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# Limit Properties of Bertrand Equilibria with Exogenous Entry

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*Abstract*: For a large class of demand and cost functions, we characterize the limit equilibrium set under Bertrand oligopoly when entry is exogenous. Unless average cost is constant, we find that the folk theorem of perfect competition necessarily fails. We also relate our results to those in Novshek and Roy Chowdhury (2003).

*Key words*: Bertrand oligopoly, limit properties, exogenous entry, folk theorem.

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# 1 Introduction

This paper examines the limit properties of Bertrand price competition when firms supply all demand[1] and the limiting procedure involves taking the number of active firms, exogenously given, to infinity.[2]

The objective is two-fold. First, we want to characterize the limit equilibrium set for a general class of demand and cost functions. Second, we want to examine if, for the Bertrand framework with exogenous entry, the folk theorem of perfect competition holds, in the sense that the set of limit equilibrium prices contains the perfectly competitive price(s), and no other price(s).[3]

This problem has been examined earlier by Novshek and Roy Chowdhury (2003), though only for the case when the demand function is negatively sloped and the average cost function is either 'U-shaped',[4] or increasing.[5]

[1]The assumption that firms supply all demand is appropriate when the costs of turning away customers are very high (see Dixon (1990), or Vives (1999)). Such costs may arise because of either reputational reasons, or governmental regulations. Vives (1999) argues that such regulations are operative in U.S. industries like electricity and telephone. This assumption can, in fact, be traced back to Chamberlin (1933). It has also been adopted, among others, by authors like Bulow, Geanakoplos and Klemperer (1985), Dastidar (1995), Novshek and Roy Chowdhury (2003), and Vives (1990, 1999).

[2]Alternatively, one can examine the limiting outcome under 'free entry' Bertrand equilibrium as firms become small compared to the market. This notion was first introduced, for the Cournot case, in Novshek (1980). The limit-equilibrium set under free-entry Bertrand competition was characterized by Novshek and Roy Chowdhury (2003).

[3]While the folk theorem is relatively well explored in the Cournot framework, (see, among others, Novshek (1980), Okuguchi (1973) and Ruffin (1971)), it is much less so in the Bertrand framework.

[4]Novshek and Roy Chowdhury (2003) define an average cost function to be 'U-shaped' if there exists $q^* > 0$ such that the average cost function is strictly decreasing for all $0 < q < q^*$, and strictly increasing for all $q > q^*$.

[5]Novshek and Roy Chowdhury (2003), of course, also examine the case when entry is 'free', rather than 'exogenous'. They also characterize the limit equilibrium set when average costs are constant, or decreasing, or have a capacity constraint.

They characterize the limit-equilibrium set for both classes of average cost functions. Surprisingly, the folk theorem of perfect competition fails, in the sense that the set of limit equilibrium prices either do not contain the perfectly competitive price, or contains other prices as well.

The assumptions on the demand and the cost functions imposed by Novshek and Roy Chowdhury (2003) are certainly quite reasonable. Given the importance of their results, however, it is of interest to re-examine the problem under a minimal set of restrictions on the demand and the cost functions. Hence in this paper we essentially only assume that the demand function is continuous and intersects both the axes, and that the cost function is continuous.[6] In particular, we do not assume that the demand function is negatively sloped, or that the average cost function is either increasing, or 'U-shaped'.

We characterize the limit-equilibrium set and show that, under a relatively mild set of assumptions, our characterization coincides with that by Novshek and Roy Chowdhury (2003). We also show that unless average cost is constant, the folk theorem of perfect competition necessarily fails. Thus this paper generalizes the Novshek and Roy Chowdhury (2003) results for the exogenous entry case to a significant extent.

## 2 The Model

The market $M(n)$ comprises the demand function $f(p)$ and $n$ firms, all producing a single homogeneous good, and having the same cost function, $c(q)$.[7]

The market demand function $f(p)$ satisfies the following assumption.

[6]Except possibly at the origin.

[7]For ease of comparison, the notations in this paper closely follow those in Novshek and Roy Chowdhury (2003).

**Assumption 1**: (a) $f : [0, \infty) \to [0, \infty)$.[8] Moreover, $f(p)$ is continuous.

(b) There exists a strictly positive $\hat{p}$ such that $f(p) = 0$, $\forall p \geq \hat{p}$ and $f(p) > 0$, $\forall p < \hat{p}$.

Note that we do not assume that the demand function is necessarily negatively sloped.

Let $AC(q)$ denote the common average cost function of all firms.

**Assumption 2**: (a) $c : [0, \infty) \to [0, \infty)$. Moreover, $c(0) = 0$ and $c(q) > 0$, $\forall q > 0$.

(b) The cost function is continuous, except possibly at the origin.[9]

(c) $AC : (0, \infty) \to (0, \infty)$. Moreover, there exists $p$ such that $p > AC(f(p))$.

Note that we do not assume that the average cost function is necessarily either increasing, or 'U-shaped'.[10]

We examine a game of Bertrand competition where the firms simultaneously announce their prices. Moreover, the firms supply all demand.

If the announced price vector is $(p_1, p_2, \cdots, p_n)$, then the demand facing firm $i$ is

$$D_i(p_1, \cdots, p_i, \cdots, p_n) = \begin{cases} 0, & \text{if } p_i > p_j, \text{ for some } j, \\ \frac{f(p_i)}{m}, & \text{if } p_i \leq p_j, \ \forall j, \text{ and } \#(l : p_l = p_i) = m. \end{cases}$$

Thus the lowest priced firms share the market equally, while firms charging higher prices have zero demand.

The profit of the $i$-th firm

[8]Note that this implies that $f(0)$ is finite.

[9]Note that $AC(q)$ is well defined and continuous on $(0, \infty)$.

[10]The second part of Assumption 2(c) implies that the optimal monopoly profit is strictly positive. It is equivalent to the Novshek and Roy Chowdhury (2003) assumption that $f(p)$ and $AC(q)$ intersect at least once in the $p - q$ plane.

$$\pi_i(p_1, \cdots, p_n) = \begin{cases} 0, \text{ if } p_i > p_j, \text{ for some } j, \\ (p_i - AC(D_i(p_1, \cdots, p_n)))D_i(p_1, \cdots, p_n), \text{if } p_i \leq p_j, \forall j. \end{cases}$$

We solve for the pure strategy Nash equilibrium in prices, i.e. Bertrand equilibrium.

**Definition.** A *Bertrand equilibrium* for the market $M(n)$ consists of a price vector $(p_1, \cdots, p_i, \cdots, p_n)$ such that, $\forall i$ and $\forall p_i'$,

$$\pi_i(p_1, \cdots, p_i, \cdots, p_n) \geq \pi_i(p_1, \cdots, p_i', \cdots, p_n). \tag{1}$$

We then define the notion of a limit-equilibrium set. In the quantity competition framework, Ruffin (1971) and Okuguchi (1973), among others, examine the Cournot-Nash equilibrium taking market conditions, in particular the number of active firms, as exogenously given. They then study the limiting outcome as the number of active firms goes to infinity. We call this the exogenous entry approach.

Novshek and Roy Chowdhury (2003) adapt this notion to the Bertrand context. In a Bertrand framework firms are active when they charge the minimum price. Hence Novshek and Roy Chowdhury (2003) characterize the set of all prices $p$ such that if the number of firms $n$ is large enough, then, for the market $M(n)$, there is some equilibrium where all firms are active and the equilibrium price is arbitrarily close to $p$. This notion of a limit equilibrium set is adopted in the present paper as well.

**Definition:** $S = \{p$ : there is a sequence $p(n)$ that converges to $p$ such that, for each sufficiently large $n$, all firms setting a price $p(n)$ is an equilibrium for the market $M(n)\}$.

We need some more notations before we can characterize $S$.

$b = \lim_{q \to 0} AC(q)$.[11]

[11] From Assumption 2(c), $b$ is well defined (allowing for infinity as a possible limit).

$c^* = \inf_q \; AC(q)$.[12]
$\tilde{p} = \; \mathrm{argmax}_{p \in [0, \hat{p}]} \; f(p)$.[13]
$\tilde{d} = \inf \; \{p : \; p > AC(f(p))\}$.[14]
$d$ is the minimum $p$ such that $AC(f(p)) = p$.[15]

We then impose the following regularity condition.

**Assumption 3.** If $b = \tilde{d}$, then the cost function is either linear, or there exists $t > 0$ such that $AC(q)$ is negatively sloped for all $q \in (0, t)$.

Note that generically $b \neq \tilde{d}$.[16] Thus Assumption 3 is not very strong. Recall that in the Novshek and Roy Chowdhury (2003) framework, $b = \tilde{d}$ implies that the average cost function is 'U-shaped', so that Assumption 3 is necessarily satisfied. In Remark 2, Novshek and Roy Chowdhury (2003) also consider the case where average cost is constant (so that $b = \tilde{d}$).

Proposition 1 below characterizes the set $S$.

**Proposition 1.** *Let Assumptions 1, 2 and 3 hold. Then* $S = [b, \tilde{d}]$ *if* $b \leq \tilde{d}$*, it is empty otherwise.*

*Proof:* To begin with we argue that no price less than $b$ or greater than $\tilde{d}$ can belong in the limit set $S$.

Suppose that $p(n)$ converges to $p$ as $n$ increases and for each sufficiently large $n$, all $n$ firms setting a price $p(n)$ is an equilibrium for $M(n)$. Note that the output per active firm is at most $\frac{f(\tilde{p})}{n}$, which converges to zero

[12]Given Assumption 2(c), $c^*$ is finite.

[13]Given that $f(p)$ is continuous, $\tilde{p}$ is well defined.

[14]Since $p = 0$ is a lower bound, there is a least upper bound. Given Assumption 2(c), the set $\{p : p > AC(f(p))\}$ is non-empty. Hence $\tilde{d}$ is finite.

[15]Given Assumption 2(c), $d$ is well defined.

[16]This is in the following sense. Take any pair of $f(p)$ and $AC(q)$ such that $b = \tilde{d}$. Now if either one of the functions is perturbed slightly (in an appropriate manner), then it will no longer be the case that $b = \tilde{d}$.

as $n$ goes to infinity.[17] Thus if $p < b$, then for all sufficiently large $n$, $p(n) < AC(\frac{f(p(n))}{n})$, so that $p(n)$ cannot be an equilibrium price.

Next let $p > \tilde{d}$. For a sufficiently large $n$, profit per active firm, $[p(n) - AC(\frac{f(p(n))}{n})]\frac{f(p(n))}{n}$, is less than $(\hat{p} - c^*)\frac{f(\tilde{p})}{n}$. Thus, for $n$ large, profit per active firm converges to zero. Moreover, from the definition of $\tilde{d}$, there exists $p'$ such that $\tilde{d} < p' < p(n)$ and $p' > AC(f(p'))$. Undercutting to such a price $p'$ yields a strictly positive profit that depends on $p'$, but not on $n$. Thus, for $n$ large, undercutting is strictly profitable.

We then argue that every price in the interval $[b, \tilde{d}]$ is in the limit set. If $p > b$, then, for any sufficiently large $n$, if $n$ firms set such a price then each firm will produce an output at which $p$ exceeds average cost, and thus obtain a positive profit. Undercutting is unprofitable since for any $p$ strictly less than $\tilde{d}$, an undercutting firm cannot make a positive profit as $p \leq AC(f(P))$.

The remaining case is $p = b$. If $b < \tilde{d}$, then this $p$ can be obtained as the limit of an appropriate sequence of equilibrium prices, $p(n)$, described above.

Finally, let $b = \tilde{d} < \hat{p}$.[18] If average cost is constant, then $p = b$ can be sustained as a Bertrand equilibrium for all $n$. Hence, given Assumption 3, we assume that there exists $t > 0$ such that $AC(q)$ is negatively sloped for all $q \in (0, t)$.

Consider some $p \in (AC(t), b)$. Let $q(p)$ be the unique $q$, $0 < q < t$, such that $AC(q(p)) = p$. Next, let $n(p)$ satisfy $\frac{f(p)}{n(p)} = q(p)$, where $n(p)$ can be a non-integer. Given that $f(b) > 0$ and $\lim_{p \uparrow b} q(p) = 0$, it follows that $\lim_{p \uparrow b} n(p) = \infty$. Next, let $\tilde{n}(p)$ be the largest possible integer such that $p \geq AC(\frac{f(p)}{n})$ (this is well defined for $n(p)$ large enough). Clearly, there exists some largest interval $(b', b)$, $AC(t) \leq b' < b$, such that $\tilde{n}(p)$ is well

[17] Note that the assumption that the demand function intersects both axes ensures that $\tilde{p}$ is well defined, and thus $f(\tilde{p})$ is bounded.

[18] Since $f(p)$ is negatively sloped at $\hat{p}$, it cannot be the case that $b = \tilde{d} > \hat{p}$. If $b = \tilde{d} = \hat{p}$, then all firms charging $\hat{p}$ and having zero demand and supply is an equilibrium for $M(n)$.

defined for all $p \in (b', b)$. Given that $|n(p) - \tilde{n}(p)| < 1$ and $\lim_{p\uparrow b} n(p) = \infty$, we have that $\lim_{p\uparrow b} \tilde{n}(p) = \infty$. Let $\hat{n} = \min_{p\in(b',b)} \tilde{n}(p)$.

We then construct a sequence $< p(n) >$ such that $\forall i \in \{0, 1, 2, \ldots\}$, $p(\hat{n} + i)$ is some $p \in (b', b)$ such that $\hat{n} + i = \tilde{n}(p)$. Note that for $n \geq \hat{n}$, the pair $(n, p(n))$ belongs to the graph of $\tilde{n}(p)$. Thus $p(n) \geq AC(\frac{f(p(n))}{n})$, so that all firms earn non-negative profits. Moreover, since $p(n) < b = \tilde{d}$, no firm can undercut profitably. Finally, we argue that the sequence $< p(n) >$ converges to $b$. Suppose not. Then there exists some $\epsilon > 0$ and some sub-sequence $< p(n_i) >$ such that $p(n_i) \leq b - \epsilon$, $\forall n_i$. Note that $\lim_{n_i\to\infty} \frac{f(p(n_i))}{n_i} \leq \lim_{n_i\to\infty} \frac{f(\tilde{p})}{n_i} = 0$. Hence, for $n_i$ large enough, $p(n_i) < AC(\frac{f(n_i)}{n_i})$. This, however, is a contradiction since for all $n_i$, $(n_i, p(n_i))$ belongs to the graph of $\tilde{n}(p)$. ■

We then relate the above characterization to the corresponding one in Novshek and Roy Chowdhury (2003) (i.e. Theorem 1). For the case when the demand function is negatively sloped, and the average cost function is either increasing, or 'U-shaped', they find that if $b \leq d$, then $S = [b, d]$. $S$ is empty otherwise.

Under the Novshek and Roy Chowdhury (2003) framework it is easy to see that if $b \leq \tilde{d}$, then $\tilde{d} = d$,[19] so that the two characterizations coincide. We then argue that there is a large class of demand and cost functions for which the above result goes through.

We begin by introducing the following definition.

**Definition.** $f(p)$ is said to be *tangent* to $AC(q)$ at some $\overline{p}$, if $\overline{p} = AC(f(\overline{p}))$ and there is some $\epsilon > 0$ such that for all $p \in (\overline{p} - \epsilon, \overline{p}) \cup (\overline{p}, \overline{p} + \epsilon)$, either $\overline{p} \geq AC(f(\overline{p}))$, or $\overline{p} \leq AC(f(\overline{p}))$.

[19] If $b \leq \tilde{d}$, then, under the Novshek and Roy Chowdhury (2003) formulation, the average cost function must be positively sloped at $\tilde{d}$. Thus there does not exist any $p' < \tilde{d}$ such that $p' = AC(f(p'))$. Of course if $b > \tilde{d}$, then $S$ is empty.

We need one final assumption.

**Assumption 4.** At any $p < \tilde{d}$ such that $p = AC(f(p))$, the demand and the average cost functions cannot be tangent to each other.

Clearly, Assumption 4 is generically true,[20] and is not a very strong assumption.

We are now in a position to prove Proposition 2.

**Proposition 2.** *Let Assumptions 1, 2, 3 and 4 hold. Then $S = [b, d]$ if $b \leq d$, $S$ is empty otherwise.*

*Proof.* Given Proposition 1, it is sufficient to show that, under Assumption 4, $\tilde{d} = d$.

Clearly, $\tilde{d} = AC(f(\tilde{d}))$. Since $d$ is the minimum $p$ such that $p = AC(f(p))$, $\tilde{d} \geq d$. Next suppose that $\tilde{d} > d$. By definition, $d = AC(f(d))$. Moreover, from the definition of $\tilde{d}$, $p \leq AC(f(p))$ for all $p \in [0, \tilde{d})$. Hence $f(p)$ and $AC(q)$ are tangent to each other at $d$. This, however, violates Assumption 4. ∎

We finally examine whether, in this framework, the folk theorem of perfect competition holds or not. Clearly, the folk theorem holds if and only if $b = \tilde{d} = c^*$. Given Assumption 3, $b = \tilde{d} = c^*$ if and only if average cost is constant. For all other classes of cost functions the folk theorem fails.

## 3 Conclusion

We examine the limit-properties of Bertrand price competition when entry is exogenous. Our results substantially generalize those in Novshek and Roy Chowdhury (2003) since we allow for demand functions that are not necessarily decreasing, and for average cost functions that are not necessarily

[20] In the sense of footnote 16.

either increasing, or 'U-shaped'. We also demonstrate that, under a set of relatively mild conditions, the characterization developed in this paper coincides with that in Novshek and Roy Chowdhury (2003). Finally, we show that the folk theorem of perfect competition fails for all classes of cost functions, except for the case when average cost is constant.

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