NONSINGULAR GROUP ACTIONS, FINITELY GENERATED ABELIAN GROUPS AND STABLE RANDOM FIELDS

PARTHANIL ROY, INDIAN STATISTICAL INSTITUTE

Lecture 1

These lecture notes will be uploaded in `https://sites.google.com/site/parthanilroy/lps_notes` and updated regularly during December 6-10, 2013. It will be greatly appreciated if you point out typos, mistakes, etc. In order to avoid unnecessary wastage of papers, a log of the updates will be maintained. It is suggested that the references are printed after LPS VIII.

Question 0.1 (Ice-breaker)**.** *How do we prove central limit theorem?*

Theorem 0.2. *$X_1, X_2, \dots$ are iid with $E(X_1) = 0, V(X_1) = \sigma^2 < \infty$ $\Rightarrow Z_n := \frac{X_1+X_2+\cdots+X_n}{\sqrt{n}} \xrightarrow{\mathcal{L}} N(0,\sigma^2)$.*

An Extremely Sloppy Proof. Note that $E(X_1) = 0, V(X_1) = \sigma^2 < \infty$ $\Rightarrow E(e^{tX_1}) = E\big(1 + itX + \frac{(itX)^2}{2} + \cdots\big) \approx 1 - \frac{t^2\sigma^2}{2}$ for "small" t. Therefore,

$$E(e^{i\theta Z_n}) = E\left(e^{i\frac{\theta}{\sqrt{n}}(X_1+X_2+\cdots+X_n)}\right) = \left(E\big(e^{i\frac{\theta}{\sqrt{n}}X_1}\big)\right)^n \approx \left(1 - \frac{\theta^2\sigma^2}{2n}\right)^n \to e^{-\frac{\sigma^2\theta^2}{2}},$$

which proves the result. □

Question 0.3. *What happens when $V(X_1) = \infty$?*

Here is a partial answer (see Feller (1971), pg 581).

Theorem 0.4. *$X_1, X_2, \dots$ are iid symmetric rvs with $P(|X_1| > \lambda) \sim \kappa\lambda^{-\alpha}$ as $\lambda \to \infty$ for some $\kappa > 0$ and $\alpha \in (0,2) \Rightarrow Y_n := \frac{X_1+X_2+\cdots+X_n}{n^{1/\alpha}} \xrightarrow{\mathcal{L}} Y$ with characteristic function $E(e^{i\theta Y}) = \exp(-C_\alpha^{-1}\kappa|\theta|^\alpha), \theta \in \mathbb{R}$, where C_α is the stable tail constant given by*

$$C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin x\,dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \text{if } \alpha \neq 1, \\ \frac{2}{\pi} & \text{if } \alpha = 1. \end{cases} \tag{0.1}$$

Yet Another Sloppy Proof. It is possible to show that $P(|X_1| > \lambda) \sim \kappa\lambda^{-\alpha}$ as $\lambda \to \infty \Rightarrow E(e^{itX_1}) \approx 1 - C_\alpha^{-1}\kappa|t|^\alpha$ for "small" t

$$\Rightarrow E(e^{i\theta Y_n}) = E\left(e^{i\frac{\theta}{n^{1/\alpha}}(X_1+X_2+\cdots+X_n)}\right)$$

$$= \left(E\left(e^{i\frac{\theta}{n^{1/\alpha}}X_1}\right)\right)^n \approx \left(1 - \frac{C_\alpha^{-1}\kappa|\theta|^\alpha}{n}\right)^n \to \exp\left(-C_\alpha^{-1}\kappa|\theta|^\alpha\right). \square$$

1. Symmetric α-stable Distribution

Definition 1.1. *A rv X is said to follow symmetric α-stable ($S\alpha S$) distribution ($\alpha \in (0,2]$ is called the index of stability) with scale parameter $\sigma > 0$ if its characteristic function is of the form*

$$E(e^{i\theta X}) = e^{-\sigma^\alpha|\theta|^\alpha},\ \theta \in \mathbb{R}.$$

Note that because of Theorem 0.4 above and Levy's continuity theorem, it follows that this is indeed a valid characteristic function. SαS distribution is a subclass of a more general class of distributions called stable distribution.

Notation. $X \sim S\alpha S(\sigma)$.

Property 1.2 (Known Distributions)**.** *(a) $\alpha = 1 \Rightarrow X \sim$ Cauchy distribution with density function $f_X(x) = \frac{\sigma}{\pi(x^2+\sigma^2)}$, $-\infty < x < \infty$.*
(b) $\alpha = 2 \Rightarrow X \sim N(0, 2\sigma^2)$.

These are the only two cases in which the density functions are known in closed form. For the other values of α, X is supported on $\mathbb{R}$ with continuous density function that can be written in a series. See, for example, Ibragimov and Linnik (1971), Feller (1971) and Zolotarev (1986). We shall assume from now on that $0 < \alpha < 2$.

Property 1.3. *$X_i \sim S\alpha S(\sigma_i)$, $i = 1, 2$, $X_1 \perp\!\!\!\perp X_2$*
$\Rightarrow a_1X_1 + a_2X_2 \sim S\alpha S\big((|a_1|^\alpha\sigma_1^\alpha + |a_2|^\alpha\sigma_2)^{1/\alpha}\big)$. In particular, $X \stackrel{\mathcal{L}}{=} -X$.

Property 1.4. *$X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} S\alpha S(\sigma) \Rightarrow \sum_{i=1}^n X_i \stackrel{\mathcal{L}}{=} n^{1/\alpha}X_1$.*

Property 1.5. *$X \sim S\alpha S(\sigma), \alpha \in (0,2) \Rightarrow P(|X| > \lambda) \sim \sigma^\alpha C_\alpha \lambda^{-\alpha}$ as $\lambda \to \infty$, where C_α is as in (0.1).*

Sketch of Proof for $0 < \alpha < 1$. **Step 1.** The Laplace transform of $|X|$ is $E(e^{-\gamma|X|}) = \exp\left(-\frac{\sigma^\alpha}{\cos(\pi\alpha/2)}\gamma^\alpha\right)$, $\gamma \geq 0$. (Use Proposition 1.2.12 and Property 1.2.13 of Samorodnitsky and Taqqu (1994).)

Step 2. Using integration by parts,

$$\int_0^\infty e^{-\gamma\lambda}P(|X| > \lambda)d\lambda = \frac{1 - E(e^{-\gamma|X|})}{\gamma} \sim \frac{\sigma^\alpha}{\cos(\pi\alpha/2)}\gamma^{\alpha-1}$$

as $\gamma \to 0$.

Step 3. Step 2 + Theorem XIII.5.4 of Feller (1971) $\Rightarrow P(|X| > \lambda) \sim \frac{\sigma^\alpha}{\cos(\pi\alpha/2)\Gamma(1-\alpha)}\lambda^{-\alpha} = \sigma^\alpha C_\alpha \lambda^{-\alpha}$ since $0 < \alpha < 1$. □

See Feller (1971) and Samorodnitsky and Taqqu (1994) for the details in the $0 < \alpha < 1$ case and the proof in the $1 < \alpha < 2$ case.

Exercise 1. *Prove Property 1.5 for $\alpha = 1$.*

Corollary 1.6. *For $0 < \alpha < 2$, $E|X|^p < \infty$ if $0 < p < \alpha$ and $E|X|^p = \infty$ if $p \geq \alpha$.*

The following series representation of an Sα S random variable will be extremely useful for us later in this mini course.

Theorem 1.7. *Let $\{\epsilon_i\}_{i\geq 1}$, $\{\Gamma_i\}_{i\geq 1}$, $\{W_i\}_{i\geq 1}$ be three independent sequences of rvs, where $\epsilon_1, \epsilon_2, \ldots \overset{iid}{\sim} \pm 1$ with probability $1/2$ each, $\Gamma_1 < \Gamma_2 < \cdots$ are the arrival times of a homogeneous Poisson process with unit arrival rate, and $W_1, W_2, \ldots$ are iid satisfying $E|W_1|^\alpha < \infty$. Then the series*

$$\sum_{i=1}^{\infty} \epsilon_i \Gamma_i^{-1/\alpha} W_i \tag{1.1}$$

converges almost surely to a rv $X \sim S\alpha S\left((C_\alpha^{-1} E|W_1|^\alpha)^{1/\alpha}\right)$.

Remark 1.8. It can be shown that $P\left(|\epsilon_1 \Gamma_1^{-1/\alpha} W_1| > \lambda\right) \sim E|W_1|^\alpha \lambda^{-\alpha}$ as $\lambda \to \infty$ whereas $P\left(\sum_{i=2}^{\infty} \epsilon_i \Gamma_i^{-1/\alpha} W_i > \lambda\right) = o(\lambda^{-\alpha})$ as $\lambda \to \infty$; see pg 26-28 of Samorodnitsky and Taqqu (1994). This means that the first term $\epsilon_1 \Gamma_1^{-1/\alpha} W_1$ is the dominating term of the series (1.1) that provides "correct asymptotics to its tail" and the rest of the terms provide the "necessary corrections" for the sum to have an SαS distribution. This is regarded as the "one large jump" heuristics for an SαS rv.

Sketch of Proof of Theorem 1.7. **Step 1.** Three series theorem (Feller (1971), Theorem IX.9.3) can be used to show that the series (1.1) converges almost surely as $n \to \infty$. This is not completely straightforward but somewhat routine; see pg 24-25 of Samorodnitsky and Taqqu (1994).

Step 2. Use a "cool trick" from elementary probability theory to identify the distribution of the (almost surely) convergent series (1.1). Take a sequence of $U_1, U_2, \ldots \overset{iid}{\sim} Unif(0,1)$ independent of $\{\epsilon_i\}_{i\geq 1}$ and $\{W_i\}_{i\geq 1}$. Recall that for each n,

$$\left(\frac{\Gamma_1}{\Gamma_{n+1}}, \frac{\Gamma_2}{\Gamma_{n+1}}, \ldots, \frac{\Gamma_n}{\Gamma_{n+1}}\right) \overset{\mathcal{L}}{=} (U_{(1)}, U_{(2)}, \ldots, U_{(n)}),$$

where $U_{(1)} < U_{(2)} < \cdots < U_{(n)}$ are the order-statistics obtained from the random sample $(U_1, U_2, \ldots, U_n)$. Using this equality of distribution and an exchangeability argument,

$$\left(\frac{\Gamma_{n+1}}{n}\right)^{1/\alpha} \sum_{i=1}^{n} \epsilon_i \Gamma_i^{-1/\alpha} W_i \stackrel{\mathcal{L}}{=} \frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} \epsilon_i U_{(i)}^{-1/\alpha} W_i \stackrel{\mathcal{L}}{=} \frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} \epsilon_i U_i^{-1/\alpha} W_i \stackrel{\mathcal{L}}{\longrightarrow} X$$

(here $X \sim S\alpha S\left((C_\alpha^{-1} E|W_1|^\alpha)^{1/\alpha}\right)$) by Theorem 0.4 and the following exercise.

Exercise 2. *$\{\epsilon_i U_i^{-1/\alpha} W_i\}_{i\geq 1}$ is a sequence of iid symmetric rvs such that $P(|\epsilon_1 U_1^{-1/\alpha} W_1| > \lambda) \sim E|W_1|^\alpha \lambda^{-\alpha}$ as $\lambda \to \infty$.*

From the above convergence in distribution, Theorem 1.7 follows because $\Gamma_{n+1}/n \stackrel{a.s.}{\longrightarrow} 1$ as $n \to \infty$ by the strong law of large numbers. □

2. SαS Random Fields and Long Range Dependence

Definition 2.1. *A random vector $\mathbf{X} := (X_1, X_2, \ldots, X_k)$ is said to follow multivariate $S\alpha S$ distribution if each nondegenerate linear combination $\sum_{i=1}^k c_i X_i$ $(c_1, c_2, \ldots, c_k \in \mathbb{R})$ follows $S\alpha S$ distribution. In this case, $\mathbf{X}$ is called an $S\alpha S$ random vector.*

The following result gives a very nice and useful characterization of an SαS random vector.

Theorem 2.2. *$\mathbf{X} \in \mathbb{R}^k$ is an $S\alpha S$ random vector with $0 < \alpha < 2$ if and only if there exists a unique finite symmetric measure Γ on the unit sphere $S_k := \{\mathbf{x} : \|\mathbf{x}\|_2 = 1\}$ such that*

$$E(e^{i\theta^T \mathbf{x}}) = \exp\left\{-\int_{S_k} |\theta^T \mathbf{x}|^\alpha \,\Gamma(d\mathbf{x})\right\}. \tag{2.1}$$

Proof. See Kuelbs (1973). □

Γ is called the spectral measure of the SαS random vector $\mathbf{X}$.

Definition 2.3. *A stochastic process $\{X_t\}_{t\in T}$ is called an $S\alpha S$ process (indexed by T) if all of its finite-dimensional distributions are multivariate $S\alpha S$ distributions. When $T = \mathbb{Z}^d$ or $\mathbb{R}^d$ for some $d \in \mathbb{N}$, $\{X_t\}_{t\in T}$ is called an $S\alpha S$ random field.*

We shall give examples of SαS random fields later. We first present the big picture of this mini course.

2.1. **The Big Picture.** The main goal of this mini course is to formalize the phrase "long range dependence" for SαS random fields. We start with a brief discussion on this terminology.

Long range dependence (also known as long memory), a property observed in many real life processes, refers to dependence between observations X_t far separated in t. Historically, it was first observed by a famous British hydrologist Harold Edwin Hurst, who noticed an empirical phenomenon (now known as Hurst phenomenon; see Hurst (1951) and Hurst (1955)) while looking at measurements of the water flow in the Nile River. In the 1960s a series of papers of Benoit Mandelbrot and his co-workers tried to explain Hurst phenomenon using long range dependence. See Mandelbrot and Wallis (1968) and Mandelbrot and Wallis (1969). From then on processes having long memory have been used in many different areas including economics, internet modelling, climate studies, linguistics, DNA sequencing etc. For a detailed discussion on long range dependence, see Samorodnitsky (2006) and the references therein.

Most of the classical definitions of long range dependence appearing in literature are based on the second order properties (e.g.- covariances, spectral density, and variances of partial sums etc) of stochastic processes mainly because of their simplicity and statistical tractability. For example, one of the most widely accepted definitions of long range dependence for a stationary Gaussian process is that a stationary Gaussian process has long range dependence if its correlation function decays slowly enough to make it not summable. In the heavy tails context, however, this definition becomes ambiguous because correlation function may not even exist in heavy tails case and even if it exists it may not have enough information about the dependence structure of the process.

Lecture 2

In the context of stationary $S\alpha S$ processes $(0 < \alpha < 2)$ indexed by $\mathbb{Z}$, instead of looking for a substitute for correlation function, Samorodnitsky (2004a) suggested a new approach through phase transition phenomena as follows. Suppose that $(P_\theta, \theta \in \Theta)$ is a family of laws of a stationary stochastic process, where θ is a parameter of the process lying in a parameter space Θ. If Θ can be partitioned into Θ_0 and Θ_1 in such a way that a significant number of functionals of this stochastic process change dramatically as we pass from Θ_0 to Θ_1, then this phase transition can be thought of as a change from short memory to long memory. The aforementioned paper investigates the rate of growth of the partial maxima of the stationary $S\alpha S$ process indexex by $\mathbb{Z}$. A transition boundary is observed based on the ergodic theoretical properties of the underlying nonsingular group action obtained from the seminal work of Rosiński (1995). In this minicourse, we shall discuss this work and its extension to the SαS random fields.

3. SαS Random Measures and Integrals

We shall now introduce SαS random measures and integral wrt such measures. In fact, we shall first introduce the integral and then define the random measure. Let $(E, \mathcal{E}, m)$ be a σ-finite measure space, $0 < \alpha < 2$ and

$$F := L^{\alpha}(E, \mathcal{E}, m) = \left\{ f : E \to \mathbb{R} : \|f\|_{\alpha} := \left(\int_E |f|^{\alpha} dm \right)^{1/\alpha} < \infty \right\}.$$

Note that F is a Banach space when $1 \le \alpha < 2$ (but not a Hilbert space) with the norm $\|\cdot\|_{\alpha}$. However for $0 < \alpha < 1$, $\|\cdot\|_{\alpha}$ is not even a norm and hence F has very little structure. It is a metric space with the distance function $d_{\alpha}(f,g) := \|f - g\|_{\alpha}^{\alpha}$. In particular, F is a very rigid space for all $\alpha \in (0,2)$ in the sense that it has very few isometries. We shall exploit this rigidity in the second half of this mini course.

Goal. Define an SαS process $\{I(f) : f \in F\}$ indexed by F so that $M(A) := I(\mathbb{1}_A)$, $A \in \mathcal{E}_0 := \{A \in \mathcal{E} : m(A) < \infty\}$ becomes an "SαS random measure" and $I(f)$ becomes the "integral wrt M".

We attain this goal as follows. Given $f_1, f_2, \ldots, f_k \in F$, we define a probability measure $P_{f_1, f_2, \ldots, f_k}$ on $\mathbb{R}^k$ by its characteristic function as follows

$$\psi_{f_1, f_2, \ldots, f_k} = \exp\left\{ - \left\| \sum_{j=1}^{k} \theta_j f_j \right\|_{\alpha}^{\alpha} \right\}. \tag{3.1}$$

Proposition 3.1. *For any $f_1, f_2, \ldots, f_k \in F$, $\psi_{f_1, f_2, \ldots, f_k}$ is the characteristic function of an SαS random vector. In particular, $P_{f_1, f_2, \ldots, f_k}$ is well-defined.*

Proof. The proof follows from Theorem 2.2 and the following exercise.

Exercise 3. *Let $E_+ := \left\{ x \in E : \sum_{j=1}^{k} \left(f_j(x) \right)^2 > 0 \right\}$. Define a measure Γ on the unit sphere S_k as*

$$\Gamma(A) := \frac{1}{2} \int_{\pi(A)} \left(\sum_{j=1}^{k} f_j^2 \right)^{\alpha/2} dm + \frac{1}{2} \int_{\pi(-A)} \left(\sum_{j=1}^{k} f_j^2 \right)^{\alpha/2} dm, \;\; A \subseteq S_k,$$

where

$$\pi(A) := \left\{ x \in E_+ : \left(\frac{f_1(x)}{\sqrt{\sum_{j=1}^k \big(f_j(x)\big)^2}}, \dots, \frac{f_k(x)}{\sqrt{\sum_{j=1}^k \big(f_j(x)\big)^2}} \right) \in A \right\}.$$

Then show that Γ *is a symmetric finite measure on* S_k *such that* $\psi_{f_1,f_2,\dots,f_k}$ *is of the form* (2.1). □

From Proposition 3.1 and Kolmogorov extention theorem, it follows that there exists an SαS process $\{I(f) : f \in F\}$ with finite-dimensional distributions of the form (3.1). In particular, each $I(f) \sim S\alpha S(\|f\|_\alpha)$.

Exercise 4 (I is linear and independently scattered)**.** *For all functions* $f_1, f_2, \dots, f_k \in F$ *and for all* $a_1, a_2, \dots a_k \in \mathbb{R}$,

$$I(a_1 f_1 + a_2 f_2 + \dots + a_k f_k) = a_1 I(f_1) + a_2 I(f_2) + \dots + a_k I(f_k)$$

almost surely. If further $f_1, f_2, \dots, f_k$ *have pairwise disjoint support, then* $I(f_1), I(f_2), \dots, I(f_k)$ *are independent.*

Definition 3.2. *Let* $(E, \mathcal{E}, m)$ *be a* σ*-finite measure space. A set function* M *defined on* $\mathcal{E}_0$ *is called an* $S\alpha S$ *random measure on* E *with control measure* m *if*

1. $\{M(A) : A \in \mathcal{E}_0\}$ *is a collection of rvs defined on the same probability space,*
2. *each* $M(A) \sim S\alpha S\big((m(A))^{1/\alpha}\big)$,
3. *if* $A_1, A_2, \dots, A_k$ *are pairwise disjoint, then* $M(A_1), M(A_2), \dots M(A_k)$ *are independent (*M *is independently scattered),*
4. *if* $A_1, A_2, \dots$ *are pairwise disjoint such that* $\bigcup_{i=1}^\infty A_i \in \mathcal{E}_0$*, then* $M(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty M(A_i)$ *almost surely (*M *is* σ*-additive).*

Proposition 3.3. *For every* σ*-finite measure space* $(E, \mathcal{E}, m)$ *there exists an* $S\alpha S$ *random measure on* E *with control measure* m*.*

Proof. Define $M(A) := I(\mathbb{1}_A)$, $A \in \mathcal{E}_0$. All the properties of an $S\alpha S$ random measure follows from the properties of I mentioned above except the σ-additivity, which can be established as follows. Note that finite additivity follows from Exercise 4 and therefore,

$$M\Big(\bigcup_{i=1}^\infty A_i\Big) - \sum_{i=1}^n M(A_i) \overset{a.s.}{=} M\Big(\bigcup_{i=n+1}^\infty A_i\Big) \sim S\alpha S\left(\Big(\sum_{i=n+1}^\infty m(A_i)\Big)^{1/\alpha}\right).$$

The above observation yields $\sum_{i=1}^n M(A_i) \xrightarrow{p} M(\bigcup_{i=1}^\infty A_i)$, which implies $\sum_{i=1}^n M(A_i) \xrightarrow{a.s.} M(\bigcup_{i=1}^\infty A_i)$ since $M(A_1), M(A_2), \dots$ are independent. □

Lecture 3

Here is a result that gives the motivation behind thinking $I(f)$ as an "integral of f wrt M".

Theorem 3.4. *$\{I(f) : f \in F\}$ defined above satisfies the following properties.*

(1) *If $f \in F$ is a simple function of the form $f = \sum_{j=1}^{k} c_j \mathbb{1}_{A_j}$ with pairwise disjoint $A_1, A_2, \dots, A_k \in \mathcal{E}_0$, then by linearity of I,*

$$I(f) \overset{a.s.}{=} \sum_{j=1}^{k} c_j M(\mathbb{1}_{A_j}) = \sum_{j=1}^{k} c_j M(A_j).$$

(2) *Let $f \in F$ be any function (not necessarily simple). Take a sequence of simple functions $\{f_n\}_{n\geq 1}$ such that $f_n \xrightarrow{a.s.} f$ and $|f_n| \leq g$ for some $g \in F$ (such a sequence always exits for any $f \in F$), then $I(f_n) \xrightarrow{p} I(f)$.*

Proof. The first part follows trivially from linearity of I. For the second part (including existence of such a sequence), see pg 122 - 124 of Samorodnitsky and Taqqu (1994). □

In view of the above result, we shall denote $I(f)$ by $\int_E f dM$ for $f \in F$. This motivates the following definition.

Definition 3.5. *The SαS process $\{I(f)\}_{f\in F}$ is called the integral (process) wrt the random measure M and this is denoted by*

$$\{I(f)\}_{f\in F} \overset{\mathcal{L}}{=} \left\{\int_E f(x) M(dx)\right\}_{f\in F}. \tag{3.2}$$

Note that the notation (3.2) is a fancy way of writing that $\{I(f)\}_{f\in F}$ is a stochastic process indexed by F such that for any $f_1, f_2, \dots, f_k \in F$, the joint characteristic function of $\big(I(f_1), I(f_2), \dots, I(f_k)\big)$ is given by (3.1).

Remark 3.6. For any $F_0 \subseteq F$, we can use the notation $\left\{\int_E f dM\right\}_{f\in F_0}$ to denote the SαS process $\{I(f)\}_{f\in F_0}$. This remark will be useful later in this mini course because we shall always work with a "suitably chosen" proper subset of F.

Example 3.7 (SαS Levy Random Measure)**.** Take $E = [0,\infty)$, $m = Leb$ (the Lebesgue measure on $[0,\infty)$) and let M be an SαS random measure on $[0,\infty)$ with control measure *Leb*. This M is called SαS Levy Random Measure. Define $X_t := M\big([0,t]\big)$, $t \geq 0$.

Exercise 5. *Show that $\{X_t\}_{t\geq 0}$ defined above an SαS process satisfying the following properties:*

(1) $X_0 \overset{a.s.}{=} 0$.
(2) $\{X_t\}_{t\geq 0}$ *has independent increments, i.e., for all* $0 \leq t_1 < t_2 < \cdots t_k < \infty$, $X_{t_1}, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_k} - X_{t_{k-1}}$ *are independent. (Follows from the fact that* M *is independently scattered.)*
(3) *For all* $0 \leq s < t < \infty$, $X_t - X_s \sim S\alpha S\big((t-s)^{1/\alpha}\big)$.
(4) *In particular,* $\{X_t\}_{t\geq 0}$ *has stationary increments, i.e., for all* $0 \leq t_1 < t_2 < \cdots t_k < \infty$ *and for all* $\tau \geq 0$,

$$\{X_t - X_0\}_{t\geq 0} \overset{\mathcal{L}}{=} \{X_{t+\tau} - X_\tau\}_{t\geq 0}.$$

(Equality of finite-dimensional distributions.)
(5) $\{X_t\}_{t\geq 0}$ *is self-similar with index* $1/\alpha$, *i.e., for all* $c > 0$,

$$\{X_{ct}\}_{t\geq 0} \overset{\mathcal{L}}{=} \{c^{1/\alpha} X_t\}_{t\geq 0}$$

$\{X_t\}_{t\geq 0}$ defined above is called SαS Levy motion. It is the extension of Brownian motion in the SαS world.

4. Integral Representation of an SαS Random Field

From now on, we shall only deal with SαS random fields. In order to keep life simple, we shall discuss the case $T = \mathbb{Z}^d$ for some $d \geq 1$.

Definition 4.1. *A family of functions* $\{f_t\}_{t\in\mathbb{Z}^d} \subseteq L^\alpha(S, \mathcal{S}, \mu)$ *(here* $(S, \mathcal{S}, \mu)$ *is a* σ*-finite standard Borel space) is called an integral representation of an SαS random field* $\{X_t\}_{t\in\mathbb{Z}^d}$ *if*

$$\{X_t\}_{t\in\mathbb{Z}^d} \overset{\mathcal{L}}{=} \left\{\int_S f_t(s) M(ds)\right\}_{t\in\mathbb{Z}^d}, \tag{4.1}$$

where M *is an SαS random measure on* S *with control measure* μ.

Note that (4.1) simply means that for all $t_1, t_2, \ldots, t_k \in \mathbb{Z}^d$,

$$E\left(e^{i\sum_{j=1}^k \theta_j X_{t_j}}\right) = \exp\left\{-\left\| \sum_{j=1}^k \theta_j f_{t_j} \right\|_\alpha^\alpha\right\},\ \theta_1, \theta_2, \ldots, \theta_k \in \mathbb{R}.$$

Lecture 4

Theorem 4.2. *Every SαS random field* $\{X_t\}_{t\in\mathbb{Z}^d}$ *has an integral representation.*

Proof. See Bretagnolle et al. (1966), Schreiber (1972),Schilder (1970). See also Kuelbs (1973) and Hardin Jr. (1982) for a discussion of history of (4.1). □

For any integral representation $\{f_t\}_{t\in\mathbb{Z}^d} \subseteq L^\alpha(S,\mathcal{S},\mu)$, one can assume without loss of generality that

$$\bigcup_{t\in\mathbb{Z}^d} \text{Support}(f_t) \overset{a.s.}{=} S.$$

From now on, we shall assume that this full support condition holds for all of our integral representations.

The converse of Theorem 4.2 holds, i.e., given any σ-finite measure space $(S,\mathcal{S},\mu)$, a family of functions $\{f_t\}_{t\in\mathbb{Z}^d} \subseteq L^\alpha(S,\mathcal{S},\mu)$ and an SαS random measure M on S with control measure μ, we can construct an SαS random field $\{X_t\}_{t\in\mathbb{Z}^d}$ using (4.1). This follows trivially from Remark 3.6 with $F_0 = \{f_t : t \in \mathbb{Z}^d\} \subseteq L^\alpha(S,\mathcal{S},\mu)$. Using this, one can construct many SαS random fields, one of which is discussed below.

Example 4.3 (Stationary SαS Moving Average Random Field)**.** This example was introduced (in the $d = 1$ case) by Surgailis et al. (1993). Let $(W,\mathcal{W},\nu)$ be a σ-finite measure space. Define $S = W\times\mathbb{Z}^d$, $\mu = \nu\otimes\eta$, where η is the counting measure on $\mathbb{Z}^d$. Let M be an SαS random measure on $W\times\mathbb{Z}^d$ with control measure $\nu\otimes\eta$. Take a single function $f \in L^\alpha(W\times\mathbb{Z}^d,\nu\otimes\eta)$ and define a family $\{f_t\}_{t\in\mathbb{Z}^d}$ of functions as

$$f_t(w,s) = f(w,s+t),\ \ (w,s) \in W\times\mathbb{Z}^d.$$

It is easy to check that each $f_t \in L^\alpha(W\times\mathbb{Z}^d,\nu\otimes\eta)$. The S$\alpha$S random field

$$\begin{aligned}\{X_t\}_{t\in\mathbb{Z}^d} &\overset{\mathcal{L}}{:=} \left\{\int_{W\times\mathbb{Z}^d} f_t(w,s)\,dM(w,s)\right\}_{t\in\mathbb{Z}^d} \\ &\overset{\mathcal{L}}{=} \left\{\int_{W\times\mathbb{Z}^d} f(w,s+t)\,dM(w,s)\right\}_{t\in\mathbb{Z}^d} \qquad (4.2)\end{aligned}$$

is called a stationary SαS moving average random field.

Definition 4.4. *A random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is called stationary if $\{X_t\}_{t\in\mathbb{Z}^d} \overset{\mathcal{L}}{=} \{X_{t+\tau}\}_{t\in\mathbb{Z}^d}$ for all $\tau\in\mathbb{Z}^d$.*

Exercise 6. *Show that $\{X_t\}_{t\in\mathbb{Z}^d}$ defined by* (4.2) *is stationary. If W is a singleton, then show that $\{X_t\}_{t\in\mathbb{Z}^d}$ is a moving average random field with iid SαS innovations.*

In view of the above exercise, one can think of $\{X_t\}_{t\in\mathbb{Z}^d}$ defined by (4.2) as a mixture of moving averages and hence it is called a mixed moving average. This example will play a very important role in this mini course.

The following notion (introduced by Hardin Jr. (1982)) is extremely technical and yet useful. We shall first give the definition and then state a theorem that will help us understand its meaning.

Definition 4.5. *An integral representation* $\{f_t\}_{t\in\mathbb{Z}^d} \subseteq L^\alpha(S,\mathcal{S},\mu)$ *of an* SαS *random field is called a minimal representation if for all* $B \in \mathcal{S}$, *there exists* $A \in \sigma\left\{f_t/f_{t'} : t,t' \in \mathbb{Z}^d\right\}$ *such that* $\mu(A\Delta B) = 0$.

The ratio $f_t(s)/f_{t'}(s)$ is defined to be ∞ when $f_t(s) > 0$, $f_{t'}(s) = 0$ and $-\infty$ when $f_t(s) < 0$, $f_{t'}(s) = 0$. In particular, the σ-algebra $\sigma\left\{f_t/f_{t'} : t,t' \in \mathbb{Z}^d\right\}$ is generated by a bunch of extended real-valued functions.

Theorem 4.6. *Every* SαS *random field has a minimal representation.*

The following result provides better insight into the notion of minimality of integral representations.

Theorem 4.7. *Let* $\{f^*_t\}_{t\in\mathbb{Z}^d} \subseteq L^\alpha(S^*,\mathcal{S}^*,\mu^*)$ *be a minimal representation of an* SαS *random field* $\{X_t\}_{t\in\mathbb{Z}^d}$ *and* $\{f_t\}_{t\in\mathbb{Z}^d} \subseteq L^\alpha(S,\mathcal{S},\mu)$ *be any integral representation of* $\{X_t\}_{t\in\mathbb{Z}^d}$. *Then there exist measurable functions* $\Phi : S \to S^*$ *and* $h : S \to \mathbb{R} \setminus \{0\}$ *such that*

$$\mu^*(A) = \int_{\Phi^{-1}(A)} |h|^\alpha d\mu \tag{4.3}$$

and for each $t \in \mathbb{Z}^d$,

$$f_t(s) = h(s) f^*_t(\Phi(s)) \quad \textit{for } \mu\textit{-almost all } s \in S. \tag{4.4}$$

If further $\{f_t\}_{t\in\mathbb{Z}^d}$ *is also a minimal representation, then* Φ *and* h *are unique modulo* μ, Φ *is one-to-one and onto,* $\mu^* \circ \Phi \sim \mu$ *and*

$$|h|^\alpha = \frac{d(\mu^* \circ \Phi)}{d\mu} \quad \mu\textit{-almost surely.} \tag{4.5}$$

Proofs of Theorems 4.6 and 4.7. These proofs use deep analysis of L^α spaces; see Hardin Jr. (1981, 1982). Theorem 4.7 follows from the rigidity (dearth of isometry) of L^α spaces, $0 < \alpha < 2$. □

Theorem 4.7 provides some sort of uniqueness to integral representations of SαS random fields and we shall capitalize on it heavily in this mini course. Since any integral representation can be expressed in terms of a minimal representation using (4.4), $\{f^*_t\}_{t\in\mathbb{Z}^d}$ should be regarded as a minimal element in the set of all integral representations. However it should be noted that in general, it is extremely difficult to check that a given integral representation is minimal. See Rosiński (1994), Rosiński (1995) and Rosiński (2006) for various useful results on minimal representations.

Lecture 5

5. The Stationary Case

From now on, we shall assume that our SαS random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is stationary (see Definition 4.4 above). Note that this means that for all $t_1, t_2, \ldots, t_k, \tau \in \mathbb{Z}^d$ and for all $c_1, c_2, \ldots, c_k \in \mathbb{R}$, either $\sum_{i=1}^k c_i X_{t_i+\tau} \overset{a.s.}{=} 0$ or $\sum_{i=1}^k c_i X_{t_i+\tau}$ follows an $S\alpha S$ distribution whose scale parameter does not depend on τ. The mixed moving average random field defined by (4.2) serves as an important class of examples of such fields.

The ultimate goal of this mini course is to study the asymptotic behaviour of a maxima sequence of $\{X_t\}_{t\in\mathbb{Z}^d}$ as t varies in hypercubes of increasing size. More precisely, define for all $n \geq 1$,

$$B_n = \Big\{t = (t_1, t_2, \ldots, t_d) \in \mathbb{Z}^d : \text{ each } t_i \in \{0, 1, 2, \ldots, n-1\}\Big\},$$

and

$$M_n := \max_{t\in B_n} |X_t|,\ n \geq 1. \tag{5.1}$$

We shall eventually answer the following questions for most of the important cases.

Question 5.1. *What is the rate of growth of M_n (as $n \to \infty$)?*

Question 5.2. *If we know the rate of growth of M_n, can we find its scaling limit?*

If $\{X_t\}_{t\in\mathbb{Z}^d} \overset{iid}{\sim} S\alpha S(\sigma)$, then by Proposition 1.11 of Resnick (1987) and Property 1.5 above, it follows that M_n grows like $n^{d/\alpha}$ as $n \to \infty$ and $M_n/n^{d/\alpha} \xrightarrow{\mathcal{L}} aZ_\alpha$, where $a > 0$ is a deterministic constant and Z_α is a Frechet type extreme value rv with distribution function

$$P(Z_\alpha \leq z) = \begin{cases} e^{-z^{-\alpha}}, & z > 0, \\ 0, & z \leq 0. \end{cases} \tag{5.2}$$

As long as, the random field $\{X_t\}_{t\in\mathbb{Z}^d}$ has short memory, it is expected to exhibit the same rate of growth of M_n. On the other hand, if $\{X_t\}_{t\in\mathbb{Z}^d}$ has long memory, then M_n is expected to grow slowly because this strong dependence will prevent erratic changes in the value of X_t even when $\|t\|_\infty := \max_{1\leq i\leq n} |t_i|$ becomes large. We shall indeed observe a phase transition in the rate of growth of M_n as $n \to \infty$. Because of the intuitions given above, this phase transition can be regarded as a passage from short memory to long memory; see Samorodnitsky (2004a) and Roy and Samorodnitsky (2008).

In order to study the rate of growth of M_n, we need to know more about the integral representation of stationary SαS random fields. It so happens that in the stationary case, any minimal representation of $\{X_t\}_{t\in\mathbb{Z}^d}$ has a very nice form in terms of a *nonsingular* $\mathbb{Z}^d$*-action* and an *associated cocycle*. We introduce these terminologies below. See Varadarajan (1970), Zimmer (1984), Krengel (1985) and Aaronson (1997) for detailed discussions of these ergodic theoretic notions.

Definition 5.3. *Let $(S,\mathcal{S},\mu)$ be a σ-finite standard Borel space. Then a family of measurable maps $\{\phi_t : S \to S\}_{t\in\mathbb{Z}^d}$ is called a* <u>*nonsingular*</u> *(also known as* <u>*quasi-invariant*</u>*) $\mathbb{Z}^d$-action if*

(1) $\phi_0(s) = s$ *for μ-almost all $s \in S$,*
(2) $\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$ *μ-almost surely,*
(3) $\mu \circ \phi_t^{-1} \sim \mu$ *for all $t \in \mathbb{Z}^d$.*

In particular, if $\mu \circ \phi_t^{-1} = \mu$, then $\{\phi_t : S \to S\}_{t\in\mathbb{Z}^d}$ is called a *measure-preserving* $\mathbb{Z}^d$-action. Clearly *measure-preserving* $\Rightarrow$ *nonsingular* but the converse is not true. See, for example, Aaronson (1997) for an example of a nonsingular $\mathbb{Z}$-action that is not measure-preserving.

Example 5.4. Let $(W,\mathcal{W},\nu)$ be a σ-finite measure space, $S := W \times \mathbb{Z}^d$, $\mu := \nu \otimes \eta$, where η is the counting measure on $\mathbb{Z}^d$. Define a $\mathbb{Z}^d$-action $\{\psi_t\}_{t\in\mathbb{Z}^d}$ on $W \times \mathbb{Z}^d$ as follows. For all $t \in \mathbb{Z}^d$,

$$\psi_t(w,s) = (w, s+t), \quad (w,s) \in W \times \mathbb{Z}^d. \tag{5.3}$$

Clearly $\{\psi_t\}_{t\in\mathbb{Z}^d}$ is a measure-preserving (and hence nonsingular) $\mathbb{Z}^d$-action on $W \times \mathbb{Z}^d$. Note that using this action, we can rewrite (4.2) as

$$\{X_t\}_{t\in\mathbb{Z}^d} \overset{\mathcal{L}}{=} \left\{\int_{W\times\mathbb{Z}^d} f(\psi_t(w,s))\, dM(w,s)\right\}_{t\in\mathbb{Z}^d}, \tag{5.4}$$

where M is an SαS random measure on $W \times \mathbb{Z}^d$ with control measure $\nu \otimes \eta$.

We need another notion that arises from cohomology theory and is widely used in ergodic theory.

Definition 5.5. *A collection of measurable maps $\big\{c_t : S \to \{-1,+1\}\big\}_{t\in\mathbb{Z}^d}$ is called a (± 1-valued) cocycle for a nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}_{t\in\mathbb{Z}^d}$ on $(S,\mathcal{S},\mu)$ if for all $t_1, t_2 \in \mathbb{Z}^d$,*

$$c_{t_1+t_2}(s) = c_{t_2}(s)c_{t_1}(\phi_{t_2}(s)) \tag{5.5}$$

for μ-almost all $s \in S$.

Remark 5.6. Instead of a ± 1-valued cocyle, we can also define positive real-valued cocycle as a collection of maps $\{c_t : S \to (0,\infty)\}_{t\in\mathbb{Z}^d}$ satisfying (5.5). The most important example of such a cocycle is given by the following exercise.

Exercise 7. *Let* $\{\phi_t\}_{t\in\mathbb{Z}^d}$ *be a nonsingular* $\mathbb{Z}^d$*-action on* $(S,\mathcal{S},\mu)$. *Show that* $c_t := d(\mu\circ\phi_t)/d\mu$, $t\in\mathbb{Z}^d$ *is a positive real-valued cocycle.*

Hardin Jr. (1982) expressed a minimal representation of a stationary SαS process using a group of linear isometries of $L^\alpha(S,\mathcal{S},\mu)$ to itself. Extending this result, Rosiński (1994), Rosiński (1995) and Rosiński (2000) showed that any minimal representation of an SαS random field can be written in terms of a nonsingular $\mathbb{Z}^d$-action and an associated cocycle. This result is given below and should be considered as the key theorem of this mini course.

Theorem 5.7. *Let* $\{f_t\}_{t\in\mathbb{Z}^d}\subseteq L^\alpha(S,\mathcal{S},\mu)$ *be a minimal representation of a stationary SαS random field* $\{X_t\}_{t\in\mathbb{Z}^d}$. *Then there exist unique (modulo* μ*) nonsingular* $\mathbb{Z}^d$*-action* $\{\phi_t\}_{t\in\mathbb{Z}^d}$ *on* $(S,\mathcal{S},\mu)$ *and a* ±1*-valued cocycle* $\{c_t\}_{t\in\mathbb{Z}^d}$ *for* $\{\phi_t\}_{t\in\mathbb{Z}^d}$ *such that for all* $t\in\mathbb{Z}^d$,

$$f_t(s) = c_t(s)\big(f_0\circ\phi_t(s)\big)\left(\frac{d(\mu\circ\phi_t)}{d\mu}(s)\right)^{1/\alpha}\quad \mu\text{-almost surely.} \tag{5.6}$$

The next theorem is the converse of Theorem 5.7 and can be used to produce many examples of stationary SαS random fields.

Theorem 5.8. *Take any measurable space* $(S,\mathcal{S},\mu)$, *any* $f\in L^\alpha(S,\mathcal{S},\mu)$ *any nonsingular* $\mathbb{Z}^d$*-action* $\{\phi_t\}_{t\in\mathbb{Z}^d}$ *on* $(S,\mathcal{S},\mu)$, *and any* ±1*-valued cocycle* $\{c_t\}_{t\in\mathbb{Z}^d}$ *for* $\{\phi_t\}_{t\in\mathbb{Z}^d}$. *Then* $\{f_t\}_{t\in\mathbb{Z}^d}$ *defined by* (5.6) *satisfies* $\{f_t\}_{t\in\mathbb{Z}^d}\subseteq L^\alpha(S,\mathcal{S},\mu)$ *and* $\{X_t\}_{t\in\mathbb{Z}^d}$ *defined by* (4.1) *(here* M *is an SαS random measure on* S *with control measure* μ*) is a stationary SαS random field.*

Lecture 6

Exercise 8. *Prove Theorem 5.8.*

Definition 5.9. *Any integral representation (not necessarily minimal) of the form* (5.6) *of an SαS random field* $\{X_t\}_{t\in\mathbb{Z}^d}$ *is called a Rosinski representation of* $\{X_t\}_{t\in\mathbb{Z}^d}$. *In this case, we say that* $\{X_t\}_{t\in\mathbb{Z}^d}$ *is generated by the triplet* $(f_0,\{\phi_t\}_{t\in\mathbb{Z}^d},\{c_t\}_{t\in\mathbb{Z}^d})$ *on* $(S,\mathcal{S},\mu)$.

Note that the stationary mixed moving average SαS random field defined by (4.2) is generated by the triplet $(f,\{\psi_t\}_{t\in\mathbb{Z}^d},\{c_t\equiv1\}_{t\in\mathbb{Z}^d})$ on $(W\times\mathbb{Z}^d,\nu\otimes\eta)$ (here the notations are as in Example 5.4). This means that (5.4) is a Rosinski representation with unit cocycle $c_t\equiv1$ and unit Radon-Nikodym derivative $d((\nu\otimes\eta)\circ\psi_t)/d(\nu\otimes\eta)\equiv1$ for all $t\in\mathbb{Z}^d$. The unit Radon-Nikodym derivative is obtained because $\{\psi_t\}_{t\in\mathbb{Z}^d}$ is a measure-preserving $\mathbb{Z}^d$-action.

Any minimal representation is a Rosinski representation but not the converse. Also given a particular minimal representation, the underlying nonsingular $\mathbb{Z}^d$-action and the associated cocycle are unique almost surely. However since minimal representation is not unique, Rosinski representation is not unique either. Because of the rigidity result Theorem 4.7, the underlying $\mathbb{Z}^d$-actions (of different Rosinski representations) preserve many important ergodic theoretic properties. We shall introduce one such property in this mini course and discuss its implications for the length of memory (and rate of growth of the maxima sequence M_n) of a stationary SαS random field.

5.1. **Proof of Theorem 5.7.** The idea of this proof is as follows. Stationarity means the the law of $\{X_t\}_{t\in\mathbb{Z}^d}$ is invariant under the shift action of $\mathbb{Z}^d$ on $\mathbb{R}^{\mathbb{Z}^d}$. This measure-preserving $\mathbb{Z}^d$-action, when viewed at the integral representation level, naturally induces a nonsingular action on S and an associated cocycle yielding (5.6). The main steps of this proof is sketched below.

Fix $t \in \mathbb{Z}^d$. Note that because of stationarity of $\{X_\tau\}_{\tau\in\mathbb{Z}^d}$ and minimality of $\{f_\tau\}_{\tau\in\mathbb{Z}^d}$, it follows that $\{f_{\tau+t}\}_{\tau\in\mathbb{Z}^d}$ is also a minimal representation of $\{X_\tau\}_{\tau\in\mathbb{Z}^d}$. Therefore by Theorem 4.7, there exist unique (modulo μ) maps $\phi_t : S \to S$ (one-to-one and onto) and $h_t : S \to \mathbb{R}\smallsetminus\{0\}$ such that for all $\tau \in \mathbb{Z}^d$,

$$f_{\tau+t} = h_t\, f_\tau \circ \phi_t \quad \mu\text{-almost surely, and} \tag{5.7}$$

$$0 < |h_t| = \left(\frac{d(\mu\circ\phi_t)}{d\mu}\right)^{1/\alpha} \ \mu\text{-almost surely, and} \tag{5.8}$$

Define $c_t := h_t/|h_t|$, $t \in \mathbb{Z}^d$. Putting $\tau = 0$ in (5.7) and using (5.8), we get that μ-almost surely

$$f_t = c_t\, f_0 \circ \phi_t \left(\frac{d(\mu\circ\phi_t)}{d\mu}\right)^{1/\alpha},\ t \in \mathbb{Z}^d,$$

from which Theorem 5.7 follows because of the following exercise.

Exercise 9. *Fix $t_1, t_2 \in \mathbb{Z}^d$. Evaluate $f_{\tau+t_1+t_2}$ in two different ways and use Theorem 4.7 (more precisely, the uniqueness of the maps) to conclude that $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is a nonsingular $\mathbb{Z}^d$-action on $(S,\mathcal{S},\mu)$, $\{c_t\}_{t\in\mathbb{Z}^d}$ is a ±1-valued cocycle for $\{\phi_t\}_{t\in\mathbb{Z}^d}$, and they are both unique modulo μ.*

6. Conservative and Dissipative Parts

When a stationary SαS random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is generated by the triplet $(f_0, \{\phi_t\}_{t\in\mathbb{Z}^d}, \{c_t\}_{t\in\mathbb{Z}^d})$ on $(S,\mathcal{S},\mu)$, $(f_0, \{\phi_t\}_{t\in\mathbb{Z}^d}, \{c_t\}_{t\in\mathbb{Z}^d})$ can be thought of as a highly infinite-dimensional parameter that determines the dependence structure of $\{X_t\}_{t\in\mathbb{Z}^d}$ and hence has information about its length of memory. It so happens that f_0 and $\{c_t\}_{t\in\mathbb{Z}^d}$ do not have too much information about the memory (this is somewhat expected because f_0 is just one function and c_ts are just ±1-valued functions). The nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}_{t\in\mathbb{Z}^d}$, on the other hand, has a lot of information on the length of memory. The next few definitions and results are motivated by this.

Definition 6.1. *Suppose $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is a nonsingular $\mathbb{Z}^d$-action on $(S,\mathcal{S},\mu)$. A set $W_*\in\mathcal{S}$ is called a wandering set (for $\{\phi_t\}_{t\in\mathbb{Z}^d}$) if $\{\phi_t(W_*):t\in\mathbb{Z}^d\}$ is a pairwise disjoint collection of subsets of S.*

Roughly speaking, wandering sets never come back to itself under the action. In Example 5.4, take any $W_0\subseteq W$ and any $t_0\in\mathbb{Z}^d$. Then $W_*:=W_0\times\{t_0\}$ is a wandering set.

Lecture 7

The following result (see Proposition 1.6.1 in Aaronson (1997)) gives a decomposition of S into two disjoint and invariant parts.

Theorem 6.2 (Hopf Decomposition)**.** *Suppose $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is a nonsingular $\mathbb{Z}^d$-action on $(S,\mathcal{S},\mu)$. Then there exist unique (modulo μ) subsets $\mathcal{C},\mathcal{D}\in\mathcal{S}$ such that*

(1) *$\mathcal{C}\cap\mathcal{D}=\varnothing$ modulo μ,*
(2) *$\mathcal{C}\cup\mathcal{D}=S$ modulo μ,*
(3) *$\mathcal{C}$ and $\mathcal{D}$ are invariant under the action $\{\phi_t\}_{t\in\mathbb{Z}^d}$, i.e., for all $t\in\mathbb{Z}^d$, $\phi_t(\mathcal{C})=\mathcal{C}$ and $\phi_t(\mathcal{D})=\mathcal{D}$ modulo μ,*
(4) *$\mathcal{C}$ has no wandering subset of positive measure, and*
(5) *$\mathcal{D}=\bigcup_{t\in\mathbb{Z}^d}\phi_t(W_*)$ modulo μ for some wandering set W_*.*

Definition 6.3. *$\mathcal{C}$ and $\mathcal{D}$ are called the conservative and dissipative parts (of $\{\phi_t\}_{t\in\mathbb{Z}^d}$), respectively. $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is called conservative if $S=\mathcal{C}$ modulo μ and dissipative if $S=\mathcal{D}$ modulo μ.*

Roughly speaking, conservative actions keep coming back to its starting point whereas the dissipative actions keep moving away. An example of dissipative action is given by Example 5.4 with $W_*=W\times\{\mathbf{0}\}$ being a wandering set whose translates cover S (see Theorem 6.2 above). On the other hand, the following exercise yields many examples of conservative actions.

Exercise 10. *Show that any measure-preserving $\mathbb{Z}^d$-action on a finite measure space is necessarily conservative. In particular, if μ is a probability measure on $S=\mathbb{R}^{\mathbb{Z}^d}$ such that under μ, the coordinate field*

$\{\pi_t\}_{t\in\mathbb{Z}^d}$ *(defined by* $\pi_t(\mathbf{x}) = \mathbf{x}(t)$, $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}^d}$*) is stationary, then show that the shift action* $\{\zeta_t\}_{t\in\mathbb{Z}^d}$ *of* $\mathbb{Z}^d$ *on* $\mathbb{R}^{Zd}$*, defined by*

$$(\zeta_t\mathbf{x})(s) = \mathbf{x}(s+t), \ \ \mathbf{x} \in \mathbb{R}^{Zd}, s \in \mathbb{Z}^d, \tag{6.1}$$

is conservative.

The following result confirms that even though Rosinski representation is not unique, the rigidity result Theorem 4.7 is kind towards the dissipativity and conservativity of the underlying nonsingular $\mathbb{Z}^d$-actions.

Proposition 6.4. *If a stationary SαS random field is generated by a conservative (dissipative, resp.)* $\mathbb{Z}^d$*-action in one Rosinski representation, then in any other Rosinski representation of the field, the underlying action must be conservative (dissipative, resp.).*

Proof. See Rosiński (1995) (for $d = 1$) and Roy and Samorodnitsky (2008) (for $d > 1$). □

Remark 6.5. The stationary SαS random fields generated by conservative $\mathbb{Z}^d$-actions tend to have longer memory compared to the ones generated by dissipative (or more generally non-conservative) actions because conservative actions keep coming back and hence introduce stronger dependence among the X_t s. This heuristic reasoning can be validated by the growth of M_n as $n \to \infty$.

The following result gives structure to a stationary SαS random field generated by a dissipative $\mathbb{Z}^d$-action.

Theorem 6.6. *A stationary SαS random field is generated by a dissipative* $\mathbb{Z}^d$*-action if and only if it is a mixed moving average defined by* (4.2).

Main Idea of the Proof. The *if part* follows from Proposition 6.4 and the fact that the $\mathbb{Z}^d$-action (5.3) is dissipative. The *only if part* uses a very deep result (known as Krengel's Structure Theorem; see Krengel (1969) for $d = 1$, and Rosiński (2000), Roy and Samorodnitsky (2008) for $d > 1$) that states that any dissipative nonsingular $\mathbb{Z}^d$ action is "isomorphic" (in an appropriate sense) to the $\mathbb{Z}^d$-action (5.3). Exploiting this isomorphism, one can change the underlying action to (5.3). However to replace the cocycle by the unit cocycle, one has to work harder. This part of the proof is slightly technical. See pg 1176 - 1177 of Rosiński (1995) for the detailed proof. □

The Hopf decomposition of the underlying nonsingular actions induces a decomposition of the stationary SαS random field into two independent stationary components as follows. Let $\{f_t\}_{t\in\mathbb{Z}^d} \subseteq L^\alpha(S,\mu)$ be a Rosinski representation of a stationary SαS random field $\{X_t\}_{t\in\mathbb{Z}^d}$ with underlying nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}_{t\in\mathbb{Z}^d}$. Let $S = \mathcal{C} \cup \mathcal{D}$ be the Hopf decomposition for $\{\phi_t\}_{t\in\mathbb{Z}^d}$. Then

$$X_t = \int_S f_t dM = \int_{\mathcal{C}} f_t dM + \int_{\mathcal{D}} f_t dM =: X_t^{\mathcal{C}} + X_t^{\mathcal{D}},\ t \in \mathbb{Z}^d, \tag{6.2}$$

where $\{X_t^{\mathcal{C}}\}_{t\in\mathbb{Z}^d} \perp\!\!\!\perp \{X_t^{\mathcal{D}}\}_{t\in\mathbb{Z}^d}$ are two stationary SαS random fields, $\{X_t^{\mathcal{D}}\}_{t\in\mathbb{Z}^d}$ is a mixed moving average, and $\{X_t^{\mathcal{C}}\}_{t\in\mathbb{Z}^d}$ has no nontrivial mixed moving average component (since it is generated by a conservative $\mathbb{Z}^d$-action).

Theorem 6.7. *The decomposition* (6.2) *is unique is law, i.e., the (finite-dimensional) distributions of* $\{X_t^{\mathcal{C}}\}_{t\in\mathbb{Z}^d}$ *and* $\{X_t^{\mathcal{D}}\}_{t\in\mathbb{Z}^d}$ *do not depend on the choice of Rosinski representation.*

Proof. See the proof of Theorem 4.3 in Rosiński (1995). □

Thanks to the above result, we define $\{X_t^{\mathcal{C}}\}_{t\in\mathbb{Z}^d}$ and $\{X_t^{\mathcal{D}}\}_{t\in\mathbb{Z}^d}$ to be the conservative and dissipative parts of $\{X_t\}_{t\in\mathbb{Z}^d}$, respectively.

7. Maxima of Stationary SαS Random Fields

In view of the discussions in the beginning of Section 5 and Remark 6.5 above, we can expect that the maxima sequence M_n grows slowly when the underlying $\mathbb{Z}^d$-action is conservative. This is confirmed by the following result.

Theorem 7.1. *Let* $\{X_t\}_{t\in\mathbb{Z}^d}$ *be a stationary SαS random field generated by a nonsingular* $\mathbb{Z}^d$*-action* $\{\phi_t\}_{t\in\mathbb{Z}^d}$ *on* $(S,\mathcal{S},\mu)$ *with the corresponding Rosinski representation* $\{f_t\}_{t\in\mathbb{Z}^d}$ *of the form*(5.6). *Then the following results hold.*

(1) $\{\phi_t\}_{t\in\mathbb{Z}^d}$ *is conservative* $\Rightarrow M_n/n^{d/\alpha} \xrightarrow{p} 0$, *and*
(2) $\{\phi_t\}_{t\in\mathbb{Z}^d}$ *is not conservative* $\Rightarrow M_n/n^{d/\alpha} \xrightarrow{\mathcal{L}} a_{\mathbf{X}} Z_\alpha$,

where $a_{\mathbf{X}} > 0$ *is a constant determined by* $\{X_t\}_{t\in\mathbb{Z}^d}$ *and* Z_α *is a Frechet type extreme value random variable with distribution function* (5.2).

The Main Idea of the Proof. The main tool behind this proof is the deterministic sequence

$$b_n = \left(\int_S \max_{t\in B_n} |f_t(s)|^\alpha \mu(ds)\right)^{1/\alpha}, \tag{7.1}$$

where B_n is as defined in Section 5. The first step of the proof is the computation of asymptotics of b_n as $n \to \infty$ and the second step is to show that the asymptotic behaviour of the maxima sequence M_n is more or less determined by that of b_n.

Remark 7.2. By Corollary 4.4.6 of Samorodnitsky and Taqqu (1994),

$$\lim_{\lambda\to\infty} \lambda^\alpha P(M_n > \lambda) = C_\alpha b_n^\alpha,$$

where C_α is the stable tail constant (0.1). In particular, this means that the sequence b_n is solely determined by the SαS random field $\{X_t\}_{t\in\mathbb{Z}^d}$ and does not depend on the choice of integral representation $\{f_t\}_{t\in\mathbb{Z}^d}$.

The first step of the proof of Theorem 7.1 is given by the following lemma.

Lemma 7.3. *Let $\{\phi_t\}_{t\in\mathbb{Z}^d}$ be as in Theorem 7.1 and b_n be as in* (7.1)*. Then the following asymptotics hold.*

(1) $\{\phi_t\}_{t\in\mathbb{Z}^d}$ *is conservative* $\Rightarrow b_n/n^{d/\alpha} \to 0$, *and*
(2) $\{\phi_t\}_{t\in\mathbb{Z}^d}$ *is not conservative* $\Rightarrow b_n/n^{d/\alpha} \to K_{\mathbf{X}}$,

where $K_{\mathbf{X}} > 0$ is a constant determined by $\{X_t\}_{t\in\mathbb{Z}^d}$.

Proof. For the first part, see the proof of Proposition 4.1 in Roy and Samorodnitsky (2008). For the second part, see the proof in the one-dimensional case, i.e., Theorem 3.1 of Samorodnitsky (2004a) (the same proof goes through in the higher dimensional case due to Theorem 6.6 above). □

The second step of the proof of Theorem 7.1 relies on the following lemma.

Lemma 7.4. *Fix a positive integer n. The random vector $(X_t, t\in B_n)$ has a series representation (in law) of the form*

$$\left(b_n C_\alpha^{1/\alpha} \sum_{j=1}^\infty \epsilon_j \Gamma_j^{-1/\alpha} \frac{f_t(U_j^{(n)})}{\max_{v\in B_n} f_v(U_j^{(n)})}\right)_{t\in B_n},$$

where b_n is as in (7.1)*, C_α is as in* (0.1)*, $\{\epsilon_i\}_{i\ge1}$ and $\{\Gamma_i\}_{i\ge1}$ be as in Theorem 1.7 above, and $\{U_j^{(n)}\}_{j\ge1}$ is a sequence of iid S-valued random variables with common law*

$$P\big(U_1^{(n)} \in A\big) = b_n^{-\alpha} \int_A \max_{t\in B_n} |f_t(s)|^\alpha \mu(ds),\ A\in\mathcal{S}.$$

Exercise 11. *Use Theorem 1.7 above to prove Lemma 7.4.*

Sketch of Proof of Theorem 7.1. When $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is conservative, using Lemma 7.3 and Lemma 7.4 and a nice coupling argument, it is possible to show that $M_n/n^{d/\alpha} \xrightarrow{p} 0$. See pg 1450 - 1452 of Samorodnitsky (2004a) for the details.

On the other hand, when $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is not conservative, using Lemma 7.4 above, we have that for any $\lambda > 0$,

$$P\left(\frac{M_n}{b_n} > \lambda\right) = P\left(\max_{t\in B_n}\left|C_\alpha^{1/\alpha}\sum_{j=1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}\frac{f_t(U_j^{(n)})}{\max_{v\in B_n} f_v(U_j^{(n)})}\right| > \lambda\right),$$

from which by using "one large jump" principle (see Remark 1.8 above), we get

$$\begin{aligned}&\approx P\left(\max_{t\in B_n}\left|C_\alpha^{1/\alpha}\epsilon_1\Gamma_1^{-1/\alpha}\frac{f_t(U_1^{(n)})}{\max_{v\in B_n} f_v(U_1^{(n)})}\right| > \lambda\right)\\ &= P(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} > \lambda) = 1 - e^{-C_\alpha\lambda^{-\alpha}}.\end{aligned}$$

The above heuristic calculations show that $M_n/b_n \xrightarrow{\mathcal{L}} C_\alpha^{1/\alpha}Z_\alpha$ and the second part of Theorem 7.1 follows using Lemma 7.3. See pg 1454 - 1455 of Samorodnitsky (2004a) to find out how to make the above "≈" precise when $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is not conservative. □

Lecture 8

8. Examples and Extensions

As long as the underlying nonsingular action is not conservative, the exact asymptotic behaviour of M_n is given in Theorem 7.1. Therefore, more interesting examples of SαS random fields are the ones generated by conservative actions. We look at a few of those in this section.

Example 8.1. Consider the conservative action in Exercise 10. Choose μ such that under μ, the coordinate field $\{\pi_t\}_{t\in\mathbb{Z}^d}$ forms a collection of iid random variables. In this case, define an SαS random field $\{X_t\}_{t\in\mathbb{Z}^d}$ by

$$\{X_t\}_{t\in\mathbb{Z}^d} \stackrel{\mathcal{L}}{=} \left\{\int_{\mathbb{R}^{\mathbb{Z}^d}} \pi_0 \circ \zeta_t(\mathbf{x})dM(\mathbf{x})\right\}_{t\in\mathbb{Z}^d},$$

where M is an SαS random measure on $\mathbb{R}^{\mathbb{Z}^d}$ with control measure μ and other notations are as in Exercise 10.

If further, we assume that π_0 follows standard normal distribution under μ, then it would follow that $\{X_t\}_{t\in\mathbb{Z}^d}$ is a sub-Gaussian random field, i.e., there is a collection of iid standard normal random variables $\{\xi_t\}_{t\in\mathbb{Z}^d}$ and another independent positive random variable A (not following an extreme value distribution) defined on the same probability space such that

$$\{X_t\}_{t\in\mathbb{Z}^d} \stackrel{\mathcal{L}}{=} \{A\xi_t\}_{t\in\mathbb{Z}^d}.$$

See Proposition 3.7.1 in Samorodnitsky and Taqqu (1994). Using this sub-Gaussian representation and standard extreme value theory estimates (see, for example, Resnick (1987)), it follows that

$$\frac{M_n}{\sqrt{2d \log n}} \xrightarrow{\mathcal{L}} A,$$

a non-extreme value limit.

On the other hand, if π_0 follows Pareto distribution with parameter $\theta > \alpha$ (i.e., $\mu(\pi_0 > x) = x^{-\theta}$, $x \ge 1$), then it can be shown that

$$\frac{M_n}{n^{d/\theta}} \xrightarrow{\mathcal{L}} c_{\alpha,\theta} Z_\alpha,$$

for some finite positive constant $c_{\alpha,\theta}$; see Section 5 in Samorodnitsky (2004a) for the details.

The above example shows that in the conservative case, the rate of growth of the partial maxima sequence can be either polynomial or slowly varying. Heuristically, one can say that stronger conservativity of the underlying group action should imply longer memory, which in turn should give rise to slower rate of growth of M_n. Therefore, the following question becomes pertinent in the setup of Rosinski representations of stationary SαS random fields.

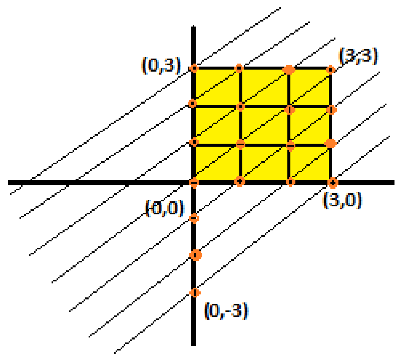
Question 8.2. *How to quantify the "strength of conservativity" of the underlying nonsingular $\mathbb{Z}^d$-action?*

In general the answer to the above question is not known. However, Roy and Samorodnitsky (2008) made further investigations on the actual rate of growth of the partial maxima sequence M_n using the theory of finitely generated abelian groups (see, for example, Lang (2002)) together with counting of the number of lattice points in dilates of rational polytopes (see De Loera (2005)). Viewing the action as a group of nonsingular transformations and studying the algebraic structure of this group, one can get better ideas about the strength of conservativity of the underlying action and hence the rate of growth of the partial maxima as well as the length of memory of the random field. We start with the following motivating example.

Example 8.3. Let $S = \mathbb{R}$, $\mu = Leb$, $d = 2$, and $\{\phi_{(i,j)}\}_{(i,j)\in\mathbb{Z}^2}$ be the measure-preserving $\mathbb{Z}^2$-action on $\mathbb{R}$ defined by $\phi_{(i,j)}(s) = s + i - j$, $s \in \mathbb{R}$. Take any $f \in L^\alpha(\mathbb{R}, Leb)$ and define a stationary SαS random field by

$$\{X_{(i,j)}\}_{(i,j)\in\mathbb{Z}^2} = \left\{ \int_{\mathbb{R}} f(\phi_{(i,j)}(s)) M(ds) \right\}_{(i,j)\in\mathbb{Z}^2},$$

where M is an SαS random measure on $\mathbb{R}$ with control measure $\mu = Leb$. Fix $k \in \mathbb{Z}$. Note that for each $(i,j) \in \mathbb{Z}^2$ situated on the line $j = i + k$, $\phi_{(i,j)} = \phi_{(0,k)}$ and therefore $X_{(i,j)} = X_{(0,k)}$ almost surely.



The Maxima Sequence for n = 4

Therefore using the above picture (in each of the lines, the random variables are equal almost surely) and stationarity of $\{X_t\}_{t\in\mathbb{Z}^d}$, we have

$$M_n = \max_{0\le i,j\le n-1} |X_{(i,j)}| \overset{a.s.}{=} \max_{1-n\le k\le n-1} |X_{(0,k)}| \overset{\mathcal{L}}{=} \max_{0\le k\le 2(n-1)} |X_{(0,k)}|$$

for all $n \ge 1$. Since $\{X_{(0,k)}\}_{k\in\mathbb{Z}}$ is a stationary SαS process generated by the dissipative $\mathbb{Z}$-action $\{\phi_{(0,k)}\}_{k\in\mathbb{Z}}$, we get that there exists a constant $a > 0$ such that

$$\frac{M_n}{n^{1/\alpha}} \xrightarrow{\mathcal{L}} aZ_\alpha.$$

Question 8.4. *What is going on in the above example?*

Here we see a reduction of "effective dimension" of the random field. Algebraically, this boils down to quotienting $\mathbb{Z}^2$ by the diagonal $K = \{(i,j) \in \mathbb{Z}^2 : i = j\}$. Note that K is the kernel of the group homomorphism $(i,j) \mapsto \phi_{(i,j)}$. Reduction of dimension occurs because $\mathbb{Z}^2/K \simeq \mathbb{Z}$.

In general, if a stationary SαS random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is generated by a nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}_{t\in\mathbb{Z}^d}$, then we need to look at the kernel K of the group homomorphism $t \mapsto \phi_t$. Clearly

$$K = \{t \in \mathbb{Z}^d : \phi_t(s) = s \text{ for } \mu\text{-almost all } s \in S\}.$$

In general, it may not happen that $\mathbb{Z}^d/K \simeq \mathbb{Z}^p$ for some $p \le d$. However by Structure Theorem for Finitely Generated Abelian Groups (see Theorem 8.5 in Chap. I of Lang (2002)),

$$\mathbb{Z}^d/K = \bar{F} \oplus \bar{N},$$

where $\bar{F} \simeq \mathbb{Z}^p$ for some $p \le d$, $\bar{N}$ is a finite group and $\oplus$ denotes the direct sum of groups. Using the fact that $\bar{F}$ is a free abelian group, it is possible to show that $\bar{F}$ has an isomorphic copy F sitting inside $\mathbb{Z}^d$; see Section 5 of Roy and Samorodnitsky (2008). In this setup, p plays the role of "effective dimension" and F plays the role of "effective index set" of the random field.

In Example 8.3, $d = 2$, $K = \{(i,j) \in \mathbb{Z}^2 : i = j\}$, $p = 1$ and $\bar{N}$ is trivial. In this case, the "effective index set" can be chosen to be $F = \{(0,k) : k \in \mathbb{Z}\}$ and since the restricted action $\{\phi_{(i,j)}\}_{(i,j)\in F} = \{\phi_{(0,k)}\}_{k\in\mathbb{Z}}$ is dissipative, we get $M_n/n^{1/\alpha} \xrightarrow{\mathcal{L}} aZ_\alpha$. The general result is as follows.

Theorem 8.5. *In the above setup, assume that* $1 \le p < d$. *Then the following results hold.*

(1) $\{\phi_t\}_{t\in F}$ *is conservative* $\Rightarrow M_n/n^{p/\alpha} \xrightarrow{p} 0$, *and*

(2) $\{\phi_t\}_{t\in F}$ *is not conservative* $\Rightarrow M_n/n^{p/\alpha} \xrightarrow{\mathcal{L}} c_{\mathbf{X}} Z_\alpha$,

where $c_{\mathbf{X}} > 0$ *is a constant determined by* $\{X_t\}_{t\in\mathbb{Z}^d}$ *and* Z_α *is a Frechet type extreme value random variable with distribution function* (5.2).

Proof. This proof is mostly algebraic with a slight touch of combinatorics in it; see Section 5 of Roy and Samorodnitsky (2008). □

8.1. **The Continuous Parameter Case.** The discrete parameter results mentioned in this mini course have been extended to the continuous parameter stationary measurable locally bounded SαS random fields $\{X_t\}_{t\in\mathbb{R}^d}$ by Rosiński (1995, 2000), Samorodnitsky (2004b) and Roy (2010b). The approach taken by these works is to approximate the continuous parameter random field $\{X_t\}_{t\in\mathbb{R}^d}$ by its discrete parameter skeletons $\{X_t\}_{t\in 2^{-i}\mathbb{Z}^d}$, $i = 0, 1, 2, \ldots$. In a recent work of Chakrabarty and Roy (2013), the notion of effective dimension has been extended to the continuous parameter case based on the following observation: the effective dimensions of $\{X_t\}_{t\in 2^{-i}\mathbb{Z}^d}$, $i = 0, 1, 2, \ldots$ are all equal and therefore can be defined as the effective dimension of $\{X_t\}_{t\in\mathbb{R}^d}$. With this definition, Theorem 8.5 can also be extended to the continuous parameter case.

8.2. **Other Related Works.** Two important examples of new classes of stationary SαS processes were introduced in Rosiński and Samorodnitsky (1996) and Cohen and Samorodnitsky (2006).

Various probabilistic aspects of stationary SαS random fields and processes have also been connected to the ergodic theoretic properties of the underlying nonsingular action. Mikosch and Samorodnitsky (2000) investigated the ruin probabilities of a negatively drifted random walk whose steps are coming from a stationary ergodic stable process and observed that ruin becomes more likely when the underlying $\mathbb{Z}$-action is conservative.

The point process induced by stationary SαS processes was considered in Resnick and Samorodnitsky (2004) and this work was extended to the random fields by Roy (2010a). It was seen that when the underlying action is not conservative, the associated point process sequence converges weakly to a Poisson cluster process. However in the conservative case, the point process sequence does not remain tight due to clustering. In many such examples, the point process sequence can be shown to converge to a random measure after proper normalization.

Using the language of positive-null decomposition of nonsingular flows (see Section 1.4 in Aaronson (1997) and Section 3.4 in Krengel (1985)) another decomposition of measurable stationary $S\alpha S$ processes was obtained in Samorodnitsky (2005) and this decomposition was used to characterize the ergodicity of such a process. This work has recently been extended to the stationary SαS random fields by Wang et al. (2013) based on the work of Takahashi (1971). See also Roy (2012) for another recent work connecting Maharam systems with various ergodic properties of stationary stable processes.

A systematic and wholesome approach to decompositions of a stationary SαS process into independent stationary SαS components is presented in Wang et al. (2012).

Decompositions based on the ergodic theory of nonsingular actions were also obtained for self-similar $S\alpha S$ processes with stationary increments in Pipiras and Taqqu (2002a) and Pipiras and Taqqu (2002b). See also Kolodyński and Rosiński (2003) for existence and rigidity results for integral representations of group self-similar stable processes.

Many of the results mentioned in this mini course have parallels in the max-stable world. See, for example, Stoev and Taqqu (2005), Stoev (2008), Kabluchko (2009), Wang and Stoev (2010a,b), Kabluchko and Schlather (2010), Wang et al. (2012), Wang et al. (2013).

Roy (2007, 2009) used the language of Poisson suspensions to obtain various decompositions (and structure results) of stationary infinitely divisible processes with no Gaussian component.

References

J. Aaronson (1997): *An Introduction to Infinite Ergodic Theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence.

J. Bretagnolle, D. Dacunha-Castelle and J. Krivine (1966): Lois stables et espaces L^p. *Ann. Inst. Henri Poincaré* 2:231–259. Section B.

A. Chakrabarty and P. Roy (2013): Group-theoretic dimension of stationary symmetric α-stable random fields. *Journal of Theoretical Probability* 26:240–258.

S. Cohen and G. Samorodnitsky (2006): Random rewards, fractional Brownian local times and stable self-similar processes. *Annals of Applied Probability* 16:1432–1461.

J. A. De Loera (2005): The many aspects of counting lattice points in polytopes. *Mathematische Semesterberichte* 52:175–195.

W. Feller (1971): *An Introduction to Probability Theory and its Applications*, volume 2. Wiley, New York, 2nd edition.

C. Hardin Jr. (1981): Isometries on subspaces of L^p. *Indiana Univ. Math. J.* 30:449–465.

C. Hardin Jr. (1982): On the spectral representation of symmetric stable processes. *Journal of Multivariate Analysis* 12:385–401.

H. Hurst (1951): Long-term storage capacity of reservoirs. *Transactions of the American Society of Civil Engineers* 116:770–808.

H. Hurst (1955): Methods of using long-term storage in reservoirs. *Proceedings of the Institution of Civil Engineers, Part I* :519–577.

I. Ibragimov and Y. V. Linnik (1971): *Independent and stationary sequences of random variables*. Wolters-Nordhoff,, The Netherlands.

Z. Kabluchko (2009): Spectral representations of sum-and max-stable processes. *Extremes* 12:401–424.

Z. Kabluchko and M. Schlather (2010): Ergodic properties of max-infinitely divisible processes. *Stochastic Processes and their Applications* 120:281–295.

S. Kolodyński and J. Rosiński (2003): Group self-similar stable processes in $\mathbb{R}^d$. *Journal of Theoretical Probability* 16:855–876.

U. Krengel (1969): Darstellungssätze für Strömungen und Halbströmungen. II. *Mathematische Annalen* 182:1–39.

U. Krengel (1985): *Ergodic Theorems*. De Gruyter, Berlin, New York.

J. Kuelbs (1973): A representation theorem for symmetric stable processes and stable measures on H. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 26:259–271.

S. Lang (2002): *Algebra*. Graduate Texts in Mathematics. Springer-Verlag, New York, revised 3rd edition.

B. Mandelbrot and J. Wallis (1968): Noah, Joseph and operational hydrology. *Water Resources Research* 4:909–918.

B. Mandelbrot and J. Wallis (1969): Some long-run properties of geophysical records. *Water Resources Research* 5:321–340.

T. Mikosch and G. Samorodnitsky (2000): Ruin probability with claims modeled by a stationary ergodic stable process. *Annals of*

Probability 28:1814–1851.
V. Pipiras and M. S. Taqqu (2002a): Decomposition of self-similar stable mixing moving averages. *Probability Theory and Related Fields* 123:412–452.
V. Pipiras and M. S. Taqqu (2002b): The structure of self-similar stable mixing moving averages. *Annals of Probability* 30:898–932.
S. Resnick (1987): *Extreme Values, Regular Variation and Point Processes*. Springer-Verlag, New York.
S. Resnick and G. Samorodnitsky (2004): Point processes associated with stationary stable processes. *Stochastic Processes and Their Applications* 114:191–210.
J. Rosiński (1994): On uniqueness of the spectral representation of stable processes. *J. Theoret. Probab.* 7:615–634.
J. Rosiński (1995): On the structure of stationary stable processes. *The Annals of Probability* 23:1163–1187.
J. Rosiński (2000): Decomposition of stationary α–stable random fields. *Annals of Probability* 28:1797–1813.
J. Rosiński (2006): Minimal integral representations of stable processes. *Probab. Math. Statist.* 26:121–142.
J. Rosiński and G. Samorodnitsky (1996): Classes of mixing stable processes. *Bernoulli* 2:3655–378.
E. Roy (2007): Ergodic properties of Poissonian ID processes. *Ann. Probab.* 35:551–576.
E. Roy (2009): Poisson suspensions and infinite ergodic theory. *Ergodic Theory and Dynamical Systems* 29:667.
E. Roy (2012): Maharam extension and stationary stable processes. *The Annals of Probability* 40:1357–1374.
P. Roy (2010a): Ergodic theory, abelian groups and point processes induced by stable random fields. *Annals of Probability* 38:770–793.
P. Roy (2010b): Nonsingular group actions and stationary $S\alpha S$ random fields. *Proceedings of the American Mathematical Society* 138:2195–2202.
P. Roy and G. Samorodnitsky (2008): Stationary symmetric α-stable discrete parameter random fields. *Journal of Theoretical Probability* 21:212–233.
G. Samorodnitsky (2004a): Extreme value theory, ergodic theory, and the boundary between short memory and long memory for stationary stable processes. *Annals of Probability* 32:1438–1468.
G. Samorodnitsky (2004b): Maxima of continuous time stationary stable processes. *Advances in Applied Probability* 36:805–823.
G. Samorodnitsky (2005): Null flows, positive flows and the structure of stationary symmetric stable processes. *Annals of Probability*

33:1782–1803.

G. Samorodnitsky (2006): Long range dependence. *Found. Trends Stoch. Syst.* 1:163–257. Also available in `http://legacy.orie.cornell.edu/gennady/techreports/LRD-NOW.pdf`.

G. Samorodnitsky and M. Taqqu (1994): *Stable Non-Gaussian Random Processes*. Chapman and Hall, New York.

M. Schilder (1970): Some structure theorems for the symmetric stable laws. *Ann. Math. Stat.* 41:412–421.

M. Schreiber (1972): Quelques remarques sur les caractéristiques des espaces L^p, $0 \le p < 1$. *Ann. Inst. Henri Poincaré* 8:83–92.

S. Stoev and M. S. Taqqu (2005): Extremal stochastic integrals: a parallel between max–stable processes and α–stable processes. *Extremes* 8:237–266.

S. A. Stoev (2008): On the ergodicity and mixing of max-stable processes. *Stochastic Process. Appl.* 118:1679–1705.

D. Surgailis, J. Rosiński, V. Mandrekar and S. Cambanis (1993): Stable mixed moving averages. *Probab. Theory Related Fields* 97:543–558.

W. Takahashi (1971): Invariant functions for amenable semigroups of positive contractions on L^1. *Kōdai Math. Sem. Rep.* 23:131–143.

V. Varadarajan (1970): *Geometry of Quantum Theory*, volume 2. Van Nostrand Reinhold, New York.

Y. Wang, P. Roy and S. A. Stoev (2013): Ergodic properties of sum- and max-stable stationary random fields via null and positive group actions. *The Annals of Probability* 41:206–228.

Y. Wang and S. A. Stoev (2010a): On the association of sum- and max-stable processes. *Statistics & probability letters* 80:480–488.

Y. Wang and S. A. Stoev (2010b): On the structure and representations of max-stable processes. *Advances in Applied Probability* 42:855–877.

Y. Wang, S. A. Stoev and P. Roy (2012): Decomposability for stable processes. *Stochastic Processes and their Applications* 122:1093–1109.

R. Zimmer (1984): *Ergodic Theory and Semisimple Groups*. Birkhäuser, Boston.

V. Zolotarev (1986): *One-dimensional Stable Distributions*, volume 65 of “Translations of mathematical monographs”. American Mathematical Society. Translation from the original 1983 Russian edition.