Exclusion processes with drift

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Preface

The purpose of these notes is to give an introduction to asymmetric exclusion processes in finite one-dimensional lattices. These lectures were written for the *Lectures in Probability and Statistics* held in ISI Kolkata from December 15–19, 2017.

The prerequisites for the text are basic courses in probability, combinatorics and linear algebra. A familiarity with some abstract algebra will be helpful, and so will a course on introductory statistical mechanics. However, these are not really necessary.

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CHAPTER 1

Introduction

The subject of these lecture notes is the asymmetric simple exclusion process in one dimension. While exclusion process first arose in the biophysical and probability literature, their entrance in the field of statistical physics arose because it was realised that they could be interpreted as a simplified model of particle transport.

1.1. Definitions

We will assume a familiarity with the basics of Markov processes. For completeness, we give the basic definitions. An introduction to Markov processes in continuous time on denumerable state spaces is given in several books. One good reference is [**Nor98**].

A *(continuous-time) Markov process* is a family of random variables taking values in some set Σ indexed by the positive reals $(X_t)_{t \geq 0}$, such that for all $s, t \geq 0$ and all $\sigma, \tau, x_u \in \Sigma$ for $0 \leq u < s$,

$$\mathbb{P}(X_{t+s} = \tau \mid X_s = \sigma, X_u = x_u, 0 \leq u < s) = \mathbb{P}(X_{t+s} = \tau \mid X_s = \sigma).$$

We will assume that Σ is finite throughout these notes. Elements of Σ will be called *states* or *configurations.*

The random time T_σ that X stays in state σ is known as the *holding time* and has the memoryless property, and is hence exponentially distributed with some rate parameter. The random time J_σ at which the process jumps from state σ is called the *jump time.*

The process is completely described by the initial distribution π_0 at time $t = 0$ and a (column-stochastic) matrix M describing the transition rates. This is called the *generator* or *Q-matrix* in the probability literature and the *Markov matrix* in the physics literature. The matrix M encodes both the rates for the jump times as well as the probabilities of choosing the target state. It satisfies the following properties:

(1) $M_{\sigma,\tau} \geq 0$ for all $\sigma, \tau \in \Sigma$ with $\sigma \neq \tau$,
(2) $M_{\sigma,\sigma} \leq 0$ for all $\sigma \in \Sigma$,
(3) $\sum_{\tau \in \Sigma} M_{\tau,\sigma} = 0$.

As one can see from the above properties, it is enough the specify the off-diagonal entries $M_{\sigma,\tau}$ to determine M completely. The diagonal entries are fixed by demanding that column-sums of M are zero[1]. Unlike the transition matrix for discrete-time Markov chains, off-diagonal entries can be arbitrarily large.

Associated to such a Markov process, one can construct the *transition graph* T_G, a directed edge-weighted graph as follows. The vertices are the states and there is a directed edge from σ to τ if and only if $M_{\tau,\sigma} \neq 0$. If such an edge exists, its weight is the value of $M_{\tau,\sigma}$. We say that X is *irreducible* if T_G is strongly connected. That is to say, there is a directed path between any two vertices. All the processes we consider in these notes will be irreducible. We say that X is *ergodic* if, for every $\tau \in \Sigma$, the probability $P_\tau(t)$ of being in state τ at time t converges to some value, which we will denote π_τ. This limiting distribution is known as the *stationary (or invariant) distribution* of the process. It is a basic theorem that an irreducible Markov process on a finite state space is ergodic.

If we write the stationary probabilities as a column-vector, $\pi = (\pi_\tau)_{\tau\in\Sigma}$, then π satisfies $M\pi = 0$. In other words, π is an eigenvector of M with eigenvalue 0. By the Perron-Frobenius theorem for irreducible Markov processes, π is the unique such eigenvector. Writing the eigenvalue equation for π in component form, we obtain

$$\sum_{\tau\in\Sigma} M_{\tau,\sigma}\pi_\sigma = \sum_{\tau\in\Sigma} M_{\sigma,\tau}\pi_\tau, \tag{1.1}$$

for every $\sigma \in \Sigma$. The left- and right-hand sides of (1.1) denotes the total outgoing and incoming weight from and to state σ respectively. Equation (1.1) is known as the *master equation for* π_σ in the physics literature and *balance condition* in the probability literature. Together, these are known as the master/balance equations. We will use the former term in these notes. The process X is said to be *reversible* if for every $\sigma, \tau \in \Sigma$, we have

$$\pi_\sigma M_{\tau,\sigma} = \pi_\tau M_{\sigma,\tau}. \tag{1.2}$$

This means that the total weight of the transition from σ to τ exactly equals that from τ to σ. The equation (1.2) is known as the *detailed balance* equation. It has the following interpretation: if X is starts at $t = 0$ in its stationary distribution, the time-reversed process is indistinguishable from the forward process. Moreover, if π satisfies (1.2), then it is easy to see that it also satisfies the master equation (1.1). A process that is not reversible is said to be *irreversible*. Most of the processes in these notes will be irreversible. Only in exceptional

[1]A word of warning for the probabilists: our matrices will be column-stochastic as opposed to the usual row-stochastic convention.

cases, when the parameters are tuned in a specific way, we will find them to be reversible.

There is a formal notion of projection of Markov processes called *lumpability* which will be useful for us. If Σ can be partitioned into equivalence classes, denoted $[\cdot]$, so that $M(x,[y]) = M(x',[y])$ whenever $x' \in [x]$, then the resulting process on the equivalence classes is also a Markov chain. Then Σ is said to be *lumpable* with respect to the equivalence relation. The restricted Markov process on $\{[x] \mid x \in \Sigma\}$ is then a *lumping* or *projection* of the original process.

We will consider a special class of Markov processes in these notes. An *interacting particle system* is a Markov process on a graph, where zero or one or many particles occupy the vertices and the transitions involve particles moving stochastically across the edges of the graph. The *exclusion process* is an interacting particle system where each vertex of the graph can either be occupied by a single particle, or be empty.

We will be interested only in graphs with a finite number of sites (L, say), which can be embedded in one-dimension. These are either the *path graphs* (trees consisting of exactly two leaves) or the *cycle graphs* (graphs with a single cycle of length L). We will also use the term *site* for a vertex. In the former case, particles will be allowed to enter and exit at the leaves, and in the latter, the number of particles will be fixed. At each occupied site which is not a leaf, the particle is allowed to jump either one site to its left or right, provided the target site is empty. In both cases, one can consider different hopping rates in different directions. The *asymmetric simple exclusion process* or ASEP is the most general such process. The ASEP can either be *totally asymmetric*, in which particles hop along a single direction, and is then called the TASEP, or *partially asymmetric*, in which particles can hop in both forward and backward directions with different rates, and is called the PASEP, or *symmetric*, in which particles hop with equal rates in both directions, and is called the *simple symmetric exclusion process* (SSEP). We refer to the general model as the ASEP, with the TASEP and SSEP being extreme limits.

The states σ of the ASEP are configurations of particles on the graph. For convenience, $\sigma_i = 1$ if the i'th site is occupied by a particle, and $\sigma_i = 0$, if the i'th site is vacant. We now describe quantities of interest in the stationary distribution. We will use the notation $\langle \cdot \rangle$ to describe averages in the stationary distribution. There are two important observables in ASEPs. The *density* (of a particle) ρ_i at a given site i is the stationary probability that the site is occupied by a particle,

$$\rho_i = \langle \sigma_i \rangle = \sum_{\substack{\sigma \in \Sigma \\ \sigma_i = 1}} \pi_\sigma. \tag{1.3}$$

Without any qualifiers, the density will always refer to that of the particle. The density is a special example of a *correlation function* or *correlation*, and is also referred to as a 1-point correlation function. The *current* J across a given edge $(i, i+1)$ is the net number of particles crossing the edge per unit time. When measuring this current, we count the particles crossing from i to $i+1$ with a positive sign, and those crossing the other way, with a negative sign. If the former has rate p and the latter, rate q, we have

$$J = \langle p\sigma_i(1-\sigma_{i+1}) - q\sigma_{i+1}(1-\sigma_i) \rangle = p \sum_{\substack{\sigma \in \Sigma \\ \sigma_i = 1, \sigma_{i+1} = 0}} \pi_\sigma - q \sum_{\substack{\sigma \in \Sigma \\ \sigma_{i+1} = 1, \sigma_i = 0}} \pi_\sigma. \tag{1.4}$$

The fact that J is independent of the edge is a consequence of the conservation of particles. The current is an example of a 2-point correlation. A general n-point correlation is of the form

$$\langle \sigma_{i_1} \cdots \sigma_{i_n} \rangle = \sum_{\substack{\sigma \in \Sigma \\ \sigma_{i_1} = 1, \dots, \sigma_{i_n} = 1}} \pi_\sigma. \tag{1.5}$$

To compute any correlation, one can write down a master equation for that particular correlation, much as in (1.1), where one only considers transitions that affect the sites. For example, the *master equation for the density at site* i will have the form

$$\begin{aligned} &\sum_j (\text{rate at which particle leaves site } i \text{ to site } j) \langle \sigma_i (1-\sigma_j) \rangle \\ &= \sum_j (\text{rate at which particle enters site } i \text{ from site } j) \langle \sigma_j (1-\sigma_i) \rangle. \end{aligned} \tag{1.6}$$

These calculations will be carried out in detail in subsequent chapters.

1.2. A short history

Exclusion processes have been reinvented several times and in different contexts. The earliest reference that we know of is in the biophysics literature, where it proposed as a prototype to describe the dynamics of ribosomes [**MGP68**]. It was Frank Spitzer who coined the term 'exclusion process' in his celebrated article initiating the study of interacting particle systems [**Spi70**]. The discussion was centered around the symmetric exclusion process. This was followed by intensive study by probabilists. Liggett's book [**Lig05**, Chapter VIII] gives a detailed account of the history.

The first nontrivial (and isolated) result on ASEPs on finite lattices probably first arose in a combinatorial paper [**SZ82**]. A systematic study of these processes was undertaken from the point of view of nonequilibrium statistical physics [**DDM92, SD93**]. The breakthrough occurred when the TASEP on the finite interval with open boundaries (i.e. entry and exit of particles from the ends) was solved exactly by inventing a new technique called the matrix ansatz [**DEHP93**]. Since then, hundreds of papers have appeared generalising the results there. In particular, this technique was used to obtain exact solutions for ASEPs with several species of particles; almost immediately, for the closed 2-species ASEP [**DJLS93**], and later, after considerable work for the general closed multispecies ASEP [**EFM09, PEM09, AAMP11, AAMP12**]. In parallel, a combinatorial interpretation of the stationary distribution for the 2-species TASEP was given [**Ang06**], which was given a complete generalization by a queueing interpretation for the closed multispecies TASEP [**FM06, FM07**]. Open 2-species ASEPs are not believed to be exactly solvable in general. Models with special boundary rates were solved using the matrix ansatz in [**EFGM95, Ari06, ALS09, ALS12, CMRV15, CEM$^+$16**]. Open multispecies ASEPs, again with restricted boundary rates, were solved in [**CGdGW16, AR16, CFRV16**].

CHAPTER 2

The asymmetric simple exclusion process on a ring

2.1. Introduction

We begin with the simplest example of an asymmetric simple exclusion process (ASEP). Consider a one-dimensional lattice of $L+1$ sites with the first and the last sites identified. Every site can either be occupied or empty. The dynamics is as follows: whenever there is a particle at a given site, it hops preferentially clockwise (i.e. forward) with rate p and counterclockwise (i.e. backward) with rate q. In either case, the hop succeeds only if the target site is empty. Without loss of generality, we take $q \leq p$. Since the dynamics conserves the number of particles, we fix the number of particles to be n, where $0 \leq n \leq L$. Figure 1 shows an example of a configuration. We call this process the *ASEP with periodic boundary conditions* or the *ASEP on a ring*. If $q = 0$, this is TASEP and if $q = p$, this is the SSEP.

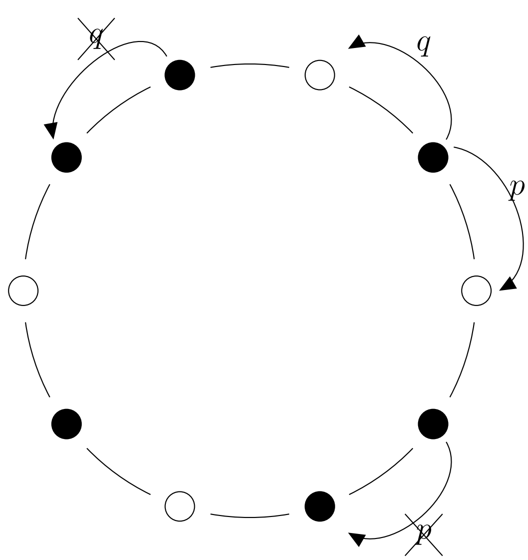

FIGURE 1. Example of a configuration in the ASEP with 10 sites and 6 particles with some allowed and disallowed hops.

It is easy to see that there are $\binom{L}{n}$ configurations. We denote configurations by words in the alphabet $\{0,1\}$ (i.e. binary words), where 0 denotes an empty site, and 1, an occupied site. The configuration space is then

$$S_{L,n} = \{w \in \{0,1\}^L \mid \text{number of 1's in } w \text{ is } n\}.$$

EXAMPLE 2.1. Let $L = 4$ and $n = 2$. There are then 6 configurations, which we order lexicographically,

$$S_{4,2} = \{0011, 0101, 0110, 1001, 1010, 1100\}.$$

The column-stochastic generator $M_{4,2}$ in this basis is then given by

$$M_{4,2} = \begin{pmatrix} -p-q & p & 0 & 0 & q & 0 \\ q & -2p-2q & p & p & 0 & q \\ 0 & q & -p-q & 0 & p & 0 \\ 0 & q & 0 & -p-q & p & 0 \\ p & 0 & q & q & -2p-2q & p \\ 0 & p & 0 & 0 & q & -p-q \end{pmatrix}.$$

2.2. Stationary Properties

We now describe the properties of the ASEP on a ring. The first is an elementary symmetry property.

PROPOSITION 2.2 (Particle-hole symmetry). *The ASEP on $S_{L,n}$ with forward (resp. backward) hopping rate p (resp. q) is isomorphic as a Markov process to the ASEP on $S_{L,L-n}$ with forward (resp. backward) hopping rate q (resp. p).*

PROOF. This obviously follows from that fact that $\phi : S_{L,n} \to S_{L,L-n}$ given by $(w_1, \dots, w_L) \mapsto (1-w_L, \dots, 1-w_1)$ is a bijection. $\square$

PROPOSITION 2.3 (Irreducibility). *The ASEP on $S_{L,n}$ is irreducible for any $p, q > 0$.*

PROOF. We will show that the TASEP is irreducible. If $q > 0$, we can only have more transitions, which cannot hinder the irreducibility. Let $\ell = (\underbrace{1, \dots, 1}_{n}, 0, \dots, 0)$, $r = (0, \dots, 0, \underbrace{1, \dots, 1}_{n})$ and $w \in S_{L,n}$. We will prove the result by showing that the transitions $\ell \to w \to r \to \ell$ are allowed.

Let the 1's be in positions $1 \leq p_1 < p_2 < \cdots < p_n$ in w. It is clear that $p_i \geq i$ for all i. To see the transition from $\ell \to w$, first move the 1 in position n in ℓ to position p_n. Then move the 1 in position $n-1$ to p_{n-1}, and continue this way, thus ending with w. Note that

$p_i \leq L + i - n$. From $w \to r$, similarly first move the 1 at position p_n to position L, followed by the 1 at position p_{n-1} to $L - 1$, and so on, ending with r. Use the periodic boundary condition to move the 1 in position L in r to n in ℓ, and so on. This completes the proof. □

An important consequence of the irreducibility of the ASEP is that the stationary distribution π is unique. Recall that the stationary distribution is defined as the solution to the master equation (1.1). Before we derive the stationary distribution, we describe a special property it satisfies. Let $\tau : S_{L,n} \to S_{L,n}$ denote the shift operator, $\tau(w) = (w_2, \dots, w_L, w_1)$.

Theorem 2.4 (Translation-invariance). *The stationary distribution satisfies, for all $w \in S_{L,n}$, $\pi(\tau(w)) = \pi(w)$.*

Proof. The key idea is that $\text{rate}(\tau(v) \to \tau(w)) = \text{rate}(v \to w)$. Now, in addition, if we assume that $\pi(\tau(v)) = \pi(v)$ for all $v \in S_{L,n}$, then we see that the master equations (1.1) for w and $\tau(w)$ are identical. Therefore, making this choice does not lead to any contradictions. Since we know that the stationary distribution is uniquely determined, we see that π has to satisfy translation invariance. □

A more algebraic way of proving Proposition 2.4 is to lift the operator τ to $S_{L,n}$ (where it is an involution) and show that $\tau M_{L,n} \tau = M_{L,n}$, or equivalently, to show that τ commutes with $M_{L,n}$.

Theorem 2.5 (Stationary distribution). *The stationary distribution of the ASEP is uniform on $S_{L,n}$.*

Proof. If $n = 0$ or $n = L$, there is a unique configuration, and there is nothing to prove. Now, suppose $1 \leq n < L$. We will show that the uniform distribution satisfies the master equation (1.1) for every w. Using Proposition 2.4, we can translate w so that $w_1 = 1$ and $w_L = 0$. Write w in block form as $w = 1^{m_1} 0^{n_1} \dots 1^{m_b} 0^{n_b}$, where for each i, $n_i, m_i \geq 1$. There are then b transitions each with rates p and q out of w. Now, let us count the transitions into w. By the definition of the process, one of the 1's in w must have moved to its present position in w, exchanging with a neighbouring 0. If we focus on the i'th block of 1's, we see that only two of them could have moved, the leftmost with rate p or the rightmost one with rate q.This happens for all b blocks, and there are no other transitions. Therefore, there are $2b$ configurations making transitions into w, half with rate p and half with rate q. As a result, if we choose all of them to have the same probability, the master equation for w gets satisfied. We now appeal to the uniqueness of π from Proposition 2.3 to complete the proof. □

Note that Theorem 2.5 does not follow from Proposition 2.4. See Exercise 2 for a counterexample.

REMARK 2.6. An equivalent way of stating Theorem 2.5 is that the column-stochastic generator $M_{L,n}$ is also row-stochastic.

In Example 2.1, one can check that the all 1's column vector is an eigenvector of $M_{4,2}$, or equivalently that $M_{4,2}$ is row-stochastic.

REMARK 2.7 (Irreversibility). If $p > q$, the ASEP on a ring of size $L > 2$ is irreversible .

Recall that the *density* at site i, $\langle w_i \rangle$, is the stationary probability that site i is occupied by a particle. The *current* J is the rate at which particles cross an edge between two neighbouring sites. Since particles move both forward and backward, it is given by

$$J = p\langle w_i(1 - w_{i+1})\rangle - q\langle w_{i+1}(1 - w_i)\rangle.$$

Note that the current is independent of i because of particle conservation. The average density and current can then be immediately derived from Theorem 2.5.

COROLLARY 2.8. *For the ASEP on the ring with L sites and n particles, the stationary density and current are given by*

$$\begin{aligned} \langle w_i \rangle &= \rho, \\ J &= (p - q)\rho(1 - \rho), \end{aligned}$$

where $\rho = n/L$.

2.3. Out of equilibrium: Bethe ansatz

For Markov processes, typically the first quantity one considers beyond the stationary distribution is the time taken to approach stationarity. For finite state irreducible Markov processes, it is a standard fact that, starting from any initial distribution, one converges exponentially fast to the stationary distribution. Then the natural quantity to consider is the rate of convergence. From a probabilistic standpoint, there are various ways of defining this rate, the most common being the notion of *mixing time*.

We will, however, restrict ourselves to a weaker notion in this section. Recall that the largest eigenvalue of the generator M of a Markov process is 0 and it occurs with multiplicity one. All other eigenvalues have negative real part. The *spectral gap* of M is given by

$$\lambda_* = \max\{-\Re(\lambda) \mid \lambda \text{ is an eigenvalue of } M\}.$$

The *relaxation time* of the process is then $\tau = 1/\lambda_*$. The intuition is that the distribution X_t of the process at time t approaches π as $\exp(-t/\tau)$. While this is a bit vague, numerics suggest that this is a good proxy for something as rigorous as the mixing time.

The material in this section is not at the level of rigour that one expects for a mathematical text. We will give a quick introduction to the idea of the Bethe ansatz in its simplest form and end with some applications.

The Bethe ansatz is a wonderful tool to calculate the spectrum of certain special operators. Fortunately, the TASEP on the ring ($q = 0$) falls in this category. We will illustrate it completely for single and two-particle situations and then mention the general results. It will be convenient for us to describe configurations by the positions of the particles rather than binary vectors.

2.3.1. Single particle: $n = 1$. Let $(\Psi(x))_{1\le x\le L}$ be the eigenvector of the generator $M_{L,1}$ with eigenvalue λ. Then Ψ satisfies the eigenvalue equation

$$\begin{aligned}\lambda\Psi(x) &= \Psi(x-1) - \Psi(x), \quad 2 \le x \le L,\\ \lambda\Psi(1) &= \Psi(L) - \Psi(1).\end{aligned}$$

We guess that $\Psi(x) = az^x$ for some yet-to-be-determined constants a and z. The constant a is just a scaling factor and can be ignored. Then the first equation above gives $\lambda z = 1 - z$ and the second gives $\lambda = z^{L-1} - 1$. Thus we have determined λ if we can solve for z. Plugging the solution for λ from the first equation into the second gives us that $z^L = 1$, from which it follows that z is an L'th root of unity, which gives L possible solutions, exactly what we want. When $z = 1$, we obtain the stationary distribution, $\Psi(x) = 1$. Thus, the eigenvalues are $\{\exp(2\pi i(L-1)/L) - 1 \mid 0 \le i \le L-1\}$.

Another way to derive the eigenvalues is to write down the generator explicitly in the lexicographically ordered basis, $\{0\ldots01, \ldots, 10\ldots0\}$. Then we see that

$$M_{L,1} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

from which it is clear that it is a circulant matrix with $a_0 = -1$ and $a_{L-1} = 1$, whose eigenvalues are explicitly known to be the same as written above.

2.3.2. Two particles: $n = 2$. This is the first test of whether the Bethe ansatz has any chance of giving a solution of the general problem. As in the previous case, we first write the eigenvalue equations for $\Psi(x_1, x_2)$,

$$\lambda\Psi(x_1,x_2) = \Psi(x_1-1,x_2) + \Psi(x_1,x_2-1) - 2\Psi(x_1,x_2) \quad \text{if } 1 < x_1, x_1+1 < x_2, \tag{2.1a}$$

$$\lambda\Psi(x_1,x_1+1) = \Psi(x_1-1,x_1+1) - \Psi(x_1,x_1+1) \quad \text{if } 1 < x_1, \tag{2.1b}$$

$$\lambda\Psi(1,x_2) = \Psi(x_2,L) + \Psi(1,x_2-1) - 2\Psi(1,x_2) \quad \text{if } 2 < x_2, \tag{2.1c}$$

$$\lambda\Psi(1,2) = \Psi(2,L) - \Psi(1,2). \tag{2.1d}$$

The ansatz that we make for the solution is that

$$\Psi(x_1,x_2) = a_{1,2} z_1^{x_1} z_2^{x_2} + a_{2,1} z_1^{x_2} z_2^{x_1}, \tag{2.2}$$

where $a_{1,2}, a_{2,1}, z_1$ and z_2 are unknowns to be determined. Of course, λ will be a function of these quantities. The equation (2.2) is the reformulation of the Bethe ansatz for the exclusion process. The intuition for this ansatz is that it will work perfectly for (2.1a) and (2.1c), which are the eigenvalue equations for the discrete Laplacian. The exclusion conditions in (2.1b) and (2.1d) are what could potentially cause the ansatz to fail. Plugging in the ansatz (2.2) into the eigenvalue equations (2.1) and simplifying, we obtain

$$\begin{aligned} &a_{1,2}((\lambda+2)z_1 z_2^{x_2-x_1+1} - z_2^{x_2-x_1+1} - z_1 z_2^{x_2-x_1}) \\ &+a_{2,1}((\lambda+2)z_1^{x_2-x_1+1} z_2 - z_1^{x_2-x_1+1} - z_1^{x_2-x_1} z_2) \\ &\qquad = 0 \quad \text{if } 1 < x_1, x_1+1 < x_2, \end{aligned} \tag{2.3a}$$

$$a_{1,2}((\lambda+1)z_1 z_2^2 - z_2^2) + a_{2,1}((\lambda+1)z_1^2 z_2 - z_1^2) = 0 \quad \text{if } 1 < x_1, \tag{2.3b}$$

$$\begin{aligned} &a_{1,2}((\lambda+2)z_2^{x_2-1} - z_1^{x_2-1} z_2^{L-1} - z_2^{x_2-2}) \\ &+a_{2,1}((\lambda+2)z_1^{x_2-1} - z_1^{L-1} z_2^{x_2-1} - z_1^{x_2-2}) = 0 \quad \text{if } 2 < x_2, \end{aligned} \tag{2.3c}$$

$$a_{1,2}((\lambda+1)z_2 - z_1 z_2^{L-1}) + a_{2,1}((\lambda+1)z_1 - z_1^{L-1} z_2) = 0. \tag{2.3d}$$

We now analyze the equation (2.3) one at a time. Rewrite (2.3a) to get

$$-\frac{a_{2,1} z_1^{x_2-x_1}}{a_{1,2} z_2^{x_2-x_1}} = \frac{(\lambda+2)z_1 z_2 - z_1 - z_2}{(\lambda+2)z_1 z_2 - z_1 - z_2}.$$

The quantity on the right hand side is 1 if $(\lambda+2)z_1 z_2 - z_1 - z_2 \neq 0$, which would imply that the ratio $a_{1,2}/a_{2,1}$ depends on $x_2 - x_1$, which is clearly nonsense. Therefore, we are forced to set the numerator on the right hand size equal to 0, from which we obtain a formula for the eigenvalue,

$$\lambda = \frac{1}{z_1} + \frac{1}{z_2} - 2. \tag{2.4}$$

Plug this formula for λ into (2.3b) and simplify to get

$$-\frac{a_{2,1}}{a_{1,2}} = \frac{1 - z_2}{1 - z_1}. \tag{2.5}$$

Since the eigenvector $\Psi(x_1, x_2)$ in (2.2) depends only on the overall scaling, we have essentially determined everything if we can solve for z_1, z_2.

Now plug in both (2.4) and (2.5) into (2.3c) to get

$$\left(\frac{z_1}{z_2}\right)^{x_2} = \frac{1 - z_1 + z_1^L(1 - z_2)}{1 - z_2 + z_2^L(1 - z_1)},$$

after considerable simplification. Arguing as before, we see that the right hand side is independent of x_2, and the only way this equation is sensible is if

$$-\frac{1 - z_1}{1 - z_2} = z_1^L = z_2^{-L}, \quad \text{which implies} \quad (z_1 z_2)^L = 1.$$

The last nontrivial check is that these equations verify (2.3d). A uniform way to write these equations is

$$z_i^{-L}(1 - z_i)^2 = -\prod_{j=1}^{2}(1 - z_j), \quad i = 1, 2. \tag{2.6}$$

These are known as the *Bethe equations* and the z_i's as the *Bethe roots*. At this point, the problem reduces to solving the Bethe equations, which requires another bag of tricks. We give a

2.3.3. Three particles: $n = 3$. The success of the Bethe ansatz for three particles is a very strong indicator of its success for an arbitrary number of particles. The ansatz that we now make is

$$\Psi(x_1, x_2, x_3) = \sum_{\pi \in S_3} a_{\pi_1, \pi_2, \pi_3} z_1^{x_{\pi_1}} z_2^{x_{\pi_2}} z_3^{x_{\pi_3}}, \tag{2.7}$$

where S_3 is the set of permutations of $\{1, 2, 3\}$. We will not redo the calculations that we did for $n = 2$, but we stress one more nontrivial check here that needs to be done. Just like we found the condition

(2.5) for the coefficients when $n = 2$, we will find the conditions,

$$
\begin{aligned}
-\frac{a_{2,1,3}}{a_{1,2,3}} &= \frac{1-z_2}{1-z_1}, & -\frac{a_{2,3,1}}{a_{2,1,3}} &= \frac{1-z_3}{1-z_1}, & -\frac{a_{3,2,1}}{a_{2,3,1}} &= \frac{1-z_3}{1-z_2}, \\
-\frac{a_{1,3,2}}{a_{1,2,3}} &= \frac{1-z_3}{1-z_2}, & -\frac{a_{3,1,2}}{a_{1,3,2}} &= \frac{1-z_3}{1-z_1}, & -\frac{a_{3,2,1}}{a_{3,1,2}} &= \frac{1-z_2}{1-z_1}.
\end{aligned} \tag{2.8}
$$

The important fact is that the ratio $a_{3,2,1}/a_{1,2,3}$ is the same whether we take the route $(1,2,3) \to (1,3,2) \to (3,1,2) \to (3,2,1)$ or the route $(1,2,3) \to (2,1,3) \to (2,3,1) \to (3,2,1)$.

At the end of the calculations, the eigenvalue is given by

$$
\lambda = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} - 3, \tag{2.9}
$$

and the Bethe equations turn out to be

$$
z_i^{-L}(1-z_i)^3 = \prod_{j=1}^{3}(1-z_j), \quad i = 1,2,3. \tag{2.10}
$$

2.3.4. n **particles.** Let S_n denote the set of permutations of the set $\{1,\ldots,n\}$. The ansatz is then

$$
\Psi(x_1,\ldots,x_n) = \sum_{\pi \in S_n} a_{\pi_1,\ldots,\pi_n} z_1^{x_{\pi_1}} \cdots z_n^{x_{\pi_n}}. \tag{2.11}
$$

We will not write the explicit formula for the ratio $a_{\pi_1,\ldots,\pi_n}/a_{1,\ldots,n}$ since they do not play an important role in the computation

The important parts are the eigenvalue formula which generalises (2.4) and (2.9),

$$
\lambda = \sum_{j=1}^{n} \frac{1}{z_j} - n, \tag{2.12}
$$

and the Bethe equations which generalise (2.6) and (2.10),

$$
z_i^{-L}(1-z_i)^n = (-1)^{n-1}\prod_{j=1}^{n}(1-z_j), \quad i = 1,\ldots,n. \tag{2.13}
$$

Note that $z_1 = \cdots = z_n = 1$ is a solution, whose eigenvector gives the stationary distribution.

Solving these set of equations is a different ball game!

Exercises

(1) The *ASEP with reflecting boundaries* is defined on a one-dimensional lattice of size L with n particles. The hopping rules in the bulk are the same as that of the ASEP on a ring. That is to say, if a particle is at site i for $2 \leq i \leq L-1$, it hops left with rate q and right with rate p and the hop succeeds only if the target site is empty. At the first (resp. last) site, the particle can only hop right (resp. left) with rate p (resp. q).

(a) Prove that the ASEP with reflecting boundaries is irreducible.

(b) Is the ASEP with reflecting boundaries translation-invariant?

(c) Give an explicit formula for the stationary distribution of the ASEP with reflecting boundaries.

(2) Consider a variant of the ASEP on a ring, where each particle hops with rate p_i (resp. q_i) if there are i 0's to its front (resp. back). Take $p_0 = q_0 = 0$. Show that the stationary distribution is translation invariant, but not uniform.

(3) Consider a variant of the ASEP on a ring where, if a particle is at site i, it hops forward (resp. backward) with rate p_i (resp. q_i). Show that the stationary distribution is not translation invariant.

(4) Go through the calculations of the Bethe ansatz for the case of $n = 3$ particles and derive the formula for the eigenvalue (2.9) and the Bethe equations (2.10).

CHAPTER 3

The asymmetric simple exclusion process on an open interval

3.1. Introduction

We now consider the ASEP on an open interval. The process is described by the following jump transitions, where, as usual, the jump is only successful if the target site (whenever it exists) is vacant.

- With rate p a particle at sites $1, \ldots, L-1$ hops to the right.
- With rate q a particle at sites $2, \ldots, L$ hops to the left.
- With rate α, a particle enters the leftmost site.
- With rate γ, a particle leaves the leftmost site.
- With rate β, a particle leaves the rightmost site.
- With rate δ, a particle enters the rightmost site.

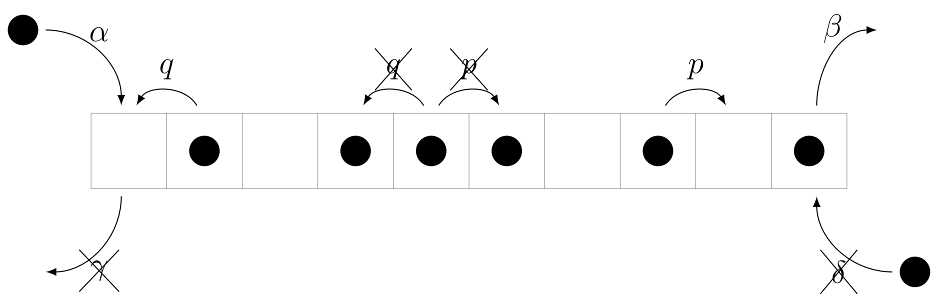

FIGURE 1. Example of a configuration in the ASEP with 10 sites as well as some allowed and disallowed transitions. The transition with rate γ is not allowed because there is no particle at the leftmost site and that with rate δ is not allowed because the rightmost site is occupied.

Throughout this chapter, whenever we refer to the ASEP without additional qualifiers, we will mean this general model. See Figure 1 for an example configuration of the ASEP on an open interval of 10 sites with some transitions drawn. A special case of the ASEP is the totally asymmetric variant, called the TASEP, where $q = \gamma = \delta = 0$.

We denote configurations in the ASEP by binary words, where 1 (resp. 0) denotes an occupied (resp. vacant) site. Then the set of configurations is $S_L = \{0,1\}^L$. We denote the column-stochastic generator of the ASEP by M_n.

EXAMPLE 3.1. Let $L = 2$. There are then 4 configurations, which we order lexicographically, $S_2 = \{00, 01, 10, 11\}$. The column-stochastic generator M_2 in this basis is then given by

$$M_2 = \begin{pmatrix} -\alpha-\delta & \beta & \gamma & 0 \\ \delta & -q-\alpha-\beta & p & \gamma \\ \alpha & q & -p-\gamma-\delta & \beta \\ 0 & \alpha & \delta & -\beta-\gamma \end{pmatrix}$$

3.2. Stationary distribution of the ASEP

PROPOSITION 3.2 (Particle-hole symmetry). *The ASEP on S_L with forward (resp. backward) hopping rate 1 (resp. q) and boundary rates $\alpha, \beta, \gamma, \delta$ as described above is isomorphic as a Markov process to the ASEP on S_L with forward (resp. backward) hopping rate q (resp. 1) and corresponding boundary rates $\beta, \alpha, \delta, \gamma$.*

PROOF. This follows from that fact that $\phi : S_L \to S_L$ given by $(w_1, \ldots, w_L) \mapsto (1 - w_L, \ldots, 1 - w_1)$ is a bijection. □

PROPOSITION 3.3 (Irreducibility). *If $p, \alpha, \beta > 0$, the ASEP on S_L is irreducible.*

PROOF. We will prove that the TASEP is irreducible, which is a stronger statement. Let $\overline{0} = (0, \ldots, 0)$ be the empty configuration and $w \in S_L$. It is clear that we can make a transition from w to $\overline{0}$ by moving particles to the right one after another starting with the rightmost one and removing them with rate β. It is also clear that we make a transition from $\overline{0}$ to w by making particles enter from the left with rate α and move to the right until they reach their positions in w. This completes the proof. □

Without loss of generality, we can set $p = 1$. This amounts to rescaling time appropriately. The stationary distribution of the ASEP can be determined explicitly by an ingenious technique originally called the *matrix ansatz* and now known as the *matrix product representation*. The idea is to suppose that the stationary probability $\pi(w)$ of the binary word $w = (w_1, \ldots, w_L)$ can be written in the form

$$\text{(3.1)} \quad \pi(w) = \frac{f(w)}{Z_L}, \quad \text{with} \quad f(w) = \langle W | \prod_{j=1}^{L} (w_i D + (1 - w_i) E) | V \rangle,$$

where D, E are linear operators (i.e. matrices) in place of 1 and 0 acting on an auxiliary space, $\langle W|$ is a row-vector and $|V\rangle$ is a column vector in that same space, and Z_L is a normalisation constant known as the *partition function* given by

$$Z_L = \langle W|(D+E)^L|V\rangle. \tag{3.2}$$

We call $f(w)$ the *stationary weight*, which is an unnormalised probability of w. Note that the order of the matrices is the same as the order of the letters in the word w.

THEOREM 3.4. *Suppose there exist (possibly infinite) matrices D, E and vectors $\langle W|$, $|V\rangle$ satisfying the equations*

$$\begin{aligned} DE - qED &= D + E, \\ \langle W|(\alpha E - \gamma D) &= \langle W|, \\ (\beta D - \delta E)|V\rangle &= |V\rangle \end{aligned} \tag{3.3}$$

with $\langle W|V\rangle \neq 0$. Then the stationary distribution of the ASEP on L sites with forward (resp. backward) hopping rate 1 (resp. q) and boundary rates $\alpha, \beta, \gamma, \delta$ as described above is given by the matrix product representation (3.1)*, where Z_L is the partition function given in* (3.2)*.*

PROOF. The method of proof is similar to what we used for the ASEP on a ring in Chapter 2. Write w in block form as $w = 0^{n_0}1^{m_1}0^{n_1} \ldots 1^{m_b}0^{n_b}1^{m_0}$, where for each $i > 0$, $n_i, m_i \geq 1$ and $n_0, m_0 \geq 0$. Then

$$\pi(w) = \frac{1}{Z_L}\langle W|E^{n_0}D^{m_1}E^{n_1}\ldots D^{m_b}E^{n_b}D^{m_0}|V\rangle.$$

We will show that using the equations in (3.3) naturally ensures that the master equation for w is satisfied.

The transitions out of w can only occur by particles hopping at the edges of the blocks. Suppose that there is a block of 1's somewhere surrounded by 0's on both sides, $w = \cdots 01^m0\cdots$. Then the transitions in which this block participates are the outgoing ones with rate $1+q$ and the incoming ones from $w' = \cdots 101^{m-1}0\cdots$ with rate 1 and $w'' = \cdots 01^{m-1}01\cdots$ with rate q. Using the matrix ansatz (3.1), the net balance from these transitions (taking incoming as positive) is

$$\begin{aligned}
&f(w') + qf(w'') - (1+q)f(w) \\
=&\langle W|\cdots DED^{m-1}E\cdots|V\rangle + q\langle W|\cdots ED^{m-1}ED\cdots|V\rangle \\
&- (1+q)\langle W|\cdots ED^{m}E\cdots|V\rangle, \\
=&\langle W|\cdots(DE-qED)D^{m-1}E\cdots|V\rangle \\
&- \langle W|\cdots ED^{m-1}(DE-qED)\cdots|V\rangle, \\
=&\langle W|\cdots(D+E)D^{m-1}E\cdots|V\rangle - \langle W|\cdots ED^{m-1}(D+E)\cdots|V\rangle, \\
=&\langle W|\cdots D^{m}E\cdots|V\rangle - \langle W|\cdots ED^{m}\cdots|V\rangle,
\end{aligned}$$

where we have used (3.3) in the penultimate line. In the last line, the two terms are $f(\cdot)$ for words with one less factor of E each. Note that the term where the block of E's to the left (resp. right) of the block of D's has one less E contributes with a positive (resp. negative) sign. As a result, if we now consider transitions for adjacent blocks of 1's, there will be a telescoping sum. The only terms that will contribute are those from the boundary. We analyse those next.

At each boundary, there are two possible terms depending on whether n_0 and m_0 in w are zero or not. We analyse the left boundary in both cases and leave the analysis of the right boundary to the reader. Suppose $n_0 = 0$. Then the leftmost block consists of 1's and the transitions it is involved in lead to

$$\begin{aligned}
&q\langle W|D^{m_1-1}ED\cdots|V\rangle + \alpha\langle W|ED^{m_1-1}E\cdots|V\rangle \\
&- (\gamma+1)\langle W|D^{m_1}E\cdots|V\rangle \\
=&\langle W|(-\gamma D+\alpha E)D^{m_1-1}E\cdots|V\rangle - \langle W|D^{m_1-1}(DE-qED)\cdots|V\rangle \\
=&\langle W|D^{m_1-1}E\cdots|V\rangle - \langle W|D^{m_1-1}(D+E)\cdots|V\rangle \\
=&-\langle W|D^{m_1}\cdots|V\rangle.
\end{aligned}$$

Therefore, this term cancels the transitions involving the block of 1's to its right. Similarly, if $n_0 > 0$, the leftmost block is a block of 0's. Now, we only consider boundary transitions that modify this block to obtain

$$\begin{aligned}
&\gamma\langle W|DE^{n_0-1}D\cdots|V\rangle - \alpha\langle W|E^{n_0}D\cdots|V\rangle \\
&= \langle W|(\gamma D-\alpha E)E^{n_0-1}D\cdots|V\rangle \\
&= -\langle W|E^{n_0-1}D\cdots|V\rangle.
\end{aligned}$$

This term again cancels the transitions involving the block of 1's to the right. Similar cancellations take place on the right boundary and the master equation for w is satisfied. □

EXAMPLE 3.5. When $L = 1$, there are two configurations and the first equation in (3.3) is irrelevant. Using the second and third equations for 0 and 1 respectively, we obtain

$$f(0) = \langle W|E|V\rangle = \frac{1}{\alpha}\langle W|V\rangle + \frac{\gamma}{\alpha}\langle W|D|V\rangle,$$
$$f(1) = \langle W|D|V\rangle = \frac{1}{\beta}\langle W|V\rangle + \frac{\delta}{\beta}\langle W|E|V\rangle.$$

We then obtain

$$f(0) = \frac{\langle W|V\rangle(\beta+\gamma)}{\alpha\beta-\gamma\delta}, \quad f(1) = \frac{\langle W|V\rangle(\alpha+\delta)}{\alpha\beta-\gamma\delta}$$

Since $\langle W|V\rangle \neq 0$, the stationary distribution is given (provided $\alpha\beta \neq \gamma\delta$) by

$$\pi(0) = \frac{\beta+\gamma}{\alpha+\beta+\gamma+\delta}, \quad \pi(1) = \frac{\alpha+\delta}{\alpha+\beta+\gamma+\delta},$$

and it is easy to see this from direct calculations.

3.2.1. Representations of the ASEP algebra. Note that Theorem 3.4 does not yet allow us to calculate the stationary distribution of the ASEP. The reason is that we have not yet shown that the algebra in (3.3) is consistent. It might turn out that this algebra has no solution. A standard way to prove the consistency of an algebra is to exhibit an explicit representation. However, it is not true that the algebra is consistent in full generality. Even for size 1 in Example 3.5, we had to use the condition that $\alpha\beta \neq \gamma\delta$. In fact, if $\alpha\beta = \gamma\delta$, then the algebra (3.3) has no representations at all! To see this, evaluate $\langle W|(\alpha E - \gamma D)|V\rangle$ in two ways. On the one hand, it is trivially equal to $\langle W|V\rangle$. On the other hand, it can be rewritten as $\gamma/\beta\langle W|(\delta E - \beta D)|V\rangle = -\gamma/\beta\langle W|V\rangle$. Since γ and β are positive and $\langle W|V\rangle \neq 0$, the algebra cannot have a solution.

It is worthwile calculating the stationary distribution of the ASEP with $L = 2$ described in Example 3.1.

EXAMPLE 3.1. *Use the matrix ansatz* (3.3) *and the formulas for the stationary distribution of the system with* $L = 1$ *in Example 3.5 to* *obtain the system of equations,*

$$\begin{pmatrix} 1 & 0 & -\gamma/\alpha & 0 \\ 0 & 1 & 0 & -\gamma/\alpha \\ 0 & -q & 1 & 0 \\ 0 & 0 & -\delta/\beta & 1 \end{pmatrix} \begin{pmatrix} \langle W|E^2|V\rangle \\ \langle W|ED|V\rangle \\ \langle W|DE|V\rangle \\ \langle W|D^2|V\rangle \end{pmatrix} = \frac{\langle W|V\rangle}{\alpha\beta - \gamma\delta} \begin{pmatrix} (\beta+\gamma)/\alpha \\ (\alpha+\delta)/\alpha \\ \alpha+\beta+\gamma+\delta \\ (\alpha+\delta)/\beta \end{pmatrix}.$$

Note that we have assumed $\alpha\beta \neq \gamma\delta$ to obtain the right hand side. It turns out that the determinant of the matrix on the left is $\alpha\beta - q\gamma\delta$. Solving this linear system, and normalising (provided $\alpha\beta \neq q\gamma\delta$), we obtain

$$\begin{aligned} \pi(00) &= (\alpha\beta\gamma + \beta^2\gamma + \beta^2 + \beta\gamma^2 + \beta\gamma\delta + \beta\gamma + \beta\gamma q + \gamma^2 q)/Z_2, \\ \pi(01) &= (\alpha\beta + \alpha\gamma\delta + \alpha\gamma + \beta\gamma\delta + \beta\delta + \gamma^2\delta + \gamma\delta^2 + \gamma\delta)/Z_2, \\ \pi(10) &= (\alpha^2\beta + \alpha\beta^2 + \alpha\beta\gamma + \alpha\beta\delta + \alpha\beta q + \alpha\gamma q + \beta\delta q + \gamma\delta q)/Z_2, \\ \pi(11) &= (\alpha^2\delta + \alpha^2 + \alpha\beta\delta + \alpha\gamma\delta + \alpha\delta^2 + \alpha\delta + \alpha\delta q + \delta^2 q)/Z_2, \end{aligned}$$

where

$$\begin{aligned} Z_2 =& \alpha^2\beta + \alpha^2\delta + \alpha^2 + \alpha\beta^2 + 2\alpha\beta\gamma + 2\alpha\beta\delta + \alpha\beta + 2\alpha\gamma\delta + \alpha\gamma + \alpha\delta^2 \\ &+ \alpha\delta + \beta^2\gamma + \beta^2 + \beta\gamma^2 + 2\beta\gamma\delta + \beta\gamma + \beta\delta + \gamma^2\delta + \gamma\delta^2 + \gamma\delta \\ &+ \alpha\beta q + \alpha\gamma q + \alpha\delta q + \beta\gamma q + \beta\delta q + \gamma^2 q + \gamma\delta q + \delta^2 q. \end{aligned}$$

In general, it turns out that the algebra can be used to calculate the stationary probabilities of an arbitrary configuration using the algebra of raising and lowering operators provided

$$\alpha\beta \neq q^j\gamma\delta \quad \text{for} \quad j \in \mathbb{N}.$$

3.2.2. Phase diagram. The phase diagram of this general model was derived in a landmark and technical paper by Uchiyama, Sasamoto and Wadati [**USW04**]. The key idea was the recognition that the ASEP is closely related to a family of q-orthogonal polynomials known as the Askey-Wilson polynomials. Let

$$\kappa^{\pm}_{u,v} = \frac{1}{2u}\left(p - q - u + v \pm \sqrt{(p-q-u+v)^2 + 4uv}\right),$$

and set $a = \kappa^+_{\alpha,\gamma}$ and $b = \kappa^+_{\beta,\delta}$. Then the phase diagram is given in Figure 2.

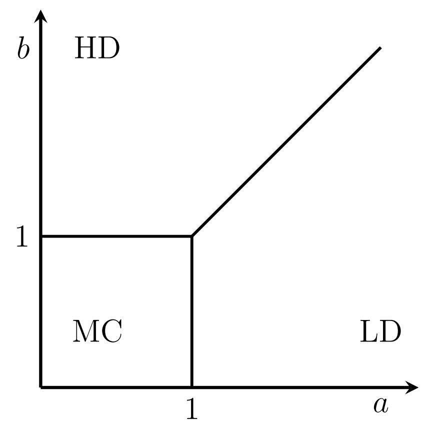

FIGURE 2. Phase diagram of the open ASEP. The labels LD, HD and MC denote the low density, high density and maximal current phases respectively. An explanation for these phases will be given in Section 3.4.

A similar phase diagram is present for the TASEP, for which the analysis is considerably simpler, and we will present all the details in Section 3.4.

3.3. Stationary distribution of the SSEP

For the special case of $q = 1$, the stationary distribution becomes somewhat simpler. While the description of the full distribution might still be complicated, correlation functions are easy to compute. We illustrate this with the computation of the density at site i for $2 \le i \le L - 1$. The master equation for the density in the general open ASEP is given, following (1.6), by

$$\langle w_i(1-w_{i+1})\rangle + q\langle w_i(1-w_{i-1})\rangle = \langle w_{i-1}(1-w_i)\rangle + q\langle w_{i+1}(1-w_i)\rangle. \tag{3.4}$$

When $q = 1$, the 2-point correlation terms in the above equation cancel and one is left with

$$2\langle w_i\rangle = \langle w_{i+1}\rangle + \langle w_{i-1}\rangle,$$

which implies that $\langle w_i\rangle$ is the solution of the discrete Laplace equation, where the boundary conditions are determined by the rates at the boundary. This general idea goes through for all correlation functions. In other words, while computing the n-point correlation, only terms for n-point and lesser correlations remain in the corresponding master equation.

When $q < 1$, this is no longer true. To compute the density, one needs to know the 2-point correlation. To compute the latter, one needs to know 3-point correlations, and so on. In the physics literature, this is known as the *BBGKY heirarchy*, named after Nikolay Bogoliubov, Max Born, Herbert Green, John Kirkwood and Jacques Yvon. They were studying the dynamics of interacting quantum particles, where the same issue came up. For the SSEP, one says that the BBGKY heirarchy is broken.

3.4. Stationary distribution of the TASEP

Recall that the TASEP is the ASEP with $q = \gamma = \delta = 0$. We have already shown in Proposition 2.3 that the TASEP is irreducible. From Theorem 3.4, we immediately obtain a formula for the stationary distribution of the TASEP.

Corollary 3.6. *Suppose there exist matrices D, E and vectors $\langle W|$, $|V\rangle$ satisfying the equations*

$$\begin{aligned} DE &= D + E, \\ \langle W|E &= \frac{1}{\alpha}\langle W|, \\ D|V\rangle &= \frac{1}{\beta}|V\rangle \end{aligned} \tag{3.5}$$

with $\langle W|V\rangle \neq 0$. Then the stationary distribution of the TASEP on L sites with forward hopping rate 1, entrance rate α on the left and exit rate β on the right is given by the matrix product representation (3.1)*, where Z_L is the partition function,*

$$Z_L = \langle W|(D + E)^L|V\rangle.$$

Unlike the ASEP, a representation of the TASEP algebra exists for all positive α, β. If we posit that D and E commute, we have

$$\begin{aligned} \frac{1}{\alpha\beta}\langle W|V\rangle &= \langle W|ED|V\rangle = \langle W|DE|V\rangle \\ &= \langle W|(D + E)|V\rangle = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\langle W|V\rangle, \end{aligned}$$

so that $\alpha + \beta = 1$. In this case, the representation of the TASEP algebra is one-dimensional with $D = 1/\beta$ and $E = 1/\alpha$ and we find the following.

Corollary 3.7. *The stationary probability in the TASEP with $\alpha + \beta = 1$ of the word w with n_1 1's and n_0 0's is $\pi(w) = \alpha^{n_1}\beta^{n_0}$.*

On the other hand, if D and E do not commute, we find the following surprising result.

PROPOSITION 3.8. *If D and E do not commute, there are no finite-dimensional representations of the TASEP algebra* (3.5).

PROOF. Let us assume, for contradiction, that there is a finite-dimensional representation of the TASEP algebra. First, suppose that 1 is an eigenvalue of E with eigenvector $|v\rangle$. Then, we have $D|v\rangle = DE|v\rangle = (D+E)|v\rangle = D|v\rangle + |v\rangle$, from which it follows that $|v\rangle = 0$, which is a contradiction. Therefore, 1 is not an eigenvalue of E and $E - \mathbb{1}$ is invertible. Then, we can use the TASEP algebra to write $D = E(E-\mathbb{1})^{-1}$ and therefore, D and E commute, which contradicts the assumption. □

We now exhibit an infinite-dimensional representation. It can be checked that the matrices

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \ddots \\ 0 & 0 & 1 & 1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \tag{3.6}$$

and the vectors

$$\langle W|_\alpha \equiv \langle W| = \kappa \left(1, \frac{1-\alpha}{\alpha}, \left(\frac{1-\alpha}{\alpha}\right)^2, \dots\right),$$

$$|V\rangle_\beta \equiv |V\rangle = \kappa \begin{pmatrix} 1 \\ \frac{1-\beta}{\beta} \\ \left(\frac{1-\beta}{\beta}\right)^2 \\ \vdots \end{pmatrix}, \tag{3.7}$$

satify the algebra (3.5). We choose $\kappa = \sqrt{(\alpha+\beta-1)/(\alpha\beta)}$ so that $\langle W|V\rangle = 1$ for convenience.

REMARK 3.9. The stationary weights for the TASEP can be calculated without any reference to the representation. Given any word w in D's and E's, use the first equation in (3.5) to write the word as a sum of words of the form E^iD^j. Now, each of these terms can be evaluated since $\langle W|E^iD^j|V\rangle = \alpha^{-i}\beta^{-j}$.

3.4.1. Partition function. The expression for the partition function Z_L in Corollary 3.6 can be used to derive a remarkably explicit formula using the TASEP algebra (3.5). Let $F = D + E$ and expand F^L as explained in Remark 3.9. For example,

$$F^2 = D^2 + DE + E^2 + D + E,$$
$$F^3 = D^3 + ED^2 + E^2D + E^3 + 2(D^2 + DE + E^2) + 2(D + E),$$

where we have sorted terms by total degree.

We will need the following notation. The n'th *Catalan number* is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \in \mathbb{N}. \tag{3.8}$$

The sequence starts $1, 1, 2, 5, 14, 42, \ldots$. The Catalan numbers form an important sequence in enumerative combinatorics and frequently occur in problems from diverse areas. As examples, the number of triangulations of an n-gon, the number of binary trees with $n+1$ leaves, the number of legal words in n pairs of parentheses, and the number of up-down paths from $(0, 0)$ to (n, n) which stay on or below the diagonal $x = y$ are all counted by the Catalan numbers. The generating function of the Catalan number is given by

$$C(x) := \sum_{n\geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}. \tag{3.9}$$

One way to verify this is to expand the right hand side using the binomial theorem. While this equality can be seen to hold at the level of formal power series (i.e. treating x as a formal variable), we will eventually be interested in $C(x)$ for complex values x. In that case, this sum converges for $|x| \leq 1/4$. See [**FI12**, Sequence A000108] for more details on the sequence. R. Stanley's treatise on enumerative combinatorics [**Sta99**, Exercise 6.19] describes 66 different interpretations of the Catalan numbers.

Among the many refinements of the Catalan numbers, an important one is the sequence of *ballot numbers*, C^n_k given by

$$C^n_k = \frac{n - k + 1}{n + 1}\binom{n + k}{k}, \quad 0 \leq k \leq n. \tag{3.10}$$

The ballot numbers C^n_k count the number of p-down paths from $(0, 0)$ to (n, n) which stay on or below the diagonal $x = y$ and which touch the diagonal $n-k+1$ times (counting both endpoints). The triangular array starts

$$\begin{array}{lllll} 1 \\ 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 5 & 5 \\ 1 & 4 & 9 & 14 & 14 \end{array}$$

The array satisfies the Pascal triangle-like recurrence

$$C_k^n = C_k^{n-1} + C_{k-1}^n, \tag{3.11}$$

which works also for $k = 0$ and $k = n$ if we assume $C_0^n = C_{n+1}^n = 0$. Notice that the row-sums $\sum_k C_k^n$ are themselves counted by the Catalan numbers C_{n+1}. Moreover, the two rightmost diagonals are also the Catalan numbers, $C_n^n = C_{n-1}^n = C_{n-1}$. Not surprisingly, the bivariate generating function of the ballot numbers is closely related to that of the Catalan numbers,

$$B(x, y) := \sum_{n \geq 0} \sum_{k=0}^{n} C_k^n x^n y^k = \frac{C(xy)}{1 - xC(xy)}. \tag{3.12}$$

This can be derived from the recurrence (3.11). The interpretation is that the generating function of the k'th left-to-right diagonal in the ballot triangle counted from the right is the k'th power of $C(z)$. See [**FI12**, Sequence A009766] for other enumerative problems counted by and properties of this sequence.

Theorem 3.10. *For any $n \in \mathbb{N}$,*

$$F^n = \sum_{k=0}^{n} C_{n-k}^{n-1} \sum_{j=0}^{k} E^j D^{k-j}. \tag{3.13}$$

As a consequence, the partition function is given by

$$Z_L = \sum_{k=1}^{L} C_{L-k}^{L-1} \frac{1/\beta^{k+1} - 1/\alpha^{k+1}}{1/\beta - 1/\alpha}. \tag{3.14}$$

Note that the index k in (3.13) starts at 0, which is not a problem because $C_n^{n-1} = 0$.

Proof. We will prove the formula for F^n by induction. For $n = 1$, the calculation is trivial. Before we begin the general calculation, we use the recurrence $D^m F = D^{m+1} + D^{m-1} F$ to obtain

$$D^m F = \frac{D^{m+2} - D}{D - 1} + E, \tag{3.15}$$

where the shorthand $(D^{m+2} - D)/(D-1)$ is a formal expression which stands for the sum $D + D^2 + \cdots + D^{m+1}$.

Now, assume (3.13) holds for n and right multiply it by F to obtain

$$F^{n+1} = \sum_{k=0}^{n} C_{n-k}^{n-1} \sum_{j=0}^{k} E^j D^{k-j} F.$$

Use (3.15) to simplify this as

$$F^{n+1} = \sum_{k=0}^{n} C_{n-k}^{n-1} \sum_{j=0}^{k} E^j \left(\frac{D^{k-j+2} - D}{D-1} + E \right).$$

Now, use the recurrence (3.11) to write $C_{n-k}^{n-1} = C_{n-k}^{n} - C_{n-k-1}^{n}$ and expand to obtain

$$\begin{aligned} F^{n+1} = &\sum_{k=0}^{n} C_{n-k}^{n} \sum_{j=0}^{k} E^j \left(\frac{D^{k-j+2} - D}{D-1} + E \right) \\ &- \sum_{k=0}^{n} C_{n-k-1}^{n} \sum_{j=0}^{k} E^j \left(\frac{D^{k-j+2} - D}{D-1} + E \right). \end{aligned}$$

Isolate the first term in the first sum, shift the index $k \to k-1$ in the second sum and combine terms to obtain

$$\begin{aligned} F^{n+1} = C_n(D+E) + \sum_{k=1}^{n} C_{n-k}^{n} \Bigg(&\sum_{j=0}^{k} E^j \left(\frac{D^{k-j+2} - D}{D-1} + E \right) \\ &- \sum_{j=0}^{k-1} E^j \left(\frac{D^{k-j+1} - D}{D-1} + E \right) \Bigg). \end{aligned}$$

The term inside the parenthesis simplifies to $E^j D^{k-j+1}$ for $j < k$ and to E^{k+1} for $j = k$. Simplifying the expression, we obtain,

$$F^{n+1} = \sum_{k=0}^{n} C_{n-k}^{n} \sum_{j=0}^{k+1} E^j D^{k+1-j},$$

completing the proof of (3.13). Now, use the formula for the partition function from Corollary 3.6 to obtain.

$$Z_L = \langle W | F^L | V \rangle = \sum_{k=1}^{n} C_{n-k}^{n-1} \sum_{j=0}^{k} \alpha^{-j} \beta^{j-k}$$

The inner sum is a geometric series, which after simplifying gives (3.14), thereby completing the proof. □

3.4.2. Current and density. For the TASEP on the interval of L sites, the current in the stationary distribution is given by $J = \langle w_i(1-w_{i+1})\rangle$, for $1 \leq i \leq L-1$. Since the dynamics conserves the number of particles in the bulk of the system, J is independent of i.

THEOREM 3.11. *The current in the stationary distribution for the TASEP on the interval of L sites is given by*

$$J = \frac{Z_{L-1}}{Z_L}. \tag{3.16}$$

PROOF. Using the matrix ansatz (3.1), we write

$$J = \frac{1}{Z_L}\langle W|F^{i-1}DEF^{L-i-1}|V\rangle,$$

which leads, via the TASEP algebra (3.5) to

$$J = \frac{1}{Z_L}\langle W|F^{i-1}(D+E)F^{L-i-1}|V\rangle = \frac{Z_{L-1}}{Z_L},$$

which proves the result. □

REMARK 3.12. The current is also equivalently given by $J = \alpha\langle 1 - w_0\rangle = \beta\langle w_L\rangle$ by considering the movement of particles at the first and last sites respectively. It is very satisfactory to see that in both cases, we naturally obtain the same answer,

$$J = \frac{\alpha}{Z_L}\langle W|EF^{L-1}|V\rangle = \frac{1}{Z_L}\langle W|F^{L-1}|V\rangle = \frac{Z_{L-1}}{Z_L},$$

and

$$J = \frac{\beta}{Z_L}\langle W|F^{L-1}D|V\rangle = \frac{1}{Z_L}\langle W|F^{L-1}|V\rangle = \frac{Z_{L-1}}{Z_L},$$

using (3.5).

The density of particles at site i in the TASEP can in principle be calculated by (3.4) with $q = 0$, but the BBGKY heirarchy makes the task daunting. One can compute the density using the matrix ansatz.

THEOREM 3.13. *The density at site j in the stationary distribution for the TASEP on the interval of L sites is given by*

$$\langle w_j\rangle = \begin{cases} \displaystyle\sum_{k=0}^{L-j-1} C_k \frac{Z_{L-k-1}}{Z_L} + \frac{Z_{j-1}}{Z_L}\sum_{k=2}^{L-j+1} C^{L-j-1}_{L-j+1-k}\frac{1}{\beta^k} & 1 \leq j < L, \\ \displaystyle\frac{1}{\beta}\frac{Z_{L-1}}{Z_L} & j = L. \end{cases} \tag{3.17}$$

PROOF. The second part of (3.17) immediately follows from the second expression for the current J in Remark 3.12. We will obtain the first part by proving the identity,

$$DF^m = \sum_{k=0}^{m-1} C_k F^{m-k} + \sum_{k=2}^{m+1} C_{m+1-k}^{m-1} D^k, \quad m \geq 1. \tag{3.18}$$

We proceed by induction. For $m = 1$, we obtain $DF = D(D+E) = F + D^2$, which agrees with (3.18). Assume the identity holds for m, and multiply both sides of (3.18) on the right to obtain

$$DF^{m+1} = \sum_{k=0}^{m-1} C_k F^{m+1-k} + \sum_{k=2}^{m+1} C_{m+1-k}^{m-1} D^k F.$$

Rewrite (3.15) as $D^k F = F + \sum_{j=2}^{k+1} D^j$ and plug it into the above equation to obtain

$$DF^{m+1} = \sum_{k=0}^{m-1} C_k F^{m+1-k} + F \sum_{k=2}^{m+1} C_{m+1-k}^{m-1} + \sum_{k=2}^{m+1} C_{m+1-k}^{m-1} \sum_{j=2}^{k+1} D^j.$$

Now replace $k \to m+1-k$ in the second sum and rearrange the summands in the last sum to get

$$DF^{m+1} = \sum_{k=0}^{m-1} C_k F^{m+1-k} + F \sum_{k=0}^{m-1} C_k^{m-1} + \sum_{j=2}^{m+2} D^j \sum_{k=0}^{m+2-j} C_k^{m-1}.$$

From the recurrence (3.11), it is easy to see that the ballot numbers satisfy the identity,

$$\sum_{k=0}^{j} C_k^m = C_j^{m+1}.$$

Use this identity twice, in the second sum as well the inner sum above to obtain

$$DF^{m+1} = \sum_{k=0}^{m} C_k F^{m+1-k} + \sum_{j=2}^{m+2} C_{m+2-j}^{m} D^j,$$

thus proving (3.18).

From the matrix ansatz (3.1), the density at site j is given by

$$\langle w_j \rangle = \frac{1}{Z_L} \langle W | F^{j-1} D F^{L-j} | V \rangle.$$

Use (3.18) to substitute for DF^{L-j} in the above equation and use the TASEP algebra to complete the proof. □

3.4.3. Phase diagram. We are interested in the steady state of the system for large sizes, the so-called thermodynamic limit. In particular, we would like to estimate the densities and currents in that limit. It will turn out that these quantities are somewhat sensitive to the values of the boundary rates, α and β.

The terminology used in the statistical physics literature is as follows. Much of this is borrowed from equilibrium statistical physics, but the correspondence is not exact. A summary of these features can be shown by what is known as a **phase diagram**. The phase diagram for a macroscopic system lives on its parameter space, and encapsulates all its gross features. More precisely, given a physical observable of interest, different regions of the phase diagram have different average values for that observable and all points in a given region have the same value of that observable. These observables are known as **order parameters** in the statistical physics literature. In principle, different order parameters can lead to different phase diagrams for the same system. Regions of the phase diagram with different values of the order parameter are known as **phases**, which are separated by **phase boundaries**. The system is said to exhibit a **phase transition** whenever the parameters are varied so that it crosses a phase boundary. The **order** of a phase transition is n if the smallest derivative of the order parameter which is discontinuous across the phase boundary is $n-1$. For example, if the order parameter is itself discontinuous, the transition is of first-order. If it's first derivative is discontinuous, the transition is second-order, and so on.

Since both the density and current depend explicitly on the partition function Z_L, we must first understand its asymptotic properties first. A useful technique to compute asymptotic expansions is via the generating function. Let

$$\mathcal{Z}(x) = \sum_{n \geq 0} Z_n x^n.$$

From the matrix algebra, it follows that

$$\mathcal{Z}(x) = \langle W | \frac{1}{1 - xF} | V \rangle.$$

Now, we use the cute observation

$$(1 - tD)(1 - tE) = 1 - (t - t^2)F$$

to write

$$\frac{1}{1 - xF} = \frac{1}{1 - tD} \frac{1}{1 - tE},$$

where $x = t(1-t)$. Plugging this expression back into the generating function, we obtain

$$\mathcal{Z}(x) = \langle W| \frac{1}{1-tD} \frac{1}{1-tE} |V\rangle = \frac{1}{1-t/\alpha} \frac{1}{1-t/\beta}.$$

We now need to express t in terms of x. There are two roots of the quadratic equation, but we have $\mathcal{Z}(0) = 1$ (because $Z_0 = 1$) and thus $t = 0$ when $x = 0$. As a result,

$$t(x) = \frac{1 - \sqrt{1-4x}}{2},$$

and we end up with

$$\mathcal{Z}(x) = \frac{4\alpha\beta}{(2\alpha - 1 + \sqrt{1-4x})(2\beta - 1 + \sqrt{1-4x})}. \tag{3.19}$$

We note that this simple expression can also be derived from (3.14) and (3.12).

A general rule-of-thumb is that the asymptotics of a sequence is determined by the singularities of the generating function in the complex plane closest to the origin. It will suffice for our purposes to restrict the singularities to be either poles or algebraic singularities. Suppose $f(x)$ is the generating function for the sequence (a_n). If x_0 is the singularity closest to the origin for $f(x)$ with order ν ($\nu \in \mathbb{R}$), then write

$$f(x) = f_r(x) \left(1 - \frac{x}{x_0}\right)^{-\nu},$$

where $f_r(x)$ is regular at x_0. It then follows that

$$a_n \sim \binom{\nu + n - 1}{n} f_r(x_0) x_0^{-n}.$$

Now,

$$\binom{\nu + n - 1}{n} \sim \frac{n^{\nu-1}}{\Gamma(\nu)}$$

for n large, where $a_n \sim b_n$ means that the ratio of the sequences tends to 1 as $n \to \infty$. Therefore, the leading order term for a_n is given by

$$a_n \sim \frac{f_r(x_0)}{\Gamma(\nu)} x_0^{-n} n^{\nu-1}. \tag{3.20}$$

A natural way to determine the regular part is using

$$f_r(x_0) = \lim_{x \to x_0} \left(1 - \frac{x}{x_0}\right)^{\nu} f(x). \tag{3.21}$$

There are three sources of singularities in (3.19). The first is the branch-cut singularity because of the square root at $x = 1/4$. The other two are poles which occur when one of the two factors in the denominator become zero, at $x = \alpha(1-\alpha)$ and $x = \beta(1-\beta)$ (i.e. $t = \alpha$ and $t = \beta$). However, note that these are singularities only for $\alpha, \beta < 1/2$ (for example, $2\alpha - 1 > 0$ if $\alpha > 1/2$). There are now several possibilities.

(1) Neither of $\alpha, \beta < 1/2$: the only singularity is at $1/4$ with order $\nu = -1/2$. Expand $\mathcal{Z}(x)$ to next-to-lowest order to obtain

$$\mathcal{Z}(x) = \frac{4\alpha\beta}{(2\alpha-1)(2\beta-1)}\left[1 - \sqrt{1-4x}\left(\frac{1}{2\alpha-1} + \frac{1}{2\beta-1}\right)\right].$$

The first term is independent of x and does not contribute to the asymptotics. We thus obtain,

$$\begin{aligned} Z_n &\sim -\frac{8\alpha\beta(\alpha+\beta-1)}{(2\alpha-1)^2(2\beta-1)^2}\frac{1}{\Gamma(-1/2)}\left(\frac{1}{4}\right)^{-n} n^{-1/2-1} \\ &= \frac{4\alpha\beta(\alpha+\beta-1)}{\sqrt{\pi}(2\alpha-1)^2(2\beta-1)^2}\frac{4^n}{n^{3/2}}, \end{aligned} \tag{3.22}$$

where we have used the fact that $\Gamma(-1/2) = -2\sqrt{\pi}$, and the regular part is computed using (3.21). The symmetry between α and β in (3.22) ensures that the relative values of α and β do not matter.

(2) Exactly one of $\alpha, \beta < 1/2$ (α, say): both $1/4$ and $\alpha(1-\alpha)$ are singularities and the closest is the latter with order $\nu = 1$. The regular part is given by

$$\begin{aligned} &\lim_{x\to\alpha(1-\alpha)}\left(1 - \frac{x}{\alpha(1-\alpha)}\right) Z(x) \\ &\quad= \lim_{x\to\alpha(1-\alpha)} \frac{-(2\alpha-1-\sqrt{1-4x})}{4\alpha(1-\alpha)}\frac{4\alpha\beta}{(2\beta-1+\sqrt{1-4x})} \\ &\quad= \frac{(1-2\alpha)\beta}{(\beta-\alpha)(1-\alpha)}. \end{aligned}$$

Plugging this in (3.20), we obtain

$$Z_n \sim \frac{(1-2\alpha)\alpha\beta}{(\beta-\alpha)}(\alpha(1-\alpha))^{-n-1}. \tag{3.23}$$

(3) Both $\alpha, \beta < 1/2$ (with $\alpha > \beta$, say): $1/4$, $\alpha(1-\alpha)$ and $\beta(1-\beta)$ are singularities and the closest is the latter. The same calculation as in the previous case is repeated and we obtain (3.23) with α and β interchanged.

(4) $\alpha = 1/2 < \beta$: In this case, the branch-cut singularity and the pole coincide at $x = 1/4$, which means that the nature of

the singularity changes. Expanding $\mathcal{Z}(x)$ to lowest order, we obtain

$$\mathcal{Z}(x) = \frac{2\beta}{(2\beta-1)} \frac{1}{\sqrt{1-4x}} \left(1 - \frac{\sqrt{1-4x}}{2\beta-1}\right),$$

and it is clear that the order is $\nu = 1/2$. The regular part is easily computed and we obtain the asymptotic formula from (3.20) to be

$$Z_n \sim \frac{2\beta}{\sqrt{\pi}(2\beta-1)} \frac{4^n}{\sqrt{n}} \tag{3.24}$$

(5) $\beta = 1/2 < \alpha$: The calculation is the same as the previous case and we obtain (3.24) with α and β interchanged.

(6) $\alpha = \beta < 1/2$: The dominant singularity now is the pole of order $\nu = 2$ at $x = \alpha(1-\alpha)$. The computation proceeds exactly as in item (2) and we obtain

$$Z_n \sim \left(\frac{1-2\alpha}{1-\alpha}\right)^2 \frac{n}{(\alpha(1-\alpha)^n}. \tag{3.25}$$

(7) $\alpha = \beta = 1/2$: At the triple point, $\mathcal{Z}(x) = (1-4x)^{-1}$, and we immediately see that $Z_n = 4^n$. This is easily seen as a special case of Corollary 3.7.

We summarize these calculations in Table 1. We now calculate the current J in the limit as the system size $L \to \infty$ using Theorem 3.11. To leading order, the exponential contribution dominates and we immediately see that

$$J = \begin{cases} \frac{1}{4} & \alpha, \beta \geq \frac{1}{2}, \\ \alpha(1-\alpha) & \alpha < \frac{1}{2}, \beta, \\ \beta(1-\beta) & \beta < \frac{1}{2}, \alpha. \end{cases} \tag{3.26}$$

In the region $\alpha, \beta > 1/2$, the current is independent of both α and β and takes the maximum possible value. This region is therefore known as the **maximal current phase**. In the region $\alpha < 1/2, \beta$, the current depends purely on α. As we will show presently, the density of particles in this region is roughly α, which is less than $1/2$. Hence this region is known as the **low density phase**. The behaviour of the system in this part of the phase diagram is completely governed by the entry rate of particles at the left boundary. As expected from Proposition 3.2, in the region $\beta < 1/2, \alpha$, the current depends purely on β, and we will

Region	Asymptotic formula for Z_n
$\alpha, \beta > \frac{1}{2}$	$\frac{4\alpha\beta(\alpha+\beta-1)}{\sqrt{\pi}(2\alpha-1)^2(2\beta-1)^2}\frac{4^n}{n^{3/2}}$
$\alpha < \frac{1}{2}, \beta$	$\frac{(1-2\alpha)\alpha\beta}{(\beta-\alpha)}\frac{1}{(\alpha(1-\alpha))^{n+1}}$
$\beta < \frac{1}{2}, \alpha$	$\frac{(1-2\beta)\alpha\beta}{(\alpha-\beta)}\frac{1}{(\beta(1-\beta))^{n+1}}$
$\alpha = \frac{1}{2} < \beta$	$\frac{2\beta}{\sqrt{\pi}(2\beta-1)}\frac{4^n}{\sqrt{n}}$
$\beta = \frac{1}{2} < \alpha$	$\frac{2\alpha}{\sqrt{\pi}(2\alpha-1)}\frac{4^n}{\sqrt{n}}$
$\alpha = \beta < \frac{1}{2}$	$\left(\frac{1-2\alpha}{1-\alpha}\right)^2\frac{n}{(\alpha(1-\alpha))^n}$
$\alpha = \beta = \frac{1}{2}$	4^n

TABLE 1. Leading order asymptotics of the partition function Z_n as functions of (α, β) in various regions of the square $[0,1]^2$.

show that the density is $1-\beta$, which is greater than $1/2$. This region is known as the **high density phase**.

The phase diagram for the open TASEP with current being the order parameter is shown in Figure 3. All the phase transitions are of second-order.

We now use Theorem 3.13 and the asymptotics of the partition function in Table 1 to calculate the density of particles in the limit of large system size. We will be interested in the density at sites far away from the boundaries and it is natural to scale the position with size. Therefore, we fix $x \in (0,1)$ and take the site to be $j = xL$. To lowest order, it suffice to take

$$Z_L \approx cL^{\nu}\lambda^L,$$

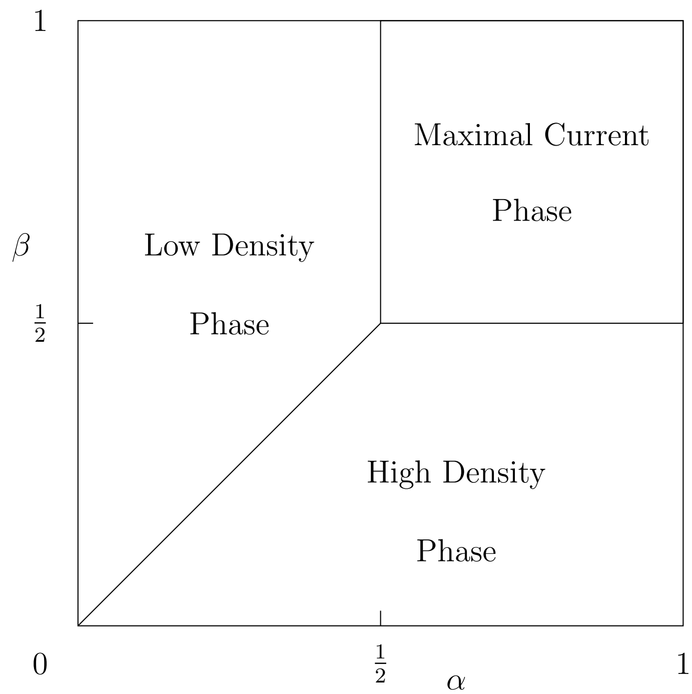

FIGURE 3. The phase diagram of the open TASEP with respect to the current, where the boundaries describe second order phase transitions.

where c, ν, λ are constants that depend on the phase as given in Table 1. Plugging in this expression into the formula for the density, we obtain

$$\langle w_{xL} \rangle \approx \sum_{k=0}^{L(1-x)} C_k \lambda^{-k-1} + \lambda^{(x-1)L-1} x^{\nu} \sum_{k=2}^{L(1-x)} C^{L(1-x)-1}_{L(1-x)+1-k} \frac{1}{\beta^k}. \tag{3.27}$$

As $L \to \infty$, the first sum in (3.27) resembles the generating function of the Catalan numbers (3.9) in the variable λ^{-1}. Note that this sum converges only if $\lambda \geq 4$. However, this causes no problems since λ takes the values $(\alpha(1-\alpha))^{-1}$, $(\beta(1-\beta))^{-1}$ or 4. We then see that the first sum approaches

$$\frac{1 - \sqrt{1 - 4\lambda^{-1}}}{2}. \tag{3.28}$$

Now, let us look at the asymptotics of the second sum in (3.27). Replace $L(1-x)$ by n to observe that we need to calculate the asymptotics of the sum,

$$y_n = \sum_{k=0}^{n-1} C_k^{n-1} \beta^{k-n-1}.$$

Using the formula for the generating function of the ballot numbers (3.12), we find that

$$Y(x) := \sum_{n=0}^{\infty} y_n x^n = \frac{1}{\beta} \frac{1-\sqrt{1-4x}}{2\beta - 1 + \sqrt{1-4x}}.$$

We now repeat the singularity analysis we performed for the partition function using (3.20). The possible singularities are at $x = 1/4$ and $x = \beta(1-\beta)$ if $\beta < 1/2$. There are thus three possibilities, depending on whether β is greater than, lesser than, or equal to $1/2$. The calculations are then essentially identical to the ones in items (1), (2) and (4) respectively. The end result is that

$$y_n = \begin{cases} \dfrac{1}{(2\beta-1)^2\sqrt{\pi}} \dfrac{4^n}{n^{3/2}} & \beta > \frac{1}{2}, \\ \dfrac{2}{\sqrt{\pi}} \dfrac{4^n}{n^{1/2}} & \beta = \frac{1}{2}, \\ \dfrac{1-2\beta}{(\beta(1-\beta))^{n+1}} & \beta < \frac{1}{2}. \end{cases} \tag{3.29}$$

Now, we need to compare the asymptotic values of the first sum in (3.28) and the second sum in (3.29) in different regions of the phase diagram to obtain the asymptotic value of the density at site xL in (3.27). In the maximal current phase, $\lambda = 4$ and $\nu = -3/2$. The first sum gives $1/2$ and the second sum goes to zero as $L \to \infty$. When $\alpha < 1/2, \beta$, the first sum gives α. There are several cases for the second sum, depending on the value of β, but in each case, it goes to zero. Therefore, the density is α, which is less than $1/2$. This justifies calling this region the *low density phase*. When $\beta < 1/2, \alpha$, the first sum gives β and the second sum gives $1 - 2\beta$. Therefore, the total is $1 - \beta$, which is greater than $1/2$. This justifies calling this region the *high density phase*. Lastly, when $\alpha = \beta < 1/2$, $\lambda = (\beta(1-\beta))^{-1}$ and $\nu = 1$. The first sum still gives β, but the second sum gives $x(1-2\beta)$. Therefore, the density depends linearly on x with value β on the left boundary and $1-\beta$ on the right boundary. This phase boundary is of special importance and is called the *shock line*. To summarise,

$$\langle w_{xL} \rangle \approx \begin{cases} 1/2 & \alpha, \beta \geq 1/2, \\ \alpha & \alpha < 1/2, \beta, \\ \beta & \beta < 1/2, \alpha, \\ \beta + x(1-2\beta) & \alpha = \beta < 1/2. \end{cases} \tag{3.30}$$

If we take the bulk density as the order parameter for our system, we obtain the same phase diagram in Figure 3. The boundaries separating the low- and high-density phases from the maximal current phase continue to be of second order, since the density also varies continuously across these boundaries. However, this is not true for the shock line, and the latter is *interpreted* as a first-order boundary. It is to be noted that this is an interpretation because we are not working within the realm of equilibrium statistical physics and, as we will see below, such phenomena do not occur there.

Except for boundary effects, the densities in generic points of the phase diagram are constant throughout, and yield no surprises. However, the density at the shock-line (as well as the terminology) needs an explanation.

3.4.4. Shock line. We have seen above that in the low-density (resp. high-density) phase, the density is determined purely by the left (resp. right) boundary. The shock line $\alpha = \beta < 1/2$ is at the boundary of these phases and thus, one would expect competition between these two boundaries in determining the density profile. It turns out that this is exactly what happens. We note that the explanation below has been amply demonstrated by simulations, but we have not seen a rigorous justification of the explanation below.

After the system has reached the stationary distribution, the density profile at any instantaneous time exhibits a sharp transition from density α on the left to $1-\alpha$ on the right, as shown in Figure 4. At a macroscopic scale, this transition, called a shock, takes place at a single point, denoted t in the figure. The location of this shock performs a simple symmetric random walk in the lattice with reflecting boundary conditions. Therefore, t is a uniform $[0,1]$ random variable. Under these assumptions, the density at a rescaled position x is given by

$$\alpha \mathbb{P}(t > x) + (1-\alpha)\mathbb{P}(t < x) = \alpha(1-x) + (1-\alpha)x,$$

which is exactly what we obtained above. Simulations also attest to this hypothesis.

This is a prime example of a stationary state that is not in equilibrium. Although the distribution of the states does not evolve in time,

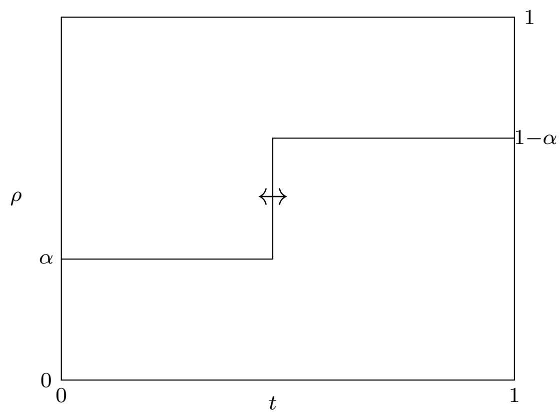

FIGURE 4. Instantaneous picture of the shock at the shock line. The x-axis is the rescaled location and the y-axis is the density of particles. The location t of the shock is random.

there are fluctuations at a macroscopic scale. Such phenomena are not expected to occur in reversible Markov processes.

One can see the instantaneous picture in Figure 4 in simulations. The way one obtains this is as follows. Starting from a large system of size L, one groups together clusters of ℓ sites, where $1 \ll \ell \ll L$. Over each cluster of ℓ sites, one computes the average density, and this is what ends up looking like the figure. This procedure is sometimes called *binning*. Notice that some form of *coarse-graining* (of which binning is one form) is necessary to observe a density profile. Otherwise, one just sees 1's and 0's. One problem with this approach is that one cannot determine the precise *microscopic* location of the shock.

An ingenious way to determine this location is the following. Relabel one of the particles to a $1'$, call it a *second-class* particle, and modify the TASEP dynamics as follows. All particles jump to the right with rate 1 and the jump succeeds if the target site is empty, as before. In addition, if a regular (i.e. first-class) particle is followed by a second-class particle, they exchange with rate 1. However, the reverse is forbidden. In other words, jumps of regular particles always succeeds.

Start this modified dynamics in the system on the shock line. If the second-class particle is to the left of the shock, the density of 1's is $\alpha < 1/2$, and there are more 0's than 1's. Therefore, the second-class particle starts to drift towards the right. On the other hand, if the second-class particle is to the right of the shock, the density of 1's is

$1 - \alpha > 1/2$ and the 1's push the second-class particle backwards. It has been shown in an infinite variant of the TASEP that the second-class particle determines the location of the shock with high probability. Therefore, one can *define* the location of the shock as that of the second-class particle.

This is a natural segue to the next chapter, where we will discuss the ASEP with two species of particles.

Exercises

(1) Show that if $\alpha\beta = q^L\gamma\delta$, the stationary distribution of the ASEP of size L is a product measure with density $\alpha/(\alpha + q^{i-1}\gamma)$ at site i.

(2) Show that if $\alpha\beta = q^j\gamma\delta$, the marginal of the stationary distribution of the ASEP of size L ($L > j$) on the leftmost $j+1$ sites is a product measure with density $\alpha/(\alpha + q^{i-1}\gamma)$ at site i.

CHAPTER 4

The two-species asymmetric simple exclusion process on a ring

The two-species asymmetric simple exclusion process (2-ASEP) is naturally motivated from the study of the shock in the single species TASEP, as explained at the end of Chapter 3.

4.1. Model definition

Consider an exclusion process with two species of particles on a finite one-dimensional lattice of size L with periodic boundary conditions. Recall that this means that the $L+1$'th site and the first site are identified. The two species of particles are called *first-class* and *second-class*, and are labelled by 2's and 1's respectively. The notation is misleading for historical reasons. To avoid confusion, we will not use this terminology at all. We will refer instead to particles by their labels, i.e. 2's and 1's. Vacant sites will be denoted by 0's.

The process is defined by the following transitions.

- With rate p, $\alpha\beta \to \beta\alpha$ on consecutive sites if $\alpha > \beta$.
- With rate q, $\alpha\beta \to \beta\alpha$ on consecutive sites if $\alpha < \beta$.

Note that the number of 2's and 1's are conserved by the dynamics. Therefore, the configuration space is determined by the number of 0's, 1's and 2's, denoted n_0, n_1 and n_2 respectively. Of course, $n_0 + n_1 + n_2 = L$. For $w \in \{0,1,2\}^L$, let $n_\alpha(w)$ be the number of α's in w for $\alpha \in \{0,1,2\}$. Then the configuration space is

$$S_{(n_0,n_1\,n_2)} = \{w \in \{0,1,2\}^L \mid n_\alpha(w) = n_\alpha \text{ for } \alpha \in \{0,1,2\}\}.$$

Without loss of generality, we set $p = 1$ and $0 \leq q \leq 1$. The interpretation is as follows. Particles of type 2 and 1 move preferentially clockwise and consequently, 0's move counterclockwise. Particles of type 2 can be interpreted as 'more aggressive'. This is easiest to see when $q = 0$, which we will call, in analogy with the single-species model, the *2-TASEP*. In that case, 2's always move clockwise, 0's always move counterclockwise, and 1's can move both ways. On consecutive sites 21 can exchange to 12, but not the other way round. An illustration of a configuration and some rates are given in Figure 1.

Let $M_{(n_0,n_1\,n_2)}$ denote the column-stochastic generator of the chain.

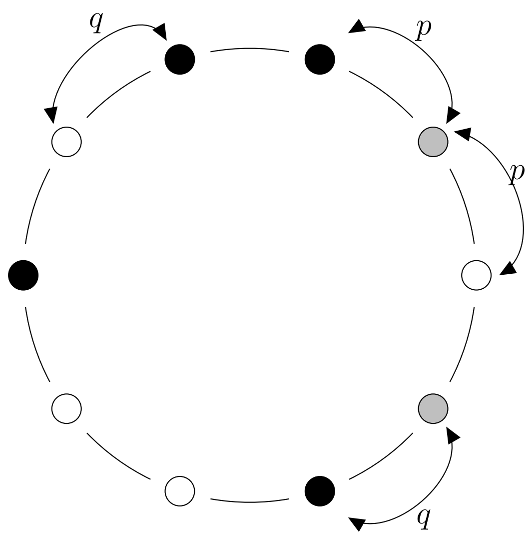

FIGURE 1. Example of a configuration in the two-species ASEP with 10 sites, 4 first-class particles (black) and 2 second-class particles (grey).

EXAMPLE 4.1. Let $n_0 = n_1 = n_2 = 1$ and order the elements of $S_{(1,1,1)}$ lexicographically, i.e.

$$S_{(1,1,1)} = \{012, 021, 102, 120, 201, 210\}.$$

Then the generator $M_{(1,1,1)}$ is given by

$$\begin{pmatrix} -2q-1 & 1 & 1 & 0 & 0 & q \\ q & -q-2 & 0 & q & 1 & 0 \\ q & 0 & -q-2 & 1 & q & 0 \\ 0 & 1 & q & -2q-1 & 0 & 1 \\ 0 & q & 1 & 0 & -2q-1 & 1 \\ 1 & 0 & 0 & q & q & -q-2 \end{pmatrix}.$$

We will now state elementary properties of the 2-ASEP. The proofs generalize those of Propositions 2.2 and 2.3 and are left as an exercise.

PROPOSITION 4.2 (Particle-hole symmetry). *The 2-ASEP on the state space $S_{(n_0,n_1,n_2)}$ with forward (resp. backward) hopping rate p (resp. q) is isomorphic as a Markov process to the 2-ASEP on $S_{(n_2,n_1,n_0)}$ with forward (resp. backward) hopping rate q (resp. p).*

PROPOSITION 4.3 (Irreducibility). *The 2-TASEP is irreducible, and as a consequence, so is the 2-ASEP.*

As a consequence of Proposition 4.3, it follows that the stationary probability distribution π is unique. We again denote the shift operator by τ on $S_{(n_0,n_1,n_2)}$. That is, $\tau(w_1,\dots,w_L) = (w_2,\dots,w_L,w_1)$. Then the following result generalises Proposition 2.4.

PROPOSITION 4.4 (Translation-invariance). *For all $w \in S_{(n_2,n_1,n_0)}$, we have that $\pi(\tau(w)) = \pi(w)$.*

EXAMPLE 4.5. For the 2-ASEP with $n_0 = n_1 = n_2 = 1$, the stationary probability distribution is given by the right eigenvector of $M_{(1,1,1)}$ with eigenvalue 0.

$$\pi(012) = \pi(120) = \pi(201) = \frac{q+2}{9(q+1)},$$
$$\pi(021) = \pi(210) = \pi(102) = \frac{2q+1}{9(q+1)}.$$

REMARK 4.6 (Irreversibility). The 2-ASEP is not a reversible process for $q < 1$.

4.2. Stationary distribution

The stationary distribution of the 2-ASEP can also be computed by the matrix ansatz, but the form is slightly different. We now suppose that the stationary probability of $w \in S_{(n_2,n_1,n_0)}$ can be written in the form

(4.1)
$$\pi(w) = \frac{f(w)}{Z_{(n_2,n_1,n_0)}}, \text{ with } f(w) = \mathrm{Tr}\left(\prod_{i=1}^{L}(D\delta_{w_i,2} + A\delta_{w_i,1} + E\delta_{w_i,2})\right),$$

where D, A, E are linear operators corresponding to $2, 1, 0$ respectively and $\delta_{i,j}$ is the Kronecker delta function, which equals 1 if $i = j$ and 0 otherwise. As before, $f(w)$ is called the *stationary weight* of w, and $Z_{(n_2,n_1,n_0)}$ is the normalisation constant.

THEOREM 4.7. *Suppose there exist (possibly infinite) matrices D, A, E satisfying the equations,*

$$\begin{aligned} DE - qED &= D + E, \\ DA - qAD &= A, \\ AE - qEA &= A, \end{aligned} \tag{4.2}$$

with $\mathrm{Tr}(A^{n_1}) \neq 0$*. Then the stationary distribution of the 2-ASEP on the configuration space* $S_{(n_2,n_1,n_0)}$ *is given by the matrix product representation* (4.1)*, where* $Z_{(n_2,n_1,n_0)}$ *is the partition function,*

$$Z_{(n_2,n_1,n_0)} = \sum_{w \in S_{(n_2,n_1,n_0)}} f(w).$$

Note that the first equation in (4.2) is the same as that of the single species open ASEP.

Proof. The strategy of proof is again similar to that of 3.4 and we will be sketchy. Write $w \in S_{(n_2,n_1,n_0)}$ in block form as $w = b_1 b_2 \cdots b_k$, where each b_i is a block of 0's, 1's or 2's and neighbouring blocks consist of distinct elements. Consider the master equation (1.1) for w. There are outgoing transitions with rate 1 whenever adjacent blocks are of the form $10, 20$ or 21, and with rate q whenever they are of the form $01, 02$ or 12. Focus on one such pair and say it is of the form 10. Then there is an outgoing transition with rate 1, and there is an incoming transition from w' to w with rate q, where w' is the same as w except that that particular pair 10 is replaced by 01. Thus, summing the total contribution for that block we obtain from (4.1) and (4.2),

$$\mathrm{Tr}(\cdots (AE - qEA) \cdots) = \mathrm{Tr}(\cdots A \cdots)$$

Therefore, on the right hand side, we obtain $f(w^-)$, where w^- is the word with one less 0 than w in that particular block. The block which comes after this block of 0's is either that of 1's or 2's. In either case, repeating the above calculation, we'll obtain $-f(w^-)$ plus another term in the latter scenario. Continuing this way and noting the cyclicity, one sees that all terms cancel and the net contribution is zero. This proves that the master equation is satisfied. $\square$

To show that the algebra in (4.2) has a solution, we will explicitly construct one. First, note that if we set $A = [D, E] = DE - ED$, then the second and third equations in (4.2) are satisfied. Therefore, we just need to find a representation for D, E satisfying the first equation. One solution is given by the infinite matrices,

$$D = \begin{pmatrix} 1-q & a_1 & 0 & 0 & \cdots \\ 0 & 1-q^2 & a_2 & 0 & \\ 0 & 0 & 1-q^3 & a_3 & \ddots \\ 0 & 0 & 0 & 1-q^4 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \text{and} \quad E = D^t, \tag{4.3}$$

where $a_n = \sqrt{(1-q^n)(1-q^{n+1})}$. One can show that $\mathrm{Tr}(A) = 1$, and in general that $\mathrm{Tr}(A^n) \neq 0$ for all $n \in \mathbb{N}$.

For the 2-TASEP, there is a further specialisation. If we set $q = 0$ in (4.3), we find that D, E satisfy the same algebra as the first equation in (3.5). They can therefore be chosen to be the same as given in (3.6). Then we find that

$$A = [D, E] = \begin{pmatrix} 1 & 0 & \dots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}, \tag{4.4}$$

which is a one-dimensional projection. Moreover, this projection can be written in bra-ket notation as $|V\rangle_1 \langle W|_1$, where $\langle W|_\alpha, |V\rangle_\beta$ are written as $\langle W|, |V\rangle$ respectively in (3.5). An explicit representation is given in (3.7). This observation has the following interesting consequence.

THEOREM 4.8 (Factorisation). *Conditioned on the presence of* 1*'s at positions j and L in the 2-TASEP on the state space $S_{(n_2,n_1,n_0)}$, the stationary weights factorise. In particular, when $n_1 = 2$, the stationary distribution in sites $1, \dots, j-1$ is independent of that in sites $j+1, \dots, L-1$.*

PROOF. The stationary weight is given by

$$\begin{aligned} & f(w_1, \dots, w_{j-1}, 1, w_{j+1}, \dots, w_{L-1}, 1) = \\ & \langle W|_1 \left(\prod_{i=1}^{j-1} (D\delta_{w_i,2} + A\delta_{w_i,1} + E\delta_{w_i,2}) \right) |V\rangle_1 \\ & \times \langle W|_1 \left(\prod_{i=j+1}^{L-1} (D\delta_{w_i,2} + A\delta_{w_i,1} + E\delta_{w_i,2}) \right) |V\rangle_1, \end{aligned}$$

from which the result immediately follows. □

Note that this factorisation property does not extend to the 2-ASEP. The 2-TASEP can be used to study the TASEP from the point of view of a single second-class particle.

Exercises

(1) Prove Proposition 4.3.

(2) Let $F = xD + zE$, where x, z are indeterminates. Show that

$$\langle W|_1 F^n |V\rangle_1 \sim \frac{(\sqrt{x} + \sqrt{z})^{2n+3}}{2\sqrt{\pi} n^{3/2} (x\,z)^{3/4}}.$$

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