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Macroeconomics II
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Problem Set 4: Please turn in your solutions in pencil.

1. In the simplified model following Romer(1986) and Arrow (1962) presented in class, growth exhibited a scale effect in that an expansion of the aggregate labor force, L, raised the per-capital growth rate for the decentralized economy and for the social planner as well.

To correct for this, assume that a firm's productivity, A_i, depends on the economy's average capital per worker, $\frac{K}{L}$, rather than the aggregate capital stock, K. The production function is assumed to be Cobb-Douglas:

$$Y_i = AK_i^{\alpha}[(\frac{K}{L})L_i]^{1-\alpha}$$

Following the CBC and utility function given in class, derive the growth rates for the decentralized economy and the social planner. Comment, in words, on how the scale effect discussed in class does not appear with this new specification.

2. Consider the model by Devarajan, Xie, and Zou, JME, Vol 41(1998), pages 319-331, entitled *Should public capital be subsidized or provided ?*. By solving (see page 325)

$$Maxmize \sum_{t=0}^{\infty} \rho^t ln(c_t)$$

subject to

$$k_{t+1}^1 = A(k_t^1)^{\alpha}(k_t^2)^{\beta}(\hat{k_t^2})^{1-\alpha-\beta}(1-\tau) - c_t - z_t$$

and

$$k_{t+1}^2 = z_t(1+s)$$

find the optimal k_{t+1}^1, k_{t+1}^2 and c_t under the case where the investment subsidy is financed by an output tax. Do this by setting up the Lagrangean, taking the first order conditions, and using the method of undetermined coefficients to derive closed form solutions for the choice variables. Derive W(s), the welfare of the representative agent as a function of the subsidy, and show that there is a unique subsidy rate that maximizes the welfare of the representative agent. Finally, in words, briefly explain the punch line of the paper.

3. This question examines savings in the neo-classical growth model. Consider an infinitely lived dynasty whose size increases over time at a constant rate, n. The objective function is now of the form

$$\int_0^\infty \frac{C^{1-\sigma}}{1-\sigma} L_t e^{-\rho t} dt$$

where $L_t = L_0 e^{nt}$ is the size of the dynasty, and C is per-capita consumption. The household maximizes the above felicity function subject to the constraint

$$\dot{K} = K^\alpha (AL)^{1-\alpha} - LC - \delta K$$

where δ is the rate of depreciation, and A grows at a constant rate, g. We will assume that the following boundedness condition holds,

$$g\sigma + \rho > n + g.$$

Following the same procedure as before, show that the necessary conditions for household optimization yield the following systems of equations

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\{\alpha Z^{\alpha-1} - (\rho + \delta)\} - g$$

and

$$\frac{\dot{Z}}{Z} = Z^{\alpha-1} - \frac{c}{Z} - (n + g + \delta)$$

where $c = \frac{C}{A}$ and $Z = \frac{K}{AL}$. Now, we will analyze the behavior of the savings rate in the model. The first step is to re-write the model in terms of the consumption ratio and the interest factor.

- Define the variables $X = \frac{c}{Z^\alpha}$ and $R = Z^{\alpha-1}$ and re-write the above system solely in terms of X and R. Solve for the steady state values of X and Z.
- Construct the log-linearization of the system obtained in the above bullet. Compute the eigenvalues of the coefficient matrix, and show that the steady state is a saddle point. Compute the

eigenvector associated with the negative eigenvalue (the stable root), and relate the slope of the saddle path to the size of the negative eigenvalue. Does anything look familiar ?

4. As in Barro (1990), a representative agent with the usual preferences

$$\int_0^\infty \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} dt$$

is endowed with an initial amount of capital, k_0 and with a production technology of the form

$$y = k^{1-\alpha} p^\alpha$$

where $\alpha \in (0,1)$, and p are government provided public services. Income is taxed at a constant rate τ. Assuming that there is no depreciation, the agent's flow budget constraint can be written as

$$\dot{k} = (1-\tau) k^{1-\alpha} p^\alpha - c.$$

- Taking the time path of p as given, write the necessary equations for a solution to the consumer's problem. Derive an equation describing the evolution of consumption over time.
- Assume that $p = \tau y$, that is, all tax revenue is used to finance public services. Substituting the production function in the last expression, solve for p as a function of τ and k. Substitute the result into the flow budget constraint and the transition equation for consumption. Call γ the growth rate of consumption, $\frac{\dot{c}}{c}$, obtained from this step, and let β be the coefficient of k in the law of motion for k (note that both γ and β are functions of τ and other parameters). Notice that β can be written as a simple function of γ.
- Observe that consumption grows at a constant exponential rate. Hence, once we determine its initial level, we have characterized its entire time path. Integrating the flow budget constraint and imposing the transversality condition, we obtain

  $$k_0 = \int_0^\infty c_t e^{-\beta t} dt$$

  Use this expression to solve for c_0.