta$ is a compact set in (Π, d).

Thoerem 2.7. *The scaled drainage network* $\overline{\chi}_\delta$ *converges in distribution to the Brownian web* $\overline{\mathcal{W}}$.

Let $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$ be such that $\mathbf{u}(1) < \mathbf{v}(1)$ and $\mathbf{u}(2) = \mathbf{v}(2)$. Consider $X^{\mathbf{u}}$ and $X^{\mathbf{v}}$ and for $t \geq \mathbf{u}(2)$ and define

$$Z_t = Z_t(\mathbf{u}, \mathbf{v}) = X_{\mathbf{v}}(t) - X_{\mathbf{u}}(t). \tag{2.7}$$

Note that, $Z_{\mathbf{u}(2)} = \mathbf{v}(1) - \mathbf{u}(1)$.

It has already been shown that $\{Z_t(\mathbf{u}, \mathbf{v}) : t \geq \mathbf{u}(2)\}$ is a non-negative $\mathbb{L}_2$ martingale. Further, $Z_t(\mathbf{u}, \mathbf{v}) \to 0$ as $t \to \infty$ almost surely. Define

$$\tau(\mathbf{u}, \mathbf{v}) = \inf\{t - \mathbf{u}(2) : t \geq \mathbf{u}(2), Z_t(\mathbf{u}, \mathbf{v}) = 0\}.$$

Thoerem 2.8. *There exists a constant* $C > 0$ *such that*

$$\mathbb{P}(\tau(\mathbf{u}, \mathbf{v}) \geq t) \leq \frac{C\big(\mathbf{v}(2) - \mathbf{u}(2)\big)}{\sqrt{t}} \tag{2.8}$$

for $t \geq 1$.

Thoerem 2.9. *Let* $(y_1, s_1), \ldots, (y_k, s_{k+1})$ *be* $(k+1)$ *distinct points in* $\mathbb{R}^2$ *such that* $s_1 \leq \cdots \leq s_k$ *and if* $s_{i-1} = s_i$ *for some* $i = 1, \ldots, k$, *then* $y_{i-1} < y_i$. *Let* $Z_n^{(i)} = \{Z_n^{(i)}(t) = \frac{n^{-1/2} X^{\lfloor y\sqrt{n}\rfloor, \lfloor sn\rfloor}}{\sigma}(\lfloor nt \rfloor) : t \geq s_i\}$ *where* $\sigma^2 = \mathbb{V}ar(I^{(1)})|$. *Then,*

$$\{Z_n^{(i)} : i = 1, \ldots, k\} \Rightarrow \{W^{(i)} : i = 1, \ldots, k\}$$

as $n \to \infty$.

2.6 Convergence of supercirtical oriented percolation

Given a fixed $p \in (p_c, 1)$, let $\alpha := \alpha(p) > 0$ and $\sigma := \sigma(p) > 0$ be as introduced earlier. We know that, conditioned on the event that $o := (0,0)$ being a percolation point, $\frac{\gamma_o(n)-\alpha n}{\sigma\sqrt{n}}$ converges in distribution to a standard normal.

For each percolation point $z = (x,i) \in \mathcal{K}$, we first extend the definition of the rightmost infinite open path γ_z from the domain $\{i, i+1, \ldots\}$ to $[i,\infty]$ such that γ_z interpolates linearly between consecutive integer times and $\gamma_z(\infty) = *$. With this extended definition of γ_z, which we still denote by γ_z for convenience, it becomes a path in the space (Π, d). We will then let $\Gamma := \{\gamma_z : z \in \mathcal{K}\}$ denote the set of extended rightmost infinite open paths in the percolation configuration. Since paths in Γ are a.s. equicontinuous, $\overline{\Gamma}$, the closure of Γ in (Π, d), is a.s. compact and hence $\overline{\Gamma}$ is a random variable taking values in $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$, the space of compact subsets of (Π, d). Note that $\overline{\Gamma}\backslash\Gamma$ only contains paths of the form $\pi : [\sigma_\pi, \infty] \to [-\infty, \infty] \cup \{*\}$ with either $\sigma_\pi \in \mathbb{R}$ and $\pi(t) \equiv \pm\infty$ for all $t \geq \sigma_\pi$; or $\sigma_\pi = \infty$; or $\sigma_\pi = -\infty$, in which case for any $t > -\infty$, there exists some $\gamma \in \Gamma$ such that $\pi = \gamma$ on $[t, \infty]$. In other words, taking the closure of Γ in (Π, d) does not alter the configuration of paths in Γ restricted to any finite space-time region. Therefore it suffices to study properties of Γ instead of $\overline{\Gamma}$ in our analysis.

To remove a common drift from all paths in Γ and perform diffusive scaling of space and time, we define for any $a \in \mathbb{R}$, $b, \epsilon > 0$, a shearing and scaling map $S_{a,b,\epsilon} : \bar{\mathbb{R}}^2 \to \bar{\mathbb{R}}^2$ with

$$S_{a,b,\epsilon}(x,t) := \begin{cases} \big(\frac{\sqrt{\epsilon}}{b}(x - at), \epsilon t\big) & \text{if } (x,t) \in \mathbb{R}^2, \\ (\pm\infty, \epsilon t) & \text{if } (x,t) = (\pm\infty, t) \text{ with } t \in \mathbb{R}, \\ (*, \pm\infty) & \text{if } (x,t) = (*, \pm\infty), \end{cases} \tag{2.9}$$

where a is the drift that is being removed by a shearing of R^2_{c}, ϵ is the diffusive scaling parameter, and b determines the diffusion coefficient in the diffusive scaling. When t is understood to be a time, we will define

$$S_{a,b,\epsilon} t := \epsilon t. \tag{2.10}$$

Note that $S_{a,b,\epsilon}$ can be obtained by first applying the shearing map $S_{a,1,1}$ and then the diffusive scaling map $S_{0,b,\epsilon}$. By identifying a path $\pi \in \Pi$ with its graph in R^2_{c}, we can also define $S_{a,b,\epsilon} : (\Pi, d) \to (\Pi, d)$ by applying $S_{a,b,\epsilon}$ to each point on the graph of π.

Similarly, if $K \subset \Pi$, then $S_{a,b,\epsilon}K := \{S_{a,b,\epsilon}\pi : \pi \in K\}$. If $K \in \mathcal{H}$, then it is clear that also $S_{a,b,\epsilon}K \in \mathcal{H}$. Therefore $S_{\alpha,\sigma,\epsilon}\overline{\Gamma}$ is also an $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variable.

We can now formulate the main result of this paper.

Thoerem 2.10. [**Convergence to the Brownian web**] *Let $p \in (p_c, 1)$ and let $\overline{\Gamma}$ be defined as above. There exist $\alpha, \sigma > 0$ such that as $\epsilon \downarrow 0$, the sequence of $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variables $S_{\alpha,\sigma,\epsilon}\overline{\Gamma}$ converges in distribution to the standard Brownian web $\mathcal{W}$.*